A sequent calculus for constructive ordered fields

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Summary: The theory of constructive ordered fields, based on a relation of strict linear order, is formalized as a proof-theoretical system, a sequent calculus extended with nonlogical rules. It is proved that structural rules, the rules of cut and contraction in particular, can be eliminated from derivations. An application of the method of extension by nonlogical rules the theory of real closed fields is presented.

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1. Introduction

The problem of failure of cut elimination for logical systems extended with axioms for specific mathematical theories has so far prevented the application of the methods of structural proof theory to the study of, even elementary, mathematical theories.

In previous work (cf. Negri 1999, Negri and von Plato 1998) a method was found for extending logical sequent calculi with nonlogical rules representing mathematical axioms while maintaining eliminability of the structural rules, cut especially.

Here the method is applied to an axiomatization of constructive ordered fields. The axiomatization is based on a single primitive notion of *constructive linear order* and uses constructions and free parameters instead of quantifiers. A well known problem with such an approach to field theory is that it goes beyond first-order logic. Namely, a proposition containing an inverse x^{-1} is well formed only if it is true that $x \neq 0$. However, the analysis of formal derivations can equally be performed if we treat the condition for the construction of inverses as a meta-level rule of well-formedness of sequents, similarly to other conditions of well-formedness occurring in first-order logic.

After giving the axiomatization for constructive ordered fields, we show how to convert it into a system of sequent calculus nonlogical rules. We then prove cut elimination for the system obtained by adding these rules to the sequent calculus for first-order intuitionistic logic **G3i**.

Similar methods can be applied to the study of the metamathematics of the theory of real closed fields. We eliminate quantifiers already in the axiomatization, by expressing the axioms of existence of square roots of positive elements and of zeros of polynomials of odd degree by means of constructions, rather than by $\forall\exists$-axioms.

2. Axioms for constructive ordered fields

We assume a set R with a primitive relation $a < b$ of strict order, instead of the relation $a \leqslant b$ of weak order. This choice of basic relation is needed in order to secure computational meaning: Intuitively, a finite approximation of a and b is sufficient for the verification of $a < b$, whereas with $a \leqslant b$ the verification may lead to an infinite computation. Negation $\sim A$ is defined as $A \supset \bot$.

The axioms for constructive ordered fields can be grouped as follows:

I. Axioms for constructive linear order:

1. $\sim(a < b \,\&\, b < a)$ (*asymmetry*)
2. $a < b \supset a < c \vee c < b$ (*split*)

Given a relation $a < b$ satisfying the above axioms, the relation $a \neq b$ defined by

$$a \neq b \equiv a < b \vee b < a$$

is an apartness relation. Equality is then defined as the negation of $a \neq b$

$$a = b \equiv \sim a \neq b$$

and it is easy to see that the relation $a = b$ satisfies reflexivity, symmetry and transitivity. Clearly, the relation $a = b$ is stable, that is, it satisfies $\sim\sim a = b \supset a = b$. Substitution in the form

$$a < b \,\&\, b = c \supset a < c$$

follows from split and definition of $a = b$. We argue similarly for $a < b \,\&\, a = c \supset c < b$. The usual weak order is defined by

$$a \leqslant b \equiv \sim b < a$$

Note that classically, the first axiom is equivalent to $a \leqslant b \vee b \leqslant a$. But constructively, any two elements a, b need not be comparable. Contraposition of the second axiom expresses transitivity of weak order.

We stipulate the existence of two distinct elements 0 and 1 in R satisfying

3. $0 < 1$.

We have a binary operation $+$ (addition) and a unary operation $-$ (opposite) satisfying:

II. Axioms of additive group:

4. $a + b = b + a$ (*commutativity*)
5. $(a + b) + c = a + (b + c)$ (*associativity*)
6. $a + 0 = a$ (*zero*)
7. $a + (-a) = 0$ (*opposite*)

We assume a *principle of strong extensionality*

8. $a + b < a + c \supset b < c$

from which the principle of extensionality,

$$b = c \supset a + b = a + c$$

follows by contraposition and definition of equality. We observe that the apparently more general form of extensionality, $a = b \,\&\, c = d \supset a + c = b + d$, follows from the previous one and transitivity of equality.

Monotonicity of addition with respect to the strict linear order, i.e.,

$$a < b \supset a + c < b + c$$

is now provable: From the antecedent, by axioms 7, 8 and substitution we obtain $a + (c + (-c)) < b + (c + (-c))$, hence by associativity and substitution again we have $(a + c) + (-c) < (b + c) + (-c)$, thus by symmetry and strong extensionality $a + c < b + c$. We also have

$$0 < a + b \supset 0 < a \vee 0 < b$$

as follows: From the antecedent, by split, we have $0 < a \vee a < a+b$. By extensionality and the axiom for opposite, the second disjunct gives $0 < b$.

Multiplication of a and b is denoted by ab. For $a \neq 0$ we have an operation of inverse, denoted a^{-1}.

III. Axioms for multiplication:

9. $ab = ba$ (*commutativity*)

10. $(ab)c = a(bc)$ (*associativity*)

11. $a1 = a$ (*unit*)

12. $aa^{-1} = 1$, provided that $a \neq 0$ (*inverse*)

13. $a(b + c) = ab + ac$ (*distributivity*)

Strong extensionality for multiplication is

14. $ab < ac \supset (a < 0 \,\&\, c < b) \vee (0 < a \,\&\, b < c)$

and gives by contraposition and definition of equality the principle of extensionality in multiplication

$$b = c \supset ab = ac$$

thus by transitivity of equality also $a = b \,\&\, c = d \supset ab = cd$.

By distributivity, from $0a = (0 + 0)a$ and extensionality for addition we obtain $0a = 0$. From this we obtain

$$0 < c \supset 0 < c^{-1}$$

as follows: From $0 < 1$, by extensionality we have $0c^{-1} < cc^{-1}$, so by commutativity and strong extensionality together with the hypothesis $0 < c$, the result $0 < c^{-1}$ follows.

We are now ready to prove monotonicity of multiplication with respect to the strict linear order, i.e.,

$$a < b \,\&\, 0 < c \supset ac < bc$$

From $a < b$, by using the unit axiom and extensionality in multiplication we obtain $a(cc^{-1}) < b(cc^{-1})$ so by associativity and substitution $(ac)c^{-1} < (bc)c^{-1}$. By commutativity and strong extensionality, since $0 < c^{-1}$, the result $ac < bc$ follows.

Standard axiomatizations of constructive ordered fields are based on a primitive notion of weak order. Our axiomatization is equivalent to other constructive axiomatizations, except for the axiom of strong extensionality for multiplication. In Palmgren (1998) the theory of contructive ordered fields is presented through a predicate $P(a)$ of strict positivity that corresponds to $0 < a$. The split property, here taken as an axiom, is instead shown to follow from the axiom

$$0 < a + b \supset 0 < a \vee 0 < b$$

The latter is in turn provable in our axiomatization using split and the axiom for opposite.

In Bridges (1999) the Archimedean axiom and a constructive version of the least upper bound principle are added to constructive ordered fields for obtaining an axiomatization of real numbers. These two axiom are not expressible in our approach since the first relies on natural numbers and the second is inherently second order (see the discussion in Truss 1997).

What are known as formal reals (cf. Negri and Soravia 1999) give a model for the axioms of the theory of constructive ordered fields. More generally, constructive reals (e.g. Bishop's reals) constitute a model for the theory.

3. Sequent calculus proof theory and its extension by nonlogical rules

We shall be using an intuitionistic multi-succedent sequent calculus in this work. Atomic formulas are denoted by $P, Q, R, \ldots$ and arbitrary formulas by $A, B, C, \ldots$. The basic expressions are *sequents* of form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite *multisets* of formulas, i.e., sets where the multiplicity of an element is counted. The left and right parts of the sequent are the *antecedent* and the *succedent*. Derivations in sequent calculus start with *logical axioms*, i.e., zero-premiss rules of two forms:

Logical axioms:

$$P, \Gamma \Rightarrow \Delta, P \qquad \bot, \Gamma \Rightarrow \Delta$$

The first axiom corresponds to making the assumption P, the second to the rule *ex falso quodlibet.* The first axiom is restricted to atomic formulas and in the second, it is essential that the false formula $\bot$ is not considered atomic, but a zero-place logical operation.

The logical rules are divided into *right* rules that correspond to the introduction rules of natural deduction, and to *left* rules that correspond to elimination rules.

Logical rules:

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A\&B, \Gamma \Rightarrow \Delta}\ L\& \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A\&B}\ R\&$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{A \supset B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$$

$$\frac{A(t/x), \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall$$

$$\frac{A(y/x), \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A(t/x)}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

In the rules for the quantifiers, $A(t/x)$ $(A(y/x))$ denotes substitution of x with the term t (the variable y) in A. The usual restrictions apply: in $R\forall$ and $L\exists$, y is not free in the conclusion; in $L\forall$ and $R\exists$ the term t has to be *free for x in A*, that is, no variable of t is in the scope of some quantifier in A.

This calculus is designated **G3i** (after Kleene's system, of which it is a simplified version). The calculus is the same as the calculus **GHPC** of Dragalin (1988).

The *height* of a derivation is its height as a tree, that is, the length of its longest branch.

The structural rules of weakening, contraction and cut are formulated as follows:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ RW$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ RC$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut$$

None of these structural rules need be assumed in **G3i** since they can be proved *admissible*, in the sense that if the premisses of a rule are derivable, its conclusion also is derivable. Weakening and contraction are admissible and *height preserving*, that is, if their premisses are derivable with a certain derivation height, then their conclusion are also derivable with the same derivation height (cf. Dragalin 1988). *Exchange* rules, permitting the permutation of order in a list of assumptions as in Gentzen's original calculus, are absent due to the use of multisets. We refer to Negri and von Plato (2001), where the above calculus is called **G3im**, for a detailed proof of admissibility of the structural rules.

A rule is said to be *invertible* if whenever the conclusion is derivable, then the premisses are derivable.

All logical propositional rules of **G3i** except $L \supset$ and $R \supset$ are invertible. The rules for implications are partially invertible in the following sense: $L \supset$ is only invertible with respect to the second premiss, i.e., from $A \supset B, \Gamma \Rightarrow \Delta$, the sequent $B, \Gamma \Rightarrow \Delta$ follows. The rule of $R \supset$ is invertible when the context Δ is empty. All the quantifier rules are invertible except $R\forall$. As for $R \supset$, the rule of $R\forall$ is invertible when the context Δ is empty.

Starting with the calculus **G3i**, the structural rules remain admissible also in extensions of **G3i** by suitably formulated nonlogical rules. Given any formula A, we can check through a formula decomposition via the invertible rules of **G3i** whether the formula is equivalent in the system to a conjunction of formulas of the form $P_1 \& \dots \& P_m \supset Q_1 \vee \dots \vee Q_n$, where the consequent is $\bot$ if $n = 0$. (This is a slightly modified version of conjunctive normal form.) It is shown in Negri and von Plato (1998) that these are precisely the formulas representable as systems of sequent calculus rules for which the structural rules are admissible. Formulas are now converted into rules through the following general

Rule-scheme:

$$\frac{Q_1, \Gamma \Rightarrow \Delta \quad \dots \quad Q_n, \Gamma \Rightarrow \Delta}{P_1, \dots, P_m, \Gamma \Rightarrow \Delta}\; \scriptstyle{Reg}$$

Γ and Δ are arbitrary multisets. Addition of a rule following the scheme will make sequents $\Rightarrow P_1 \& \dots \& P_m \supset Q_1 \vee \dots \vee Q_n$ derivable by the logical rules.

Special cases of the rule-scheme are obtained when n is 0 and the zero-premiss rule *Reg* becomes a *nonlogical axiom*, with inference line omitted,

$$P_1, \dots, P_m, \Gamma \Rightarrow \Delta$$

which translates a Hilbert style axiom of the form $\sim(P_1 \& \dots \& P_m)$.

As explained in Negri and von Plato (1998), the *principal formulas* $P_1, \dots, P_m$ of the scheme must be repeated in the antecedent of each premiss, but we leave this out for better readability. Such repetition is needed for proving the rule of contraction admissible. We also note the following subtlety:

Closure condition: Given a system with nonlogical rules, if it has a rule with an instance of form

$$\frac{Q_1, \Gamma \Rightarrow \Delta \quad \dots \quad Q_n, \Gamma \Rightarrow \Delta}{P_1, \dots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta}\; \scriptstyle{Reg}$$

then also the rule

$$\frac{Q_1, \Gamma \Rightarrow \Delta \quad \dots \quad Q_n, \Gamma \Rightarrow \Delta}{P_1, \dots, P_{m-2}, P, \Gamma \Rightarrow \Delta}\; \scriptstyle{Reg}$$

has to be included in the system. It can happen that a substitution in the atoms of a rule produces duplications of formulas the contraction of which requires the condition. But it is in principle unproblematic, since the number of rules to be added to a given system of nonlogical rules is bounded. Often the closure condition is superfluous, for example, as noted in Negri (1999), the rule expressing irreflexivity in the theory of strict linear order is derivable from the other rules. However, the rule of irreflexivity can be added to the system if height-preserving admissibility (and not just admissibility) of contraction is needed.

Theorem 3.1: *The structural rules are admissible in extensions of* **G3i** *following the rule-scheme and satisfying the closure condition. Weakening and contraction are admissible and height preserving.*

This result is proved in Negri and von Plato (1998, section 3).

The immediate subformula property fails for nonlogical rules, but a *weak subformula property*, stating that in a derivation only subformulas of the endsequent or atomic formulas occur, is enough for proof-analysis. In particular, we have the

Corollary 3.2: *If a derivable sequent* $\Gamma \Rightarrow \Delta$ *has no logical operator, then its derivation uses only logical axioms and nonlogical rules, but no logical or structural rules.*

4. Sequent calculus for constructive ordered fields

In usual sequent calculi, the rules for well-formed sequents are meta-level rules, not explicit rules in derivations. For instance, the variable conditions in quantifier rules, the conditions on terms for substitution, and even the syntactic rules for well-formed formulas are such rules. We shall handle the conditions arising in the theory of constructive ordered fields in the same way. If a sequent $\Gamma \Rightarrow \Delta$ contains a term x^{-1} we assume that the condition $x \neq 0$ has been established.

Another example of conditions is found in elementary geometry. The construction of a line through two points a and b can only be made if a and b are distinct points. In treating the conditions of wellformedness of sequents as meta-level rules we have followed von Plato (1998).

In a more formal treatment, conditions can be made into progressive contexts in the sense of type theory (see Martin-Löf 1984 and von Plato 1995).

The rules for constructive ordered fields are grouped as in the axiomatic presentation. For simplicity of notation we have omitted the repetition of the principal formula (or formulas) of each rule in its premisses. The rule of irreflexivity has to be added in order to satisfy the closure condition.

I Rules for constructive linear order:

$$a < a, \Gamma \Rightarrow \Delta \ \ {}_{irref} \qquad a < b, b < a, \Gamma \Rightarrow \Delta \ \ {}_{asym}$$

$$\frac{a < c, \Gamma \Rightarrow \Delta \quad c < b, \Gamma \Rightarrow \Delta}{a < b, \Gamma \Rightarrow \Delta} \ {}_{split}$$

The axiom $0 < 1$ becomes the rule

$$\frac{0 < 1, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ {}_{nondeg}$$

In order to express the axioms of addition and multiplication in the form of inference rules we first have to formulate the axioms for defined equality in terms of the primitive notion of strict linear order. In each axiom involving equality, $a = b$ is replaced by $\sim (a < b \vee b < a)$ which is in turn replaced by its logical equivalent $\sim a < b \,\&\, \sim b < a$. The latter form of axiom is rendered into two inference rules with zero premisses, namely $a < b, \Gamma \Rightarrow \Delta$ and $b < a, \Gamma \Rightarrow \Delta$. Following the same general method we obtain:

II Rules for additive group:

$$a + b < b + a, \Gamma \Rightarrow \Delta \quad {\scriptstyle +\text{-}comm1} \qquad b + a < a + b, \Gamma \Rightarrow \Delta \quad {\scriptstyle +\text{-}comm2}$$

$$(a + b) + c < a + (b + c), \Gamma \Rightarrow \Delta \quad {\scriptstyle +\text{-}ass1} \qquad a + (b + c) < (a + b) + c, \Gamma \Rightarrow \Delta \quad {\scriptstyle +\text{-}ass2}$$

$$a + 0 < a, \Gamma \Rightarrow \Delta \quad {\scriptstyle zero1} \qquad a < a + 0, \Gamma \Rightarrow \Delta \quad {\scriptstyle zero2}$$

$$a + (-a) < 0, \Gamma \Rightarrow \Delta \quad {\scriptstyle opp1} \qquad 0 < a + (-a), \Gamma \Rightarrow \Delta \quad {\scriptstyle opp2}$$

$$\frac{b < c, \Gamma \Rightarrow \Delta}{a + b < a + c, \Gamma \Rightarrow \Delta} \; {\scriptstyle +\text{-}ext}$$

III Rules for multiplication:

$$ab < ba, \Gamma \Rightarrow \Delta \quad {\scriptstyle \times\text{-}comm1} \qquad ba < ab, \Gamma \Rightarrow \Delta \quad {\scriptstyle \times\text{-}comm2}$$

$$(ab)c < a(bc), \Gamma \Rightarrow \Delta \quad {\scriptstyle \times\text{-}ass1} \qquad a(bc) < (ab)c, \Gamma \Rightarrow \Delta \quad {\scriptstyle \times\text{-}ass2}$$

$$a1 < a, \Gamma \Rightarrow \Delta \quad {\scriptstyle unit1} \qquad a < a1, \Gamma \Rightarrow \Delta \quad {\scriptstyle unit2}$$

$$aa^{-1} < 1, \Gamma \Rightarrow \Delta \quad {\scriptstyle inv1} \qquad 1 < aa^{-1}, \Gamma \Rightarrow \Delta \quad {\scriptstyle inv2}$$

$$a(b + c) < ab + ac, \Gamma \Rightarrow \Delta \quad {\scriptstyle distr1} \qquad ab + ac < a(b + c), \Gamma \Rightarrow \Delta \quad {\scriptstyle distr1}$$

Observe that by symmetry, the rules $+$-*comm2* and $\times$-*comm2* can be dropped.

The axiom of strong extensionality for multiplication gives rise, by using the regular decomposition described in the previous section, to the following four rules:

$$\frac{a < 0, \Gamma \Rightarrow \Delta \quad 0 < a, \Gamma \Rightarrow \Delta}{ab < ac, \Gamma \Rightarrow \Delta} \; {\scriptstyle \times\text{-}ext1} \qquad \frac{c < b, \Gamma \Rightarrow \Delta \quad b < c, \Gamma \Rightarrow \Delta}{ab < ac, \Gamma \Rightarrow \Delta} \; {\scriptstyle \times\text{-}ext2}$$

$$\frac{a < 0, \Gamma \Rightarrow \Delta \quad b < c, \Gamma \Rightarrow \Delta}{ab < ac, \Gamma \Rightarrow \Delta} \; {\scriptstyle \times\text{-}ext3} \qquad \frac{c < b, \Gamma \Rightarrow \Delta \quad 0 < a, \Gamma \Rightarrow \Delta}{ab < ac, \Gamma \Rightarrow \Delta} \; {\scriptstyle \times\text{-}ext4}$$

It follows as in Negri and von Plato (1998, pp. 421–422) that these four rules above can be reduced to the single rule

$$\frac{a < 0, c < b, \Gamma \Rightarrow \Delta \quad 0 < a, b < c, \Gamma \Rightarrow \Delta}{ab < ac, \Gamma \Rightarrow \Delta} \; {\scriptstyle \times\text{-}ext}$$

In the rules for the inverse the condition $a \neq 0$ is required. We notice that substitutions in the atomic formulas may produce duplications only in the rule containing two principal formulas in the conclusion, namely asymmetry. In this case the closure condition requires the rule of irreflexivity. Since all of our rules follow the rule-scheme, we conclude from theorem 3.1 the

Corollary 4.1: *The structural rules of weakening, contraction and cut are admissible in the sequent calculus for constructive ordered fields. Weakening and contraction are admissible and height preserving.*

It follows that our system is *complete*, in the sense that if A is derivable in the Hilbert style axiomatization given in section 2, then $\Rightarrow A$ is derivable in the sequent calculus system.

With logical systems, proofs of cut elimination have as a trivial consequence a proof of consistency for the system. With the extension by nonlogical rules, the situation is different. Here a purely proof-theoretic proof of consistency is complicated by the rule *nondeg*: In a root-first analysis of derivations, the rule *nondeg* introduces atoms in the antecedents of sequents. All the other nonlogical rules have atoms in the antecedent of their conclusion and therefore, in absence of *nondeg*, cannot produce a derivation of $\Rightarrow\bot$. These problems will be studied in a subsequent work.

5. Conclusion

Having a cut-free sequent calculus for a mathematical theory has a methodological implication: A proof-theoretic treatment based on the methods of proof analysis which are typical of structural proof theory becomes possible. For pure logical systems standard applications of cut elimination include consistency proofs as direct consequences of the subformula property. In systems with nonlogical axioms as those studied here the subformula property does not hold, but only a weaker form of the subformula property. Nevertheless the control of the structure of derivations in a cut-free system permits to answer non trivial questions of conservativity. For instance, the methods developed in Negri (1999) for showing conservativity of apartness over equality can be applied also in the context of constructive ordered fields. By these methods it is possible to show that the theory of constructive ordered fields is conservative over the theory of ordered fields based on a relation of weak partial order, defined as the negation of strict linear order.

Our method of treating constructive ordered fields can also be applied to *real closed fields* (see Delzell 1996 for references on the literature on real closed fields). The axioms asserting the existence of square roots of positive elements and the existence of zeros of polynomials of odd degree will be first formulated in terms of two constructions, $sqr(a)$ and $z(a_0, \ldots, a_{2n+1})$ satisfying the quantifier-free axioms

$$\begin{aligned} sqr(a)sqr(a) &= a, \quad \text{for } 0 \leqslant a \\ \textstyle\sum_{i=0}^{2n+1} a_i z(a_0, \ldots, a_{2n+1})^i &= 0, \quad \text{for } a_{2n+1} \neq 0 \end{aligned}$$

By adding the two constructions and the respective axioms to the theory of constructive ordered fields, we obtain a quantifier-free axiomatization for real closed fields, with all axioms conforming to the rule-scheme. Consequently, we obtain a sequent calculus for real closed fields with all structural rules admissible.

In the standard theory of real closed fields, the law of excluded middle is assumed for atoms. By a result of Tarski (see Delzell 1996), the law holds for arbitrary formulas. The assumption means that atoms are decidable, but this has constructive sense only if a field is discrete. In our proof-theoretical approach, decidability for atoms can be included in the cut-free system by adding the rule of *Gentzen excluded middle for atoms*

$$\frac{a < b, \Gamma \Rightarrow \Delta \quad \sim a < b, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle Gem\text{-}at}$$

By arguing as in von Plato (1999) it can be shown that the corresponding excluded middle rule for arbitrary quantifier-free formulas is admissible in the theory of real closed fields with decidable atoms, thus showing that, at least for the quantifier-free fragment, the theory behaves like a classical theory. Palmgren (1998) proves the decidability for all formulas using Tarski quantifier elimination and the Dragalin-Friedman translation.

We plan to make a detailed study of the proof theory of real closed fields in a subsequent work.

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