Tychonoff's theorem in the framework of formal topologies

Sara Negri **Silvio Valentini**
Dipartimento di Matematica Pura ed Applicata
Università di Padova
via G. Belzoni n.7, I–35131 Padova, Italy
e-mail: negri@pdmat1.math.unipd.it,
silvio@brouwer.math.unipd.it

1 Introduction

In this paper we give a constructive proof of the pointfree version of Tychonoff's theorem within formal topology, using ideas from Coquand's proof in [7]. To deal with pointfree topology Coquand uses Johnstone's coverages. Because of the representation theorem in [3], from a mathematical viewpoint these structures are equivalent to formal topologies but there is an essential difference also. Namely, formal topologies have been developed within Martin Löf's constructive type theory (cf. [15]), which thus gives a direct way of formalizing them (cf. [4]).

The most important aspect of our proof is that it is based on an inductive definition of the topological product of formal topologies. This fact allows us to transform Coquand's proof into a proof by structural induction on the last rule applied in a derivation of a cover. The inductive generation of a cover, together with a modification of the inductive property proposed by Coquand, makes it possible to formulate our proof of Tychonoff's theorem in constructive type theory. There is thus a clear difference to earlier localic proofs of Tychonoff's theorem known in the literature (cf. [9], [10], [12], [14]). Indeed we not only avoid to use the axiom of choice, but reach constructiveness in a very strong sense. Namely, our proof of Tychonoff's theorem supplies an algorithm which, given a cover of the product space, computes a finite subcover, provided that there exists a similar algorithm for each component space. Since type theory has been implemented on a computer (cf. [18]), an eventual strict formalization of our proof will at the same time be a computer program that executes the task of finding a finite subcover.

The paper is organized as follows. In the first part we recall the basic definitions and motivations of formal topologies and introduce in this framework the notion of topological product. Tychonoff's theorem is then proved both for the binary and arbitrary product of topological spaces. Next we relate our proof to the proof of Johnstone and Vickers (cf. [12]). In an appendix, it is shown how to define the type $\mathcal{P}_\omega(S)$ of finite subsets of the type S and the type $\Pi_\omega(I, B)$ of finite support functions from I to the indexed family of sets $B(i)$, where $i \in I$. In a second appendix, the formal definition of the coproduct of formal topologies is detailed.

2 Preliminaries

Formal topologies were introduced by Martin-Löf and Sambin ([19], [20]) as a constructive approach to topology, in the tradition of Johnstone's version of *Grothendieck topologies* [9] and Fourman and Grayson's *formal spaces* [8], but using simpler technical devices and a constructive set theory based on Martin Löf's type theory ([15], [21]).

Classically, a topological space is a couple $\langle X, \Omega(X)\rangle$, where $\Omega(X)$ is the family of the open subsets of the *collection*[1] X. The main purpose of formal topology is the study, in a constructive framework, of the properties of a topological space which can be expressed without any reference to the points, that is to the elements of the collection X. In this way we avoid thinking of open sets as collections of points (cf. [24]).

Since a point-set topology can always be presented using one of its bases, the abstract structure that we will consider is a commutative monoid $\langle S, \cdot_{\mathcal{S}}, 1_{\mathcal{S}}\rangle$ where the set S corresponds to the set of the elements of the base of the point-set topology $\Omega(X)$, $\cdot_{\mathcal{S}}$ corresponds to the operation of intersection between basic elements, and $1_{\mathcal{S}}$ corresponds to the whole collection X.

In a point-set topology any open set is obtained as a union of elements of the base, but union does not make sense if we reject any reference to points. Hence we are naturally led to think that an open set may directly correspond to a subset of the set S. Unfortunately this idea is not completely correct since there may be many different subsets of basic elements whose union is the same open set. In order to better define what corresponds to an open set, we need to introduce also an equivalence relation $\cong$ between two subsets U and V of S such that $U \cong V$ holds if and only if, denoting by c^* the element of the base which corresponds to the formal basic open c, the opens $U^* \equiv \cup_{a \in U} a^*$ and $V^* \equiv \cup_{b \in V} b^*$ are equal. To this purpose we introduce an infinitary relation $\lhd$, called *cover*, between a basic element a of S and a subset U of S whose intended meaning is that $a \lhd U$ when $a^* \subseteq U^*$, and therefore the equivalence $U \cong V$ will amount to $(\forall u \in U)\ u \lhd V$ & $(\forall v \in V)\ v \lhd U$.

Besides the relation of cover, we introduce a predicate Pos on S to express positively (that is without using negation) the fact that a basic open is not empty. The intended meaning of $Pos(a)$ is that a is inhabited, i.e., there exists at least one point in a. For instance, the negative definition of $Pos(a)$ by means of "a is not empty" would amount to $\neg(a \lhd \emptyset)$ which just says that it is impossible that there isn't any point in a. Even worse would be the impredicative choice

[1] Cf. [15] to see how the notion of set is contrasted with that of collection, there called category.

of putting $Pos(a) \equiv (\forall U \subseteq S)\ a \lhd U \rightarrow U \neq \emptyset$, i.e., any cover of a is inhabited, which is also used in the literature. These considerations lead us to the following definition.

Definition 2.1 (Formal topology) *A* formal topology *over a set S is a structure*

$$\mathcal{A} \equiv \langle S, \cdot_{\mathcal{S}}, 1_{\mathcal{S}}, \lhd_{\mathcal{A}}, Pos_{\mathcal{A}} \rangle$$

where $\langle S, \cdot_{\mathcal{S}}, 1_{\mathcal{S}} \rangle$ is a commutative monoid with unit and $\lhd_{\mathcal{A}}$ is a relation, called the cover *relation, between an element and a subset of S such that, for any $a, b \in S$ and for any $U, V \subseteq S$ the following conditions hold:*

$$\text{(reflexivity)} \quad \frac{a \in U}{a \lhd_{\mathcal{A}} U}$$

$$\text{(transitivity)} \quad \frac{a \lhd_{\mathcal{A}} U \qquad U \lhd_{\mathcal{A}} V}{a \lhd_{\mathcal{A}} V}$$

$$(\cdot\text{ - left}) \quad \frac{a \lhd_{\mathcal{A}} U}{a \cdot b \lhd_{\mathcal{A}} U}$$

$$(\cdot\text{ - right}) \quad \frac{a \lhd_{\mathcal{A}} U \qquad a \lhd_{\mathcal{A}} V}{a \lhd_{\mathcal{A}} U \cdot V}$$

where $U \lhd_{\mathcal{A}} V \equiv (\forall u \in U)\ u \lhd_{\mathcal{A}} V$ and $U \cdot V \equiv \{u \cdot v \mid u \in U, v \in V\}$.
Pos is a predicate on S, called positivity predicate, *satisfying:*

$$\text{(monotonicity)} \quad \frac{Pos(a) \qquad a \lhd_{\mathcal{A}} U}{(\exists b \in U)\ Pos(b)}$$

$$\text{(positivity)} \quad a \lhd_{\mathcal{A}} a^{+}$$

where $a^{+} \equiv \{b \in S \mid (a =_S b)\ \&\ Pos(b)\}$.

All the conditions are a straightforward rephrasing of the preceding intuitive considerations except *positivity*. The first reason to introduce positivity is that it allows to prove that any non-positive basic open is covered by anything. Technically, positivity also allows proofs by cases on $Pos(a)$ for deductions involving covers (for a detailed discussion cf. [22]). Moreover the introduction of Pos increases the expressiveness of the language for formal topology. For instance, the use of the predicate Pos is essential if one wants to represent connected spaces, the definition being (cf. the appendix of [16]): if $1 \lhd U$ then for any $a, b \in U$ such that $Pos(a)$ and $Pos(b)$ there exist $a_0 = a, \ldots, a_n = b$ in U such that $Pos(a_k \cdot a_{k+1})$ for all $k < n$.

The following lemma will be useful in the sequel.

Lemma 2.2 *Let $\mathcal{A} \equiv \langle S, \cdot_{\mathcal{S}}, 1_{\mathcal{S}}, \lhd_{\mathcal{A}}, Pos_{\mathcal{A}} \rangle$ be a formal topology and $U_1, \ldots, U_n$, $M \subseteq S$. Then*

$$(U_1 \cup M) \cdot \ldots \cdot (U_n \cup M) \lhd_{\mathcal{A}} (\bullet_{i \leq n} U_i) \cup M,$$

where $\bullet_{i \leq n} U_i \equiv U_1 \cdot \ldots \cdot U_n$.

Proof: Let $x \in (U_1 \cup M) \cdot \ldots \cdot (U_n \cup M)$, i.e., $x = y_1 \cdot \ldots \cdot y_n$ with $y_i \in U_i \cup M$ for each $i \leq n$.

If $y_i \in M$ for some $i \leq n$, then by *reflexivity* and $\cdot$ *-left*, $x \lhd_{\mathcal{A}} M$ and a fortiori $x \lhd_{\mathcal{A}} (\bullet_{i \leq n} U_i) \cup M$. Otherwise, for all $i \leq n$, $y_i \in U_i$, thus $y_i \lhd_{\mathcal{A}} U_i$. By $\cdot$ *-left* and $\cdot$ *-right*, $x \lhd_{\mathcal{A}} \bullet_{i \leq n} U_i$ and therefore $x \lhd_{\mathcal{A}} (\bullet_{i \leq n} U_i) \cup M$.

Observe that the proof does not make use of decidability of the relation $\in$, since it is done via a $\vee$-elimination. $\Box$

Since covers are defined by requiring closure under some rules, it is possible to generate them starting from some given conditions, that we conceive as axioms, and closing under the cover rules, that is *reflexivity*, *transitivity*, $\cdot$ *-left* and $\cdot$ *-right*, which can be conceived as inference rules. [2]

Given a base S and an infinitary relation $R(a, U)$, where $a \in S$ and $U \subseteq S$, we say that a cover $\lhd$ satisfies the relation R if, for all $a \in S$ and $U \subseteq S$, $R(a, U)$ implies $a \lhd U$. Then there exists the minimal cover satisfying R, which is obtained by closing R under the cover rules:

Definition 2.3 (Generated Cover) *Let $R(a, U)$ be a relation between elements and subsets of S. Then $\lhd_R$ is the relation inductively defined by*

$$\textit{(axioms)} \quad \frac{R(a, U)}{a \lhd_R U} \;,$$

the cover rules, and the rule

$$\textit{(substitution)} \quad \frac{a = b \quad a \lhd_R U}{b \lhd_R U}$$

Clearly, $\lhd_R$ is a cover on $\mathcal{S}$, since it is obviously closed under the rules defining covers; it is called the *cover generated by R*.

Observe that *substitution* is necessary in order to ensure that $\lhd_R$ is a relation, i.e., respects equality in S (whereas extensionality, which says that equality in the collection of all subsets of S is respected, is derivable by using *reflexivity* and *transitivity*).

Given an infinitary relation R and a predicate Pos on S satisfying suitable properties, it is possible to generate a formal topology, by the following:

Proposition 2.4 (Generated Topology) *Let R be a relation which satisfies $R(a, U) \rightarrow R(a \cdot b, U \cdot \{b\})$ and let Pos be a predicate on S such that the following properties hold:*

$$\begin{array}{l} Pos(a \cdot b) \rightarrow Pos(a), \\ Pos(a) \rightarrow Pos(a \cdot a), \\ Pos(a) \ \& \ R(a, U) \rightarrow (\exists b \in U) \ Pos(b). \end{array}$$

[2] The cover rules can also be conceived as introduction rules for the type corresponding to the cover relation. This will allow proofs by induction on derivations, that is, the use of the corresponding elimination rule (see concluding remarks).

Then Pos is monotone relative to the cover $\lhd_R$ generated by R.

If in addition $R(a, a^+)$ holds then $\mathcal{A} \equiv \langle S, \cdot, 1_S, \lhd_R, Pos\rangle$ is a formal topology, called the formal topology generated by R and Pos.

Proof: The proof of the first part of the proposition consists of four steps. First, the rule of $\cdot$ *-right* can be equivalently replaced by using the rules

$$\textit{(contraction)} \qquad a \lhd_R \{a \cdot a\},$$

$$\textit{(localization)} \qquad \frac{a \lhd_R U}{a \cdot b \lhd_R U \cdot \{b\}},$$

which are easily seen to be derivable from $\cdot$ *-right*, whereas for the converse one has to use *transitivity* on the following instances of *localization*:

$$\frac{a \lhd U}{a \cdot a \lhd U \cdot \{a\}} \quad \text{and, for all } u \in U, \quad \frac{a \lhd V}{u \cdot a \lhd \{u\} \cdot V}.$$

The second step consists in replacing the rule *transitivity* with its equivalent "localized" form

$$(\textit{localized transitivity}) \quad \frac{a \lhd U \cdot \{c\} \quad (\forall x \in U)\ (x \cdot c \lhd V)}{a \lhd V}.$$

Then *localization* can be eliminated from any given deduction of a cover: it can be permuted upward with all the present rules, because of the form of transitivity chosen *ad hoc*, and it is absorbed into the axioms by the condition required on R.

Finally monotonicity can be proved by induction on the length of the considered derivation, by using the conditions required on Pos for the basic cases, and the inductive hypothesis on the remaining rules, i.e., *reflexivity*, *localized transitivity*, $\cdot$ *-left* and *substitution*.

In order to show that $\mathcal{A}$ is a formal topology, only *positivity* remains to be proved. It is simply obtained by an instance of *axioms*, from the further assumption that $R(a, a^+)$ holds. $\square$

The notion of generated formal topology is particularly useful in defining the coproduct topology:

Definition 2.5 (Coproduct Topology) *Let $\mathcal{A} \equiv \langle S, \cdot, 1_S, \lhd_{\mathcal{A}}, Pos_{\mathcal{A}}\rangle$ and $\mathcal{B} \equiv \langle T, \cdot, 1_T, \lhd_{\mathcal{B}}, Pos_{\mathcal{B}}\rangle$ be formal topologies. Then the* coproduct formal topology *is defined by*

$$\mathcal{A} + \mathcal{B} \equiv \langle S \times T, \cdot, (1_S, 1_T), \lhd_{\mathcal{A}+\mathcal{B}}, Pos_{\mathcal{A}+\mathcal{B}}\rangle,$$

where $\langle S \times T, \cdot, (1_S, 1_T)\rangle$ is the monoid on the cartesian product $S \times T$ with componentwise operation, $\lhd_{\mathcal{A}+\mathcal{B}}$ is the cover generated by the relation R defined by

$$\begin{array}{rcll} R((a,b), U \times \{b\}) & \equiv & a \lhd_{\mathcal{A}} U & \textit{for any } b \in T, \\ R((a,b), \{a\} \times V) & \equiv & b \lhd_{\mathcal{B}} V & \textit{for any } a \in S, \end{array}$$

and the positivity predicate is defined by

$$Pos_{\mathcal{A}+\mathcal{B}}((a,b)) \equiv Pos_{\mathcal{A}}(a)\ \&\ Pos_{\mathcal{B}}(b).$$

This definition is sound since it is easy to see that the conditions on R and Pos for generating a formal topology are satisfied. Moreover it will follow from the proof in appendix B that $\mathcal{A}+\mathcal{B}$ is the categorical coproduct of the formal topologies $\mathcal{A}$ and $\mathcal{B}$ and therefore it yields the categorical product of the corresponding topological spaces.

We have preferred the above presentation for the coproduct topology instead of the equivalent and more symmetric

$$R((a,b), U \times V) \quad \equiv \quad a \lhd_{\mathcal{A}} U \text{ and } b \lhd_{\mathcal{B}} V$$

since the former allows a uniform extension to the infinite (co)product.

3 Tychonoff's theorem

Following the standard definition we say that, given a formal topology $\mathcal{A} \equiv \langle S, \cdot, 1_S, \lhd_{\mathcal{A}}, Pos_{\mathcal{A}} \rangle$, a basic neighborhood $a \in S$ is *compact* in $\mathcal{A}$ if, whenever $a \lhd_{\mathcal{A}} U$, there exists a finite subset U_0 of U such that $a \lhd_{\mathcal{A}} U_0$. To express the finite subset relation we will write $U_0 \subseteq_\omega U$.

In order to prove Tychonoff's theorem we first define a predicate that relies on quantifying over the set $\mathcal{P}_\omega(I)$ of finite subsets a given set I. In appendix A.1 we show how to define such a set within type theory.

Definition 3.1 *Let $x \in S$, $y \in T$, $Z \subseteq S \times T$. Then define*

$$\begin{aligned} P(x,y,Z) \equiv \quad & (\forall M \in \mathcal{P}_\omega(S))(\forall N \in \mathcal{P}_\omega(T)) \\ & \quad ((a \lhd \{x\} \cup M \ \& \ b \lhd \{y\} \cup N) \rightarrow \\ & \quad (\exists U \in \mathcal{P}_\omega(S))(\exists V \in \mathcal{P}_\omega(T)) \\ & \quad\quad (a \lhd U \cup M \ \& \ b \lhd V \cup N \ \& \ U \times V \subseteq (S \times T) \cdot Z)). \end{aligned}$$

This is a modification in two ways of the predicate proposed by Coquand in [7]. First, following a suggestion owed to Coquand himself, we restrict quantification to *finite* subsets of S and T, that is elements of the sets $\mathcal{P}_\omega(S)$ and $\mathcal{P}_\omega(T)$, instead of quantifying over the collection of all the subsets of S and T, which would have no constructive meaning. In this way we avoid being impredicative and develop the proof within a constructive framework. Then we require $U \times V \subseteq (S \times T) \cdot Z$, instead of $U \times V \subseteq Z$, in order to avoid the restriction to downward closed coverings, a restriction that is essential in Coquand's proof.

We are now in the position to prove:

Proposition 3.2 (Binary Tychonoff's theorem) *Suppose that a and b are compact in the formal topologies $\mathcal{A}$ and $\mathcal{B}$ respectively. Then (a,b) is compact in $\mathcal{A}+\mathcal{B}$.*

Proof: The claim amounts to proving that, whenever $(a,b) \lhd W$ in $\mathcal{A}+\mathcal{B}$, there exists $W_0 \subseteq_\omega W$ such that $(a,b) \lhd W_0$. In order to achieve this result, it is enough to show that:

1. If $P(a, b, W)$ holds then there exists $W_0 \subseteq_\omega W$ such that $(a, b) \lhd W_0$;

2. If $(x, y) \lhd Z$ then $P(x, y, Z)$ holds.

(1) If $P(a, b, W)$ holds, then in particular, with $M = N = \emptyset$, we obtain that there exist $U \subseteq_\omega S$ and $V \subseteq_\omega T$ such that $a \lhd U$, $b \lhd V$ and $U \times V \subseteq (S \times T) \cdot W$. Hence, by the axioms of product and some calculations, one obtains that $(a, b) \lhd U \times V$. Suppose now that $(u, v) \in U \times V$. Then there exist $(s, t) \in S \times T$ and $w \in W$ such that $(u, v) = (s, t) \cdot w$ and therefore, by $\cdot$ *-left*, $(u, v) \lhd w$. Since $U \times V$ is finite, w varies in a finite subset W_0 of W, and thus for all $(u, v) \in U \times V$, $(u, v) \lhd W_0$. Finally, by *transitivity*, $(a, b) \lhd W_0$ follows.

(2) We will prove the second condition by induction on the derivation of $(x, y) \lhd Z$, by analyzing the last rule applied.

Suppose the last rule applied is an axiom, say

$$\frac{x \lhd U}{(x, y) \lhd U \times \{y\}},$$

so that $Z \equiv U \times \{y\}$, and suppose that for arbitrary $M \subseteq_\omega S$, $a \lhd \{x\} \cup M$. Then, since $x \lhd U$, we get $a \lhd U \cup M$; therefore, by the assumption of compactness of a, there exists $U_0 \subseteq_\omega U$ such that $a \lhd U_0 \cup M$. Suppose moreover that for arbitrary $N \subseteq_\omega T$, $b \lhd \{y\} \cup N$. As $U_0 \times \{y\} \subseteq_\omega U \times \{y\} \equiv Z$ and $Z \subseteq (S \times T) \cdot Z$, since $(1_S, 1_T) \in S \times T$, we have proved that $P(x, y, Z)$ holds.

Reflexivity:

$$\frac{(x, y) \in Z}{(x, y) \lhd Z}.$$

The claim is evident by putting $U \equiv \{x\}$ and $V \equiv \{y\}$ since, by the premise, $\{x\} \times \{y\} \subseteq Z$ and we already showed that $Z \subseteq (S \times T) \cdot Z$.

$\cdot$ *-Left*:

$$\frac{(x_1, y_1) \lhd Z}{(x_1 \cdot x_2, y_1 \cdot y_2) \lhd Z} \qquad \text{where } (x_1 \cdot x_2, y_1 \cdot y_2) = (x, y).$$

Suppose $a \lhd \{x\} \cup M$; then since $x = x_1 \cdot x_2 \lhd x_1$ we have $a \lhd \{x_1\} \cup M$. In the same way we obtain $b \lhd \{y_1\} \cup N$ from the premise $b \lhd \{y\} \cup N$. By the inductive hypothesis applied to $(x_1, y_1) \lhd Z$, there exist finite subsets U of S and V of T such that $a \lhd U \cup M$, $b \lhd V \cup N$, with $U \times V \subseteq (S \times T) \cdot Z$.

$\cdot$ *-Right*:

$$\frac{(x, y) \lhd Z_1 \qquad (x, y) \lhd Z_2}{(x, y) \lhd Z_1 \cdot Z_2}.$$

Suppose $a \lhd \{x\} \cup M$ and $b \lhd \{y\} \cup N$. By the inductive hypothesis applied to $(x, y) \lhd Z_1$, there exist $U_1 \subseteq_\omega S$, $V_1 \subseteq_\omega T$ such that $a \lhd U_1 \cup M$, $b \lhd V_1 \cup N$ and $U_1 \times V_1 \subseteq (S \times T) \cdot Z_1$. Moreover, by inductive hypothesis applied to $(x, y) \lhd Z_2$, there exist $U_2 \subseteq_\omega S$, $V_2 \subseteq_\omega T$ such that $a \lhd U_2 \cup M$, $b \lhd V_2 \cup N$

and $U_2 \times V_2 \subseteq (S \times T) \cdot Z_2$. Then, by $\cdot$ *-right* and lemma 2.2, $a \lhd (U_1 \cdot U_2) \cup M$ and similarly $b \lhd (V_1 \cdot V_2) \cup N$, where $(U_1 \cdot U_2) \times (V_1 \cdot V_2) = (U_1 \times V_1) \cdot (U_2 \times V_2) \subseteq_\omega ((S \times T) \cdot Z_1) \cdot ((S \times T) \cdot Z_2)$; since $((S \times T) \cdot Z_1) \cdot ((S \times T) \cdot Z_2) = (S \times T) \cdot (Z_1 \cdot Z_2)$ the claim follows.

Transitivity:

$$\frac{(x, y) \lhd W \quad W \lhd Z}{(x, y) \lhd Z}.$$

Suppose $a \lhd \{x\} \cup M$ and $b \lhd \{y\} \cup N$. By the inductive hypothesis applied to $(x, y) \lhd W$, there exist $U \subseteq_\omega S$, $V \subseteq_\omega T$ such that $a \lhd U \cup M$, $b \lhd V \cup N$ and $U \times V \subseteq (S \times T) \cdot W$. Let $U \equiv \{u_1, \ldots, u_h\}$ and $V \equiv \{v_1, \ldots, v_k\}$. Then for all $i \leq h$, $j \leq k$, there exist $(s, t) \in S \times T$ and $(w^1_{i,j}, w^2_{i,j}) \in W$ such that $(u_i, v_j) = (s, t) \cdot (w^1_{i,j}, w^2_{i,j})$. By the componentwise definition of product and $\cdot$ *-left*, $u_i \lhd w^1_{i,j}$ and $v_j \lhd w^2_{i,j}$, thus by *transitivity* we obtain $a \lhd w^1_{i,j} \cup \{u_1, \ldots, \hat{u}_i, \ldots, u_h\} \cup M$ and $b \lhd w^2_{i,j} \cup \{v_1, \ldots, \hat{v}_j, \ldots, v_k\} \cup N$, where $\{u_1, \ldots, \hat{u}_i, \ldots, u_h\}$ is short for $\{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_h\}$. Since $(w^1_{i,j}, w^2_{i,j}) \in W$, then, by the minor premise of the rule, $(w^1_{i,j}, w^2_{i,j}) \lhd Z$. Again by inductive hypothesis there exist $W^1_{i,j} \subseteq_\omega S$ and $W^2_{i,j} \subseteq_\omega T$ such that $a \lhd W^1_{i,j} \cup \{u_1, \ldots, \hat{u}_i, \ldots, u_h\} \cup M$, $b \lhd W^2_{i,j} \cup \{v_1, \ldots, \hat{v}_j, \ldots, v_k\} \cup N$ and $W^1_{i,j} \times W^2_{i,j} \subseteq (S \times T) \cdot Z$. By letting j vary in $\{1, \ldots, k\}$ and repeatedly applying $\cdot$ *-right* and lemma 2.2, we have $a \lhd \bullet_{j \leq k} W^1_{i,j} \cup \{u_1, \ldots, \hat{u}_i, \ldots, u_n\} \cup M$ and hence, by repeating the same argument for all $i \leq h$, $a \lhd \bullet_{j \leq k} W^1_{1,j} \cup \ldots \cup \bullet_{j \leq k} W^1_{h,j} \cup M$. Similarly we obtain $b \lhd \bullet_{i \leq h} W^2_{i,1} \cup \ldots \cup \bullet_{i \leq h} W^2_{i,k} \cup N$. In order to achieve the conclusion we have to prove that $(\bullet_{j \leq k} W^1_{1,j} \cup \ldots \cup \bullet_{j \leq k} W^1_{h,j}) \times (\bullet_{i \leq h} W^2_{i,1} \cup \ldots \cup \bullet_{i \leq h} W^2_{i,k}) \subseteq (S \times T) \cdot Z$. So let $(u, v) \in (\bullet_{j \leq k} W^1_{1,j} \cup \ldots \cup \bullet_{j \leq k} W^1_{h,j}) \times (\bullet_{i \leq h} W^2_{i,1} \cup \ldots \cup \bullet_{i \leq h} W^2_{i,k})$. Then there exist i^* and j^* such that $u \in \bullet_{j \leq k} W^1_{i^*,j}$ and $v \in \bullet_{i \leq h} W^2_{i,j^*}$. Thus $u \in S \cdot W^1_{i^*,j^*}$ and $v \in T \cdot W^2_{i^*,j^*}$ and therefore $(u, v) \in (S \times T) \cdot (W^1_{i^*,j^*} \times W^2_{i^*,j^*})$. Since by hypothesis $W^1_{i^*,j^*} \times W^2_{i^*,j^*} \subseteq (S \times T) \cdot Z$, we also have $(S \times T) \cdot (W^1_{i^*,j^*} \times W^2_{i^*,j^*}) \subseteq (S \times T) \cdot Z$ and therefore $(u, v) \in (S \times T) \cdot Z$, thus proving the claim. □

Tychonoff's theorem for the topological product of two spaces follows as a corollary, by considering in the above proposition 1_S and 1_T in place of a and b.

It is worth noting that the proof works as well if one takes away the predicate Pos in the definition of formal space.

3.1 Arbitrary product

The case of an arbitrary product of spaces can be dealt with similarly via an inductive definition of the coproduct topology. Given a set-indexed family of formal topologies $\mathcal{A}_i \equiv \langle S_i, \cdot_i, 1_i, \lhd_i, Pos_i \rangle$, $i \in I$, let

$$\coprod_{i \in I} \mathcal{A}_i \equiv \langle \Pi_\omega(I, S), \diamond, \lambda x.1_x, \lhd, Pos \rangle$$

be the formal topology thus defined: the base monoid consists of the type $\Pi_\omega(I,S)$ of finite support functions from the set of indices I to the indexed family S_i where $i \in I$, $\diamond$ is the componentwise product between finite support functions and $\lambda x.1_x$ is its unit.[3]

By generalizing the axioms for the binary product, the cover $\lhd$ is inductively generated by the infinitary relation R defined by

$$R(f, \{\lambda_\omega(i,u,f) \mid u \in U\}) \equiv f[i] \lhd_i U$$

where $\lambda_\omega(i,u,f)$ is the finite support function whose value on j is u if $j = i$ and $f[j]$ otherwise.

Finally, the positivity predicate is defined by

$$Pos(f) \equiv (\forall i \in I)\ Pos_i(f[i]).$$

The reader is referred to appendix B for the proof that $\coprod_{i\in I} \mathcal{A}_i$ is the categorical coproduct of the formal topologies $\mathcal{A}_i$.

Proposition 3.3 (Tychonoff's theorem) *Let*

$$\coprod_{i\in I} \mathcal{A}_i \equiv \langle \Pi_\omega(I,S), \diamond, \lambda x.1_x, \lhd, Pos\rangle$$

be the formal topology defined as above, and let $a \in \Pi_\omega(I,S)$. Suppose that for all $i \in I$, $a[i]$ is compact in $\mathcal{A}_i$. Then a is compact in $\coprod_{i\in I} \mathcal{A}_i$.

Proof: Consider, for $f \in \Pi_\omega(I,S)$ and $Z \subseteq \Pi_\omega(I,S)$ the following type-theoretical proposition:

$$\begin{array}{rl} P(f,Z) \equiv & (\forall M \in (\Pi i \in I)(\mathcal{P}_\omega(S_i))) \\ & \quad (a[i] \lhd M(i) \cup \{f[i]\} \rightarrow \\ & \quad (\exists T \in (\Pi i \in I)(\mathcal{P}_\omega(S_i))) \\ & \qquad (\exists K \in \mathcal{P}_\omega(I))(\forall i \in I - K)\ T(i) =_{\mathcal{P}_\omega(S_i)} \{1_i\} \\ & \qquad \&\ (\forall i \in I)\ a[i] \lhd T(i) \cup M(i) \\ & \qquad \&\ \oplus_{i\in I} T(i) \subseteq \Pi_\omega(I,S) \diamond Z)) \end{array}$$

where $\oplus_{i\in I}T(i)$ is the subset of $\Pi_\omega(I,S)$ such that $f \in \oplus_{i\in I}T(i)$ if and only if, for all $i \in I$, $f[i]\varepsilon T(i)$.

Observe that in the above property intuitionistic type theory allows us to express a family of subsets as a member of a function type, thus avoiding second order quantification.

If the index set I consists of two elements this property reduces to the one given for the binary case; the additional demand here is that the finite subsets $T(i)$ are different from $\{1_i\}$ only on a finite number of indices, which is plain for the binary product. This ensures that $\oplus_{i\in I}T(i)$ is a finite set, which is essential in order to prove that the property yields Tychonoff's theorem.

[3]See appendix A.2 for these and other notions in this section.

As far as the technical part of the proof is concerned, we just observe that it goes on as in the binary case, by showing first that if $P(a, W)$ holds then there exists $W_0 \subseteq_\omega W$ such that $a \lhd W_0$ and then that if $f \lhd Z$ then $P(f, Z)$ holds. The only nontrivial step of the latter is the case where the last rule applied is *transitivity*. So, suppose $f \lhd Z$ is obtained by *transitivity* from $f \lhd V$ and $V \lhd Z$. Let $M \in (\Pi i \in I)\ \mathcal{P}_\omega(S_i)$ and suppose that for all $i \in I$, $a[i] \lhd M(i) \cup \{f[i]\}$. By $P(f, V)$, holding by inductive hypothesis, there exists $T \in (\Pi i \in I)\ \mathcal{P}_\omega(S_i)$ such that for all $i \in I$, $a[i] \lhd M(i) \cup T(i)$, $T(i) = \{1_i\}$ for every $i \in I$ except a finite subset, and $\oplus_{i \in I} T(i) \subseteq \Pi_\omega(I, S) \diamond V$. Then consider, for all $t \in \oplus_{i \in I} T(i)$, the decomposition $t = s_t \diamond v_t$, where $s_t \in \Pi_\omega(I, S)$ and $v_t \in V$. Now, supposing $T(i) = \{t^i_1, \ldots, t^i_{h_i}\}$, for all $i \in I$, $a[i] \lhd M(i) \cup \{v_t[i]\} \cup \{t^i_1, \ldots, \widehat{t[i]}, \ldots, t^i_{h_i}\}$. By applying inductive hypothesis to $v_t \lhd Z$, we obtain that there exists $Z_{v_t} \in (\Pi i \in I)\ \mathcal{P}_\omega(S_i)$ such that for all $i \in I$, $a[i] \lhd M(i) \cup Z_{v_t}(i) \cup \{t^i_1, \ldots, \widehat{t[i]}, \ldots, t^i_{h_i}\}$ and $\oplus_{i \in I} Z_{v_t}(i) \subseteq \Pi_\omega(I, S) \diamond Z$. By proceeding as in the binary case, we have, for all $i \in I$,

$$a[i] \lhd M(i) \cup \bullet_{\substack{t \in \oplus_{i \in I} T(i) \\ t[i] = t^i_1}} Z_{v_t}(i) \cup \ldots \cup \bullet_{\substack{t \in \oplus_{i \in I} T(i) \\ t[i] = t^i_{h_i}}} Z_{v_t}(i).$$

Introducing the shorthand

$$Z(i) \equiv \bullet_{\substack{t \in \oplus_{i \in I} T(i) \\ t[i] = t^i_1}} Z_{v_t}(i) \cup \ldots \cup \bullet_{\substack{t \in \oplus_{i \in I} T(i) \\ t[i] = t^i_{h_i}}} Z_{v_t}(i),$$

in order to conclude we have to prove that $Z(i) = \{1_i\}$ for every $i \in I$ except a finite subset and that $\oplus_{i \in I} Z(i) \subseteq \Pi_\omega(I, S) \diamond Z$. The former follows from the fact that $\oplus_{i \in I} T(i)$ is finite and that $Z_{v_t}(i) = \{1_i\}$ holds for every $i \in I$ except a finite subset. To prove the latter, let $x \in \oplus_{i \in I} Z(i)$. Then for all $i \in I$, there exists j_i such that

$$x[i] \in \bullet_{\substack{t \in \oplus_{i \in I} T(i) \\ t[i] = t^i_{j_i}}} Z_{v_t}(i)$$

Let t^* be defined by $t^*[i] \equiv t^i_{j_i}$. Then $x[i] \in Z_{v_{t^*}}(i) \cdot S_i$ and therefore $x \in (\oplus_{i \in I} Z_{v_{t^*}}(i)) \diamond \Pi_\omega(I, S)$. Thus

$$\oplus_{i \in I} Z(i) \subseteq (\cup_{t \in \oplus_{i \in I} T(i)} \oplus_{i \in I} Z_{v_t}(i)) \diamond \Pi_\omega(I, S)$$

Since for all $t \in \oplus_{i \in I} T(i)$, $\oplus_{i \in I} Z_{v_t}(i) \subseteq \Pi_\omega(I, S) \diamond Z$ holds, the conclusion follows. $\Box$

4 Concluding remarks

The algorithmic character of our proof becomes clearer if it is compared with other proofs of the localic Tychonoff's theorem.

We shall concentrate here on the comparison with the proof by Johnstone and Vickers [12], which is one of the most recent in the literature. Whereas our proof is based on sup-lattices, which are constructively representable by means of formal topologies (cf. [3]), the proof by Johnstone and Vickers is based on the use of the structure of *preframe*. Preframes are posets with all finite meets and directed joins, where binary meets distribute over directed joins, and homomorphisms preserve finite meets and directed joins. The fundamental result used in the proof is that the coproduct of frames is the tensor product of the underlying preframes. Compactness of a frame A is defined as the characteristic function of 1 from A to $\mathbb{2}$ being a preframe homomorphism, so that it is immediate to pass from compactness of two frames A and B to compactness of $A \coprod B$, since $A \coprod B \cong A \otimes B$ and $\mathbb{2} \otimes \mathbb{2} \cong \mathbb{2}$. Technically, the proof uses transfinite induction for the construction of nuclei out of prenuclei, whereas our proof never uses transfinite methods.

Our proof is so conceived that it can be formalized within constructive type theory. The possibility of such a formalization means that the proof has a direct computational meaning. Indeed, in a constructive approach to topology, a topological space is compact if there is an algorithm that from any given cover computes a finite subcover. By means of a proof analysis, our result provides such an algorithm for the product space, assuming the algorithms for the component spaces are given. Therefore our approach has a clear proof-theoretic character, as opposed to the topos-theoretic one in [12].

Formal topology has been developed so that it could be expressed in terms of type theory (cf. the introduction of [19]). As for the formalization proper, it has been given in [4]. For instance, one can proceeds as follows. First the proposition $a \lhd U$, where $a \in S$ and $U \subseteq S$ is formed (here a formal way to treat with subsets within type theory is needed [21]). Thinking of this proposition as a type, the cover rules are its introduction rules. For example, the rule of *reflexivity* is effected by a function $refl$ on proofs c (intended as proof-objects) of the proposition expressing that a is an element of U, with a proof $refl(a, U, c)$ of $a \lhd U$ as value. All the other rules can be treated in a similar way.

The elimination rule for the type $a \lhd U$ then allows to prove properties $C(c)$ of proofs c of $a \lhd U$ starting from the proofs of the aforementioned introduction rules. The inductive proof we gave for the property P can thus be seen as the proof of the premises of such an elimination rule.

Furthermore, the proof for the infinite topological product uses the same machinery as the proof for the finite product, once we have provided the type-theoretic definitions for finite support functions.

We conclude with observing that similar inductive techniques have been employed to get constructive proofs of other classical non-constructive basic results in mathematics, like the Heine-Borel and the Hahn-Banach theorems (cf. [5], [6], [17]).

Acknowledgments We are indebted to Giovanni Sambin who suggested the topic of this paper and helped in many ways, in particular by providing the notion of inductively generated cover, the proof of elimination of *localization* and suggesting the definition of product of formal topologies.

A On some types beyond basic Type Theory

In this first appendix, which is due to the second author, we will provide just the general ideas allowing to define the types used in the paper. Moreover we assume the reader is already familiar with Martin-Löf's type theory and therefore we will not explain the machinery to define a new type, which is worked out as in [15].

A.1 The type $\mathcal{P}_\omega(I)$ of finite subsets of I.

The type $\mathcal{P}_\omega(I)$ we are going to introduce in this section corresponds to the set of finite subsets of I. The idea we want to develop is that the elements of this new type are just the elements of the type of the lists on I (cf. [2]). Of course we have to impose a suitable equality on such elements which makes equal those lists which represent extensionally equal subsets. We are thus lead to use the same formation and introduction rules of the lists on I, and add two rules expressing the fact that sets are defined extensionally, and thus repeating an element (*contr*) or exchanging two elements (*exchg*) does not change the set.

Formation

$$\frac{I\ set}{\mathcal{P}_\omega(I)\ set} \qquad \frac{I = J}{\mathcal{P}_\omega(I) = \mathcal{P}_\omega(J)}$$

Introduction

$$\emptyset : \mathcal{P}_\omega(I) \qquad \frac{i : I \qquad U : \mathcal{P}_\omega(I)}{\{i\} * U : \mathcal{P}_\omega(I)}$$

$$\emptyset = \emptyset : \mathcal{P}_\omega(I) \qquad \frac{i = j : I \qquad U = V : \mathcal{P}_\omega(I)}{\{i\} * U = \{j\} * V : \mathcal{P}_\omega(I)}$$

$$\frac{i : I \quad U : \mathcal{P}_\omega(I)}{\{i\} * \{i\} * U = \{i\} * U : \mathcal{P}_\omega(I)}\ contr$$

$$\frac{i : I \quad j : I \quad U : \mathcal{P}_\omega(I)}{\{i\} * \{j\} * U = \{j\} * \{i\} * U : \mathcal{P}_\omega(I)}\ exchg$$

As far as the elimination rule is concerned, we cannot just innocently borrow the elimination rule from the type of lists. Indeed, the fact that the rules of *contr* and *exchg* identify sets which would not be equal as lists, leads easily to contradictions. This is the reason why the elimination rules for lists have to be modified in order to take these additional equalities into account.

Thus, in order to obtain $S(U, d, e) : C(U)$, besides $U : \mathcal{P}_\omega(I)$, $d : C(\emptyset)$ and $e(x, Y, z) : C(\{x\} * Y)\ [x : I, Y : \mathcal{P}_\omega(I), z : C(Y)]$, inherited from the lists-elimination rule, we have to add also the following four assumptions:

$e(x, \{x\} * Y, e(x, Y, z)) = e(x, Y, z) : C(\{x\} * Y)\ [x : I, Y : \mathcal{P}_\omega(I), z : C(Y)]$

and

$$e(x_1, \{x_2\} * Y, e(x_2, Y, z)) = e(x_2, \{x_1\} * Y, e(x_1, Y, z)) : \\ C(\{x_1\} * \{x_2\} * Y)\ [x_1 : I, x_2 : I, Y : \mathcal{P}_\omega(I), z : C(Y)],$$

which express the fact that the function $e(-,-,-)$ is a congruence w.r.t. the additional equalities;

$C(\{x\} * \{x\} * Y) = C(\{x\} * Y)\ [x : I, Y : \mathcal{P}_\omega(I)]$

and

$C(\{x_1\} * \{x_2\} * Y) = C(\{x_2\} * \{x_1\} * Y)\ [x_1 : I, x_2 : I, Y : \mathcal{P}_\omega(I)]$

which say that the propositional function $C(-)$ respects the additional equalities.

Of course the equality rules have to be modified in the same way in order to obtain $S(\emptyset, d, e) = d : C(\emptyset)$ and $S(\{x\} * Y, d, e) = e(x, Y, S(Y, d, e)) : C(\{x\} * Y)$.

With this definition it is not possible any longer to build up the solution for all the inductive definitions one can give on the lists, but only for those in which the involved functions are "well behaved" w.r.t. the equalities we have introduced. For instance, we can give the definition of the proposition $a \varepsilon U$ which, given $I\ set$, $U : \mathcal{P}_\omega(I)$ and $a : I$, says that a is an element of U. In order to do that it is useful to use the universe Ω_0 of small types. An inductive definition is thus

$$\begin{cases} a\varepsilon\emptyset = \bot : \Omega_0 \\ a\varepsilon\{i\} * U = (a =_I i) \vee a\varepsilon U : \Omega_0 \end{cases}$$

which admits the solution

$$a\varepsilon U \equiv S(U, \bot, (x, Y, z)\ (a =_I x) \vee z).$$

By using this proposition, provided that $I\ set$ and $K \in \mathcal{P}_\omega(I)$, we can define the type $I - K$, whose elements are those elements of I which do not belong to K, by the following introduction rules and the obvious elimination and equality rules.

$$\frac{i : I \qquad \neg(i\varepsilon K)\ true}{compl(K, i) : I - K} \qquad \frac{i = j : I \qquad \neg(i\varepsilon K)\ true}{compl(K, i) = compl(K, j) : I - K}$$

One can obtain an element of I out of an element of $I - K$ by using an instance of the elimination rule. For this reason, in order to simplify the notation, given $i : I$ we will sometime write $i : I - K$ instead of $compl(K, i) : I - K$ and will use elements of $I - K$ as they were element of I.

In the paper we also use the function $\cup$ which given two finite subsets constructs their union. Supposing $I\ set$ and $U, V \in \mathcal{P}_\omega(I)$, an inductive definition is

$$\begin{cases} U \cup \emptyset = U : \mathcal{P}_\omega(I) \\ U \cup (\{i\} * V) = \{i\} * (U \cup V) : \mathcal{P}_\omega(I) \end{cases}$$

which admits the solution

$$U \cup V \equiv S(V, U, (x, Y, z)\ \{i\} * z).$$

A.2 The type $\Pi_\omega(I, B)$ of finite support functions.

In this section we define the type $\Pi_\omega(I, B)$ of functions with finite support. We assume the hypothesis, which holds when dealing with formal topologies, that

$B(i)$ is a family of sets each containing a special element, denoted with 1_i. So we have to modify the type of functions by considering only those functions which differ from the functions $\lambda x.1_x$ only on a finite number of inputs.

Formation

$$\frac{I\ set \quad \begin{array}{c}[i:I]_1\\ \vdots\\ B(i)\ set\end{array}}{\Pi_\omega(I,B)\ set}\ 1 \qquad \frac{I=J \quad \begin{array}{c}[i:I]_1\\ \vdots\\ B(i)=D(i)\end{array}}{\Pi_\omega(I,B)=\Pi_\omega(J,D)}\ 1$$

Introduction

$$\lambda x.1_x : \Pi_\omega(I,B) \qquad \frac{f:\Pi_\omega(I,B) \qquad i:I \qquad b:B(i)}{\lambda_\omega(i,b,f):\Pi_\omega(I,B)}$$

Thus $\lambda x.1_x$ is the function whose output is $1_i : B(i)$ for each $i : I$, whereas $\lambda_\omega(i,b,f)$ has output $b : B(i)$ when applied to $i : I$ and the output of f applied to j when applied to j different from i. With this intuitive explanation of the introduction rules at hand, the following rules of equality between canonical elements are clear.

$$\lambda x.1_x = \lambda x.1_x : \Pi_\omega(I,B) \qquad \frac{f=g:\Pi_\omega(I,B) \qquad i=j:I \qquad b=d:B(i)}{\lambda_\omega(i,b,f)=\lambda_\omega(j,d,g):\Pi_\omega(I,B)}$$

$$\lambda x.1_x = \lambda_\omega(i,1_i,\lambda x.1_x):\Pi_\omega(I,B)\ [i:I]\ contr_0$$

$$\frac{f:\Pi_\omega(I,B) \quad i:I \quad b:B(i) \quad d:B(i)}{\lambda_\omega(i,b,\lambda_\omega(i,d,f))=\lambda_\omega(i,b,f):\Pi_\omega(I,B)}\ contr$$

$$\frac{f:\Pi_\omega(I,B) \quad i:I \quad b:B(i) \quad j:I \quad d:B(j) \quad \neg(i=_I j)\ true}{\lambda_\omega(i,b,\lambda_\omega(j,d,f))=\lambda_\omega(j,d,\lambda_\omega(i,b,f)):\Pi_\omega(I,B)}\ exchg$$

The elimination rule and the equality rules follow the same pattern we have already explained for the type of finite subsets. The former introduces the constant F_ω such that $F_\omega(c,d,e):C(c)$ provided that $c:\Pi_\omega(I,B)$, $d:C(\lambda x.1_x)$ and $e(x,y,z,t) : C(\lambda_\omega(x,y,z))$ $[z:\Pi_\omega(I,B), x:I, y:B(x), t:C(z)]$. The latter shows that $F_\omega(\lambda x.1_x,d,e) = d : C(\lambda x.1_x)$ and $F_\omega(\lambda_\omega(x,y,z),d,e) = e(x,y,z,F_\omega(z,d,e)):C(\lambda_\omega(x,y,z))$.

We can now define the function which applies the function with finite support $f:\Pi_\omega(I,B)$ to an element of I in order to obtain the element $f[i]:B(i)$. The simplest idea to perform this consists in transforming f in the "corresponding" function of the type $\Pi(I,B)$ and then to have recourse to the usual application operator Ap. We point out that the solution we propose here requires the existence of a *decision function* $\simeq$ from $I\times I$ to $Boole$ such that $i =_I j$ iff $i \simeq j =_{Boole} true$. In [23] this requirement is shown to be equivalent to decidability of equality in I by using the so called *intuitionistic axiom of choice* which is provable in ITT (cf. [15]) thanks to the constructive meaning of the quantifiers. The latter assumption seems to be common in the proofs of localic Tychonoff's theorem in the literature[4]. Assuming this hypothesis, the inductive definition of the function we are looking for is the following:

$$\begin{cases} F(\lambda x.1_x) = \lambda x.1_x : \Pi(I,B) \\ F(\lambda_\omega(i,b,f)) = \lambda x. \text{ if } x \simeq i \text{ then } b \text{ else } Ap(F(f),x) : \Pi(I,B) \end{cases}$$

which admits the solution

$$F(f) \equiv F_\omega(f, \lambda x.1_x, (z,i,b,t)\ \lambda x. \text{ if } x \simeq i \text{ then } b \text{ else } Ap(t,x))$$

so that $f[i] \equiv Ap(F(f),i)$.

Due to the following result the equality rules we give for finite support functions are the right ones.

Proposition A.1 *Let $f,g \in \Pi_\omega(I,B)$ and assume that for all $i \in I$, $f[i] = g[i] : B(i)$. Then $f = g : \Pi_\omega(I,B)$.*

Let us illustrate the functions on the elements of this type that we use in this paper. We begin with the formal definition of the operation $\diamond$ which, given two elements $f,g \in \Pi_\omega(I,B)$, yields the finite support function corresponding to their componentwise product. So, let us suppose that, for all $i \in I$, $\langle B(i), \cdot_i, 1_i\rangle$ is a commutative monoid; then an inductive definition of the function $\diamond$ is:

$$\begin{cases} f \diamond \lambda x.1_x = f : \Pi_\omega(I,B) \\ f \diamond \lambda_\omega(i,b,g) = \lambda_\omega(i, b \cdot f[i], f \diamond g) : \Pi_\omega(I,B) \end{cases}$$

which is solved by

$$f \diamond g \equiv F_\omega(g, f, (z,x,y,t)\ \lambda_\omega(x, y \cdot f[x], t)).$$

Another function we use in the paper is the function *supp* which, when applied to $f : \Pi_\omega(I,B)$, yields a finite subset of I containing the subset on which f differs from the function $\lambda x.1_x$. An inductive definition of the function *supp* is the following:

$$\begin{cases} supp(\lambda x.1_x) = \emptyset : P_\omega(I) \\ supp(\lambda_\omega(i,b,f)) = \{i\} * supp(f) : P_\omega(I) \end{cases}$$

which is satisfied by putting

$$supp(f) \equiv F_\omega(f, \emptyset, (z,x,y,t)\ \{x\} * t).$$

It is easy to check that, supposing $f,g \in \Pi_\omega(I,B)$, $supp(f \diamond g) = supp(f) \cup supp(g)$.

[4] Indeed J. Vermeulen has a proof of Tychonoff's theorem which does not assume decidability of equality between elements of I, cf. [26]

B From the categorical coproduct of formal topologies to topological product

In this appendix we will prove that the coproduct of formal topologies, as defined in sections 2 and 3, is indeed a categorical coproduct and yields the product of the corresponding topological spaces.

In order to make a category **FTop** out of formal topologies, we only have to recall the definition of the morphisms [19].

Let $\mathcal{A} = \langle S, \cdot_{\mathcal{A}}, 1_{\mathcal{A}}, \lhd_{\mathcal{A}}, Pos_{\mathcal{A}} \rangle$, and $\mathcal{B} = \langle T, \cdot_{\mathcal{B}}, 1_{\mathcal{B}}, \lhd_{\mathcal{B}}, Pos_{\mathcal{B}} \rangle$ be two formal topologies and let $Pt(\mathcal{A})$ and $Pt(\mathcal{B})$ be the respective collections of formal points. Then a map $f : \mathcal{A} \longrightarrow \mathcal{B}$ has to correspond to the inverse of a continuous function $f^* : Pt(\mathcal{B}) \longrightarrow Pt(\mathcal{A})$ between topological spaces, and hence it has to map a basic open in S into an open of $\mathcal{B}$. Since we reject any reference to points, hence to unions, an open of $\mathcal{B}$ can be specified only by means of subsets of T, with the proviso that different subsets could specify the same open. To cope with this problem, in the preliminaries we have introduced the equivalence relation

$$U \cong_{\mathcal{B}} V \equiv U \lhd_{\mathcal{B}} V \ \&\ V \lhd_{\mathcal{B}} U.$$

We are thus able to express in our framework the conditions we require on f from S to $\mathcal{P}(T)$:

1. $f(1_{\mathcal{A}}) \cong_{\mathcal{B}} \{1_{\mathcal{B}}\}$;
2. $f(a \cdot_{\mathcal{A}} b) \cong_{\mathcal{B}} f(a) \cdot_{\mathcal{B}} f(b)$;
3. $\dfrac{a \lhd_{\mathcal{A}} U}{(\forall x \in f(a))\ x \lhd_{\mathcal{B}} f(U)}$, where $f(U) \equiv \cup_{b \in U} f(b)$;
4. $\dfrac{(\exists x \in f(a))\ Pos_{\mathcal{B}}(x)}{Pos_{\mathcal{A}}(a)}$.

Yet, we cannot identify the morphisms from $\mathcal{A}$ to $\mathcal{B}$ with the maps as above since two maps f and g yielding the same open for all $a \in S$ have to coincide. Hence we define a morphism as an equivalence class of maps, modulo the equivalence relation

$$f \sim_{\mathcal{B}} g \equiv (\forall a \in S)\ f(a) =_{\mathcal{B}} g(a).$$

In this way, supposing f and g to be maps as above, by defining composition of the equivalence classes $[f]$ and $[g]$ as

$$[f] \circ [g] \equiv [f \circ g]$$

it is easy to check that a category is obtained. Of course, in the following we will adopt the usual mathematical practice to forget about equivalence classes and to work directly with their representatives. As announced we have:

Proposition B.1 $\coprod_{i\in I}\mathcal{A}_i \equiv \langle \Pi_\omega(I,S), \diamond, \lambda x.1_x, \lhd, Pos\rangle$ *is the coproduct in* **FTop** *of the formal topologies* $\mathcal{A}_i$*.*

Proof: The canonical injections are given by the maps:

$$e_i : \mathcal{A}_i \longrightarrow \coprod_{i\in I}\mathcal{A}_i$$

such that

$$e_i(a) \equiv \{\lambda_\omega(i, a, \lambda x.1_x)\}$$

The maps e_i satisfy the defining conditions for morphisms in **FTop**; in fact, let a, $b \in S_i$ and $U \subseteq S_i$, then:

1. $e_i(1_i) \equiv \{\lambda_\omega(i, 1_i, \lambda x.1_x)\} = \{\lambda x.1_x\}$.
2. $e_i(a \cdot b) \equiv \{\lambda_\omega(i, a\cdot b, \lambda x.1_x)\} = \{\lambda_\omega(i, a, \lambda x.1_x)\} \diamond \{\lambda_\omega(i, b, \lambda x.1_x)\} \equiv e_i(a) \diamond e_i(b)$.
3. If $a \lhd_i U$, then, by the axiom, we have $\lambda_\omega(i, a, \lambda x.1_x) \lhd \{\lambda_\omega(i, u, \lambda x.1_x) : u \in U\}$, and thus $e_i(a) \lhd e_i(U)$ by definition.
4. If $Pos(e_i(a))$ holds, then in particular the i-th component of $e_i(a)$ is inhabited, that is $Pos(a)$ holds.

We now come to the universal property. Suppose that $\mathcal{C}$ denotes a formal topology and for all $i \in I$ a morphism $\phi_i : \mathcal{A}_i \longrightarrow \mathcal{C}$ is given. We prove that there exists a unique morphism $\oplus_{i\in I}\phi_i : \coprod_{i\in I}\mathcal{A}_i \longrightarrow \mathcal{C}$ such that $(\oplus_{i\in I}\phi_i) \circ e_i = \phi_i$ for all $i \in I$.

We first prove uniqueness. If $f \in \Pi_\omega(I,S)$, then f can be written as $f = \diamond_{i\in supp(f)} e_i(f[i])$ (if *supp*(f) is empty we soundly define the above product to be equal to $\lambda x.1_x$). Then, if $\oplus_{i\in I}\phi_i$ has to respect product and to make the diagram commute, the following definition is forced:

$$(\oplus_{i\in I}\phi_i)(f) \equiv \bullet_{i\in supp(f)}\phi_i(f[i])$$

Thus, to prove existence, it is enough to prove that the above defined map is a morphism in **FTop**. We have:

1. $(\oplus_{i\in I}\phi_i)(\lambda x.1_x) \equiv \bullet_{i\in supp(\lambda x.1_x)}\phi_i(\lambda x.1_x[i]) \equiv \bullet_{i\in\emptyset}\phi_i(1_i) =_{\mathcal{C}} \{1_{\mathcal{C}}\}$.
2. $(\oplus_{i\in I}\phi_i)(f \diamond g) \equiv \bullet_{i\in supp(f\diamond g)}\phi_i(f[i]\cdot g[i])$; then using the fact that the ϕ_i are morphisms, hence respect product, and that product is associative and commutative, we obtain $\bullet_{i\in supp(f\diamond g)}\phi_i(f[i]\cdot g[i]) \equiv \bullet_{i\in supp(f)\cup supp(g)}\phi_i(f[i])\cdot \phi_i(g[i]) = \bullet_{i\in supp(f)}\phi_i(f[i]) \cdot \bullet_{i\in supp(g)}\phi_i(g[i]) \equiv (\oplus_{i\in I}\phi_i)(f)\cdot(\oplus_{i\in I}\phi_i)(g)$.
3. Suppose $f \lhd W$. We prove that $(\oplus_{i\in I}\phi_i)(f) \lhd_{\mathcal{C}} (\oplus_{i\in I}\phi_i)(W)$ by induction on the derivation of $f \lhd W$. If it derives from an axiom,

$$\frac{f[i] \lhd_i U}{f \lhd \{\lambda_\omega(i,u,f) : u \in U\}}$$

i.e., $W \equiv \{\lambda_\omega(i,u,f) : u \in U\}$, then from the premise we get, since all the ϕ_i are morphisms, $\phi_i(f[i]) \lhd_{\mathcal{C}} \phi_i(U)$. By (repeated applications of) $\cdot$ *-left* we have $\bullet_{i\in supp(f)}\phi_i(f[i]) \lhd_{\mathcal{C}} \phi_i(U)$. Since for all $i \in I$, $\phi_i(1_i) =_{\mathcal{C}} \{1_{\mathcal{C}}\}$ we obtain $\phi_i(U) =_{\mathcal{C}} \{(\oplus_{i\in I}\phi_i)(\lambda_\omega(i,u,f)) : u \in U\}$, and therefore $(\oplus_{i\in I}\phi_i)(f) \lhd_{\mathcal{C}} (\oplus_{i\in I}\phi_i)(W)$.

If $f \lhd W$ is obtained by *reflexivity*, then from the premise $f \in W$ we get $\{f\} \subseteq W$ and therefore $(\oplus_{i\in I}\phi_i)(\{f\}) \subseteq (\oplus_{i\in I}\phi_i)(W)$, which gives the claim by applying *reflexivity* again.

If the last rule applied is $\cdot$ *-left* ($\cdot$ *-right* or *transitivity* resp.), then by inductive hypothesis applied to the premises and $\cdot$ *-left* ($\cdot$ *-right* or *transitivity* resp.) again, we conclude.

4. Suppose $Pos((\oplus_{i\in I}\phi_i)(f))$, that is $Pos(\bullet_{i\in supp(f)}\phi_i(f[i]))$, then, by monotonicity and $\cdot$ *-left*, $(\forall i \in supp(f))\ Pos(\phi_i(f[i]))$ holds, and therefore, since all the ϕ_i are morphisms, we have that $(\forall i \in supp(f))\ Pos(f[i])$, i.e., $Pos(f)$ holds. $\Box$

We will now prove that coproduct of formal topologies yields product of formal spaces. Categorically, formal points can be defined as "generalized elements" in **FTop**. First observe that $\mathbb{1} \equiv \langle\{1\}, \cdot, 1, \in, \{1\}\rangle$ is the initial object of the category **FTop**. Thus a formal point x of $\mathcal{A}$ is a morphism from $\mathcal{A}$ to $\mathbb{1}$. With the identification of points on $\mathcal{A}$ with morphisms, i.e.,

$$Pt(\mathcal{A}) \equiv \mathbf{FTop}(\mathcal{A}, \mathbb{1})$$

we obtain a simple characterization of the points of the coproducts. In fact, by the universal property of coproduct we have

$$Pt(\coprod_{i\in I}\mathcal{A}_i) \equiv \mathbf{FTop}(\coprod_{i\in I}\mathcal{A}_i, \mathbb{1}) \cong \prod_{i\in I}\mathbf{FTop}(\mathcal{A}_i, \mathbb{1})$$

that is, formal points of the coproduct of the formal topologies $\mathcal{A}_i$, $i \in I$ are the cartesian product of formal points of each formal topology.

References

[1] P. Aczel, *An Introduction to Inductive Definitions*, in Handbook of Mathematical Logic, J. Barwise ed., North-Holland, 1977, pp. 739-782.

[2] R. Backhouse, P. Chisholm, G. Malcom, E. Saaman, *Do-it-yourself Type Theory*, Formal Aspects of Computing, vol. 1, 1989, pp.19-84.

[3] G. Battilotti, G. Sambin, *Pretopologies and a uniform presentation of sup-lattices, quantales and frames*, ms., University of Padua, Dept. of Mathematics, 1993.

[4] J. Cederquist. *A machine assisted formalization of pointfree topology in type theory*, Chalmers University of Technology and University of Göteborg, Sweden, Licentiate Thesis, 1994.

[5] J. Cederquist, S. Negri *A constructive proof of the Heine-Borel covering theorem for formal reals*, Lecture Notes in Computer Science, Proceedings of the Turin BRA-TYPES meeting, in press.

[6] J. Cederquist, T. Coquand, S. Negri *Helly-Hahn-Banach in formal topology*, forthcoming.

[7] T. Coquand, *An intuitionistic proof of Tychonoff's theorem*, The Journal of Symbolic Logic vol. 57, no. 1, 1992, pp. 28-32.

[8] M. P. Fourman, R.J. Grayson, *Formal Spaces*, in "The L. E. J. Brouwer Centenary Symposium", A. S. Troelstra and D. van Dalen (eds), North Holland, Amsterdam, 1982, pp. 107-122.

[9] P. T. Johnstone, "Stone Spaces", Cambridge University Press, 1982.

[10] P. T. Johnstone, *Tychonoff's theorem without the axiom of choice*, Fundamenta Mathematicae, vol. 113, 1981, pp. 21-35.

[11] P. T. Johnstone, *The point of pointless topology*, Bulletin of the American Mathematical Society, vol. 8, 1983, pp. 41-43.

[12] P. T. Johnstone, S. Vickers, *Preframe presentations present*, in " Category Theory" (Proceedings, Como 1990), Carboni, Pedicchio, Rosolini (eds), Lecture Notes in Mathematics, vol. 1488, 1991, pp. 193-212.

[13] J. L. Kelley, *The Tychonoff product theorem implies the axiom of choice*, Fundamenta Mathematicae, vol. 37, 1950, pp. 75-76.

[14] I. Kriz, *A constructive proof of Tychonoff's theorem for locales*, Commentationes Mathematicae Universitatis Carolinae, vol. 26/3, 1985, pp. 619-630.

[15] P. Martin-Löf, "Intuitionistic type theory", notes by Giovanni Sambin of a series of lectures given in Padua, June 1980, Bibliopolis, Naples, 1984.

[16] I. Moerdijk, *Continuous fibrations and inverse limits of toposes*, Compositio Mathematica, vol. 58, 1986, pp. 45-72.

[17] S. Negri, "Dalla Topologia Formale all'Analisi", Ph.D. thesis, University of Padua, Dept. of Mathematics, 1996.

[18] B. Nordström, K. Petersson, J. Smith, "Programming in Martin-Löf's Type Theory", Oxford University Press, 1990.

[19] G. Sambin, *Intuitionistic formal spaces – a first communication*, in Mathematical logic and its applications, D. Skordev ed., Plenum, 1987, pp. 187-204.

[20] G. Sambin, *Intuitionistic formal spaces and their neighborhood*, in "Logic Colloquium 88", Ferro, Bonotto, Valentini, Zanardo (eds), North-Holland, 1989, pp. 261-285.

[21] G. Sambin, S. Valentini, *Building up a tool-box for Martin-Löf intuitionistic type theory*, to appear.

[22] G. Sambin, S. Valentini, P. Virgili, *Constructive Domain Theory as a branch of Intuitionistic Pointfree Topology*, Theoretical Computer Science, in press.

[23] S. Valentini, *Decidability in Intuitionistic Theory of Types is functionally decidable*, Mathematical Logic Quarterly, in press.

[24] Paulus Venetus (alias G. Sambin, S. Valentini), *Propositum Cameriniense sive etiam itinera certaminis inter rationem insiemes aedificandi analisym situs intuitionisticamque et mathematicae artium reliquas res*, in "Atti degli incontri di Logica Matematica", vol. 8, Gerla, Toffalori, Tulipani (eds), 1993, pp. 115-143.

[25] J. J. C. Vermeulen, "Constructive techniques in functional analysis", Ph.D. Thesis, University of Sussex, 1986.

[26] J. J. C. Vermeulen, *Proper maps of locales*, Journal of Pure and Applied Algebra, vol. 92, 1994, pp. 79-107.

[27] S. Vickers. "Topology via logic", Cambridge Tracts in Theoretical Computer Science, vol. 5, Cambridge University Press, 1989.