The duality of classical and constructive notions and proofs

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Abstract: Previous work gave a method of converting mathematical axioms into rules that extend the logical rules of Gentzen's sequent calculus. The method reveals a perfect duality between classical and constructive basic notions, such as equality and apartness, and between the respective rules for these notions. Derivations with the mathematical rules of a constructive theory are specular duals of corresponding classical derivations. The class of geometric theories is among those convertible into rules, through the use of variable conditions, and the duality defines a new class of "co-geometric" theories. Examples of such theories are projective and affine geometry with the standard basic notions of equality and incidence.

The rules of classical sequent calculus are invertible, which has for quantifier-free theories the effect that logical rules in derivations can be permuted to apply after the mathematical rules. In the case of mathematical rules involving variable conditions, this separation of logic does not always hold, because quantifier rules may fail to permute down. A sufficient condition for the permutability of mathematical rules is determined and applied to give an extension of Herbrand's theorem from universal to geometric and co-geometric theories.

1. Introduction

A constructive approach to the real numbers uses the apartness of two real numbers as a basic relation. The axioms for this relation, written $a \neq b$, are as follows:

AP1. $\sim a \neq a$,
AP2. $a \neq b \supset a \neq c \vee b \neq c$.

Substituting a for c in AP2, we get $a \neq b \supset a \neq a \vee b \neq a$, so that symmetry of apartness follows by AP1. Equality is a defined notion:

EQDEF $\quad a = b \equiv \sim a \neq b$.

By AP1, equality is reflexive. By the contraposition of symmetry of apartness, we have also symmetry of equality. By AP2 and symmetry of apartness, we have $a \neq b \supset a \neq c \vee c \neq b$, so contraposition gives transitivity of equality.

If instead of the constructively motivated notion of apartness we take equality as a basic notion, with its standards properties of reflexivity, symmetry, and transitivity, apartness can be defined by

APDEF $\quad a \neq b \equiv \sim a = b$.

Irreflexivity and symmetry of apartness follow. For the "splitting" property of an apartness $a \neq b$ into two cases $a \neq c \vee b \neq c$, the contraposition of transitivity of equality gives

$$\sim a = b \supset \sim (a = c \ \&\ c = b)$$

To distribute negation inside the conjunction, classical logic is needed.

The above game with classical and constructive notions can be carried further. In von Plato (1995), the basic relations of constructive elementary geometry were treated. (Incidentally, apartness relations were used in geometry already in Heyting's doctoral dissertation of 1925, see Heyting 1927.) The parallelism of two lines is a classical basic relation, and its constructive counterpart is the "convergence" of two lines l and m, written $l \nparallel m$. The axioms are as for the apartness relation above.

The intuition with constructive basic notions is that the classical notions such as equality are "infinitely precise," whereas apartness, if it obtains, can be verified by a finite computation. Something of this intuition can be seen already in Brouwer's first ideas on the topic of apartness relations of 1924, where it is required that the set of objects considered be continuous. This was certainly the intention with Brouwer's constructive real numbers and with Heyting's constructive synthetic geometry. A set is defined as discrete if it has a decidable equality relation, otherwise it is continuous. The constructive interpretation of the law of excluded middle for equality, $a = b \vee \sim a = b$, is precisely that the basic set of objects is discrete. With such sets, it makes no difference which relations are used as basic, the constructive or classical ones, as the axioms are interderivable. (Incidentally, we have here an argument against the creation of a special "intuitionistic notation," such as $a \# b$, parallel to the standard classical one. Such extra symbolism turns out redundant in the discrete case. All we need is to slash the standard symbols for relations.)

In a contribution to the previous Venice conference of 1999, von Plato (2001), the constructivization of elementary axiomatics was extended to lattice theory. It then seemed that proofs that use apartness relations would be harder to find than corresponding classical proofs (see especially theorem 7.1 and the discussion on p. 196.) It has turned out, however, that there is an automatic bridge between classical and constructive notions and proofs. The matter is best seen on a formal level if for the representation of proofs Gentzen's sequent calculus is used. The method we shall apply was found in connection with a proof-theoretical investigation of apartness relations Negri (1999), and generalized to theories with universal axioms in Negri and von Plato (1998) (a later paper that appeared earlier). The duality of classical and constructive notions and proofs was used first in a study of order relations, in Negri, von Plato, and Coquand (2004).

2. From mathematical axioms to mathematical rules

Sequents are expressions of the form $\Gamma \to \Delta$ in which Γ and Δ are finite lists of formulas with order disregarded (i.e., finite multisets). The reading is that Δ gives the possible (open) cases that are derivable under the (open) assumptions Γ. In logical symbolism, with $\&\Gamma$ the conjunction of formulas in Γ and $\vee\Delta$ the disjunction of formulas in Δ, the sequent $\Gamma \to \Delta$ expresses the derivablity of the formula $\&\Gamma \supset \vee\Delta$.

The logical rules of sequent calculus show how assumptions in the left, antecedent part, and cases in the right, succedent part of a sequent can be modified by the logical operations. We show only the rules for the connectives:

$$\frac{A, B, \Gamma \to \Delta}{A\&B, \Gamma \to \Delta}\,L\& \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A\&B}\,R\&$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\,L\vee \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\,R\vee$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\,L\supset \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\,R\supset$$

$$\frac{\Gamma \to \Delta, A}{\sim A, \Gamma \to \Delta}\,L\sim \qquad \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \sim A}\,R\sim$$

Table 1. The logical rules of classical sequent calculus.

We observe that the rule pairs $L\& - R\vee$, $L\vee - R\&$, and $L\sim - R\sim$ display a left-right mirror image duality.

The most remarkable property of these rules of classical logic is their invertibility: If a sequent of any of the forms given in a conclusion of a rule is derivable, the sequents we find in the corresponding premisses are also derivable. The other way, from the premisses to the conclusion is licensed by the rules themselves. Therefore, given a sequent $\Gamma \to \Delta$, we can decompose its formulas and get simpler sequents that together are equiderivable with the given sequent, until we arrive at topsequents of a derivation tree in which there is nothing to decompose left. If each of these leaves is an initial sequent, one that has a common atomic formula (atom) on both sides of the arrow, the endsequent $\Gamma \to \Delta$ is derivable by the rules given in Table 1, otherwise it is underivable.

Many axiom systems consist of universal formulas or, equivalently, of formulas of propositional logic with free parameters in the atoms. Each such formula is (at least classically) equivalent to a finite number of implications of the form

$$P_1 \& \ldots \& P_m \supset Q_1 \vee \ldots \vee Q_n \qquad (2.1)$$

with P_i, Q_j atoms and $m, n \geq 0$. Special cases are $m = 0$, with (2.1) reduced to $Q_1 \vee \ldots \vee Q_n$, and $n = 0$, with (2.1) reduced to $\sim(P_1 \& \ldots \& P_m)$. Formula (2.1) can be converted into a sequent calculus rule in two ways: One is based on the idea that if each of the Q_j, together with other assumptions Γ, is sufficient to derive a number of cases Δ, as expressed formally by the sequent $Q_j, \Gamma \rightarrow \Delta$, then the P_i together are sufficient. We have the schematic rule

$$\frac{Q_1, \Gamma \rightarrow \Delta \quad \ldots \quad Q_n, \Gamma \rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \rightarrow \Delta}\textit{L-rule} \qquad (2.2)$$

A dual scheme says that if each of the P_i follows as a case from some assumptions Γ, then the Q_j follow as cases:

$$\frac{\Gamma \rightarrow \Delta, P_1 \quad \ldots \quad \Gamma \rightarrow \Delta, P_m}{\Gamma \rightarrow \Delta, Q_1, \ldots, Q_n}\textit{R-rule} \qquad (2.3)$$

The rules that act on the left, or antecedent, part of sequents are best seen in a root-first order. Assume given a system of such left rules and assume that $\Gamma \rightarrow \Delta$ contains only atoms. Now try matching $\Gamma \rightarrow \Delta$ as a conclusion to the rules. Whenever there is a match, we have premisses that each get one additional atom in the antecedent. Thus, in the end, we obtain the deductive closure of Γ relative to the rules, with branchings into several possible closures each time there is more than one premiss. The succedent Δ remains untouched by the rules.

One special case merits attention, namely that of $n = 1$. We can then limit the rules to have just one formula in the succedent. A sequent $\Gamma \rightarrow P$ is derivable if and only if P belongs to the deductive closure of Γ relative to the rules.

Two more details need to be added before we can make an overall statement:

1. The formulas $P_1, \ldots, P_m$ in the assumption part of the conclusion of scheme (2.2) have to be repeated in each of the premisses, and dually for $Q_1, \ldots, Q_n$ in scheme (2.3). The intuitive justification is that if $P_1, \ldots, P_m$ are among the assumptions in the endsequent, they can be permitted as assumptions anywhere else, and dually for $Q_1, \ldots, Q_n$.

2. It can happen that instantiation of free parameters in atoms produces a duplication (two identical atoms in the conclusion of a rule instance), say

$$P_1, \ldots, P, P, \ldots, P_m, \Gamma \rightarrow \Delta$$

By condition 1, each premiss has the duplication. We now require that the rule with the duplication P, P *contracted* into a single P is added to the system of rules. For each axiom system, there is only a bounded number of possible cases of contracted rules to be added, very often none at all.

Detailed explanations of the need for conditions 1 and 2 can be found in Negri and von Plato (1998, 2001).

Let us assume given a system Ax with a finite number of axioms of the form (2.1). Let HAx be the axiomatic system Ax together with a standard axiomatic system of classical logic. Let G3* be Gentzen's system G3 of Table 1 extended with the (left or right) sequent calculus rules and their contracted forms as determined by the axioms Ax. We have:

Theorem 1. *The system* G3* *is complete, i.e.,* $\Gamma \rightarrow \Delta$ *is derivable in* G3* *if and only if* $\&\Gamma \supset \vee\Delta$ *is derivable in* HAx.

A proof can be found in Negri and von Plato (1998, 2001). It is easily seen, by the invertibility of the logical rules of G3, that instances of logical rules permute down relative to the mathematical rules.

3. Derivations in left and right rule systems

We shall show the duality of classical and constructive notions and proofs through examples that are easily seen to be representative of the general situation. Consider the theory of apartness. Its two axioms convert into the system of left rules

$$\frac{}{a \neq a, \Gamma \rightarrow \Delta}\,\mathit{Irref} \qquad \frac{a \neq c, a \neq b, \Gamma \rightarrow \Delta \quad b \neq c, a \neq b, \Gamma \rightarrow \Delta}{a \neq b, \Gamma \rightarrow \Delta}\,\mathit{Split}$$

Symmetry of apartness is expressed by the sequent $\rightarrow a \neq b \supset b \neq a$ and has the derivation

$$\frac{\dfrac{\dfrac{}{a \neq a, a \neq b \rightarrow b \neq a}\,\mathit{Irref} \qquad b \neq a, a \neq b \rightarrow b \neq a}{a \neq b \rightarrow b \neq a}\,\mathit{Split}}{\rightarrow a \neq b \supset b \neq a}\,R\supset \tag{3.1}$$

Now take rules *Irref* and *Split* and move all atoms to the other side by rule $R\sim$ of classical sequent calculus. Next write $a = b$ for $\sim a \neq b$, etc. The result can be written as the two rules for equality

$$\frac{}{\Gamma \rightarrow \Delta, a = a}\,\mathit{Ref} \qquad \frac{\Gamma \rightarrow \Delta, a = b, a = c \quad \Gamma \rightarrow \Delta, a = b, b = c}{\Gamma \rightarrow \Delta, a = b}\,\mathit{ETr}$$

Here *ETr* stands for "Euclidean transitivity," from the way transitivity is expressed by Euclid.

With our example derivation, switch atoms on the left and right sides of the arrow, erase the slashes, and change the rule names to get

$$\frac{\dfrac{\dfrac{}{b = a \rightarrow a = b, a = a}\,\mathit{Ref} \qquad b = a \rightarrow a = b, b = a}{b = a \rightarrow a = b}\,\mathit{ETr}}{\rightarrow b = a \supset a = b}\,R\supset \tag{3.2}$$

The sequents in the mathematical part of derivation (3.2) are perfect mirror images of those in derivation (3.1).

Next we convert the two axioms of an apartness relation into a system of right rules:

$$\frac{\Gamma \rightarrow \Delta, a \neq a}{\Gamma \rightarrow \Delta}\,{\scriptstyle Irref} \qquad \frac{\Gamma \rightarrow \Delta, a \neq c, b \neq c, a \neq b}{\Gamma \rightarrow \Delta, a \neq c, a \neq b}\,{\scriptstyle Split}$$

The symmetry of apartness now has the derivation

$$\cfrac{\cfrac{\cfrac{a \neq b \rightarrow b \neq a, a \neq a, a \neq b}{a \neq b \rightarrow b \neq a, a \neq a}\,{\scriptstyle Split}}{a \neq b \rightarrow b \neq a}\,{\scriptstyle Irref}}{\rightarrow a \neq b \supset b \neq a}\,{\scriptstyle R\supset} \tag{3.3}$$

The mirror image left rules for equality are

$$\frac{a = a, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\,{\scriptstyle Ref} \qquad \frac{a = b, a = c, b = c, \Gamma \rightarrow \Delta}{a = c, b = c, \Gamma \rightarrow \Delta}\,{\scriptstyle ETr}$$

Symmetry is derived by the mirror image of derivation (3.3):

$$\cfrac{\cfrac{\cfrac{a = b, a = a, b = a \rightarrow a = b}{a = a, b = a \rightarrow a = b}\,{\scriptstyle Etr}}{b = a \rightarrow a = b}\,{\scriptstyle Ref}}{\rightarrow b = a \supset a = b}\,{\scriptstyle R\supset} \tag{3.4}$$

There are thus two kinds of systems of rules of equality, and the same for apartness. Euclidean equality has axioms that are Harrop formulas, i.e., have no disjunctions in their positive parts. As a consequence, derivations with the two rules of this theory are linear, with just one premiss. Also the mirror image right theory of apartness has linear derivations. It could be called a "co-Harrop" theory, with axioms that have no conjunctions in their negative parts. (See Negri and von Plato 2001 for the notions of Harrop formula and theory, and positive and negative parts of formulas.)

The above examples of rules and derivations are fully representative of the general situation: We can take the left rule scheme (2.2) and convert it into a right rule scheme (2.3) in exactly the same way as in the examples, with a change in the basic notions from constructive to classical or the other way around. The question remains what, if anything, is gained by the constructivization of classical elementary axiomatic theories; Combinatorially, for each derivation in a constructive system of rules, there is a dual classical derivation and vice versa.

4. Geometric and co-geometric axioms and rules

It is possible to extend the left and right rule schemes (2.2) and (2.3) by allowing in their active formulas the occurrence of free variables subject to a variable condition, and thus to express the role of quantifiers in a "logic free" way.

The left rule scheme takes the form

$$\frac{\overline{Q}_1(y_1/x_1), \overline{P}, \Gamma \to \Delta \quad \dots \quad \overline{Q}_n(y_n/x_n), \overline{P}, \Gamma \to \Delta}{\overline{P}, \Gamma \to \Delta} {\scriptstyle GRS} \tag{4.1}$$

where $\overline{Q}_j$ and $\overline{P}$ indicate the multisets of atomic formulas $Q_{j_1}, \dots Q_{j_{k_j}}$ and $P_1, \dots, P_m$, respectively, and the eigenvariables $y_1, \dots, y_n$ of the premisses satisfy the condition of not having free occurrence in the conclusion of the scheme. This variable condition is the same as for the rules $R\forall$ and $L\exists$ of first-order logic.

A rule scheme of the form (4.1) expresses as a rule an axiom, called a *geometric axiom*, of the form

$$\forall \overline{z}(P_1 \& \dots \& P_m \supset \exists x_1 M_1 \vee \dots \vee \exists x_n M_n)$$

where M_j is the conjunction of the multiset $\overline{Q}_j$, $Q_{j_1} \& \dots \& Q_{j_{k_j}}$. Finite conjunctions of geometric axioms lead to the class of formulas, called *geometric implications*, that are sentences of the form

$$\forall \overline{z}(A \supset B)$$

where A and B are formulas not containing $\supset$ or $\forall$.

The geometric rule scheme, introduced in Negri (2003), permits the extension of structural proof analysis from the first-order system G3c to *geometric theories*, that is, theories axiomatized by geometric implications. In particular, this method permits to present geometric theories as contraction- and cut-free sequent systems and to obtain what is undoubtedly the simplest possible proof of *Barr's theorem*: If a geometric implication is provable classically in a geometric theory, then it is provable intuitionistically. The proof of this conservativity result consists in noting that a classical cut-free proof of a geometric implication is already an intuitionistic proof in sequent systems with rules for geometric theories.

We observe that Barr's theorem is not a characterization of the intuitionistic fragment of geometric theories, because we can go beyond geometric implications and maintain the conservativity result. First, following Dragalin's suggestion (cf. sec. 3.7.3 in Troelstra and Schwichtenberg 2000) we can modify the intuitionistic left rule for implication by admitting a multi-succedent conclusion in the left premiss

$$\frac{A \supset B, \Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta} {\scriptstyle L\supset}$$

Rule $L\supset$ of the classical calculus (without $A \supset B$ in the left premiss) is then admissible in the modified intuitionistic calculus, thus the difference between the intuitionistic and classical sequent systems is confined to $R\supset$ and $R\forall$. An operational definition of formulas for which the conservativity of classical derivations holds can be given: If a formula is derivable classically in a geometric theory and the derivation contains no steps of $R\supset$ and $R\forall$ with a nonempty context in the premiss, then the derivation is an intuitionistic derivation. However, this is an empty characterization, stating nothing but that "an intuitionistic derivation is an intuitionistic derivation." A characterization in terms of the form of the formulas alone, not of their derivations, would be desirable. There are classes of formulas, such as geometric implications, the form of which forces the derivation to be of the stated kind. The same is true, for example, if the formula does not contain in its positive part implications or universally quantified formulas as components of a disjunction. Even so, there are still formulas outside the mentioned classes for which the conservativity holds.

Examples of geometric theories include the theory of real-closed fields, Robinson arithmetic, and constructive affine geometry as given in 6.6 (e) of Negri and von Plato (2001).

In order to obtain a geometric axiomatization some care is needed when formulating the axioms: For instance, the axiom stating the existence of inverses on nonzero elements in the theory of fields,

$$\sim x = 0 \supset \exists y\ x \cdot y = 1$$

is not geometric as it contains an implication the antecedent of which is an implication ($x = 0 \supset \bot$), but it can be replaced by the geometric axiom

$$x = 0 \vee \exists y\ x \cdot y = 1$$

that can be given as a rule following the geometric rule scheme

$$\frac{x = 0, \Gamma \rightarrow \Delta \quad x \cdot y = 1, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\ \text{L-}inv$$

with the variable condition y not free in Γ, Δ.

Alternatively, we can take inequality $\neq$ as the primitive relation and turn L-*inv* into the following right rule with the same variable condition on y,

$$\frac{\Gamma \rightarrow \Delta, x \neq 0 \quad \Gamma \rightarrow \Delta, x \cdot y \neq 1}{\Gamma \rightarrow \Delta}\ \text{R-}inv$$

corresponding to the axiom $\sim \forall y(x \neq 0\ \&\ x \cdot y \neq 1)$.

All the other axioms for fields and real-closed fields can be given in terms of right rules for the primitive relation of inequality.

A similar transformation can be done with the axioms of constructive affine geometry. These axioms, presented in 6.6.(e) of Negri and von Plato (2001), are based on the primitive notions of distinct points, $a \neq b$, distinct lines, $l \neq m$, convergent lines, $l \nparallel m$, and of a point outside a line, $a \notin l$, and on the constructions of a line $ln(a,b)$ connecting two distinct points a and b, and of a point $pt(l,m)$ obtained as the intersection of two convergent lines l and m.

We observed in Negri (2003) that the extension with the axiom stating the existence of three non-collinear points,

$$\exists x \exists y \exists z (x \neq y \ \& \ z \notin ln(x,y))$$

maintains the theory geometric as the axiom corresponds to the following instance of the geometric rule scheme

$$\frac{x \neq y, z \notin l(x,y), \Gamma \to \Delta}{\Gamma \to \Delta}$$

with the variable condition x, y, z not free in Γ, Δ.

If the axiomatization is instead based on the primitive relation of equality of points, equality of lines, parallelism of lines, and incidence a point with a line, the axiom becomes

$$\exists x \exists y \exists z (\sim x = y \ \& \ \sim z \in ln(x,y))$$

which is no longer a geometric implication. Thus, as concluded in Negri (2003), "classical geometry is not a geometric theory." However, the axiom can be given in the form of the right rule, with the condition x, y, z not free in Γ, Δ,

$$\frac{\Gamma \to \Delta, x = y, z \in l(x,y)}{\Gamma \to \Delta}$$

by which

$$\sim \forall x \forall y \forall z (x = y \vee z \in ln(x,y))$$

is derivable. All the other axioms can also be uniformly presented as right rules for the primitive relations $a = b$, $l = m$, $l \parallel m$, and $a \in l$.

The above examples illustrate a general result:

Theorem 2. *Let* T *be a geometric theory based on the primitive relations* R_i*, with rules following the geometric rule scheme GRS, and let* T' *be the theory obtained by formulating the axioms in terms of the dual relations* R_i'*. Then a contraction- and cut-free system for the theory* T' *is obtained by turning all the instances of GRS into the form*

$$\frac{\Gamma \to \Delta, \overline{P}', \overline{Q}_1'(y_1/x_1) \quad \dots \quad \Gamma \to \Delta, \overline{P}', \overline{Q}_n'(y_n/x_n)}{\Gamma \to \Delta, \overline{P}'} \ \textit{co-GRS}$$

where the apices indicate the atoms transformed in terms of the dual relations R'_i.

We can ask what kinds of axioms are captured by the scheme *co-GRS*. Clearly, the scheme is interderivable with an axiom of the form

$$\forall \overline{z}(\forall x_1 M'_1 \& \ldots \& \forall x_n M'_n \supset P'_1 \vee \ldots \vee P'_m) \qquad co\text{-}GA$$

where $M'_j \equiv Q'_{j_1} \vee \ldots \vee Q'_{j_{k_j}}$.

It is easy to verify that any formula of the form

$$\forall \overline{z}(A \supset B)$$

with A and B formulas not containing $\supset$ or $\exists$, can be brought to a canonical form consisting of conjunctions of formulas of the form given by *co-GA*. Formulas A, B not containing $\supset$ or $\exists$ will be called *co-geometric* and the implication $A \supset B$ a *co-geometric implication.* A theory axiomatized by co-geometric implications will be called a *co-geometric theory.* Classical projective and affine geometry with the axiom of non-collinearity included constitute examples of co-geometric theories.

The above examples have shown how the duality between geometric and co-geometric theories can be used for changing the primitive notions in the sequent formulation of a theory. Meta-theoretical results can be imported from one theory to its dual by exploiting the symmetry of their associated sequent calculi.

In Negri and von Plato (2001) an extension of Herbrand's theorem to universal theories is presented. We recall the statement:

Theorem 3. Herbrand's theorem for universal theories. *Let* T *be a theory with a finite number of purely universal axioms and let* G3cT *be the sequent system obtained by turning the theory into a system of mathematical rules. If the sequent* $\rightarrow \forall x \exists y_1 \ldots \exists y_k A$*, with* A *quantifier free, is derivable in* G3cT, *then there are terms* t_{i_j} *with* $i \leqslant n, j \leqslant k$ *such that*

$$\rightarrow \bigvee_{i=1}^{n} A(t_{i_1}/y_1, \ldots, t_{i_k}/y_k)$$

is derivable in G3cT.

Clearly, the theorem does not extend to geometric theories. In fact, if $\exists x P$ is an axiom of the theory T, then $\rightarrow \exists x P$ is derivable in G3cT but there is no finite disjunction such that $\rightarrow P(t_1) \vee \ldots \vee P(t_n)$ is derivable in G3cT.

The crucial ingredient in the proof of Herbrand's theorem is the possibility to assume a derivation in which the quantifier rules come last. In first-order logic and in universal theories, this is unproblematic. With mathematical rules involving variable conditions, like the geometric or the co-geometric rule scheme, the quantifier rules cannot in general be permuted last in a derivation. Suppose we have a derivation containing the steps

$$\cfrac{\cfrac{\overline{Q}_1(y_1/x_1),\overline{P},\Gamma\to\Delta,\exists xA,A(t/x)}{\overline{Q}_1(y_1/x_1),\overline{P},\Gamma\to\Delta,\exists xA}\,R\exists \quad \dots \quad \overline{Q}_n(y_n/x_n),\overline{P},\Gamma\to\Delta,\exists xA}{\overline{P},\Gamma\to\Delta,\exists xA}\,GRS$$

If the term t contains the variable y_1, the permutation of $R\exists$ to below GRS fails because the variable condition for a correct application of GRS would no longer be satisfied. This is the exact structural reason for the failure of Herbrand's theorem for existential theories. We can nevertheless impose an additional hypothesis that makes the permutation possible. The hypothesis ensures that a fresh variable substitution *limited to the atoms* $\overline{Q}_1$ is possible.

Lemma 4. *Let* T *be a geometric theory and let* G3cT *be the sequent system obtained by turning the theory into a system of left nonlogical rules. Suppose that the sequent* $\overline{Q}_i(y_i/x_i),\overline{P},\Gamma\to\Delta,A(t/x)$ *is derivable in* G3cT, *that* y_i *is not free in* Γ,Δ, *and that no atom* $\overline{Q}_i$ *occurs positively in* A. *Then* $\overline{Q}_i(z/x_i),\overline{P},\Gamma\to\Delta,A(t/x)$ *is derivable for an arbitrary fresh variable* z.

Proof: Consider the initial sequents in a derivation of the given sequent. By the assumptions that y_i does not occur free in Δ and that no atom among the $\overline{Q}_i$ is in the positive part of A, it follows that the principal atoms of the axioms are not among the $\overline{Q}_i$. Thus, after the substitution of the variable y_i with a fresh variable z in the atoms $\overline{Q}_i(y_i/x_i)$, the leaves of the tree remain initial sequents, and the logical steps remain correct because the atoms in $\overline{Q}_i$ are never principal in logical rules. Since z is a fresh variable, also the instances of the geometric rule scheme remain correct, thus the sustitution produces a derivation of $\overline{Q}_i(z/x_i),\overline{P},\Gamma\to\Delta,A(t/x)$ in G3cT. QED

By the lemma, under the additional hypothesis of non-occurrence of the atoms $\overline{Q}_i$ in positive parts of A, we can assume a derivation where the mathematical rules come first, followed by propositional rules, followed by a linear part consisting of quantifier rules. The rest of the proof of Herbrand's theorem is then a routine matter. Thus we have:

Theorem 5. Herbrand's theorem for geometric theories. *Let* T *be a geometric theory and let* G3cT *be the sequent system obtained by turning the theory into a system of nonlogical rules following the geometric rule scheme* GRS. *If the sequent* $\to\forall x\exists y_1\dots\exists y_kA$, *with* A *quantifier-free, is derivable in* G3cT *and no atom* $\overline{Q}_i$ *occurs positively in* A, *then there are terms* t_{i_j} *with* $i\leqslant n, j\leqslant k$ *such that*

$$\to\bigvee_{i=1}^{n}A(t_{i_1}/y_1,\dots,t_{i_k}/y_k)$$

is derivable in G3cT.

By exploiting the symmetry between a left and a right rule system we obtain the corresponding results for co-geometric theories.

Lemma 6. *Let* T *be a co-geometric theory and let* G3cT *be the sequent system obtained by turning the theory into a system of right nonlogical rules. Suppose the sequent* $\Gamma \rightarrow \Delta, \overline{Q}_i(y_i/x_i), \overline{P}, A(t/x)$ *is derivable in G3cT,* y_i *is not free in* Γ, Δ*, and no atom* $\overline{Q}_i$ *occurs negatively in* A*. Then* $\Gamma \rightarrow \Delta, \overline{Q}_i(z/x_i), \overline{P}, A(t/x)$ *is derivable for an arbitrary fresh variable* z*.*

Theorem 7. Herbrand's theorem for co-geometric theories. *Let* T *be a co-geometric theory and let* G3cT *be the sequent system obtained by turning the theory into a system of right nonlogical rules following the co-geometric rule scheme co-GRS. If the sequent* $\rightarrow \forall x \exists y_1 \ldots \exists y_k A$*, with* A *quantifier-free, is derivable in* G3cT *and no atom* $\overline{Q}_i$ *occurs negatively in* A*, then there are terms* t_{i_j} *with* $i \leqslant n, j \leqslant k$ *such that*

$$\rightarrow \bigvee_{i=1}^{n} A(t_{i_1}/y_1, \ldots, t_{i_k}/y_k)$$

is derivable in G3cT.

5. Duality of dependent types and degenerate cases

The axiomatization of elementary geometry with constructive basic notions leads in a natural way to *dependent typing*: A formula with a constructed line $ln(a,b)$, such as the incidence axiom $a \in ln(a,b)$, is well-formed only if the condition of non-degeneracy $a \neq b$ is satisfied. In a first-order formulation, incidence axioms with conditions of nondegeneracy can be given as implications, as in von Plato (1995). For projective geometry, we have

$$a \neq b \supset {\sim} a \notin ln(a,b), \quad a \neq b \supset {\sim} b \notin ln(a,b),$$

and similarly for intersection points. The corresponding left rule for the first axiom is the zero-premiss rule

$$\overline{a \neq b, a \notin ln(a,b), \Gamma \rightarrow \Delta}^{\,Inc}$$

By the duality of left and right rules, we have for the classical notions of equality and incidence the rule

$$\overline{\Gamma \rightarrow \Delta, a = b, a \in ln(a,b)}^{\,Inc}$$

Thus, the incidence axioms for connecting lines in a classical formulation are

$$a = b \vee a \in ln(a,b), \quad a = b \vee b \in ln(a,b),$$

and similarly for the rest of the incidence axioms. The *degenerate cases* $a = b$ in these axioms are the classical duals of dependent typings in constructive geometry. The phenomenon is quite general; similar observations could be made about the condition for the inverse operation.

The use of constructions is strictly necessary for the conversion of mathematical axioms into systems of cut-free rules, be it a system based on classical or constructive notions. To see this, we formulate elementary geometry as a *relational theory* with existential axioms in place of construtions, as in

$$\forall x \forall y \exists z (x \in z \ \& \ y \in z).$$

(The sorts of the variables are determined from their places in the incidence relation: x and y points, z a line.) Next, uniqueness axioms are added, such as

$$\forall x \forall y \forall z \forall v (x \in z \ \& \ y \in z \ \& \ x \in v \ \& \ y \in v \supset z = v).$$

As mentioned above, it is possible to formulate geometry, the axiom of noncollinearity included, either as a constructive geometric theory, or as a classical co-geometric theory. This result refers to a formulation with geometric constructions. With a relational formulation, a comparison of the form of the existential axioms that replace constructions with the form of the axiom of noncollinearity leads instead to the following result:

If noncollinearity is formulated as a geometric implication, the existence axioms are co-geometric; if the existence axioms instead are geometric, noncollinearity is co-geometric.

There is thus a fundamental incompatibility in both approaches, and it can be overcome only through the use of constructions. This phenomenon is quite general and is met in, for example, lattice theory (as in Negri and von Plato 2004), and in field theory.

References

Brouwer, L. (1924) Intuitionistische Zerlegung mathematischer Grundbegriffe, as reprinted in Brouwer's Collected Works, vol. 1, pp. 275–280.

Heyting, A. (1927) Zur intuitionistischen Axiomatik der projektiven Geometrie, *Mathematische Annalen*, vol. 98, pp. 491–538.

Negri, S. (1999) Sequent calculus proof theory of intuitionistic apartness and order relations, *Archive for Mathematical Logic*, vol. 38, pp. 521–547.

Negri, S. (2003) Contraction-free sequent calculi for geometric theories with an application to Barr's theorem, *Archive for Mathematical Logic*, vol. 42, pp. 389–401.

Negri, S. and J. von Plato (1998) Cut elimination in the presence of axioms, *The Bulletin of Symbolic Logic*, vol. 4 , pp. 418–435.

Negri, S. and J. von Plato (2001) *Structural Proof Theory.* Cambridge University Press.

Negri, S. and J. von Plato (2004) Proof systems for lattice theory, *Mathematical Structures in Computer Science*, vol. 14, pp. 000–000.

Negri, S., J. von Plato, and T. Coquand (2004) Proof-theoretical analysis of order relations, *Archive for Mathematical Logic*, vol. 43, pp. 297–309.

von Plato, J. (1995) The axioms of constructive geometry, *Annals of Pure and Applied Logic*, vol. 76, pp. 169–200.

von Plato, J. (2001) Positive lattices, in P. Schuster et al., eds, *Reuniting the Antipodes*, pp. 185–197, Kluwer, Dordrecht 2001.

Troelstra, A. and H. Schwichtenberg (2000) *Basic Proof Theory*, 2nd ed, Cambridge.