Non-normal modal logics: bi-neighbourhood semantics and its labelled calculi

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Abstract

The classical cube of non-normal modal logics is considered, and an alternative neighbourhood semantics is given in which worlds are equipped with sets of pairs of neighbourhoods. The intuition is that the two neighbourhoods of a pair provide independent positive and negative evidence (or support) for a formula. This bi-neighbourhood semantics is significant in particular for logics without the monotonicity property. It is shown that this semantics characterises the cube of non-normal modal logics and that there is a mutual correspondence between models in the standard and in the bi-neighbourhood semantics. On the basis of this alternative semantics, labelled sequent calculi are developed for all the logics of the classical cube. The calculi thus obtained are fully modular and have good structural properties, first of all, syntactic cut elimination. Moreover, they provide a decision procedure and an easy countermodel extraction, both in the bi-neighbourhood and in the standard semantics.

Keywords: Non-normal modal logics, (bi)-neighbourhood semantics, labelled sequent calculi.

1 Introduction

Non-normal modal logics are called in this way because they do not satisfy all the axioms and rules of the minimal normal modal logic K. They have been studied since the seminal work of C.I. Lewis, Scott, Lemmon, and Chellas (for an introduction see [1] and [5]) and can be seen as generalisations of standard modal logics. Non-normal modal logics have found an interest in several areas such as epistemic and deontic reasoning, reasoning about games, and reasoning about probabilistic notions such as 'truth in most of the cases'. In all these contexts the $\Box$ modality is better understood as non-normal. For instance, an epistemic interpretation of $\Box A$ as 'the agent knows A' for a non-omniscient agent would reject the rule of monotonicity (RM), that $A \to B$ implies $\Box A \to \Box B$, and possibly the rule of necessitation, the latter meaning in this case that the agent would know every logical validity. In deontic logic, where $\Box A$ is

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interpreted as 'A is obligatory', some paradoxes like the 'gentle murder' can be avoided if $\Box$ is non-normal. Furthermore, if we interpret $\Box A$ as 'A is true in most of the cases' or 'A is highly probable', the modality $\Box$ will not be likely to satisfy axiom (C) $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$. Validity of this axiom would also fail in a game-theoretical interpretation, where $\Box A$ is interpreted as the agent's availability of a winning strategy to bring about A.

Non-normal modal logics have been studied essentially from a semantical point of view. The standard semantics (Chellas [1]) for these systems is defined in terms of *neighbourhood* models: these are possible world models, where each world w is equipped with a set of neighbourhoods $\mathcal{N}(w)$, each one of them being a set of worlds/states. The loose intuition is that each neighbourhood provides sufficient or relevant evidence to establish the truth of a formula of type $\Box A$. A formula $\Box A$ is forced by a world w if the truth-set of A belongs to $\mathcal{N}(w)$. By imposing further closure conditions on $\mathcal{N}(w)$, various non-normal modal logics can be obtained. The classical cube of non-normal modal logics is determined by considering any combination of the following three conditions: for any world w, (M) $\mathcal{N}(w)$ is closed under supersets, (C) it is closed under intersection, (N) it contains the whole set of possible worlds.

The study of proof systems for non-normal modal logics does not have a state of the art comparable with the one of proof systems for normal modal logics, for which there exist well-understood proof methods of many kinds.

There are several desiderata on proof systems: [2] they should be *standard*, that is, they should contain only a finite number of rules, each with a fixed number of premisses; logical operators should be dealt with dual rules (for the antecedent and the succedent) that introduce a *single* occurrence of a formula; the rules should be *analytic* and allow for a *syntactic* proof of cut elimination; the calculi should be *modular*, with stronger systems obtained simply by adding rules to a basic system; finally, they should provide a *decision procedure* (possibly of optimal complexity) whenever the logic is decidable, and from a failed proof it should be possible to extract directly a *countermodel* of the formula the validity of which is being checked.

Cut-free sequent calculi for non-normal modal logics have been studied by Lavendhomme and Lucas [9]; in their calculi, however, rules allow several formulas as principal and modularity does not obtain; further, a decision procedure is given but it is rather complicated in the non-monotonic case. Indrzejczak [6] has further developed the calculi by Lavendhomme and Lucas [9] extending them with standard axioms of normal modal logics (the non-normal counterpart of logics from **K** to **S5**). Gilbert and Maffezioli [3] investigate labelled calculi using three modalities, on the basis of the translation of non-normal modal logics into normal modal logics given by Gasquet and Herzig [2] and Kracht and Wolter [8]. As a bi-product of the general methodology employed, their calculi are also fully modular (i.e., modular with no exceptions)

[2] For general desiderata on proof systems see [7,15]; for modularity see [11], and for the extraction of countermodels from failed proof search see [12].

but computational issues are not discussed. Recently, Lellmann and Pimentel [10] have proposed linear *nested* sequent calculi for non-normal modal logics; their calculi are fully modular and allow for syntactic cut elimination, but it is not obvious how to get countermodels and a decision procedure out of them.

In this work, we propose labelled sequent calculi for the basic non-normal modal logics. Our calculi are based on *bi-neighbourhood* models, an alternative semantics more general than the standard one. Differently from the standard semantics, worlds in a bi-neighbourhood model are equipped with sets of *pairs* of neighbourhoods rather than single neighbourhoods. The intuition is that the two components of a pair provide independent positive and negative evidence (or support) for a proposition. Standard models correspond exactly to bi-neighbourhood models in which the two neighbourhoods of a pair are complement of each other. The bi-neighbourhood semantics is significant mostly for logics without the monotonicity property, as it collapses into the standard one in the monotonic case. We show *directly* that this semantics characterises non-normal modal logics, being sound and complete with respect to them. Moreover, each bi-neighbourhood model gives rise (in an effective way) to a standard model, providing thereby a mutual correspondence between models of the two kinds.

The new semantics is the starting point for developing labelled sequent calculi for non-normal modal logics. Our aim is to define calculi that satisfy all the above desiderata. The calculi presented in this work are standard (in the sense specified above) and are based on the same approach of Negri [13] of importing the semantics into the syntax by making use of labels; however, they differ significantly from those for non-monotonic systems. The main difference is that the calculi presented here make use of *pseudo-complement* neighbourhoods (corresponding to pairs in the bi-neighbourhood semantics) instead of the *covering* relation to express the inclusion of the truth-set of a formula in a neighbourhood. The new semantic element has the effect that the calculi presented here do not introduce relational formulas in the consequent of a sequent and thus avoid exponential branching in proof search. Departing from the standard neighbourhood semantics gives, as a further bonus, calculi that cover in a modular way the whole cube of non-normal modal logics.

We shall first present a version of the calculi with good proof-theoretical properties, the most important being syntactic cut elimination, from which syntactic completeness of the calculi follows. We then present a second version of the calculi with optimised rules for *closure under intersection.* We show that proof search in these calculi is always terminating, just by adopting a very simple strategy (with no additional mechanism needed). We then prove semantic completeness with respect to bi-neighbourhood models, whence also with respect to the standard semantics by virtue of the correspondence mentioned above. This means that from a failed proof search it is possible to extract directly a countermodel both in the bi-neighbourhood semantics and in the standard one. Since the models obtained in this way are finite, the semantic completeness proof provides in itself also a constructive proof of the finite model property. We finally give a syntactic proof of the fact that bi-neighbourhood semantics coincides with the standard semantics: if we force the two neighbourhoods of each pairs to be complements of each other, we do not get more provable formulas.

2 Non-normal modal logics and bi-neighbourhood semantics

In this section, we present the modal logic **E** and its extensions. We also present its standard semantics in terms of neighbourhood models and a more general semantics in terms of bi-neighbourhood models. We show that bi-neighbourhood semantics characterises logic **E** and its extensions and is equivalent to the standard neighbourhood semantics.

Let $\mathcal{L}$ be a propositional modal language based on countably many propositional variables, the Boolean connectives, and $\Box$. We use A, B, C and p, q as metavariables for arbitrary formulas and atoms of $\mathcal{L}$. $\Diamond A$ is an abbreviation for $\neg\Box\neg A$. Logic **E** is obtained by adding to classical propositional logic the rule of inference

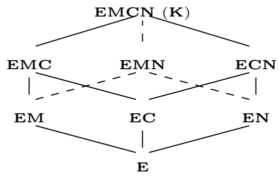
$$\mathsf{RE}\ \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$$

and can be extended further by choosing any combination of axioms M, C and N (below left), thus producing eight distinct logics. The resulting systems are denoted by $\mathbf{ES_1...S}_n$, where $\mathrm{S}_i \in \{\mathrm{M,C,N}\}$[3] (see the *classical cube* below on the right). We write $\mathbf{E}^*$ ($\mathbf{EM}^*$, $\mathbf{EC}^*$, $\mathbf{EN}^*$) to indicate any extension of **E** (**EM**, **EC**, **EN**) with some of these axioms and recall that the top extension coincides with **K**.

M $\Box(A \wedge B) \rightarrow \Box A \wedge \Box B$

C $\Box A \wedge \Box B \rightarrow \Box(A \wedge B)$

N $\Box\top$



Definition 2.1 A *standard neighbourhood model* (just *standard model* in the following) is a triple $\mathcal{F} = \langle W, \mathcal{N}, V\rangle$, where W is a non-empty set, $\mathcal{N}$ is a function $W \longrightarrow \mathcal{PP}(W)$ and V is a valuation function for propositional variables of $\mathcal{L}$. A model is said to be *supplemented* if for all $\alpha, \beta \subseteq W$, $\alpha \in \mathcal{N}(w)$ and $\alpha \subseteq \beta$ implies $\beta \in \mathcal{N}(w)$; it is *closed under intersection* if $\alpha \in \mathcal{N}(w)$ and $\beta \in \mathcal{N}(w)$ implies $\alpha \cap \beta \in \mathcal{N}(w)$; and it *contains the unit* if for all $w \in W, W \in \mathcal{N}(w)$. The forcing relation $\mathcal{M}, w \models_{st} A$ is defined in the usual way for atomic formulas and Boolean connectives. For the modality we have $\mathcal{M}, w \models_{st} \Box A$ iff $[A]_{\mathcal{M}} \in \mathcal{N}(w)$, where $[A]_{\mathcal{M}}$ denotes the set $\{v \mid \mathcal{M}, v \models_{st} A\}$ of the worlds v that force A, also called the *truth set* of A.

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[3] In the literature, in the presence of axiom M the letter E is sometimes omitted from the name of the systems, that are instead denoted by $\mathbf{MS_1...S}_n$, where $\mathrm{S}_i \in \{\mathrm{C,N}\}$.

As a consequence of the above definition, we obtain the following truth condition for $\Diamond A$: $\mathcal{M}, w \models_{st} \Diamond A$ iff $[\neg A]_{\mathcal{M}} \notin \mathcal{N}(w)$.

Theorem 2.2 (Chellas [1]) *Logic* $\mathbf{E}(\mathbf{M}, \mathbf{C}, \mathbf{N})$ *is sound and complete with respect to standard models (which in addition are, respectively, supplemented, closed under intersection, or contain the unit).*

We now introduce a new semantics where pairs of neighbourhood are used to evaluate the truth of a modal formula.

Definition 2.3 A *bi-neighbourhood model* is a triple $\mathcal{M} = \langle W, \mathcal{N}, V \rangle$, where W is a non-empty set, V is a valuation function and $\mathcal{N}$ is a function that assigns to each world w a subset of $\mathcal{P}(W) \times \mathcal{P}(W)$ such that if $(\alpha, \beta) \in \mathcal{N}(w)$, then $\alpha \cap \beta = \emptyset$. Moreover, $\mathcal{M}$ is a N-model if for all $w \in W$, $(W, \emptyset) \in \mathcal{N}(w)$; it is a C-model if $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{N}(w)$ implies $(\alpha_1 \cap \alpha_2, \beta_1 \cup \beta_2) \in \mathcal{N}(w)$; and it is an M-model if for all $w \in W$, $(\alpha, \beta) \in \mathcal{N}(w)$ implies $\beta = \emptyset$.

The forcing relation $\mathcal{M}, w \models_{bi} A$ is defined as in Definition 2.1 except for the modality, for which the clause is as follows:

$\mathcal{M}, w \models_{bi} \Box A$ iff for some $(\alpha, \beta) \in \mathcal{N}(w)$ and all $v \in W$,
$v \in \alpha$ implies $\mathcal{M}, v \models_{bi} A$, and $v \in \beta$ implies $\mathcal{M}, v \not\models_{bi} A$.

Observe that in case the considered model does not satisfy condition M (*i.e.* in the non-monotonic case), if α and β are complementary, this definition becomes equivalent to the standard one. From Definition 2.3 we obtain the following truth condition for $\Diamond A$: $\mathcal{M}, w \models_{bi} \Diamond A$ iff for all $(\alpha, \beta) \in \mathcal{N}(w)$, there is $v \in \alpha$ such that $\mathcal{M}, v \models_{bi} A$, or there is $u \in \beta$ such that $\mathcal{M}, u \not\models_{bi} A$. Notice also that bi-neghbourhood models satisfying condition M collapse into standard models, where $\Box$ coincides with the modality $\langle\ ]$ considered by Pacuit [14].

Theorem 2.4 *Logic* $\mathbf{E}$ $(\mathbf{M}, \mathbf{C}, \mathbf{N})$ *is sound with respect to bi-neighbourhood (M,C,N-)models.*

Proof. It can be easily shown that each axiom is valid in the respective class of models and that all the rules preserve validity. □

Even if completeness of all logics $\mathbf{E}^*$ with respect to bi-neighbourhood models follows from Theorem 2.2 and the fact that standard models are particular cases of bi-neighbourhood models, it can be interesting to prove it directly by the canonical model construction. In the proof we do not consider the case of M as we saw it is standard. First of all, for any logic L based on the language $\mathcal{L}$ and for any set X of formulas of $\mathcal{L}$, we say that X is L-consistent if $X \not\vdash_{\mathsf{L}} \bot$, and that it is L-maximal consistent if it is L-consistent and for any formula $A \in \mathcal{L}$ such that $A \notin X$, $X \cup \{A\}$ is not L-consistent. We denote by Max_{L} the class of all L-maximal consistent sets of formulas of $\mathcal{L}$, and for any formula A we denote by $\uparrow A$ the set $\{Y \in Max_{\mathsf{L}} \mid A \in Y\}$. Before defining canonical models, we recall some basic properties of L-maximal consistent sets.

Lemma 2.5 (a) *Any* L*-consistent set of formulas* Γ *can be extended to an* L*-maximal consistent set.* (b) *If* $\Gamma \not\vdash_{\mathsf{L}} A$*, there is* X *in* Max_{L} *such that* $\Gamma \subseteq X$

and $X \notin \uparrow A$. *(c) If* $\nvdash_{\mathsf{L}} B \to A$, *there is* X *in* Max_{L} *such that* $X \in \uparrow B$ *and* $X \notin \uparrow A$.

Lemma 2.6 *Let* X *be an* L*-maximal consistent set. The usual properties of maximal consistent sets hold, in particular: (a) If* $\vdash_{\mathsf{L}} A$, *then* $A \in X$; *(b) if* $Y \vdash_{\mathsf{L}} A$ *and* $Y \subseteq X$, *then* $A \in X$; *(c) if* $\vdash_{\mathsf{L}} A \leftrightarrow B$ *and* $\Box A \in X$, *then* $\Box B \in X$; *(d)* $\uparrow(A \wedge B) = \uparrow A \cap \uparrow B$; *and (e)* $\uparrow(A \vee B) = \uparrow A \cup \uparrow B$.

Lemma 2.7 *Let the* canonical model $\mathcal{M}^c = \langle W^c, \mathcal{N}^c, V^c \rangle$ *for* L *be defined as follows:* $W^c = Max_{\mathsf{L}}$; *for any* $p \in \mathcal{L}$, $V^c(p) = \{X \in W^c \,|\, p \in X\}$; *for all* $X \in W$,

$$\mathcal{N}^c(X) = \{(\uparrow A, \uparrow \neg A) | \Box A \in X\}.$$

Then for any formula $B \in \mathcal{L}$ *we have* $\mathcal{M}^c, X \models B$ *iff* $B \in X$. *Moreover, (N) if* L *contains axiom* N, *then* $\mathcal{M}^c$ *is an N-model, and (C) if* L *contains axiom* C, *then* $\mathcal{M}^c$ *is a C-model.*

Proof. By induction on B. If B is p the claim holds by definition of V^c. If B is $\bot$, we have $\bot \notin X$ for every X, because X is consistent. If B is $C \circ D$, the proof is immediate by applying the inductive hypothesis and properties of maximal consistent sets. If B is $\Box C$: ($\Rightarrow$) Assume $\mathcal{M}^c, X \models \Box C$. Then for some $(\alpha, \beta) \in \mathcal{N}^c(X)$ and all $Y \in W^c$, $Y \in \alpha$ implies $\mathcal{M}^c, Y \models C$ and $Y \in \beta$ implies $\mathcal{M}^c, Y \not\models C$. By definition of $\mathcal{N}^c$, there is a formula D such that $\alpha = \uparrow D$, $\beta = \uparrow \neg D$ and $\Box D \in X$. Since by inductive hypothesis $[C]_{\mathcal{M}^c} = \uparrow C$, it holds that for all $Z \in W^c$, $Z \in \uparrow D$ implies $Z \in \uparrow C$ and $Z \in \uparrow C$ implies $Z \in \uparrow D$ (if $\mathcal{M}^c, Z \models C$, then $Z \notin \uparrow \neg D$, then $Z \in \uparrow D$); that is $\uparrow D = \uparrow C$. By the properties of maximal consistent sets, $\vdash_{\mathsf{L}} D \leftrightarrow C$. Since $\Box D \in X$, by Lemma 2.6 we have $\Box C \in X$. ($\Leftarrow$) Assume $\Box C \in X$. By definition, $(\uparrow C, \uparrow \neg C) \in \mathcal{N}^c(X)$. Since, by inductive hypothesis, $\uparrow C = [C]_{\mathcal{M}^c}$, we have that for all $Y \in W^c$, $Y \in \uparrow C$ implies $\mathcal{M}^c, Y \models C$ and $Y \in \uparrow \neg C$ implies $\mathcal{M}^c, Y \not\models C$ (because $Y \in \uparrow \neg C$ iff $Y \notin \uparrow C$). Thus $\mathcal{M}^c, X \models \Box C$.

(N) Since $\vdash_{\mathsf{L}} \Box \top$, for all $X \in W^c$ we have $\Box \top \in X$. Thus by definition, $(\uparrow \top, \uparrow \neg \top) \in \mathcal{N}^c(X)$, and by Lemma 2.6, $([\top]_{\mathcal{M}^c}, [\neg \top]_{\mathcal{M}^c}) = (W^c, \emptyset) \in \mathcal{N}^c(X)$.

(C) Assume $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{N}^c(X)$. Then, by definition, for some $C, D \in \mathcal{L}$, $\alpha_1 = \uparrow C$, $\beta_1 = \uparrow \neg C$, $\alpha_2 = \uparrow D$, $\beta_2 = \uparrow \neg D$ and $\Box C, \Box D \in X$. Thus by the properties of maximal consistent sets we have $\Box C \wedge \Box D \in X$ and, since X contains axiom C, also $\Box(C \wedge D) \in X$. Then $(\uparrow (C \wedge D), \uparrow \neg (C \wedge D)) \in \mathcal{N}^c(X)$, where $\uparrow(C \wedge D) = \alpha_1 \cap \alpha_2$ and $\uparrow \neg (C \wedge D) = \beta_1 \cup \beta_2$. $\Box$

Theorem 2.8 (Completeness of $\mathbf{E}^*$) *A formula* A *is a theorem of* $\mathbf{E}^*$ *if and only if is valid in the corresponding class of bi-neighbourhood models.*

We now show that from any bi-neighbourhood model we can build an equivalent standard model. As a matter of fact, we can relativise the construction and the equivalence to an arbitrary set of formulas $\mathcal{S}$ provided that it is closed under subformulas. In this way we have an effective procedure to transform a finite bi-neighbourhood model satisfying a given formula into a standard one satisfying the same formula. Because of the obvious equivalence of the two semantics in the monotonic case, the latter is not considered in the lemma.

Lemma 2.9 *Let $\mathcal{M} = \langle W, \mathcal{N}, V \rangle$ be a bi-neighbourhood model and $\mathcal{S}$ be a set of formulas of $\mathcal{L}$ closed under subformulas. We define the standard model $\mathcal{M}^{\mathcal{S}} = \langle W, \mathcal{N}^{\mathcal{S}}, V \rangle$ with the same W and V and by taking, for all $w \in W$,*

$$\mathcal{N}^{\mathcal{S}}(w) = \{[C]_{\mathcal{M}} \mid C \in \mathcal{S} \text{ and } \mathcal{M}, w \models \Box C\}.$$

Then for any formula $A \in \mathcal{S}$ and any world $w \in W$, $\mathcal{M}^{\mathcal{S}}, w \models A$ iff $\mathcal{M}, w \models A$. Moreover, (N) if $\top \in \mathcal{S}$ and $\mathcal{M}$ is a N-model, then $\mathcal{M}^{\mathcal{S}}$ contains the unit; and (C) if $\mathcal{S}$ is closed under conjunction and $\mathcal{M}$ is a C-model, then $\mathcal{M}^{\mathcal{S}}$ is closed under intersection.

Proof. By induction on the complexity of any formula B it can be easily shown that $[B]_{\mathcal{M}^{\mathcal{S}}} = [B]_{\mathcal{M}}$. Moreover it can be proved that (N) $\mathcal{M}^{\mathcal{S}}$ contains the unit whenever $\mathcal{M}$ is a N-model and $\top \in \mathcal{S}$ and (C) $\mathcal{M}^{\mathcal{S}}$ is closed under intersection whenever $\mathcal{M}$ is a C-model and $\mathcal{S}$ is closed under conjunction. $\Box$

Theorem 2.10 *A formula A is valid in bi-neighbourhood models if and only if it is valid in the standard models satisfying the corresponding model conditions (N, C and M).*

Proof. From right to left, the claim follows from Lemma 2.9. From left to right, observe that given a standard model $\mathcal{M}_{st}$, we obtain an equivalent bi-neighbourhood model $\mathcal{M}_{bi}$ by taking, for all $w \in W$, $\mathcal{N}_{bi}(w) = \{(\alpha, W \setminus \alpha) \mid \alpha \in \mathcal{N}_{st}(w)\}$. Moreover, $\mathcal{M}_{bi}$ is a N-model if $\mathcal{M}_{st}$ contains the unit, and $\mathcal{M}_{bi}$ is a C-model if $\mathcal{M}_{st}$ is closed under intersection. $\Box$

3 The calculi LSE*

In this section, we define our labelled calculi **LSE***. We first present their language and rules, then prove soundness with respect to bi-neighbourhood semantics and syntactic completeness.

Let $\mathbb{WL} = \{x, y, z, ...\}$ and $\mathbb{NL} = \{a, b, c, ...\}$ be two infinite sets, respectively of *world labels* and of *neighbourhood labels*. *Positive neighbourhood terms* (or just *terms*) are finite sets of neighbourhood labels, and are written $[a_1 \dots a_n]$. If t is a positive term, then $\bar{t}$ is a *negative* term. The term τ and its negative counterpart $\bar{\tau}$ are neighbourhood constants. If a (positive or negative) term contains exactly one label or it is τ or $\bar{\tau}$, then it is *atomic*, otherwise it is *complex*.

Intuitively, a positive complex term represents the intersection of its constituents, whereas a negative complex term represents the union of the negative counterparts of its constituents. Moreover, t and $\bar{t}$ are the two members of a pair of neighbourhoods in bi-neighbourhood models. Observe that the operation of overlining a term cannot be iterated: it can be applied only once for turning a positive term into a negative one. Two operations over terms are defined as follows: (a) Composition of positive terms:

$$[a_1 \dots a_n][b] = \begin{cases} [a_1 \dots a_n] & \text{if } b = a_i \text{ for some } i,\ 1 \leq i \leq n; \\ [a_1 \dots a_n b] & \text{otherwise.} \end{cases}$$

$$[a_1 \dots a_n][b_1 \dots b_m] = (...([a_1 \dots a_n][b_1])...[b_{m-1}])[b_m]$$

(b) Substitution of a positive term for a neighbourhood label inside a term:

$$[a](t/b) = \begin{cases} t & \text{if } b = a \\ [a] & \text{if } b \neq a \end{cases} \quad [a_1 \dots a_n](t/b) = [a_1](t/b)...[a_n](t/b) \quad \overline{s}(t/b) = \overline{s(t/b)}$$

Observe that these operations do not introduce multiple occurrences of the same label, thus their results are still neighbourhood terms. We write $\Gamma(t/a)$ to indicate that the substitution applies to all formulas in Γ. As immediate consequences of the definition we have: $\tau(t/a) = \tau$ and $\overline{\tau}(t/a) = \overline{\tau}$, $(sr)(t/a) = s(t/a)r(t/a)$, and $\overline{sr}(t/a) = \overline{s(t/a)r(t/a)}$.

Definition 3.1 The formulas of $\mathcal{L}_{\text{LS}}$ are of the following kinds:

$$\phi ::= x : A \mid x : t \mid x : \overline{t} \mid t : A \mid \overline{t} : A \mid t : x.$$

The semantic interpretation of formulas of $\mathcal{L}_{\text{LS}}$ is given in Definition 3.3. Intuitively, $x : A$ means that x forces A, $x : t$ (resp. $x : \overline{t}$) means that x is a world in neighbourhood t (resp. $\overline{t}$), $t : A$ (resp. $\overline{t} : A$) means that every world in t (resp. some world in $\overline{t}$) forces A, and $t : x$ means that the pair $(t, \overline{t})$ is a bi-neighbourhood of x.

We have chosen a polymorphic notation, in which the colon has a meaning that depends on the type of its arguments, because of its compactness. As we shall see the interpretation of a formula ϕ is uniquely determined.

Sequents are defined as usual as pairs $\Gamma \Rightarrow \Delta$ of finite multisets of formulas, however they must satisfy some restrictions in order to assure cut admissibility.

Definition 3.2 A sequent is a pair $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas of $\mathcal{L}_{\text{LS}}$, that respect the following conditions: (1) Δ contains only formulas of the kinds $x : A$, $t : A$ and $\overline{t} : A$ (whereas Γ may contain any formula of $\mathcal{L}_{\text{LS}}$); (2) If Γ is non-empty, then all world labels and all neighbourhood labels occurring in Δ occur also in Γ.[4] (3) If Γ is empty, then Δ contains only formulas of the kind $x : A$, and all these formulas are labelled by the same world label x. (4) If $x : t$ is in Γ, then there is a world label y such that $t : y$ is in Γ.

The calculi **LSE*** are defined by the rules in Figure 1. Observe that, in analogy with the calculi based on standard possible world semantics, the left-right rules are meaning conferring and directly derive from the semantic explanation of logical constants in terms of bi-neighbourhood semantics, whereas the rules that manipulate only labels provide modular extensions of the basic systems to yield all the systems of the modal cube.

In Figure 2, the derivations of rule RE and axioms M, N and C in the respective calculi will be shown (for RE we assume sequents $y : A \Rightarrow y : B$ and $y : B \Rightarrow y : A$ derivable for any label y). Observe that considering rule applications backwards, the restrictions on sequents of Definition 3.2 are necessarily satisfied: If the conclusion of an instance of a rule satisfies conditions (1)-(4), then its premisses also satisfy (1)-(4). On the other hand, if we consider forward applications of the rules, these must be obviously restricted in such

[4] A neighbourhood label a occurs in (or belongs to) a labelled formula ϕ (set of formulas, sequent) if there is a (positive or negative) term containing a in ϕ.

Initial sequents: $x : p, \Gamma \Rightarrow \Delta, x : p$ $\quad x : \bot, \Gamma \Rightarrow \Delta$ $\quad \Gamma \Rightarrow \Delta, x : \top$

Propositional rules: As for **G3K**.

$$\frac{x : t, x : A, t : A, \Gamma \Rightarrow \Delta}{x : t, t : A, \Gamma \Rightarrow \Delta}\,\text{L}\Vdash^{\forall} \qquad \frac{x : t, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, t : A}\,\text{R}\Vdash^{\forall} \qquad \frac{x : \bar{t}, x : A, \Gamma \Rightarrow \Delta}{\bar{t} : A, \Gamma \Rightarrow \Delta}\,\text{L}\Vdash^{\exists}$$

$$\frac{x : \bar{t}, \Gamma \Rightarrow \Delta, x : A, \bar{t} : A}{x : \bar{t}, \Gamma \Rightarrow \Delta, \bar{t} : A}\,\text{R}\Vdash^{\exists} \qquad \frac{[a] : x, [a] : A, \Gamma \Rightarrow \Delta, \overline{[a]} : A}{x : \Box A, \Gamma \Rightarrow \Delta}\,\text{L}\Box$$

$$\frac{t : x, \Gamma \Rightarrow \Delta, x : \Box A, t : A \qquad t : x, \bar{t} : A, \Gamma \Rightarrow \Delta, x : \Box A}{t : x, \Gamma \Rightarrow \Delta, x : \Box A}\,\text{R}\Box$$

$$\frac{}{t : x, y : \bar{t}, \Gamma \Rightarrow \Delta}\,\text{M} \qquad \frac{\tau : x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\text{N}\tau \qquad \frac{}{x : \overline{\tau}, \Gamma \Rightarrow \Delta}\,\text{N}\overline{\tau}$$

$$\frac{ts : x, t : x, s : x, \Gamma \Rightarrow \Delta}{t : x, s : x, \Gamma \Rightarrow \Delta}\,\text{C} \qquad \frac{x : t, x : s, x : ts, \Gamma \Rightarrow \Delta}{x : ts, \Gamma \Rightarrow \Delta}\,\text{dec}$$

$$\frac{x : \bar{t}, x : \overline{ts}, \Gamma \Rightarrow \Delta \qquad x : \bar{s}, x : \overline{ts}, \Gamma \Rightarrow \Delta}{x : \overline{ts}, \Gamma \Rightarrow \Delta}\,\overline{\text{dec}}$$

Application conditions:
x is fresh in R$\Vdash^{\forall}$ and L$\Vdash^{\exists}$, a is fresh in L$\Box$, and x occurs in the conclusion of Nτ.

Fig. 1. The calculi **LSE***.

a way that they satisfy (1)-(4). Notice also that if rule M is added to the basic calculus, our rules L$\Box$ and R$\Box$ become interderivable with the rules for monotonic $\Box$ given in [13]; the latter rules, rewritten with the present notation, are as follows:

$$\frac{[a] : x, [a] : A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta}\,\text{L}\Box^{\text{M}}\ (a \text{ fresh}) \qquad \frac{t : x, \Gamma \Rightarrow \Delta, x : \Box A, t : A}{t : x, \Gamma \Rightarrow \Delta, x : \Box A}\,\text{R}\Box^{\text{M}}$$

It can be shown that these calculi are sound with respect to bi-neighbourhood semantics. For this purpose, we need to introduce the notion of realisation.

Definition 3.3 Given a model $\mathcal{M} = \langle W, \mathcal{N}, V\rangle$, a *realisation* is a pair of functions (ρ, σ), where $\rho : \mathbb{WL} \longrightarrow W$, and $\sigma : \mathbb{NT} \longrightarrow \mathcal{P}(W)$ such that $\sigma(\tau) = W$, $\sigma(t) \cap \sigma(\bar{t}) = \emptyset$, $\sigma(ts) = \sigma(t) \cap \sigma(s)$ and $\sigma(\overline{ts}) = \sigma(\bar{t}) \cup \sigma(\bar{s})$. The relation $\mathcal{M} \models_{\rho,\sigma} \phi$ is defined by cases as follows:
$\mathcal{M} \models_{\rho,\sigma} x : t$ iff $\rho(x) \in \sigma(t)$, and $\mathcal{M} \models_{\rho,\sigma} x : \bar{t}$ iff $\rho(x) \in \sigma(\bar{t})$;
$\mathcal{M} \models_{\rho,\sigma} x : A$ iff $\mathcal{M}, \rho(x) \models A$;
$\mathcal{M} \models_{\rho,\sigma} t : A$ iff for all $w \in \sigma(t)$, $\mathcal{M}, w \models A$;
$\mathcal{M} \models_{\rho,\sigma} \bar{t} : A$ iff there is a $w \in \sigma(\bar{t})$ such that $\mathcal{M}, w \models A$;
$\mathcal{M} \models_{\rho,\sigma} t : x$ iff $(\sigma(t), \sigma(\bar{t})) \in \mathcal{N}(\rho(x))$.
Then given a sequent $\Gamma \Rightarrow \Delta$ we stipulate that $\mathcal{M} \models_{\rho,\sigma} \Gamma \Rightarrow \Delta$ iff whenever $\mathcal{M} \models_{\rho,\sigma} \phi$ for all formulas ϕ in Γ we also have $\mathcal{M} \models_{\rho,\sigma} \psi$ for a formula ψ in Δ. Moreover, $\Gamma \Rightarrow \Delta$ is valid in $\mathcal{M}$ iff for all realisations (ρ, σ) we have $\mathcal{M} \models_{\rho,\sigma} \Gamma \Rightarrow \Delta$, and it is valid in bi-neighbourhood (N,C,M)-models iff it is

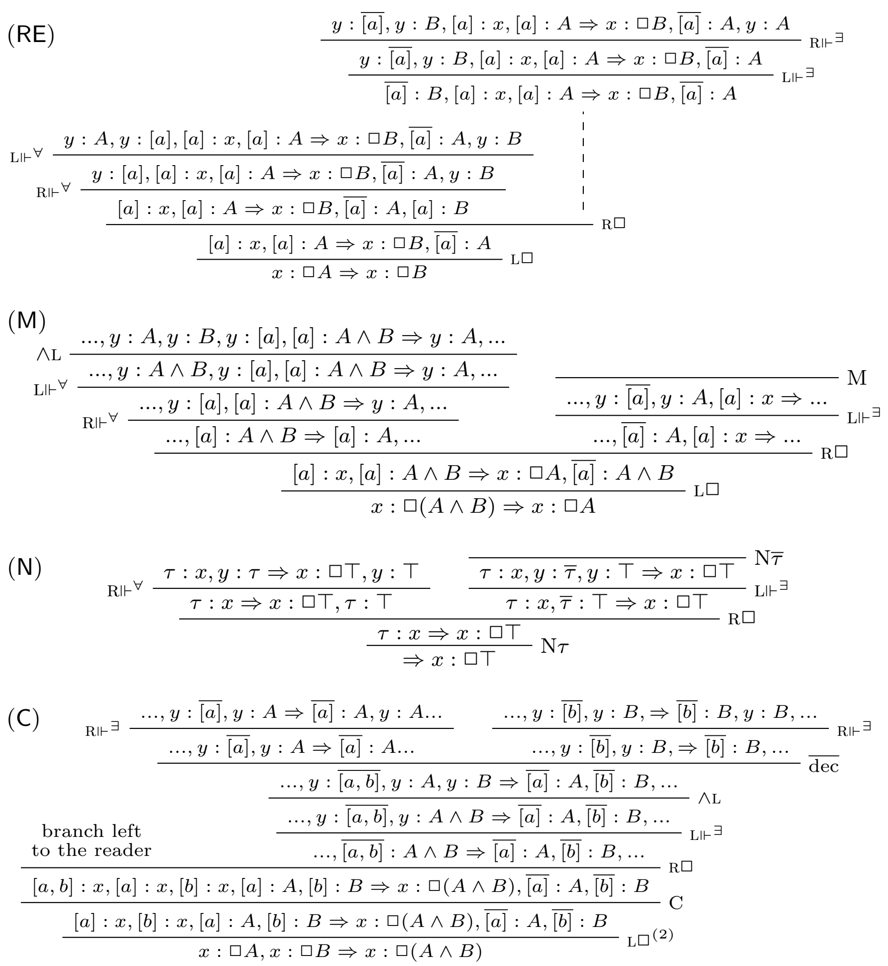

Fig. 2. Derivation of rule RE and axioms M, N and C in the respective calculi.

valid in every model $\mathcal{M}$ of the corresponding class.

By an easy induction on derivations we can prove the soundness of the calculi.

Theorem 3.4 *If a sequent* $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathbf{LSE}(\mathbf{N}, \mathbf{C}, \mathbf{M})$*, then it is valid in the class of all bi-neighbourhood (N,C,M-)models.*

Observe that all rules are also sound in standard models in which $\bar{t}$ is interpreted as the real complement of t, with the exception of rule M which is incompatible with such an interpretation. In what follows, we prove the main structural properties of the calculus, most importantly admissibility of *cut*, from which we obtain the syntactic completeness of the calculus.

Proposition 3.5 (a) *Substitution of world labels and* (b) *substitution of pos-*

itive terms for neighbourhood labels are height-preserving admissible (hp-admissible) in **LSE***. *Moreover,* (*c*) *the rules of left and right weakening are hp-admissible in* **LSE***; (*d*) *all rules of* **LSE*** *are hp-invertible; and* (*e*) *the rules of left and right contraction are hp-admissible in* **LSE***.

We aim to prove admissibility of the following cut rule:

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{cut}$$

where ϕ is any formula of $\mathcal{L}_{\text{LS}}$ that can occur on both sides of a sequent. Observe that any application of *cut* respects the restrictions on sequents of Definition 3.2. In order to prove admissibility of *cut* we need to define the weight of a labelled formula. Then by admissibility of *cut* it is easy to prove completeness of **LSE***.

Definition 3.6 The weight $w(\phi)$ of a formula ϕ of the form $x : A$, $t : A$ or $\bar{t} : A$ is the pair $\langle w(f(\phi)), w(l(\phi))\rangle$, where $f(\phi)$ and $l(\phi)$ are, respectively, the $\mathcal{L}$ formula A and the world label or neighbourhood term occurring in ϕ; $w(x) = 0$ and $w(t) = w(\bar{t}) = card(t)$, where $card(t)$ is the number of neighbourhood labels occurring in t; $w(p) = 1$, $w(A \circ B) = w(A)+w(B)+1$, $w(\Box A) = w(A)+1$. We consider weights of formulas lexicographically ordered.

Theorem 3.7 *Cut is admissible in* **LSE***.

Proof. By double induction, with primary induction on the weight of the cut formula and subinduction on the cut height. Observe that, because of Definition 3.2, cut formulas can be only of the kinds $x : A$, $t : A$ and $\bar{t} : A$. We only show some significant cases. (*i*) The last rule applied in the derivation of the left premiss of *cut* is Nτ. The derivation on the left is converted into the one on the right (in this and in the other cases we implicitly use hp-admissibility of structural rules). Observe that the restrictions on sequents guarantee that in the right derivation the label condition on the application of Nτ is respected, *i.e.* it is not the case that ϕ contains the only occurrence of x.

$$\dfrac{\dfrac{\tau : x, \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \phi}\,\text{N}\tau \qquad \phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\tau : x, \Gamma \Rightarrow \Delta, \phi \qquad \dfrac{\phi, \Gamma \Rightarrow \Delta}{\tau : x, \phi, \Gamma \Rightarrow \Delta}\,\mathit{wk}}{\tau : x, \Gamma \Rightarrow \Delta}\,\mathit{cut}}{\Gamma \Rightarrow \Delta}\,\text{N}\tau$$

(*ii*) The cut formula is $x : \Box A$, principal in the last rule of the derivation of both premisses of *cut*:

$$\dfrac{\dfrac{\begin{array}{c} t : x, \Gamma \Rightarrow \Delta, x : \Box A, t : A \\ \vdots \\ t : x, \bar{t} : A, \Gamma \Rightarrow \Delta, x : \Box A \end{array}}{t : x, \Gamma \Rightarrow \Delta, x : \Box A}\,\text{R}\Box \qquad \dfrac{\dfrac{\mathcal{D}}{[a] : x, [a] : A, t : x, \Gamma \Rightarrow \Delta, \overline{[a]} : A}}{x : \Box A, t : x, \Gamma \Rightarrow \Delta}\,\text{L}\Box}{t : x, \Gamma \Rightarrow \Delta}\,\mathit{cut}$$

with a fresh in the application of L$\Box$. The derivation is converted into the following, with four applications of *cut*, each one having smaller height or a cut formula of smaller weight:

$$
\dfrac{\dfrac{\dfrac{\mathcal{D}(t/a)}{t : x, t : A, t : x, \Gamma \Rightarrow \Delta, \bar{t} : A}}{t : x, t : A, \Gamma \Rightarrow \Delta, \bar{t} : A}\,ctr \qquad \begin{array}{c}\dfrac{\dfrac{t : x, \Gamma \Rightarrow \Delta, x : \Box A, t : A \qquad \dfrac{t : x, x : \Box A, \Gamma \Rightarrow \Delta}{t : x, x : \Box A, \Gamma \Rightarrow \Delta, t : A}\,wk}{t : x, \Gamma \Rightarrow \Delta, t : A}\,cut}{t : x, \Gamma \Rightarrow \Delta, t : A, \bar{t} : A}\,wk \\ \vdots \end{array}}{t : x, \Gamma \Rightarrow \Delta, \bar{t} : A}\,cut \qquad \dfrac{t : x, \bar{t} : A, \Gamma \Rightarrow \Delta, x : \Box A \qquad \begin{array}{c}\dfrac{t : x, x : \Box A, \Gamma \Rightarrow \Delta}{t : x, \bar{t} : A, x : \Box A, \Gamma \Rightarrow \Delta}\,wk \\ \vdots \end{array}}{t : x, \bar{t} : A, \Gamma \Rightarrow \Delta}\,cut
$$

$$
\dfrac{t : x, \Gamma \Rightarrow \Delta, \bar{t} : A \qquad t : x, \bar{t} : A, \Gamma \Rightarrow \Delta}{t : x, \Gamma \Rightarrow \Delta}\,cut
$$

□

Theorem 3.8 *The calculus* **LSE*** *is complete with respect to the logic* **E***.

Proof. Straightforward by showing that any instance of the axioms and all the rules of **E*** are derivable in **LSE*** (cf. Figure 2), using *cut* when needed.□

4 The calculi $\mathbb{T}$LSE*

In this section, we present the calculi $\mathbb{T}$**LSE*** (where $\mathbb{T}$ stays for *terms*) which are refinements of the calculi **LSE*** for the cases in which complex terms are present. We show that these calculi are terminating and thereby provide a decision procedure for the respective logics, and we prove semantic completeness of the calculi with respect to bi-neighbourhood semantics. By simulating derivations in **LSE***, we also show that these calculi are syntactically complete, although, as explained below, a direct proof of cut elimination cannot be given, what justifies a separate presentation of the two calculi.

Observe that in **LSE*** it may happen that if the starting sequent contains n atomic terms $[a_1], ..., [a_n]$, a derivation branch - by application of rule C and repeated applications of dec - may take $\mathcal{O}(2^n)$ steps to generate a complex term t containing an arbitrary subset of $a_1, ..., a_n$. To prevent this situation we reformulate the rules for complex terms as follows:

Simplified rules for C:

$$
\dfrac{[a_1] : x, ..., [a_n] : x, [a_1 \dots a_n] : x, \Gamma \Rightarrow \Delta}{[a_1] : x, ..., [a_n] : x, \Gamma \Rightarrow \Delta}\,\mathbb{C}
$$

$$
\dfrac{x : [a_1], ..., x : [a_n], \Gamma \Rightarrow \Delta}{x : [a_1 \dots a_n], \Gamma \Rightarrow \Delta}\,\mathtt{dec} \qquad \dfrac{x : \overline{[a_1]}, \Gamma \Rightarrow \Delta \quad ... \quad x : \overline{[a_n]}, \Gamma \Rightarrow \Delta}{x : \overline{[a_1 \dots a_n]}, \Gamma \Rightarrow \Delta}\,\overline{\mathtt{dec}}
$$

Since these rules are easily derivable in **LSE***, it turns out that $\mathbb{T}$**LSE*** is sound. The rules for decomposition of terms are modified as follows: a complex term can be decomposed only into its atomic components and is not copied into the premiss; moreover by the simplified rule for C complex terms can be formed only by joining atomic terms. However, the calculi with the restricted rules are complete only with respect to sequents of a special form, as described in the next definition.

Definition 4.1 A sequent $\Gamma \Rightarrow \Delta$ of $\mathcal{L}_{\text{LS}}$ is *proper* if it satisfies all the following additional conditions: (1) If $t : A$ is in Γ, then t is atomic and different from τ; (2) $t : A$ is in Γ if and only if $\bar{t} : A$ is in Δ; (3) If $[a]$ occurs in $\Gamma \Rightarrow \Delta$, then there is exactly one formula A such that $[a] : A$ is in Γ; (4) If $[a_1 \dots a_n] : x$ is in Γ, then $[a_1] : x, ..., [a_n] : x$ are in Γ.

It follows from Definition 4.1 that if a formula $\bar{t} : A$ occurs in the right-hand side of a proper sequent $\Gamma \Rightarrow \Delta$, then $\bar{t}$ is atomic and different from $\overline{\tau}$, and $\bar{t} : A$ is the only formula of this kind labelled by $\bar{t}$ occurring in Δ. Trivially, since a sequent of the form $\Rightarrow x_0 : A$ is proper, restricting consideration to proper sequents is sufficient to prove the validity of any formula of $\mathbf{E}^*$.

It can be shown that the calculi $\mathbb{T}\mathbf{LSE}^*$ are syntactically complete as they can simulate $\mathbf{LSE}^*$ derivations restricted to proper sequents. As a preliminary condition, observe that any sequent occurring in a derivation of a proper sequent in $\mathbf{LSE}^*$ or $\mathbb{T}\mathbf{LSE}^*$ is proper, since whenever the conclusion of a rule of $\mathbf{LSE}^*$ or $\mathbb{T}\mathbf{LSE}^*$ is proper its premisses are also proper. The need of such an indirect proof is due to the fact that proper sequents are not preserved by substitution of neighbourhood terms, as it is needed for a direct proof of cut elimination. Although we do not have a syntactic proof of cut admissibility, we have a semantic proof of it: by the completeness of the calculi, the cut rule turns out to be admissible in each system.

By the restrictions of Definition 4.1 we obtain the following property, that will be needed in the proof of Theorem 4.11.

Proposition 4.2 *Every proper sequent of the form* $x : [a], x : \overline{[a]}, \Gamma \Rightarrow \Delta$ *is derivable in* $\mathbb{T}\mathbf{LSE}^*$.

Proof. Since $x : [a], x : \overline{[a]}, \Gamma \Rightarrow \Delta$ is proper, by definition there is a formula A such that $[a] : A$ is in Γ and $\overline{[a]} : A$ is in Δ. Then the sequent has the form $x : [a], x : \overline{[a]}, [a] : A, \Gamma' \Rightarrow \Delta', \overline{[a]} : A$ and is derivable as follows:

$$\dfrac{\dfrac{x : [a], x : \overline{[a]}, [a] : A, x : A, \Gamma' \Rightarrow \Delta', \overline{[a]} : A, x : A}{x : [a], x : \overline{[a]}, [a] : A, \Gamma' \Rightarrow \Delta', \overline{[a]} : A, x : A}\ \text{L}\Vdash^{\forall}}{x : [a], x : \overline{[a]}, [a] : A, \Gamma' \Rightarrow \Delta', \overline{[a]} : A}\ \text{R}\Vdash^{\exists}$$

□

The adequacy of rules $\mathbb{C}$, $\mathbb{dec}$ and $\overline{\mathbb{dec}}$ is proved by the following proposition.

Proposition 4.3 (a) *Rules* $\mathbb{dec}$ *and* $\overline{\mathbb{dec}}$ *are invertible in* $\mathbb{T}\mathbf{LSE}^*$ *with respect to derivations of proper sequents.* (b) *Contraction is hp-admissible in* $\mathbb{T}\mathbf{LSE}^*$.

Theorem 4.4 *Any proper sequent derivable in* $\mathbf{LSE}^*$ *is derivable also in* $\mathbb{T}\mathbf{LSE}^*$*, whence the calculi* $\mathbb{T}\mathbf{LSE}^*$ *are complete for the corresponding logic.*

Proof. We just consider the most significant cases. If the last rule applied is C, then S has the form $t : x, s : x, \Gamma \Rightarrow \Delta$ and it was derived from the proper sequent $ts : x, t : x, s : x, \Gamma \Rightarrow \Delta$, that by inductive hypothesis is derivable in $\mathbb{T}\mathbf{LSE}^*$. Let t and s be the terms $[a_1 \dots a_n]$ and $[b_1 \dots b_m]$. Then ts is $[a_1 \dots a_n b_1 \dots b_m]$ (without possible repetitions). By definition of proper sequent, Γ contains $[a_i] : x$ and $[b_j] : x$ for all $1 \leq i \leq n$, $1 \leq j \leq m$. Then we can apply $\mathbb{C}$ and obtain $t : x, s : x, \Gamma \Rightarrow \Delta$.

If the last rule applied is $\overline{\text{dec}}$, then S has the form $x : \overline{ts}, \Gamma \Rightarrow \Delta$ and it was derived from the proper sequents $x : \overline{t}, x : \overline{ts}, \Gamma \Rightarrow \Delta$ and $x : \overline{s}, x : \overline{ts}, \Gamma \Rightarrow \Delta$, that by inductive hypothesis are derivable in $\mathbb{T}$**LSE***. Let t and s be the terms $[a_1 \dots a_n]$ and $[b_1 \dots b_m]$. Then ts is $[a_1 \dots a_n b_1 \dots b_m]$ (without possible repetitions). Consider the first premiss, that is $x : \overline{[a_1 \dots a_n]}, x : \overline{[a_1 \dots a_n b_1 \dots b_m]}, \Gamma \Rightarrow \Delta$. By invertibility of $\overline{\mathbb{dec}}$ in $\mathbb{T}$**LSE***, $x : \overline{[a_i]}, x : \overline{[a_1 \dots a_n b_1 \dots b_m]}, \Gamma \Rightarrow \Delta$ is derivable for all $1 \leq i \leq n$. Again by invertibility of $\overline{\mathbb{dec}}$, $x : \overline{[a_i]}, x : \overline{[a_k]}, \Gamma \Rightarrow \Delta$ and $x : \overline{[a_i]}, x : \overline{[b_l]}, \Gamma \Rightarrow \Delta$ are derivable for all $1 \leq k \leq n$, $1 \leq l \leq m$. By applying the same procedure to the second premiss, we obtain that sequents $x : \overline{[b_j]}, x : \overline{[a_k]}, \Gamma \Rightarrow \Delta$ and $x : \overline{[b_j]}, x : \overline{[b_l]}, \Gamma \Rightarrow \Delta$ are derivable for all $1 \leq k \leq n$, $1 \leq j, l \leq m$. Now take all sequents $x : \overline{[a_i]}, x : \overline{[a_k]}, \Gamma \Rightarrow \Delta$ and $x : \overline{[b_j]}, x : \overline{[b_l]}, \Gamma \Rightarrow \Delta$ where $i = k$ and $j = l$. By contraction we obtain $x : \overline{[a_i]}, \Gamma \Rightarrow \Delta$ and $x : \overline{[b_j]}, \Gamma \Rightarrow \Delta$. Then by an application of $\overline{\mathbb{dec}}$ with all these sequents as premisses we derive $x : \overline{ts}, \Gamma \Rightarrow \Delta$. □

We now show that by adopting a simple strategy, proof search in $\mathbb{T}$**LSE*** always terminates in a finite number of steps, thereby providing a decision procedure for the corresponding logic. This is basically proved by showing that the set of labelled formulas which can occur in any sequent in any derivation branch is finite. In order to define the strategy, we introduce saturation conditions associated to the rules and the notion of saturated branch.

Definition 4.5 Let $\mathcal{B} = \{\Gamma_i \Rightarrow \Delta_i\}$ be a (finite or infinite) branch in a proof search in $\mathbb{T}$**LSE*** for $\Gamma \Rightarrow \Delta$. We define $\Gamma^* = \bigcup \Gamma_i$ and $\Delta^* = \bigcup \Delta_i$. The saturation conditions associated to each rule of $\mathbb{T}$**LSE*** are as follows: ($Init$) for all i, there is no $x : p$ in $\Gamma_i \cap \Delta_i$; $x : \bot$ is not in Γ_i and $x : \top$ is not in Δ_i. Standard for propositional rules (omitted). (L$\Vdash^\forall$) If $t : A$ and $x : t$ are in Γ^*, then $x : A$ is in Γ^*. (R$\Vdash^\forall$) If $t : A$ is in Δ^*, then for a label x, $x : t$ is in Γ^* and $x : A$ is in Δ^*. (L$\Vdash^\exists$) If $\overline{t} : A$ is in Γ^*, then for a label x, $x : \overline{t}$ and $x : A$ are in Γ^*. (R$\Vdash^\exists$) If $\overline{t} : A$ is in Δ^* and $x : \overline{t}$ is in Γ^*, then $x : A$ is in Δ^*. (L$\Box$) If $x : \Box A$ is in Γ^*, then for a label a, $[a] : x$ and $[a] : A$ are in Γ^* and $\overline{[a]} : A$ is in Δ^*. (R$\Box$) If $x : \Box A$ is in Δ^* and $t : x$ is in Γ, then either $t : A$ is in Δ^* or $\overline{t} : A$ is in Γ^*. (Nτ) For every world label x occurring in $\Gamma^* \cup \Delta^*$, $\tau : x$ is in Γ^*. (N$\overline{\tau}$) $x : \overline{\tau}$ is not in Γ^*. (M) $t : x$ and $y : \overline{t}$ are not both in Γ^*. ($\mathbb{C}$) If $[a_1] : x$, ..., $[a_n] : x$ are in Γ^*, then $[a_1 \dots a_n] : x$ is in Γ^*. ($\mathbb{dec}$) If $x : [a_1 \dots a_n]$ is in Γ^*, then $x : [a_1]$, ..., $x : [a_n]$ are in Γ^*. ($\overline{\mathbb{dec}}$) If $x : \overline{[a_1 \dots a_n]}$ is in Γ^*, then $x : \overline{[a_1]}$ or, ..., or $x : \overline{[a_n]}$ is in Γ^*.

We say that $\mathcal{B}$ is saturated with respect to an application of a rule if the corresponding condition holds, and it is saturated with respect to $\mathbb{T}$**LSE*** if it is saturated with respect to all possible applications of any rule of $\mathbb{T}$**LSE***.

The *strategy* for constructing a root-first proof search tree in $\mathbb{T}$**LSE*** of the sequent $\Rightarrow x_0 : A$ obeys the following conditions: (i) No rule can be applied to an initial sequent; (ii) A specific application of a rule R to a formula ϕ (or to a pair of formulas ϕ and ψ) in a sequent $\Gamma_i \Rightarrow \Delta_i$ is not allowed if the branch from $\Rightarrow x_0 : A$ to $\Gamma_i \Rightarrow \Delta_i$ already fulfills the saturation condition for that application of R; (iii) If rules for N are present, as first step apply Nτ to x_0.

We now show that for each sequent $\Rightarrow x_0 : A$ this strategy produces either a proof of it or a finite tree in which all open branches are saturated.

Definition 4.6 Let $\mathcal{B}$ be a branch of a proof search in $\mathbb{T}$**LSE*** for $\Rightarrow x_0 : A$, t a neighbourhood term and x, y world labels occurring in $\mathcal{B}$, and let $k(x) = min\{i \in \mathbb{N} \mid x \text{ is in } \Gamma_i\}$. The relations $\rightarrow_1 \subseteq \mathbb{WL} \times \mathbb{NT}$, $\rightarrow_2 \subseteq \mathbb{NT} \times \mathbb{WL}$, and $\rightarrow_w \subseteq \mathbb{WL} \times \mathbb{WL}$ are defined as follows:

$\rightarrow_1$) (*i*) $x \rightarrow_1 t$ if $t \neq \tau$ and $t : x$ is in Γ^*; (*ii*) $x_0 \rightarrow_1 \tau$; (*iii*) $y \neq x_0$ implies $y \not\rightarrow_1 \tau$; and (*iv*) $x \rightarrow_1 \bar{t}$ if $x \rightarrow_1 t$.

$\rightarrow_2$) $t \rightarrow_2 x$ if for a $i \in \mathbb{N}$, $k(x) = i$ and $x : t$ is in Γ_i (for t positive or negative).

$\rightarrow_w$) $x \rightarrow_w y$ if for some (positive or negative) term t, $x \rightarrow_1 t$ and $t \rightarrow_2 y$.

Lemma 4.7 *Given a branch $\mathcal{B}$ in a proof search tree for $\Rightarrow x_0 : A$ built in accordance with the strategy we have that* (a) *the graph $\mathcal{T}_w$ determined by x_0 and the relation $\rightarrow_w$ is a tree with root x_0, and* (b) *all the world labels occurring in $\mathcal{B}$ are nodes of $\mathcal{T}_w$.*

Lemma 4.8 *Let for any world label x and any (positive or negative) term t, $md(x) = max\{md(A) \mid x : A$ is in $\Gamma^* \cup \Delta^*\}$ and $md(t) = max\{md(A) \mid t : A$ is in $\Gamma^* \cup \Delta^*\}$, where $md(A)$ is the modal degree of A defined in the standard way. Then for any x, y in $\mathcal{T}_w$ we have that $x \rightarrow_w y$ implies $md(y) < md(x)$.*

Proposition 4.9 *Given a branch $\mathcal{B}$ of a proof search for $\Rightarrow x_0 : A$,* (a) *any world label occurring in $\mathcal{B}$ generates at most finitely many terms, and* (b) *any term occurring in $\mathcal{B}$ generates at most finitely many world labels. Whence* (c) *$\mathcal{T}_w$ is finite.*

Proof. (a) Consider first atomic terms: A world label x generates an atomic term $[a]$ by an application of L$\Box$. By its saturation clause, L$\Box$ can be applied to each formula $x : \Box B$ at most once. Therefore the problem is reduced to counting how many different formulas $x : \Box B$ can occur in the branch. If x is x_0, *i.e.* the label occurring in the sequent $\Rightarrow x_0 : A$ at the root, then the number of these formulas is smaller than the length of A. If x is generated by a term t, then it is generated by an application of R$\Vdash^{\forall}$ with a formula $[b] : C$ in Δ^* principal in the rule application (or by an application of L$\Vdash^{\exists}$ with a formula $[b] : C$ in Γ^* principal in the rule application). Thus all formulas $\Box B$ such that $x : \Box B$ is in the branch are subformulas of C or - if t is atomic and different from τ - subformulas of D, where D is the only formula such that $t : D$ is in Γ^* (or $\bar{t} : D$ is in Δ^*), whose existence is guaranteed by definition of proper sequents. For complex terms: If x generates n atomic (positive) terms, then - by means of $\mathbb{C}$ - it generates at most $2^n - 1$ positive terms. Therefore the terms generated by x are in any case finitely many.

(b) A term t generates a world label y by an application of R$\Vdash^{\forall}$ or L$\Vdash^{\exists}$. By the saturation clauses of these rules, every expression $t : B$ produces at most one world label. Therefore the problem is reduced to counting how many different expressions $t : B$ can occur in the branch. First assume $t \neq \tau$ and t generated by x. Then the number of these expressions depends directly on the number of formulas $x : \Box B$ in Δ^*, which - as shown in point (a) - are finitely many. If $t = \tau$: By the properties of the calculus, if $\tau : B$ is in Δ^*, then B is a subformula of A, where A is the formula labelled by x_0 at the root. Thus the possible expressions $\tau : B$ in Δ^* are finitely many. Observe also that there is no $\overline{\tau} : B$ in Γ^*. In fact, by an application of $\mathsf{L}\Vdash^{\exists}$ this would give a formula $y : \overline{\tau}$ in Γ^*, against the saturation clause for $\mathrm{N}\overline{\tau}$.

(c) By the decrease in modal depth stated stated by Lemma 4.8 it follows that any branch of $\mathcal{T}_w$ has a finite length. Moreover, $\mathcal{T}_w$ is finitary: if $x \overset{w}{\rightarrow} y$, then by definition there is a term t such that $x \rightarrow_1 t \rightarrow_2 y$; but by points ($a$) and ($b$) x is related to finitely many terms and t is related to finitely many world labels. □

Theorem 4.10 *Any branch $\mathcal{B}$ of a proof search for $\Rightarrow x_0 : A$ built in accordance with the strategy is finite, therefore proof search for any sequent of the form $\Rightarrow x_0 : A$ always comes to an end after a finite number of steps. Furthermore, each branch is either closed or saturated.*

Proof. By Proposition 4.9, $\mathcal{B}$ contains finitely many world labels and neighbourhood terms. Moreover, by the properties of the calculus, in any formula $x : B$ (or $t : B$, $\overline{t} : B$) that can occur in $\mathcal{B}$, B is a subformula of A, where A is the formula labelled by x_0 in the root sequent. Therefore only a finite number of labelled formulas can occur in $\mathcal{B}$. Thus, since by the saturation conditions a rule is not applied more than once to the same labelled formula ϕ (or the same pair of formulas ϕ and ψ), there are always only finitely many possible rule applications. □

We now prove semantic completeness of the calculi. This result shows that given an unprovable formula we can extract a finite countermodel of it in the bi-neighbourhood semantics. Moreover, by Lemma 2.9 we can also get a standard countermodel. Observe that this result, combined with the soundness of $\mathbb{T}$**LSE***, provides a constructive proof of the finite model property both in the bi-neighbourhood and in the standard semantics.

Theorem 4.11 $\mathbb{T}$**LSE*** *is complete with respect to the corresponding class of bi-neighbourhood models.*

Proof. Given a saturated branch $\mathcal{B}$ in a proof search in $\mathbb{T}$**LSE*** for the proper sequent $\Gamma \Rightarrow \Delta$, we build a bi-neighbourhood countermodel $\mathcal{M}$ to $\Gamma \Rightarrow \Delta$ that makes all formulas in Γ^* true and all formulas in Δ^* false. Model $\mathcal{M} = \langle W, \mathcal{N}, V \rangle$ is defined as follows: $W = \{x \in \mathbb{WL} \mid x \text{ occurs in } \Gamma^* \cup \Delta^*\}$; $\alpha_{[a_1 \ldots a_n]} = \{x \in W \mid \text{for all } 1 \leq i \leq n,\ x : [a_i] \text{ is in } \Gamma^*\}$; $\alpha_{\overline{[a_1 \ldots a_n]}} = \{x \in W \mid \text{for some } 1 \leq i \leq n,\ x : \overline{[a_i]} \text{ is in } \Gamma^*\}$; $\alpha_\tau = W$; $\alpha_{\overline{\tau}} = \emptyset$; for any $x \in W$, $\mathcal{N}(x) = \{(\alpha_t, \alpha_{\overline{t}}) \mid t : x \text{ is in } \Gamma^*\}$; for any $p \in \mathcal{L}$, $V(p) = \{x \in W \mid x : p \text{ is in } \Gamma^*\}$. Then we define the realisation (ρ, σ) by choosing $\rho(x) = x$ for any world label x, and $\sigma(t) = \alpha_t$ for any positive or negative term t occurring in $\Gamma^* \cup \Delta^*$.

First of all observe that $\mathcal{M}$ and σ are well defined: By the definition of $\alpha_{[a_1 \ldots a_n]}$ and $\alpha_{\overline{[a_1 \ldots a_n]}}$ it follows immediately that $\sigma(ts) = \sigma(t) \cap \sigma(s)$ and $\sigma(\overline{ts}) = \sigma(\overline{t}) \cup \sigma(\overline{s})$. Moreover, $\sigma(t) \cap \sigma(\overline{t}) = \emptyset$. In fact, assume $t = [a_1 \ldots a_n]$

and $\alpha_t \cap \alpha_{\overline{t}} \neq \emptyset$. By definition, for some $1 \leq i \leq n$ and some $y \in W$, $y : [a_i]$ and $y : \overline{[a_i]}$ are in Γ^*. Since such expressions are never deleted, this means that there is a sequent $\Gamma_j \Rightarrow \Delta_j$ in the brach $\mathcal{B}$ such that $y : [a_i]$ and $y : \overline{[a_i]}$ are in Γ_j. Then by Proposition 4.2, $\Gamma_j \Rightarrow \Delta_j$ is derivable, against the hypothesis that $\mathcal{B}$ is saturated. Finally, from this it follows that $(\alpha, \beta) \in \mathcal{N}(x)$ implies $\alpha \cap \beta = \emptyset$. By considering all possible cases, it is easy to prove by induction on the weight of ϕ that if ϕ is in Γ^*, then $\mathcal{M} \models_{\rho,\sigma} \phi$, and if ϕ is in Δ^*, then $\mathcal{M} \not\models_{\rho,\sigma} \phi$. Moreover, it can be shown that if $\mathbb{T}\mathbf{LSE}^*$ contains the rules for C, then $\mathcal{M}$ is a C-model, if it contains the rules for N, then $\mathcal{M}$ is a N-model, and if it contains the rules for M, then $\mathcal{M}$ is a M-model. $\Box$

Example 4.12 Here is a failed derivation of an instance of axiom M in $\mathbb{T}\mathbf{LSE}$:

$$\dfrac{\dfrac{\mathcal{A}: \text{closed} \qquad \dfrac{\dfrac{\dfrac{\mathcal{A}': \text{closed} \qquad \overset{\text{saturated branch } \mathcal{B}}{y : \overline{[a]}, y : p, [a] : x, [a] : p \wedge q \Rightarrow x : \Box p, \overline{[a]} : p \wedge q, y : q}}{y : \overline{[a]}, y : p, [a] : x, [a] : p \wedge q \Rightarrow x : \Box p, \overline{[a]} : p \wedge q, y : p \wedge q}\, \wedge\text{L}}{y : \overline{[a]}, y : p, [a] : x, [a] : p \wedge q \Rightarrow x : \Box p, \overline{[a]} : p \wedge q}\, \text{R}\Vdash^{\exists}}{[a] : x, [a] : p \wedge q, \overline{[a]} : p \Rightarrow x : \Box p, \overline{[a]} : p \wedge q}\, \text{L}\Vdash^{\exists}}{[a] : x, [a] : p \wedge q \Rightarrow x : \Box p, \overline{[a]} : p \wedge q}\, \text{R}\Box}{x : \Box(p \wedge q) \Rightarrow x : \Box p}\, \text{L}\Box$$

The bi-neighbourhood model $\mathcal{M} = \langle W, \mathcal{N}, V \rangle$ defined directly from the saturated branch $\mathcal{B}$ is the following: $W = \{x, y\}$, $\mathcal{N}(x) = \{(\emptyset, \{y\})\}$, $\mathcal{N}(y) = \emptyset$, $V(p) = \{y\}$ and $V(q) = \emptyset$. Then we have $\mathcal{M}, x \not\models \Box p$ and, since $[p \wedge q]_{\mathcal{M}} = \emptyset$, we also have $\mathcal{M}, x \models \Box(p \wedge q)$, thus the sequent at the root is falsified.

If we now consider the set $\mathcal{S} = \{\Box(p \wedge q), \Box p, p \wedge q, p, q\}$ and we follow the definition in Lemma 2.9, we obtain the standard model $\mathcal{M}^{\mathcal{S}}$ in which $\mathcal{N}^{\mathcal{S}}(x) = \{[p \wedge q]_{\mathcal{M}}\} = \{\emptyset\}$ and $\mathcal{N}^{\mathcal{S}}(y) = \emptyset$. It is immediate to verify that also $\mathcal{M}^{\mathcal{S}}$ falsifies the sequent.

5 Proof-theoretic equivalence of the semantics

In the previous section, we have shown that $\mathbb{T}\mathbf{LSE}^*$ is sound and complete with respect to bi-neighbourhood semantics, thus by virtue of Theorem 2.10 also with respect to the standard semantics. For the non-monotonic case we now give a proof-theoretical argument to show that the two semantics coincide (therefore we do not consider rule M in this section). More precisely, we show that interpreting the negative terms as true complements (as it happens in standard semantics) does not extend the set of provable formulas, whence the set of valid formulas. To this purpose we consider the following rule:

$$\dfrac{x : [a], \Gamma \Rightarrow \Delta \qquad x : \overline{[a]}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{cmp}\ (x, a \in \Gamma \cup \Delta)$$

and we show that it is admissible in $\mathbb{T}\mathbf{LSE}^*$. Moreover, we also show easily that by using this rule we can directly build countermodels in the standard semantics. As before, the analysis is restricted to proper sequents. Observe that the application of cmp respects (backwards) the constraints of proper sequents.

Proposition 5.1 *Rule* cmp *is admissible in* $\mathbb{T}$**LSE*** *for derivations of proper sequents.*

Proof. First of all, by induction on the height of the derivations one can prove that (a) if $x : [a], [a] : B, x : B, \Gamma \Rightarrow \Delta$ is proper and derivable, then $[a] : B, x : B, \Gamma \Rightarrow \Delta$ is proper and derivable with a derivation of the same height; and (b) if $x : \overline{[a]}, \Gamma \Rightarrow \Delta, \overline{[a]} : B, x : B$ is proper and derivable and x is in Γ, then $\Gamma \Rightarrow \Delta, \overline{[a]} : B, x : B$ is proper and derivable with a derivation of the same height. Then by induction on the height of the application of cmp it is possible to show how to remove all its applications. We only show the most significant case, in which $x : [a]$ and $x : \overline{[a]}$ are both principal in the last rule of the derivation of the respective premisses. The only possibility is that the applied rules are $\text{L}\Vdash^{\forall}$ for the left premiss and $\text{R}\Vdash^{\exists}$ for the right premiss:

$$\dfrac{\dfrac{x : [a], x : B, [a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : C}{x : [a], [a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : C}\,\text{L}\Vdash^{\forall} \qquad \dfrac{x : \overline{[a]}, [a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : C, x : C}{x : \overline{[a]}, [a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : C}\,\text{R}\Vdash^{\exists}}{[a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : C}\,\text{cmp}$$

However, since $[a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : C$ is a proper sequent, we have that $B \equiv C$. Therefore the case under consideration is as follows:

$$\dfrac{\dfrac{x : [a], x : B, [a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : B}{x : [a], [a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : B}\,\text{L}\Vdash^{\forall} \qquad \dfrac{x : \overline{[a]}, [a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : B, x : B}{x : \overline{[a]}, [a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : B}\,\text{R}\Vdash^{\exists}}{[a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : B}\,\text{cmp}$$

where the premisses of $\text{L}\Vdash^{\forall}$ and $\text{R}\Vdash^{\exists}$ are proper. Then by (a) and (b) we have that also $x : B, [a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : B$, and $[a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : B, x : B$ are proper and are derivable with derivations of the same heights. Observe in particular that point (b) is here applicable because of the condition on the application of cmp and the definition of sequents, that guarantee that x is in Γ. By an application of *cut* to these sequents with $x : B$ as cut formula we then obtain $[a] : B, \Gamma \Rightarrow \Delta, \overline{[a]} : B$. $\square$

Theorem 5.2 $\mathbb{T}$**LSE*** *is complete with respect to the corresponding class of standard models.*

Proof. Let $\mathcal{B}$ be a saturated branch in a proof search in $\mathbb{T}$**LSE*** for the proper sequent $\Gamma \Rightarrow \Delta$ satisfying also the saturation condition for rule cmp: If x and $[a]$ are in Γ^*, then $x : [a]$ is in Γ^* or $x : \overline{[a]}$ is in Γ^*. We then build a standard countermodel $\mathcal{M}$ to $\Gamma \Rightarrow \Delta$ that makes all formulas in Γ^* true and all formulas in Δ^* false. Let the realisation (ρ, σ) and the model $\mathcal{M}$ be defined as in Theorem 4.11 with the minor modification that for all $x \in W$, $\mathcal{N}(x) = \{\alpha_t \mid t : x \text{ is in } \Gamma^*\}$. We only need to prove that $\mathcal{M}$ is now a standard model, that is $\sigma(\bar{t}) = W \setminus \sigma(t)$. We already know that $\sigma(t) \cap \sigma(\bar{t}) = \emptyset$; we show that $\sigma(t) \cup \sigma(\bar{t}) = W$. If $t = \tau$, this holds by definition of $\sigma(\tau)$. Assume $t = [a_1 \dots a_n]$. By saturation of cmp, for all $1 \leq i \leq n$, $x : [a_i]$ or $x : \overline{[a_i]}$ is in Γ^*. If for some i, $x : \overline{[a_i]}$ is in Γ^*, then by definition $x \in \alpha_{\bar{t}}$. Otherwise $x : [a_i]$ is in Γ^* for all i, and by definition $x \in \alpha_t$. In addition observe also that by saturation of rules $\mathbb{C}$ and Nτ we have that if $\mathbb{T}\mathbf{LSE}^*$ contains the rules for C, then $\mathcal{M}$ is closed under intersection, and if $\mathbb{T}\mathbf{LSE}^*$ contains the rules for N, then $\mathcal{M}$ contains the unit. $\square$

Example 5.3 This example shows how to obtain directly a standard countermodel from a failed branch of a proof search in $\mathbb{T}\mathbf{LSE}$ which is saturated also with respect to rule cmp. In the derivation below we extend the branch $\mathcal{B}$ of the proof search of Example 4.12 in order to get such a saturation.

saturated branch $\mathcal{C}_3$

$$x : \overline{[a]}, y : \overline{[a]}, y : p, [a] : x, [a] : p \wedge q \Rightarrow x : \Box p, \overline{[a]} : p \wedge q, y : q, x : p$$

saturated branch $\mathcal{C}_2$

$$\dfrac{x : \overline{[a]}, y : \overline{[a]}, y : p, [a] : x, [a] : p \wedge q \Rightarrow x : \Box p, \overline{[a]} : p \wedge q, y : q, x : q}{\dfrac{x : \overline{[a]}, y : \overline{[a]}, y : p, [a] : x, [a] : p \wedge q \Rightarrow x : \Box p, \overline{[a]} : p \wedge q, y : q, x : p \wedge q}{x : \overline{[a]}, y : \overline{[a]}, y : p, [a] : x, [a] : p \wedge q \Rightarrow x : \Box p, \overline{[a]} : p \wedge q, y : q}\ \text{R}\Vdash^{\exists}}\ \wedge\text{R}$$

saturated branch $\mathcal{C}_1$

$$\dfrac{\dfrac{\dfrac{x : [a], x : p, x : q, y : \overline{[a]}, y : p, [a] : x, [a] : p \wedge q \Rightarrow x : \Box p, \overline{[a]} : p \wedge q, y : q}{x : [a], x : p \wedge q, y : \overline{[a]}, y : p, [a] : x, [a] : p \wedge q \Rightarrow x : \Box p, \overline{[a]} : p \wedge q, y : q}\ \wedge\text{L}}{x : [a], y : \overline{[a]}, y : p, [a] : x, [a] : p \wedge q \Rightarrow x : \Box p, \overline{[a]} : p \wedge q, y : q}\ \text{L}\Vdash^{\forall}}{y : \overline{[a]}, y : p, [a] : x, [a] : p \wedge q \Rightarrow x : \Box p, \overline{[a]} : p \wedge q, y : q}\ \text{cmp}$$

On the basis of the three open branches we define three models following the definition of Theorem 5.2. The branch $\mathcal{C}_1$ gives the model $\mathcal{M}_1 = \langle W, \mathcal{N}_1, V_1 \rangle$, where $W = \{x, y\}$, $\mathcal{N}_1(x) = \{\{x\}\}$, $\mathcal{N}_1(y) = \emptyset$, $V_1(p) = \{x, y\}$ and $V_1(q) = \{x\}$. The branch $\mathcal{C}_2$ gives the model $\mathcal{M}_2 = \langle W, \mathcal{N}_2, V_2 \rangle$, where $W = \{x, y\}$, $\mathcal{N}_2(x) = \{\emptyset\}$, $\mathcal{N}_2(y) = \emptyset$, $V_2(p) = \{y\}$, and $V_2(q) = \emptyset$. Finally, $\mathcal{C}_3$ gives the same model of $\mathcal{C}_2$. It is immediate to verify that they are countemodels to the sequent at the root. Observe that $\mathcal{M}_2$ is the model $\mathcal{M}^{\mathcal{S}}$ of Example 4.12.

It is instructive to compare this example with the countermodels provided by the (rather complicated) decision procedure given by Lavendhomme and Lucas [9] (Example pp. 137-139). The first model they obtain is the following (after renaming variables): $\mathfrak{M} = \langle W, \mathcal{N}, V \rangle$ where $W = \{x, y\}$, $\mathcal{N}(x) = \{\{x\}\}$, $\mathcal{N}(y) = \{\{x, y\}, \{x\}\}$, $V(p) = \{x, y\}$ and $V(q) = \{x\}$. The second model is the same as $\mathfrak{M}$ except for $\mathcal{N}(y) = \{\{x\}\}$. Both models are very similar to our model $\mathcal{M}_1$, however $\mathcal{M}_1$ is simpler as $\mathcal{N}_1(y) = \emptyset$. This is essentially due to the fact that we do not need to saturate worlds with respect to boxed subformulas as in the procedure given in [9].

6 Conclusion

In this paper, we have proposed labelled calculi for the cube of basic non-normal modal logic. The calculi are based on bi-neighbourhood models, a variation of the standard neighbourhood models, where each world is equipped with a set of pairs of neighbourhoods. The two components of a pair provide separate positive and negative support for a formula. This semantics might be of independent interest, being perhaps more natural for logics without monotonicity. We have shown that this semantics characterises all non-normal modal logics and (in the non-monotonic case) a standard model can be directly built from a bi-neighbourhood one. The sequent calculi we propose are fully modular and standard. For logics containing axiom C we actually propose two versions of the calculi: the first allows a syntactic proof of cut admissibility, whereas the second handles a more restricted form of sequents and comprises more efficient rules for handling intersections of neighbourhoods. In any case, the calculi provide a decision procedure for the respective logics and they are semantically complete: from any failed derivation of a formula one can effectively (and easily) extract a countermodel, both a bi-neighbourhood and a standard one, of the formula. A number of issues deserve to be further investigated: first we aim to study how to get optimal decision procedures from the calculi. We then plan to study how our calculi are related to other proof systems known in the literature, in particular the calculi proposed in [9] and the structural calculi proposed recently in [10]. We also intend to extend our approach, both the bi-neighbourhood semantics and the calculi, to stronger non-normal modal logics determined by the analogous ones of the normal cube from **K** to **S5** and to logical systems below **E**. Finally, it might be useful to draw a detailed comparison between bi-neighbourhood semantics and *bi-lattice semantics* since there is a resemblance between the two and the latter has recently been provided with a display proof system in [4]. All these topics will be object of our future work.

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