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Semantical observations on the embedding of Intuitionistic Logic into Intuitionistic Linear Logic

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1. Introduction

It is well known that Intuitionistic Logic can be faithfully embedded into (Intuitionistic) Linear Logic. For the embedding, or translation, different solutions have been proposed (cf. Girard (1987), Troelstra (1992), Danos et al. (1993)).

The purpose of this paper is to study the embedding from a semantical viewpoint, by investigating into the relationship between various models for Intuitionistic Logic and for Intuitionistic Linear Logic. We will follow this pattern: given a structure which is a model for Intuitionistic Linear Logic we explain how to reconstruct inside it a structure which is a model for Intuitionistic Logic.

In the first section this procedure is worked out for the algebraic semantics of quantales with modality (cf. Rosenthal (1990), Troelstra (1992), Yetter (1990)). Here it is proved in detail that every quantale with modality gives rise to a frame included in it; furthermore the isomorphism between the complete lattice of modalities on a quantale and the complete lattice of subframes of a quantale is established. Finally, the largest subframe included in a quantale is described.

By a result due to G. Sambin, frames and quantales are representable by means of formal topologies and pretopologies respectively (cf. Battilotti and Sambin (199?)). In the appendix, by an extension of these representation theorems to quantales with modality, we show that any quantale with modality is isomorphic to the class of saturated subsets of a suitable pretopology endowed with an operator, already introduced in Sambin (1989), and called *stable interior operator.* As an application, we show, given a formal pretopology with a stable interior operator representing a quantale, how to obtain a formal topology representing the subframe determined by the modality.

All these semantical considerations lead to consider, in the second section, "natural" requirements on translations from Intuitionistic Logic into Intuitionistic Linear Logic. More explicitly, we define a faithful translation corresponding to the insertion of a subframe into its matching quantale with modality, and such that translated formulas are equivalent to their exclamation. Furthermore we define a faithful translation which is also a translation of schemes, that is which commutes with substitution.

In the final section we deal with categorical semantics, extensively studied in Linear Logic literature (cf. Barr (1991), Martí-Oliet and Meseguer (1991), de Paiva (1989), Seely(1989), etc.). We study sufficient (and in a way necessary) conditions so that the co-Kleisli construction, applied to a category which is a model for Linear Logic, gives rise to a categorical model for Intuitionistic Logic. In particular, we propose an answer to a question, raised in Seely (1989), concerning the existence of coproducts in the co-Kleisli category. Finally, we show that the algebraic construction developed in the first part is just a particular case of this categorical one, namely its reduction to partial orders.

2. Frames and quantales with modality

In what follows we will show how a quantale endowed with a modality gives rise to a frame and that every frame included in a quantale can be obtained in this way.

We first review some definitions.

Definition 2.1. A *quantale* Q is a quadruple $(Q, \cdot, 1, \bigvee)$ such that:

(1) $(Q, \cdot, 1)$ is a commutative monoid with unity 1;
(2) $(Q, \bigvee)$ is a complete lattice;
(3) for any family $(b_i)_{i\in I}$ of elements of Q and any $a \in Q$, infinite distributivity $a \cdot \bigvee_{i\in I} b_i = \bigvee_{i\in I} a_i \cdot b$ holds.

One of the basic properties of quantales, which shall be needed in the sequel, is *stability*:

(S) $a \leq a'$ & $b \leq b' \to a \cdot a' \leq b \cdot b'$.

Stability can be easily proved, in fact, by distributivity applied to the right hand side, we have $(a \cdot a') \vee (b \cdot b') \leq (a \vee b) \cdot (a' \vee b')$; by hypothesis $a \vee b = b$, $a' \vee b' = b'$, thus $(a \cdot a') \vee (b \cdot b') \leq b \cdot b'$, which is the claim.

In a more general framework, e.g. in Rosenthal (1990), commutativity and the presence of unity are special features and are therefore explicitly stated. Since we deal with *commutative* linear logic, our quantales will be assumed to be commutative, even if many of the following results continue to hold in the non-commutative case.

Definition 2.2. Let S be a subset of a quantale Q. We say that S satisfies *weakening* if $a \leq 1$ for all $a \in S$ and that S satisfies *contraction* if $a \cdot a = a$ for all $a \in S$.

A complete lattice where meet is distributive over arbitrary join is a special kind of quantale, in which the monoid and the meet operation coincide; it is known in the literature as *locale*, *frame* or *complete Heyting algebra*, according to the notion of morphism (cf. Mac Lane and Moerdijk (1992)). Here we shall consider morphisms preserving arbitrary join and finite meet, namely frame morphisms.

The following proposition provides a useful characterization of frames.

Proposition 2.3. A quantale Q satisfies weakening and contraction iff $a \wedge b = a \cdot b$ for all $a, b \in Q$, i.e. iff Q is a frame.

Proof. By stability applied to $a \leq a$ and $b \leq 1$ we get $a \cdot b \leq a$; in the same way $a \cdot b \leq b$. Thus $a \cdot b \leq a \wedge b$. By stability applied to $a \wedge b \leq a$ and $a \wedge b \leq b$ we have $(a \wedge b) \cdot (a \wedge b) \leq a \cdot b$, thus, by contraction, $a \wedge b \leq a \cdot b$. The converse is obvious. □

Definition 2.4. (cf. Rosenthal (1990)) In any poset P, a map $\mu : P \to P$ is called a *coclosure operator* if it satisfies:

(a) $\mu a \leq a$;
(b) $a \leq b \to \mu a \leq \mu b$;
(c) $\mu a \leq \mu\mu a$.

A coclosure operator on a quantale Q such that $\mu a \cdot \mu b \leq \mu(a \cdot b)$ for all $a, b \in Q$ is called a *quantic conucleus*.
A coclosure operator on a frame L such that $\mu a \wedge \mu b \leq \mu(a \wedge b)$ (hence by monotonicity $\mu(a \wedge b) = \mu a \wedge \mu b$) for all $a, b \in L$ is called a *conucleus*.
A coclosure operator such that

(d) $\mu(a \wedge b) = \mu a \cdot \mu b$,
(e) $\mu\top = 1$, where $\top = \vee Q$

is called a *modality*, (cf. Troelstra (1992)).

It is easy to prove that μ is a coclosure operator iff it satisfies the following:

$(*)$ $\mu a \leq b \leftrightarrow \mu a \leq \mu b$.

In the language of categories conditions (a), (b), (c) define a *comonad* in the partial order Q.

A quantale Q with such a modality μ is called in Troelstra (1992) a *complete IL-algebra with storage*. If moreover Q is provided with an *involution*, that is a unary operation $\sim$ such that for all $a, b \in Q$, $a = \sim\sim a$ and $a \leq b \to \sim b \leq \sim a$, we obtain a structure which is easily seen to be equivalent to a *modal Girard structure* as defined in Avron (1988).
We say that $a \in Q$ is a μ-*element* if $a = \mu c$, for some $c \in Q$; by idempotency of μ, a is a μ-element iff $\mu a = a$. We shall denote by Q_μ the set of μ-elements of Q.

We can give an alternative definition of the modality μ involving only the monoid operation.

Proposition 2.5. A coclosure operator μ on Q is a modality iff it satisfies the following conditions:

(1) $\mu a \cdot \mu a = \mu a$;
(2) $\mu a \cdot \mu b \leq \mu(\mu a \cdot \mu b)$ (hence $\mu a \cdot \mu b = \mu(\mu a \cdot \mu b)$);
(3) $\mu\top = 1$.

Proof. If μ is a modality on Q then (1) is an instance of (d). Moreover, since $\mu a \cdot \mu b$ is a μ-element by (d), then $\mu a \cdot \mu b = \mu(\mu a \cdot \mu b)$, thus (2) holds.

Conversely, from monotonicity and stability applied to $a \wedge b \leq a$ and $a \wedge b \leq b$, we get $\mu(a \wedge b) \cdot \mu(a \wedge b) \leq \mu a \cdot \mu b$, which gives $\mu(a \wedge b) \leq \mu a \cdot \mu b$, since by (1) the product of μ-elements is idempotent. As for the opposite inequality, first observe that monotonicity applied to $b \leq \top$ gives, by (3), $\mu b \leq 1$. Thus, from $\mu a \leq a$ and stability, we obtain $\mu a \cdot \mu b \leq a$. In the same way we get $\mu a \cdot \mu b \leq b$, and therefore $\mu a \cdot \mu b \leq a \wedge b$. Thus by monotonicity and assumption (2) we have $\mu a \cdot \mu b \leq \mu(a \wedge b)$. □

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We have a useful presentation of μ-elements as suprema, that is:

Lemma 2.6. If μ is a coclosure operator on Q, then for all $a \in Q$, $\mu a = \vee\{\mu c : \mu c \leq a\}$.

Proof. Since $\mu a \leq a$, then $\mu a \leq \vee\{\mu c : \mu c \leq a\}$; conversely, if $\mu c \leq a$ then $\mu c \leq \mu a$ and therefore $\vee\{\mu c : \mu c \leq a\} \leq \mu a$. □

The following lemma lists some technical results we shall need in the sequel and shows in particular that any modality is a quantic conucleus.

Lemma 2.7. For any coclosure operator the following four conditions are equivalent:

(1) Q_μ is closed under $\cdot$

(2) $\mu a \cdot \mu b = \mu c$ for some c

(3) $\mu a \cdot \mu b = \mu(\mu a \cdot \mu b)$

(4) $\mu a \cdot \mu b \leq \mu(a \cdot b)$

Moreover each of the previous ones together with :

(5) $\mu c \leq \mu c \cdot \mu c$

is equivalent to

(6) $\mu(a \wedge b) = \mu a \cdot \mu b$.

Proof. (1)$\Rightarrow$(2),(1)$\Rightarrow$(3) Since $\mu a, \mu b \in Q_\mu$, then $\mu a \cdot \mu b \in Q$ i.e. $\mu(\mu a \cdot \mu b) = \mu a \cdot \mu b$ with $c = \mu a \cdot \mu b$.

(2)$\Rightarrow$(1) and (3)$\Rightarrow$(1) are obvious.

(3)$\Rightarrow$(4) $\mu a \cdot \mu b = \mu(\mu a \cdot \mu b) \leq \mu(a \cdot b)$.

(4)$\Rightarrow$(1) For all $a, b \in Q_\mu$, $a \cdot b = \mu a \cdot \mu b$, thus by hypothesis $a \cdot b \leq \mu(a \cdot b)$; since the opposite inequality always holds, we get $a \cdot b = \mu(a \cdot b)$, i.e. $a \cdot b \in Q_\mu$.

As to the second statement, it is obvious that (6) implies (2) and (5); we are now going to show that (3) and (5) imply (6): by stability applied to $\mu a \leq a$ and $\mu b \leq 1$ we get $\mu a \cdot \mu b \leq a$ and similarly $\mu a \cdot \mu b \leq b$, hence $\mu a \cdot \mu b \leq a \wedge b$, therefore by monotonicity $\mu(\mu a \cdot \mu b) \leq \mu(a \wedge b)$ which together with the hypothesis gives $\mu a \cdot \mu b \leq \mu(a \wedge b)$; conversely $\mu(a \wedge b) = \vee\{\mu c : \mu c \leq a \wedge b\} = \vee\{\mu c : \mu c \leq a \ \& \ \mu c \leq b\} \leq \vee\{\mu c : \mu c \cdot \mu c \leq \mu a \cdot \mu b\} = \vee\{\mu c : \mu c \leq \mu a \cdot \mu b\} \equiv \mu(\mu a \cdot \mu b)$, where inequality follows from stability and the last equality from hypothesis. □

For any modality μ on Q, the relation

$$a \leq_\mu b \ \equiv \ \mu a \leq b$$

is a preorder on Q. Indeed, by $(*)$, $a \leq_\mu b$ iff $\mu a \leq \mu b$, and thus $\leq_\mu$ is reflexive and transitive because so is $\leq$.

The preorder $\leq_\mu$ is not antisymmetric in general, and actually antisymmetry of $\leq_\mu$ is clearly equivalent to: $\mu a = \mu b$ implies $a = b$, i.e. injectivity of μ. Thus, in order to get antisymmetry, we consider as usual the equivalence relation $=_\mu$ induced by the preorder, i.e. $a =_\mu b \ \equiv \ a \leq_\mu b \ \& \ b \leq_\mu a$, which then becomes

$$a =_\mu b \ \equiv \ \mu a = \mu b.$$

Let $Q/_{=_\mu}$ be the quotient set of Q with respect to $=_\mu$ and $[a]_\mu \equiv \{b \in Q : \mu a = \mu b\}$. Then it is obvious that

$$[a] \leq_\mu [b] \ \equiv \ a \leq_\mu b$$

is a (well defined) partial order on $Q/_{=_\mu}$, which is also denoted by $\leq_\mu$, since no confusion is likely to arise[†].

The following lemma shows that the supremum of μ-elements is a μ-element. This fact will be used to prove that $Q/_{=_\mu}$ can be given the structure of frame.

Lemma 2.8. If μ is a coclosure operator, $\mu(\vee_{i\in I}\mu a_i) = \vee_{i\in I}\mu a_i$, thus the set of μ-elements of Q is a complete sub-lattice of Q.

Proof. One inequality is an instance of (a); conversely, $\mu a_i \leq \vee_{i\in I}\mu a_i$ gives, by $(*)$, $\mu a_i \leq \mu(\vee_{i\in I}\mu a_i)$ for any index $i \in I$, hence $\vee_{i\in I}\mu a_i \leq \mu(\vee_{i\in I}\mu a_i)$. $\square$

Proposition 2.9. For every quantale Q with modality μ, the quotient set $Q/_{=_\mu}$ is a frame with the following operations:

$$\begin{aligned} \vee^\mu_{i\in I}[a_i] &\equiv [\vee_{i\in I}\mu a_i] \\ [a] \wedge^\mu [b] &\equiv [a \wedge b]. \end{aligned}$$

Actually $Q/_{=_\mu}$ is a quantale where the product

$$[a] \cdot^\mu [b] \equiv [\mu a \cdot \mu b]$$

and the meet coincide.

Proof. It is straightforward that $\vee^\mu$ and $\wedge^\mu$ are respectively the join and the meet w.r.t. the partial order $\leq_\mu$. We show that $Q/_{=_\mu}$ is a quantale where the monoid and the meet operation coincide. The operation $\cdot^\mu$ is associative since $\mu(\mu a \cdot \mu b) = \mu a \cdot \mu b$ and makes $Q/_{=_\mu}$ a commutative monoid with unity $[\top] = [1]$. By the definitions and the previous lemma the distributive law $[a] \cdot^\mu \vee^\mu_{i\in I}[b_i] = \vee^\mu_{i\in I}[a] \cdot^\mu [b_i]$ holds, so that $Q/_{=_\mu}$ is a quantale. Lastly, since $\mu(a \wedge b) = \mu a \cdot \mu b$, we have $[a] \wedge^\mu [b] = [a] \cdot^\mu [b]$, for all $[a], [b] \in Q/_{=_\mu}$. $\square$

In the frame $Q/_{=_\mu}$, an operation of implication is defined as usual by:

$$[a] \to [b] \equiv \vee^\mu\{[c] \in Q/_{=_\mu} : [a] \wedge^\mu [c] \leq_\mu [b]\}.$$

There exists an interesting connection with the implication $a \multimap b \equiv \vee\{c \in Q : a \cdot c \leq b\}$ of the quantale Q :

Proposition 2.10. For all $[a], [b] \in Q/_{=_\mu}$, $[a] \to [b] = [\mu a \multimap b]$.

Proof. We have the following chain of equalities: $\vee^\mu\{[c] \in Q/_{=_\mu} : [a] \wedge^\mu [c] \leq_\mu [b]\} = \vee^\mu\{[c] \in Q/_{=_\mu} : [a] \cdot^\mu [c] \leq_\mu [b]\} = [\vee\{\mu c \in Q : \mu c \leq \mu a \multimap b\}]$, by definition of the frame structure of $Q/_{=_\mu}$, (2) of proposition 2.5 and the adjunction existing in Q between the functors $a \cdot -$ and $a \multimap -$. Finally, since $\mu(\mu a \multimap b) \in \{\mu c \in Q : \mu c \leq \mu a \multimap b\}$, then $\mu(\mu a \multimap b) \leq \vee\{\mu c \in Q : \mu c \leq \mu a \multimap b\}$, that is $[\mu a \multimap b] \leq_\mu [a] \to [b]$. Conversely, from $\vee\{\mu c \in Q : \mu c \leq \mu a \multimap b\} \leq \mu a \multimap b$ one gets $\mu(\vee\{\mu c \in Q : \mu c \leq \mu a \multimap b\}) \leq \mu a \multimap b$, i.e. $[a] \to [b] \leq_\mu [\mu a \multimap b]$. [‡] $\square$

[†] We could say that $Q/_{=_\mu}$ is the abstraction obtained from (Q, μ) disregarding the difference between a and μa, that is, via the interpretation, between a formula A and its exclamation $!A$.

[‡] Proposition 2.10 can be proved in a more general, but not shorter, way, using categories. Via the

An equivalent, and somehow simpler, approach is to consider instead of $Q/_{=_\mu}$, the set Q_μ of μ-elements of Q. We will show that Q_μ can be provided with the structure of a frame isomorphic to $Q/_{=_\mu}$. We first need a preliminary lemma.

Lemma 2.11. Let μ be a coclosure operator on Q. Then Q_μ satisfies weakening iff $\mu\top \leq 1$; Q_μ satisfies contraction iff $\mu a \cdot \mu a = \mu a$, for all $a \in Q$.

Proof. If Q_μ satisfies weakening, then in particular $\mu\top \leq 1$. The converse follows by monotonicity, since $a \leq \top$ for all $a \in Q$. The second part holds by definitions. $\square$

We are now in a position to prove that Q_μ is a frame.

Proposition 2.12. Let Q be a quantale and μ a unary operation on Q, then:

(1) If μ is a coclosure operator, Q_μ is a complete sublattice of Q;
(2) If μ is a quantic conucleus, Q_μ is a subquantale of Q;
(3) If μ is a modality, Q_μ is a frame.

If μ is a conucleus on a frame L, L_μ is a subframe of L.

Proof. The first two statements and the last are also in [R], proposition 1.1.3 and theorem 3.1.3, but we prove them all the same, for the sake of completeness.

If μ is a coclosure operator, by lemma 2.8, Q_μ is closed under arbitrary joins, i.e. Q_μ is a sublattice of Q. Moreover, if μ is a quantic conucleus, by lemma 2.7, Q_μ is closed under binary products. If μ is a modality, by lemma 2.11, Q_μ satisfies weakening and contraction, thus, by proposition 2.3, Q_μ is a frame.

Lastly, let μ be a conucleus on L. Then μ is also a quantic conucleus on L, hence L_μ is a subquantale, therefore a subframe, of L. $\square$

As already announced we have:

Proposition 2.13. For every quantale Q and modality μ, the map

$$\begin{array}{rcl} \phi : Q/_{=_\mu} & \longrightarrow & Q_\mu \\ [a] & \mapsto & \mu a \end{array}$$

is a frame isomorphism.

Proof. The map ϕ is well defined and injective by definition of $=_\mu$ and is surjective by definition of Q_μ. Moreover, ϕ preserves arbitrary sups because $\phi(\vee^\mu_{i\in I}[a_i]) \equiv \phi([\vee_{i\in I}\mu a_i]) \equiv \mu(\vee_{i\in I}\mu a_i) = \vee_{i\in I}\mu a_i \equiv \vee_{i\in I}\phi([a_i])$ and preserves finite meets because $\phi([a]\cdot^\mu [b]) \equiv \phi([\mu a \cdot \mu b]) \equiv \mu(\mu a \cdot \mu b) = \mu a \cdot \mu b \equiv \phi([a]) \cdot \phi([b])$ and $\phi([\top]) \equiv \mu\top = 1$. $\square$

equality holding for quantic conuclei $\mu(\mu a \multimap c) = \mu(\mu a \multimap \mu c)$, which can be proved directly or as a consequence of Cor. 1 to prop. 3.1.1 in Rosenthal (1990), one shows that the map

$$\begin{array}{rcl} Q/_{=_\mu} & \longrightarrow & Q/_{=_\mu} \\ [c] & \longmapsto & [\mu a \multimap c] \end{array}$$

is well defined and monotone, i.e. is an endofunctor of the partial order $Q/_{=_\mu}$. Then, by straightforward application of definitions, one shows that for all $a, b, c \in Q$, $[a] \wedge^\mu [b] \leq_\mu [c]$ iff $[b] \leq_\mu [\mu a \multimap c]$, which means that the functor $[\mu a \multimap \cdot\,]$ is right adjoint to $[a] \wedge^\mu \cdot$. Thus, by the uniqueness of the adjoint functor, we have $[a] \to [b] = [\mu a \multimap b]$, for all $[a], [b] \in Q/_{=_\mu}$, since, by its definition, also $[a] \to \cdot$ is right adjoint to $[a] \wedge^\mu \cdot$.

The following lemma, which will be useful in subsequent sections, defines the operations of the frame Q_μ in terms of the operations of the quantale Q.

Lemma 2.14. Let μ be a modality on Q and $L = Q_\mu$. Then for all $a, b, a_i \in L, i \in I$:

(a) $a \wedge^L b = \mu(a \wedge^Q b)$;
(b) $a \vee^L b = a \vee^Q b$;
(c) $\wedge^L_{i \in I} a_i = \mu(\wedge^Q_{i\in I} a_i)$;
(d) $\vee^L_{i\in I} a_i = \vee^Q_{i\in I} a_i$;
(e) $a \to^L b = \mu(\mu a \multimap b) = \mu(a \multimap b)$;
(f) $0_L = 0_Q$.

Proof. (a) Since $a \wedge^Q b \leq a$ and $\mu(a \wedge^Q b) \leq a \wedge^Q b$, we have $\mu(a \wedge^Q b) \leq a$; similarly $\mu(a \wedge^Q b) \leq b$. If c is an element of Q_μ such that $c \leq a$ and $c \leq b$, then $c \leq a \wedge b$, hence $c = \mu c \leq \mu(a \wedge^Q b)$.

(b) It is enough to observe that if $a, b \in Q_\mu$, then by lemma 2.8, $a \vee^Q b \in Q_\mu$.

(c) and (d) can be proved generalizing (a) and (b).

(e) By uniqueness of the adjoint functor, it is enough to prove that, for all $a, b, c \in L$,

$$c \leq \mu(a \multimap b) \text{ iff } a \wedge^L c \leq b$$

in fact $c \leq \mu(a \multimap b)$ iff $\mu c \leq a \multimap b$ iff, by adjointness, $\mu c \cdot a \leq b$ iff, since $a = \mu a$, $\mu c \cdot \mu a \leq b$ iff $\mu(c \wedge^Q a) \leq b$ iff, by (a), $a \wedge^L c \leq b$.

(f) is obvious since $\mu 0 = 0$. □

Given a modality μ on a quantale Q, we have constructed the frame $Q_\mu \subseteq Q$. Conversely, given a frame $L \subseteq Q$, we obtain an operator μ_L on Q, defined by

$$\mu_L a \equiv \vee\{b \in L : b \leq a\}$$

As we shall prove below, this operator is in fact a modality. Moreover this procedure defines a biunivocal correspondence between frames included in quantales and modalities over quantales. Indeed this can be proved even with weaker hypotheses, as proposition 2.16 shows.

We have already proved (cf. lemma 2.8) that if μ is a coclosure operator, then Q_μ is closed under arbitrary joins. In fact also the converse holds (cf. Rosenthal (1990), proposition 1.1.3):

Proposition 2.15. If L is a subset closed under arbitrary joins of the quantale Q, μ_L is a coclosure operator.

Proof. The inequality $\mu_L a \leq a$ holds by definition of supremum. If $a \leq b$ then $\{c \in L : c \leq a\} \subseteq \{c \in L : c \leq b\}$, therefore $\mu_L a \leq \mu_L b$, so that μ_L is monotone. Since L is closed under joins, $\mu_L(a) \in \{b \in L : b \leq \mu_L a\}$ therefore $\mu_L(a) \leq \vee\{b \in L : b \leq \mu_L a\}$, which by definition is $\mu_L \mu_L(a)$, and idempotence is proved as well. □

The following proposition show that the correspondence thus obtained is biunivocal.

Proposition 2.16. Let μ be a coclosure operator on Q and L a subset of Q closed under joins. Then:

(i) $\mu_{Q_\mu} = \mu$;
(ii) $Q_{\mu_L} = L$.

Proof. (i) For all $a \in Q$, $\mu_{Q_\mu}(a) \equiv \vee\{b \in Q_\mu : b \leq a\} = \vee\{b \in Q : \mu b = b \,\&\, b \leq a\} = \mu a$, where the last equality holds by definition of coclosure operator.

(ii) $Q_{\mu_L} \equiv \{a \in Q : \mu_L a = a\} \equiv \{a \in Q : \vee\{b \in L : b \leq a\} = a\} = L$, since L is a subset of Q closed under arbitrary joins. □

This biunivocal correspondence extends to subquantales-quantic conuclei and subframes-conuclei (cf. Rosenthal (1990)). If μ is a quantic conucleus, then by lemmas 2.7 and 2.8, Q_μ is a submonoid of Q closed under arbitrary joins, i.e. a subquantale of Q. Moreover it follows from definitions that that if μ is a conucleus on L, then L_μ is a subframe of L. Conversely:

Proposition 2.17. If S is a subquantale of Q, then μ_S is a quantic conucleus. If S is a subframe of a frame L , then μ_S is a conucleus.

Proof. By proposition 2.15, μ_S is a coclosure operator, thus the claim amounts to prove the inequality $\mu(a) \cdot \mu(b) \leq \mu(a \cdot b)$ for arbitrary $a, b \in Q$. By lemma 2.7 this is equivalent to prove that Q_{μ_S} is closed under $\cdot$, which holds by proposition 2.16 from hypotheses on S. As for the second statement, just observe that the first part gives $\mu_S(a) \wedge \mu_S(b) \leq \mu_S(a \wedge b)$. Since the converse inequality holds by monotonicity of the coclosure operator, μ_S is a conucleus. □

At last we can prove:

Proposition 2.18. If L is a subframe of the quantale Q, μ_L is a modality.

Proof. By propositions 2.15, to reach the conclusion there remain (d) and (e) to be proved. As for (d), by lemma 2.7 we have to show that Q_{μ_L} is closed under $\cdot$ and that $\mu_L a \leq \mu_L a \cdot \mu_L a$. The first statement holds since we have already seen that $Q_{\mu_L} = L$. The second one amounts to prove that

$$\vee\{d \in L : d \leq a\} \leq \vee\{b \in L : b \leq a\} \cdot \vee\{c \in L : c \leq a\}$$

i.e., using distributive property twice

$$\vee\{d \in L : d \leq a\} \leq \vee\{b \cdot c : b \in L, c \in L, b \leq a, c \leq a\}$$

which is now evident since product in the frame L is idempotent.

As for (e), $\mu_L \top \equiv \vee\{b \in L : b \leq \top\} = 1$ holds since for all $a \in L$, $a \leq 1$. □

We thus may summarize the above results with the following:

Theorem 2.19. Let Q be a quantale. Then any modality (resp. quantic conucleus, coclosure operator) μ gives rise to a frame (resp. subquantale, complete sublattice) $Q_\mu \subseteq Q$, given by $Q_\mu \equiv \{a \in Q : \mu a = a\}$. Conversely any frame (resp. subquantale, complete sublattice) $L \subseteq Q$ gives rise to a modality (resp. quantic conucleus, coclosure operator) μ_L, defined by $\mu_L a \equiv \vee\{b \in L : b \leq a\}$. Such a correspondence is biunivocal. In the case of frames, it gives a biunivocal correspondence between conuclei μ on a frame L and subframes of L.

A partial result in the same direction was announced in Avron (1988).

As an application we can show how this construction works in a special case.

If we consider the pointwise partial order on the set of modalities on Q, i.e. $\mu_1 \leq \mu_2 \equiv \mu_1(a) \leq \mu_2(a)$ for all $a \in Q$, then the biunivocal correspondence stated above is easily seen to be order preserving, thus becoming an isomorphism of partially ordered sets. It is known (cf. Rosenthal (1990), proposition 3.1.3) that the set of subframes of a given quantale Q is a complete lattice, where the meet of a family $(L_i)_{i\in I}$ of subframes of L is their intersection, and the join is the meet of the family of subframes of Q containing $\cup_{i\in I} L_i$, which we shall denote by $\vee_{i\in I} L_i$. By the isomorphism we get:

Corollary 2.20. The set of modalities on a quantale Q is a complete lattice. [§]

Through the top element in the lattice of subframes of Q, we can now give an explicit description of the top element in the lattice of modalities on Q.

Definition 2.21. Let Q be a commutative quantale; we define:

$$C(Q) \equiv \{a \in Q : a \leq 1 \;\&\; a \leq a \cdot a\}$$

and call it the *centre* of Q.

We have:

Proposition 2.22. For any quantale Q , $C(Q)$ is a frame and is the greatest frame included in Q.

Proof. $C(Q)$ is closed under $\cdot$, since if $a \leq 1$, $b \leq 1$, then by stability $a \cdot b \leq 1$; if $a \leq a \cdot a$ and $b \leq b \cdot b$, then by stability and commutativity we have $a \cdot b \leq (a \cdot b)(a \cdot b)$. $C(Q)$ is closed under $\vee$: if for all $i \in I$, $b_i \leq 1$, then $\vee_{i\in I} b_i \leq 1$; moreover, if for all $i \in I$, $b_i \leq b_i \cdot b_i$, we have $\vee_{i\in I} b_i \leq \vee_{i\in I} b_i \cdot b_i \leq \vee_{i\in I, j\in I} b_i \cdot b_j = (\vee_{i\in I} b_i) \cdot (\vee_{i\in I} b_i)$, by distributive property. The subquantale $C(Q)$ is a frame because $a \cdot b = a \wedge b$ in $C(Q)$, by proposition 2.3. $C(Q)$ is the greatest frame included in Q because, if L is a frame included in Q, for all $a \in L$ we have $a \leq 1$ and $a \leq a \cdot a$. $\square$

The previous construction allows us to describe explicitly the modality corresponding to $C(Q)$. By definition of induced modality, $\mu_{C(Q)}(a) = \vee\{b \in C(Q) : b \leq a\}$, hence, by definition of $C(Q)$, $\mu_{C(Q)}(a) = \vee\{b \in Q : b \leq 1, b \leq b \cdot b, b \leq a\}$. Moreover, by the biunivocal correspondence stated by theorem 2.19, we can characterize elements of $C(Q)$ as follows: $a \in C(Q)$ iff $\mu_{C(Q)}(a) = a$, thus $a \in C(Q)$ iff $a = \vee\{b \in Q : b \leq 1, b \leq b \cdot b, b \leq a\}$.

We can give now a simpler characterization of modalities on Q. By theorem 2.19 the set $Mod(Q)$ of modalities on Q is in biunivocal correspondence with the set of frames included in Q, which is the same as the set of frames included in $C(Q)$, by proposition 2.22. Since $C(Q)$ is a frame, the set of frames included in $C(Q)$ is, by theorem 2.19, in biunivocal correspondence with the set of conuclei on $C(Q)$, $CN(C(Q))$. We therefore have the following chain of biunivocal correspondences:

$$Mod(Q) \longleftrightarrow \text{ subframes of } Q \longleftrightarrow \text{ subframes of } C(Q) \longleftrightarrow CN(C(Q)).$$

Thus, since the composition of left to right arrows is the identity map, any modality on

[§] We could ask whether it is also a frame, i.e. whether distributivity holds. But the following example shows that the inclusion $\langle \cup_{i\in I}(L_0 \cap L_i)\rangle \subseteq L_0 \cap \langle \cup_{i\in I} L_i\rangle$ may be proper. Let L be the frame $\{0, a, b, \top\}$, with $0 \leq a \leq \top$ and $0 \leq b \leq \top$; consider the subframes $L_0 = \{0, \top\}$, $L_1 = \{0, a\}$, $L_2 = \{0, b\}$. Then $L_0 \cap \langle L_1 \cup L_2\rangle = L_0$, while $\langle (L_0 \cap L_1) \cup (L_0 \cap L_2)\rangle = \{0\}$.

Q is a conucleus g, when restricted to $C(Q)$. Conversely, any conucleus g on $C(Q)$ gives a modality on Q, defined by composing from right to left:

$$\mu_g(a) = \vee\{b \in C(Q) : g(b) = b \ \& \ b \leq a\}.$$

We thus may conclude with the following

Theorem 2.23. If μ is a modality on Q, its restriction to $C(Q)$ is a conucleus on $C(Q)$. If g is a conucleus on $C(Q)$, μ_g, as defined above, is a modality on Q. The correspondence between modalities on Q and conuclei on $C(Q)$ is biunivocal.

Since $\mu = \mu_{Q_\mu}$, for any μ, if a is an element of Q, by definition of modality induced by a subframe, $\mu(a) \in Q_\mu$, hence, a fortiori, $\mu(a) \in C(Q)$. We thus obtain:

Corollary 2.24. For any modality μ on Q, $\mu(Q) \subseteq C(Q)$, hence also $\mu(Q) \subseteq \mu(C(Q))$.

3. Semantics of the translation from IL into ILL

¶

In 2.14 (e), we have seen how the implication of the locale Q_μ is related to the implication of the quantale Q via the modality μ: if $a, b \in Q_\mu$,

$$a \to b = \mu(\mu a \multimap b) = \mu(a \multimap b).$$

Since the valuation for ! in a quantale with modality is given by $V(!A) \equiv \mu V(A)$, this suggests a comparison with Girard's translation of the intuitionistic implication into the linear one, given by

$$(A \to B)^g = !A^g \multimap B^g.$$

In general, as it is known that frames and quantales with modality provide us with a complete semantics for intuitionistic and linear logic respectively, we would like to investigate more deeply the analogy between the insertion of the frame $L = Q_\mu$ in the given quantale Q with modality μ and the translation from IL into ILL. We can draw the following diagram:

$$\begin{tikzcd} IL \arrow[r, "(\cdot)^\circ"] \arrow[d, "V_L"'] & ILL \arrow[d, "V_Q"'] \\ L \arrow[r, "(\cdot)_\circ"] & (Q, \mu) \end{tikzcd} \qquad (1)$$

where V_L and V_Q are the valuations of intuitionistic and linear logic respectively into the frame L and into the quantale Q, $(\cdot)^\circ$ is a translation from IL into ILL and $(\cdot)_\circ$ is the embedding of the frame into the quantale.

Our purpose now is to find a suitable faithful translation that makes the diagram commute. Let us recall that Girard's translation (cf. Girard (1987), Schellinx (1991)) is defined by induction on formulas via the following clauses:

¶ See appendix B for a sequent calculus definition of IL and ILL.

$P^g \equiv P$, for any atomic formula P
$\bot^g \equiv 0$
$(A \wedge B)^g \equiv A^g \& B^g$
$(A \vee B)^g \equiv !A^g \oplus !B^g$
$(A \to B)^g \equiv !A^g \multimap B^g$
$(\forall x A)^g \equiv \forall x A^g$
$(\exists x A)^g \equiv \exists x !A^g$
This translation satisfies faithfulness in the form:

$$\Gamma \vdash_{IL} A \text{ iff } !\Gamma^g \vdash_{ILL} A^g.$$

One can easily realize that it does not make the diagram commute, since, for instance, $V_Q((A \wedge B)^g) \equiv V_Q(A^g \& B^g) \equiv V_Q(A^g) \wedge^Q V_Q(B^g)$, while $(V_L(A \wedge B))_\circ = (V_L(A) \wedge^L V_L(B))_\circ = \mu(V_L(A) \wedge^Q V_L(B))$ which in general are not the same, assuming as inductive hypotheses that $V_Q(A^g) = V_L(A)$ and $V_Q(B^g) = V_L(B)$.

There is another requirement on translations to point out. In Gödel-Gentzen double negation translation from Classical into Intuitionistic Logic, translated formulas are *stable*, i.e. they are equivalent to their double negation, and this is perfectly sound, since a translation should not distinguish more than its source does. In the same way, since intuitionistic formulas can be *weakened* and *contracted*, i.e. they are the same as their exclamation, a translation of an intuitionistic formula should be a formula which is *exclamative*, i.e. a formula A such that $A =_{ILL} !A$. Since $!A \vdash A$ always holds by $!L$, A is exclamative iff $A \vdash !A$. To find such a translation, note that:

Proposition 3.1.
(i) The constant 0 is an exclamative formula.
(ii) For any formula A, $!A$ is an exclamative formula.
(iii) If B and C are exclamative formulas, then $B \otimes C$, $B \oplus C$, $\exists x B$ are exclamative formulas.

Proof. (i) It holds since $0 \vdash A$ for all A, hence $0 \vdash !0$.
(ii) This is an immediate consequence of the $!R$ rule.
(iii) Suppose that $B \vdash !B$, $C \vdash !C$; by $\otimes R$ we get $B, C \vdash !B \otimes !C$. Since $!B \otimes !C \vdash !(B \otimes C)$ always holds, by cut and $\otimes L$ we obtain $B \otimes C \vdash !(B \otimes C)$. As for the connective $\oplus$, by applying $\oplus R$ to the hypotheses we have $B \vdash !B \oplus !C$ and $C \vdash !B \oplus !C$, hence, by $\oplus L$, $B \oplus C \vdash !B \oplus !C$. Moreover we can easily prove $!B \oplus !C \vdash !(B \oplus C)$, thus, by cut, $B \oplus C \vdash !(B \oplus C)$. Lastly, since $B \vdash \exists x B$, by $!L$ and $!R$, we have $!B \vdash !\exists x B$. By hypothesis and cut, $B \vdash !\exists x B$, thus $\exists x B \vdash !\exists x B$. ☐

These considerations lead us to propose the following translation, which satisfies the above requirement but is more economical than prefixing every subformula with !:
$P^* \equiv !P$, for any atomic formula P
$\bot^* \equiv 0$
$(A \wedge B)^* \equiv !(A^* \& B^*)$
$(A \vee B)^* \equiv A^* \oplus B^*$
$(A \to B)^* \equiv !(A^* \multimap B^*)$
$(\forall x A)^* \equiv !\forall x A^*$

$(\exists x A)^* \equiv \exists x A^*$

It is clear that for any formula A, its translation A^* is exclamative. Moreover observe that since $!(A\&B) =_{ILL} !A \otimes !B$, we could equivalently define $(A \wedge B)^* \equiv !A^* \otimes !B^*$, or, by proposition 3.1, $(A \wedge B)^* \equiv A^* \otimes B^*$, but we prefer the former since it allows a simpler translation of proofs.

We now show that $(\cdot)^*$ is a faithful translation.

Proposition 3.2. For any set of formulas Γ and any formula A of IL,

$$\Gamma \vdash_{IL} A \text{ iff } \Gamma^* \vdash_{ILL} A^*.$$

Proof. Suppose that $\Gamma \vdash_{IL} A$. We prove that $\Gamma^* \vdash_{ILL} A^*$ by induction on the lenght of the derivation of $\Gamma \vdash_{IL} A$.

Let $\Gamma \vdash A$ be an axiom: if it is $A \vdash A$, then $A^* \vdash A^*$ is an axiom of ILL; $\Gamma, \bot \vdash A$ becomes $\Gamma^*, 0 \vdash A^*$, which is an axiom of ILL as well.

Let us now examine the last rule applied in a sequent calculus derivation of $\Gamma \vdash_{IL} A$. If $A = B \wedge C$ and the sequent $\Gamma \vdash_{IL} A$ is obtained by means of $\wedge R$, let γ and δ be the derivations in IL of $\Gamma \vdash B$ and $\Gamma \vdash C$ respectively; by inductive hypothesis we obtain the derivations in ILL γ^* and δ^* of $\Gamma^* \vdash B^*$ and $\Gamma^* \vdash C^*$ which by $\&R$ give $\Gamma^* \vdash B^* \& C^*$ and therefore, since all formulas in Γ^* are exclamative, by $!R$, $\Gamma^* \vdash !(B^* \& C^*)$. For $\wedge L_1$ we have:

$$\dfrac{\dfrac{\gamma}{\Gamma, A \vdash C}}{\Gamma, A \wedge B \vdash C}\ \wedge L_1 \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\gamma^*}{\Gamma^*, A^* \vdash C^*}}{\Gamma^*, A^* \& B^* \vdash C^*}}{\Gamma^*, !(A^* \& B^*) \vdash C^*}$$

The rule $\wedge L_2$ has a similar translation.
The $\vee$-rules are translated into the corresponding $\oplus$-rules. As we have done for $\wedge R$, also in the translation of $\rightarrow R$ and $\forall R$, we use the fact that all formulas in Γ^* are exclamative,

and the dots denote repeated applications of $!L$, $!R$ and *cuts*.

$$\dfrac{\dfrac{\gamma}{\Gamma, B \vdash C}}{\Gamma \vdash B \to C} \to R \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\gamma^*}{\Gamma^*, B^* \vdash C^*}}{\Gamma^* \vdash B^* \multimap C^*}}{\Gamma^* \vdash !(B^* \multimap C^*)} \vdots$$

$$\dfrac{\dfrac{\gamma}{\Gamma \vdash A} \quad \dfrac{\delta}{\Gamma, B \vdash C}}{\Gamma, A \to B \vdash C} \to L \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\gamma^*}{\Gamma^* \vdash A^*} \quad \dfrac{\delta^*}{\Gamma^*, B^* \vdash C^*}}{\Gamma^*, A^* \multimap B^* \vdash C^*}}{\Gamma^*, !(A^* \multimap B^*) \vdash C^*}$$

$$\dfrac{\dfrac{\gamma}{\Gamma \vdash Bx}}{\Gamma \vdash \forall x Bx} \forall R \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\gamma^*}{\Gamma^* \vdash Bx^*}}{\Gamma^* \vdash \forall x Bx^*}}{\Gamma^* \vdash !\forall x Bx^*} \vdots$$

$$\dfrac{\dfrac{\gamma}{\Gamma, At \vdash B}}{\Gamma, \forall x Ax \vdash B} \forall L \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\gamma^*}{\Gamma^*, At^* \vdash B^*}}{\Gamma^*, \forall x Ax^* \vdash B^*}}{\Gamma^*, !(\forall x Ax^*) \vdash B^*}$$

$$\dfrac{\dfrac{\gamma}{\Gamma \vdash Bt}}{\Gamma \vdash \exists x Bx} \exists R \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\gamma^*}{\Gamma^* \vdash Bt^*}}{\Gamma^* \vdash \exists x Bx^*}$$

$$\dfrac{\dfrac{\gamma}{\Gamma, At \vdash B}}{\Gamma, \exists x Ax \vdash B} \exists L \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\gamma^*}{\Gamma^*, At^* \vdash B^*}}{\Gamma^*, \exists x Ax^* \vdash B^*}$$

The translation of structural rules is:

$$\dfrac{\dfrac{\gamma}{\Gamma \vdash A}}{\Gamma, B \vdash A} w \qquad \rightsquigarrow \qquad \dfrac{B^* \vdash !B^* \quad \dfrac{\dfrac{\gamma^*}{\Gamma^* \vdash A^*}}{\Gamma^*, !B^* \vdash A^*}}{\Gamma^*, B^* \vdash A^*}$$

$$\dfrac{\dfrac{\gamma}{\Gamma, A, A \vdash B}}{\Gamma, A \vdash B} c \qquad \rightsquigarrow \qquad \dfrac{A^* \vdash !A^* \quad \dfrac{\dfrac{!A^* \vdash A^* \quad \dfrac{!A^* \vdash A^* \quad \dfrac{\gamma^*}{\Gamma^*, A^*, A^* \vdash B^*}}{\Gamma^*, !A^*, A^* \vdash B^*}}{\Gamma^*, !A^*, !A^* \vdash B^*}}{\Gamma^*, !A^* \vdash B^*}}{\Gamma^*, A^* \vdash B^*}$$

Finally, translation of cut is unproblematic.

Conversely, assume a deduction in ILL of $\Gamma^* \vdash A^*$ is given. Then, by replacing $!A$ with A, for any formula A, $\oplus$ with $\vee$, $\otimes$ and & with $\wedge$, $\multimap$ with $\to$, 0 with $\perp$, we obtain a deduction in IL of $\Gamma \vdash A$. To get convinced about this, just observe that the axioms and the rules for the constants and the connectives of ILL which are translations of constants and connectives of IL, after these substitutions, turn into valid inferences of IL. Let us give full details:

$\Gamma, 0 \vdash A$ becomes $\Gamma, \perp \vdash A$ which is an axiom of IL.
$\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$ gives $\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$ which is easily proved in IL.
$\dfrac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}$ turns into $\dfrac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$ which is obtained from $\wedge R$ with several applications of weakening.
The rules for the connectives $\&, \oplus, \multimap$ become the rules for $\wedge, \vee, \rightarrow$ respectively.
$\dfrac{\Gamma \vdash B}{\Gamma, !A \vdash B}$ becomes weakening, $\dfrac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$ and $\dfrac{!\Gamma \vdash C}{!\Gamma \vdash !C}$ after substitutions are tautologies.
$\dfrac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$ yields contraction.
The rules for quantifiers, cut and exchange remain the same. □

We are now in the position to show that the above translation from IL into ILL is, by means of semantics, the syntactical counterpart to the algebraic construction of a frame inside a quantale provided with a modality, i.e. that diagram (1) commutes. Observe that for all A, $V_Q(A^*) \in Q_\mu$, since $A^* =_{ILL} !A^*$ implies $V_Q(A^*) = \mu V_Q(A^*)$.

Theorem 3.3. Let V_L and V_Q be valuations of IL and ILL into the frame L and the quantale Q, respectively. If $V_L(P) = V_Q(P)$ for all atomic formula P, then for any formula A,

$$V_Q(A^*) = V_L(A)_\circ$$

i.e. if diagram (1) commutes on atomic formulas, then it commutes on all formulas.

Proof. In what follows we will omit the explicit writing of $(\cdot)_\circ$. The proof is by induction on formulas.

If A is the atomic formula P, $V_Q(P^*) \equiv V_Q(!P) \equiv \mu V_Q(P) = \mu V_L(P) = V_L(P)$, by hypothesis and definition of $L = Q_\mu$. If $A = \perp$, $V_Q(\perp^*) = V_Q(0) \equiv 0_Q = 0_L \equiv V_L(\perp)$. If $A = B \wedge C$, $V_Q(A^*) \equiv V_Q(!(B^* \& C^*)) \equiv \mu(V_Q(B^*) \wedge V_Q(C^*))$, thus by inductive hypothesis $V_Q(A^*) \equiv \mu(V_L(B) \wedge_Q V_L(C))$ and, by lemma 2.14, $V_Q(A^*) = V_L(B) \wedge_L V_L(C) \equiv V_L(A \wedge B)$. For all other connectives the proof is completely similar. □

We say that a translation $(\cdot)^\star$ is a translation of schemes if it commutes with substitution, that is it satisfies the following diagram

$$\begin{array}{ccc} A(P) & \xrightarrow{(\cdot)^\star} & A(P)^\star \\ {\scriptstyle P/B}\downarrow & & \downarrow{\scriptstyle P/B^\star} \\ A(B) & \xrightarrow{(\cdot)^\star} & A(B)^\star = A(B^\star)^\star \end{array} \qquad (2)$$

If $P^\star = !P$, for atomic P, the previous condition is not satisfied, e.g. taking for A and B the formula P itself, since, in this case, $A(B)^\star = !P$ while $A(B^\star)^\star = !!P$.

This counterexample also shows a general fact: if $(\cdot)^\star$ has to be a translation of schemes it has to be the identity on atomic formulas. We see that this condition is also sufficient for a translation to be a translation of schemes. This amounts to prove that

$$A(B)^\star = A^\star(B^\star) \tag{3}$$

for all formulas A and B. In fact, if the previous equality holds for all B, in particular, for P atomic, it gives $A(P)^\star = A^\star(P^\star) = A^\star(P)$ and the commutativity of $(\cdot)^\star$ with substitution becomes:

$$\begin{tikzcd} A(P) \arrow[r, "(\cdot)^\star"] \arrow[d, "P/B"'] & A^\star(P) \arrow[d, "P/B^\star"'] \\ A(B) \arrow[r, "(\cdot)^\star"] & A(B)^\star = A^\star(B^\star) \end{tikzcd} \tag{4}$$

Thus, to show that $(\cdot)^\star$ is a translation of schemes, we just have to prove (3). If A is the atomic formula P, it collapses to the identity $P^\star = P^\star$. If A is the constant $\perp$ there is nothing to prove. Otherwise (3) is satisfied by any translation inductively defined, thus, in particular, it holds for Girard's translation.

Let $(\cdot)^t$ be the translation which is defined like $(\cdot)^g$, except for $(A \wedge B)^t \equiv !(A^t \& B^t)$, $(A \to B)^t \equiv !(!A^t \multimap B^t)$, $(\forall x A)^t \equiv !\forall x A^t$. By the previous remark, $(\cdot)^t$ is a translation of schemes. It is easy to prove that theorem 3.3 holds also for the translation $(\cdot)^t$. Moreover $(\cdot)^t$ satisfies the same faithfulness property of Girard's translation:

Proposition 3.4. For any set of formulas Γ and any formula A of IL,

$$\Gamma \vdash_{IL} A \text{ iff } !\Gamma^t \vdash_{ILL} A^t.$$

Proof. The proof follows the same pattern of 3.2, and is straightforward except for the three following cases:

$$\dfrac{\dfrac{\gamma}{\Gamma, A \vdash C}}{\Gamma, A \wedge B \vdash C}\ \wedge L_1 \qquad \rightsquigarrow \qquad \dfrac{\dfrac{!(A^t \& B^t) \vdash !A^t \otimes !B^t \qquad \dfrac{\dfrac{\dfrac{\gamma^t}{!\Gamma^t, !A^t \vdash C^t}}{!\Gamma^t, !A^t, !B^t \vdash C^t}}{!\Gamma^t, !A^t \otimes !B^t \vdash C^t}}{!\Gamma^t, !(A^t \& B^t) \vdash C^t}}{!\Gamma^t, !!(A^t \& B^t) \vdash C^t}$$

$$\dfrac{\dfrac{\gamma}{\Gamma \vdash A} \quad \dfrac{\delta}{\Gamma, B \vdash C}}{\Gamma, A \to B \vdash C}\ \to L \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!A^t \vdash !A^t \quad B^t \vdash B^t}{!A^t \multimap B^t, !A^t \vdash B^t}}{!(!A^t \multimap B^t), !A^t \vdash B^t}}{!(!A^t \multimap B^t), !A^t \vdash !B^t}}{!(!A^t \multimap B^t) \vdash !A^t \multimap !B^t} \qquad \dfrac{\dfrac{\dfrac{\gamma^t}{!\Gamma^t \vdash A^t}}{!\Gamma^t \vdash !A^t} \quad \dfrac{\delta^t}{!\Gamma^t, !B^t \vdash C^t}}{!\Gamma^t, !A^t \multimap !B^t \vdash C^t}}{!\Gamma^t, !(!A^t \multimap B^t) \vdash C^t}}{!\Gamma^t, !!(!A^t \multimap B^t) \vdash C^t}$$

$$\dfrac{\dfrac{\gamma}{\Gamma, At \vdash B}}{\Gamma, \forall x Ax \vdash B}\ \forall L \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall x A^t \vdash A^t}{!\forall x A^t \vdash !A^t}}{!\forall x A^t \vdash \forall x !A^t} \qquad \dfrac{\dfrac{\gamma^t}{!\Gamma^t, !A^t \vdash B^t}}{!\Gamma^t, \forall x !A^t \vdash B^t}}{!\Gamma^t, !\forall x A^t \vdash B^t}}{!\Gamma^t, !!\forall x A^t \vdash B^t}$$

□

Since $(\cdot)^t$ is the identity on atomic formulas, in general A^t is not exclamative. However we can prove:

Proposition 3.5. Let A be a formula of IL. If $P =_{ILL} !P$ for any atomic formula P occurring in A, then $!A^t =_{ILL} A^t$.

Proof. Immediate by induction on formulas (cf. 3.1). □

Observe that the previous result does not hold for Girard's translation: for instance, if A is $B \to C$, we have $A^g = !B^g \multimap C^g$, $!A^g = !(!B^g \multimap C^g)$. In this case the assumption on atomic formulas does not imply that $!A^g =_{ILL} A^g$, since there is no cut-free proof of $!B^g \multimap C^g \vdash_{ILL} !(!B^g \multimap C^g)$ under the inductive hypotheses $B^g \vdash !B^g$, $C^g \vdash !C^g$.

Before giving an "internal" formulation of the above proposition, a remark on notation is in order: if $\overline{P}$ denotes the multiset of formulas $(P_i)_{i=1,\ldots,n}$, then $\overline{P} \multimap !\overline{P}$ denotes the multiset of formulas $(P_i \multimap !P_i)_{i=1,\ldots,n}$.

Proposition 3.6. If $\overline{P}$ are the atomic formulas occurring in A, then $!(\overline{P} \multimap !\overline{P}) \vdash_{ILL} A^t \multimap !A^t$.

Proof. It follows from 3.5 and the proposition on p.70 in Troelstra (1992). □

Corollary 3.7. Let $\overline{P}$ be the multiset of atomic formulas in Γ. Then

$$\Gamma \vdash_{IL} A \text{ iff } !(\overline{P} \multimap !\overline{P}), \Gamma^t \vdash_{ILL} A^t.$$

Proof. If $\Gamma \vdash_{IL} A$, then by proposition 3.4 $!\Gamma^t \vdash_{ILL} A^t$. By the above result, for all formulas B in Γ, we have $!(\overline{P} \multimap !\overline{P}) \vdash_{ILL} B^t \multimap !B^t$, that is $!(\overline{P} \multimap !\overline{P}), B^t \vdash_{ILL} !B^t$. Hence, through several applications of cut, we get the claim.

Conversely, with the same replacements of proposition 3.2, we obtain $\overline{P} \to \overline{P}, \Gamma \vdash_{IL} A$ and thus, by axioms and cuts, $\Gamma \vdash_{IL} A$. □

4. The co-Kleisli category associated to a comonad

We are now going to see how the construction of the first part is a particular case of a categorical one. Let us first recall some definitions concerning the categorical semantics of linear logic (cf. Mac Lane (1971), Troelstra (1992)).

Definition 4.1. A *symmetric monoidal category* (SMC) $(\mathcal{C}, \otimes, I, \alpha, \gamma, \lambda)$ consists of
- a category $\mathcal{C}$
- a bifunctor $\cdot \otimes \cdot : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
- an object $I \in Ob(\mathcal{C})$
such that for all A, B, $C \in Ob(\mathcal{C})$ there are the following natural isomorphisms

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C$$

$$\gamma_{A,B} : A \otimes B \longrightarrow B \otimes A$$

$$\lambda_A : I \otimes A \longrightarrow A$$

satisfying the *Kelly-Mac Lane coherence conditions.*
A *closed symmetric monoidal category* $(CSMC)$ is a symmetric monoidal category $\mathcal{C}$ such that for all $A \in \mathcal{C}$, the functor $\cdot \otimes A$ has a right adjoint $A \multimap \cdot$.
A closed symmetric monoidal category with products, coproducts, terminal $(\top)$ and initial (O) objects $(\mathcal{C}, \otimes, I, \alpha, \gamma, \lambda, \phi, \prod, \coprod, \top, O)$ is called an *intuitionistic linear category***.
A *comonad* on $\mathcal{C}$ is a triple (T, r, s), where $T : \mathcal{C} \to \mathcal{C}$ is a functor and $s : T \to T^2$, $r : T \to id$ are natural transformations making the following diagrams commute

$$\begin{tikzcd} TA \arrow[r, "s_A"] \arrow[d, "s_A"'] & T^2A \arrow[d, "Ts_A"] \\ T^2A \arrow[r, "s_{TA}"] & T^3A \end{tikzcd} \tag{5}$$

|| This result is, with the same convention on notation, the analogue of

$$\Gamma \vdash_{CL} A \text{ iff } \neg\neg\overline{P} \to \overline{P}, \Gamma^G \vdash_{IL} A^G,$$

holding for Gödel-Gentzen double negation translation, with $P^G \equiv P$ in order to get a translation of schemes.

** As done elsewhere (cf. Martí-Oliet and Meseguer (1991)), instead of $\prod$ and $\coprod$, the more suggestive symbols & and $\oplus$ are used for binary product and coproduct respectively.

$$\begin{tikzcd} & TA \arrow[dl, "1_{TA}"'] \arrow[d, "s_A"] \arrow[dr, "1_{TA}"] & \\ TA & T^2A \arrow[l, "Tr_A"'] \arrow[r, "r_{TA}"] & TA \end{tikzcd} \tag{6}$$

To get a suitable interpretation of the modality ! of linear logic, we need further assumption on T, namely we suppose that

$$T(\top) \cong I \tag{7}$$

and that for all A, B there exists a natural isomorphism

$$\psi_{A,B} : TA \otimes TB \xrightarrow{\simeq} T(A\&B). \tag{8}$$

It is a standard construction (cf. Mac Lane (1971)) to associate to a comonad (T, r, s) on $\mathcal{C}$ the "co-Kleisli" category $K(\mathcal{C})$, whose objects are those of $\mathcal{C}$ itself, while the arrows are given by

$$K(\mathcal{C})(A, B) \equiv \mathcal{C}(TA, B)$$

If $f \in K(\mathcal{C})(A, B), g \in K(\mathcal{C})(B, C)$, then their composition is defined by

$$g \circ_{K(\mathcal{C})} f \equiv g \circ Tf \circ s_A.$$

As usual, let π_A and π_B be the canonical projections and let $\langle \cdot, \cdot \rangle$ be the universal morphism of coproduct. Then if we add the following coherence condition (cf. Hoofman (1992))

$$\begin{tikzcd} T(A\&B) \arrow[r, "\psi^{-1}_{A,B}"] \arrow[d, "s_{A\&B}"'] & TA \otimes TB \arrow[r, "s_A \otimes s_B"] & T^2A \otimes T^2B \arrow[d, "\psi_{TA,TB}"] \\ T^2(A\&B) \arrow[rr, "T\langle T\pi_A, T\pi_B \rangle"] & & T(TA\&TB) \end{tikzcd} \tag{9}$$

we get a a basic property of this construction:

Proposition 4.2. Let $\mathcal{C}$ be a $CSMC$ with terminal object and binary products, (T, r, s) a comonad on $\mathcal{C}$, satisfying the above conditions (5)-(9), $K(\mathcal{C})$ the corresponding co-Kleisli category. Then $K(\mathcal{C})$ is a cartesian closed category, with terminal object $\top$, binary product $A \times B \equiv A\&B$, projections $p_A \equiv r_A \circ T\pi_A$, $p_B \equiv r_B \circ T\pi_B$, universal arrow of product as in $\mathcal{C}$, exponentiation $A \Rightarrow B \equiv TA \multimap B$.
If moreover $\mathcal{C}$ has arbitrary products, then also $K(\mathcal{C})$ has arbitrary products.

Proof. Cf. Hoofman (1992) for the first part; the second is a straightforward generalization of the proof of existence of binary products. □

We can require additional structure on the $CSMC$ $\mathcal{C}$, e.g. the presence of finite and arbitrary coproducts. To get this structure inherited by the co-Kleisli category $K(\mathcal{C})$ we need further hypotheses on the comonad T. The following fact is useful to find necessary conditions.

By a general property of the co-Kleisli construction (cf. Mac Lane (1971)), there exists a couple of adjoint functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "G"] \arrow[r, phantom, "\top"] & K(\mathcal{C}), \arrow[l, shift left, "F"] \end{tikzcd}$$

where $G(A) \equiv A$, $G(f) \equiv f \circ r_A$, if $f \in \mathcal{C}(A, B)$, $F(A) \equiv TA$, and $F(f) \equiv Tf \circ s_A$, if $f \in K(\mathcal{C})(A, B)$. Thus F preserves coproducts, as every left adjoint does.

We are now ready to characterize existence of initial object in $K(\mathcal{C})$.

Proposition 4.3. Let $\mathcal{C}$ and $K(\mathcal{C})$ be as in the hypotheses of 4.2. Then $K(\mathcal{C})$ has initial object A iff $TA \cong_{\mathcal{C}} 0$.

Proof. Suppose that A is initial in $K(\mathcal{C})$, then $FA \equiv TA \cong 0$, since the functor F preserves coproducts. Conversely, for each B in $\mathcal{C}$ there is a unique arrow $TA \cong 0 \to B$, thus A is initial in $K(\mathcal{C})$. □

Now, let us suppose that $\mathcal{C}$ has binary coproduct $A \oplus B$ with injections ϵ_A, ϵ_B, and universal morphism of $f : A \to C$ and $g : B \to C$, $[f, g] : A \oplus B \to C$. We can define in a natural way a coproduct $A \oplus^K B$ in $K(\mathcal{C})$, by imposing suitable conditions on T, to be specified below. Indeed these conditions are also necessary if we require that this position defines a coproduct, since they are equivalent to conditions arising from the fact that the functor F preserves coproducts. These facts are stated by the following couple of propositions.

Proposition 4.4. Let $\mathcal{C}$ be a $CSMC$ with comonad, as in the hypotheses of 4.2, with binary coproducts. Suppose that for all $A, B \in \mathcal{C}$ there exists an isomorphism

$$\phi : T(TA \oplus TB) \to TA \oplus TB$$

such that the following diagrams are commutative

$$\begin{tikzcd} T^2A \arrow[r, "T\epsilon_{TA}"] & T(TA \oplus TB) \arrow[d, "\phi"] & T^2B \arrow[r, "T\epsilon_{TB}"] & T(TA \oplus TB) \arrow[d, "\phi"] \\ TA \arrow[u, "s_A"] \arrow[r, "\epsilon_{TA}"] & TA \oplus TB & TB \arrow[u, "s_B"] \arrow[r, "\epsilon_{TB}"] & TA \oplus TB \end{tikzcd} \tag{10}$$

and moreover

$$\phi^{-1} = [T\epsilon_{TA} \circ s_A, T\epsilon_{TB} \circ s_B]. \tag{11}$$

Then $K(\mathcal{C})$ has binary coproduct $A \oplus^K B \equiv TA \oplus TB$, with injections $\epsilon^K_A \equiv \epsilon_{TA}$, $\epsilon^K_B \equiv \epsilon_{TB}$ and universal morphism $[\cdot, \cdot]^K \equiv [\cdot, \cdot] \circ \phi$.

Proof. Routine checking of definitions. □

The existence of the isomorphism ϕ and the coherence conditions to be satisfied are also necessary, and moreover ϕ is in a way determined by the coproduct. Indeed the previous proposition admits the following converse.

Proposition 4.5. Suppose that $K(\mathcal{C})$ has binary coproduct given by $A \oplus^K B \equiv TA \oplus TB$, with canonical injections ϵ_{TA} and ϵ_{TB}. Then there exists an isomorphism $\phi : T(TA \oplus$

$TB) \to TA \oplus TB$ satisfying conditions (10) and (11) of proposition 4.4 and such that the universal morphism of coproduct in $K(\mathcal{C})$ is given by $[\cdot,\cdot]^K = [\cdot,\cdot] \circ \phi$.

Proof. The isomorphism ϕ exists because the above defined functor F preserves coproducts, thus $F(A \oplus^K B) \cong FA \oplus FB$, that is $T(TA \oplus TB) \cong TA \oplus TB$. Conditions (10) and (11) are obtained writing up what it means for the functor F to preserve coproducts. In particular, (11) yields easily the last statement. □

These propositions can be generalized to arbitrary coproducts in the obvious way, that is:

Proposition 4.6. Let $\mathcal{C}$ be a $CSMC$ with comonad, as in the hypotheses of 4.2, with arbitrary coproducts $\coprod_\lambda A_\lambda$. Suppose that there exists an isomorphism $\phi : T(\coprod_\lambda TA_\lambda) \to \coprod_\lambda TA_\lambda$, such that for all $\lambda \in \Lambda$ the following diagram is commutative

$$\begin{tikzcd} T^2 A_\lambda \arrow[r, "T\epsilon_{TA_\lambda}"] & T(\coprod\limits_\lambda TA_\lambda) \arrow[d, "\phi"'] \\ TA_\lambda \arrow[u, "s_{A_\lambda}"] \arrow[r, "\epsilon_{TA_\lambda}"] & \coprod\limits_\lambda TA_\lambda \end{tikzcd} \tag{12}$$

and moreover

$$\phi^{-1} = [(T\epsilon_{TA_\lambda} \circ s_{A_\lambda})_\lambda]. \tag{13}$$

Then $K(\mathcal{C})$ has arbitrary coproduct $\coprod_\lambda^K A_\lambda \equiv \coprod_\lambda TA_\lambda$, with injections $\epsilon^K_{A_\lambda} \equiv \epsilon_{TA_\lambda}$, and universal morphism $[\cdot]^K \equiv [\cdot] \circ \phi$.

Conversely:

Proposition 4.7. Suppose that $K(\mathcal{C})$ has arbitrary coproduct given by $\coprod_\lambda^K A_\lambda \equiv \coprod_\lambda TA_\lambda$, with canonical injections ϵ_{TA_λ}. Then there exists an isomorphism $\phi : T(\coprod_\lambda TA_\lambda) \to \coprod_\lambda TA_\lambda$ satisfying conditions (12) and (13) of proposition 4.6 and such that the universal morphism of coproduct in $K(\mathcal{C})$ is given by $[\cdot]^K = [\cdot] \circ \phi$.

It is well known that a CMSC with arbitrary coproducts which is a partial order is in fact a quantale and that a CCC with arbitrary coproducts is a frame. The natural question arising now is whether the construction of the first part is a particular case of the co-Kleisli construction applied to partial order categories.

To make these ideas precise we need a "partial order" functor between the given category and the corresponding partial order. First of all, to get a preorder, we have to identify all the morphisms having the same domain and codomain. Then, to force antisymmetry, we have to identify isomorphic objects, that is to consider the skeleton of the category thus obtained (this is essentially a quotient on morphisms and on objects). The first step is a particular case of the construction in Mac Lane (1971), p. 51, which we recall below:

Proposition 4.8. For a given category $\mathcal{C}$, let R be a function which assigns to each pair of objects A, B of $\mathcal{C}$ a binary relation $R_{A,B}$ on the set $\mathcal{C}(A,B) \times \mathcal{C}(A,B)$. Then there exists a category $\mathcal{C}/_R$ and a functor $Q = Q_R : \mathcal{C} \to \mathcal{C}/_R$ such that:

(i) If $fR_{A,B}g$ then $Qf = Qg$;
(ii) If $H : \mathcal{C} \to \mathcal{D}$ is any functor from $\mathcal{C}$ for which $fR_{A,B}g$ implies $Hf = Hg$, then there exists a unique functor $G : \mathcal{C}/_R \to \mathcal{D}$ with $G \circ Q_R = H$. Moreover the functor Q_R is a bijection on objects.

This proposition yields the desired result taking for $R_{A,B}$ the whole set $\mathcal{C}(A,B) \times \mathcal{C}(A,B)$, for every pair of objects of $\mathcal{C}$.

Let S be the projection functor of the preorder $\mathcal{C}/_R$ on its skeleton and let P be the composition of this functor with Q. Then $P(\mathcal{C})$ becomes a partial order with the position

$$PA \leq PB \;\equiv\; \textit{there exists a morphism } f \in \mathcal{C}(A,B).$$

Actually, $P(\mathcal{C})$ is a quantale with product

$$PA \cdot PB \equiv P(A \otimes B) \tag{14}$$

and lattice operations

$$PA \wedge PB \equiv P(A \& B) \tag{15}$$

$$\vee_{\lambda\in\Lambda} PA_\lambda \equiv P(\coprod_{\lambda\in\Lambda} A_\lambda) \tag{16}$$

In fact these are good definitions by the meaning of skeleton. The product is associative, commutative and with unity $P(\top)$ since the bifunctor $\otimes$ is associative, commutative and with unity up to isomorphism. Furthermore it is easy to prove that (15) and (16) give the meet and the join w.r.t. the above defined partial order. Distributivity holds since, for all A, the functor $A \otimes \cdot$ is a left adjoint and therefore preserves coproducts. Of course $P(\top)$ is the top element of the quantale.

The comonad T induces in $P(\mathcal{C})$ the modality defined by

$$\mu_T(PA) \equiv P(TA).$$

Indeed μ_T is well defined, since if A and B are isomorphic in $\mathcal{C}$, thus are TA and TB, because T is a functor. The comonad natural trasformations give $\mu_T PA \leq PA$ and $\mu_T PA \leq \mu_T \mu_T PA$, while monotonicity follows from T being a functor. Equality between $\mu_T(PA \wedge PB)$ and $\mu_T PA \cdot \mu_T PB$ holds by the isomorphism $T(A\&B) \cong TA \otimes TB$, while $T(\top) \cong I$ yields $\mu_T(P(\top)) = PI$.

Therefore, as we have seen in the first part, we can get - with suitable smallness conditions - the frame of the μ_T-stable objects of $P(\mathcal{C})$. On the other way round, also the partial order obtained from $K(\mathcal{C})$, $P(K(\mathcal{C}))$, is a frame, since $K(\mathcal{C})$ is a CCC with arbitrary coproducts. As one could expect, these frames are isomorphic. More explicitly:

Proposition 4.9. In the above hypotheses, the map

$$\begin{array}{rcl} \phi : P(K(\mathcal{C})) & \longrightarrow & (P(\mathcal{C}))_{\mu_T} \\ PA & \longmapsto & P(TA) \equiv \mu_T PA \end{array}$$

is a frame isomorphism.

Before proceeding to the proof the following consequence of coherence conditions is useful.

Lemma 4.10. Let $\mathcal{C}$ be a category with comonad T, $K(\mathcal{C})$ the associated co-Kleisly category. Then for any pair of objects A, $B \in \mathcal{C}$

$$A \cong_{K(\mathcal{C})} B \text{ iff } TA \cong_{\mathcal{C}} TB$$

where $\cong_{K(\mathcal{C})}$ and $\cong_{\mathcal{C}}$ denote isomorphism in $K(\mathcal{C})$ and in $\mathcal{C}$ respectively.

Proof. (of 4.9) By lemma 4.10, ϕ is well defined and injective, and it is surjective by definition of $(P(\mathcal{C}))_{\mu_T}$. The map ϕ preserves infima, in fact $\phi(PA \wedge PB) = \phi P(A\&B) \equiv PT(A\&B)$; on the other way round, $\phi PA \wedge^{\mu_T} \phi PB \equiv \mu_T(\phi PA \wedge \phi PB)$, and by definitions $\mu_T(\phi PA \wedge \phi PB) = PT(A\&B)$. As for arbitrary suprema, observe that $\phi(\vee_\lambda PA_\lambda) = \phi(P(\coprod_\lambda^{K(\mathcal{C})} A_\lambda))$ and $\phi(P(\coprod_\lambda^{K(\mathcal{C})} A_\lambda)) \equiv \phi(P(\coprod_\lambda^{\mathcal{C}} TA_\lambda)) \equiv P(T(\coprod_\lambda^{\mathcal{C}} TA_\lambda))$ by definition of coproduct in $K(\mathcal{C})$. By the hypothesis on T ensuring existence of coproducts in $K(\mathcal{C})$, $P(T(\coprod_\lambda^{\mathcal{C}} TA_\lambda)) = P(\coprod_\lambda^{\mathcal{C}} TA_\lambda)$ and moreover, $\vee_\lambda^{\mu_T} \phi P(A_\lambda) \equiv \vee_\lambda^{\mu_T} \mu_T(PA_\lambda) = \vee_\lambda \mu_T(PA_\lambda)$ by 2.14 (d). Furthermore $\vee_\lambda \mu_T(PA_\lambda) = \coprod_\lambda \mu_T(PA_\lambda) \equiv \coprod_\lambda PTA_\lambda$, thus $\phi(\vee_\lambda PA_\lambda) = \vee_\lambda^{\mu_T} \phi(PA_\lambda)$. Lastly, $\phi PI \equiv \mu_T PI = PI$, since μ_T is a modality on $P(\mathcal{C})$. $\square$

We thus obtain the diagram

$$\begin{tikzcd}
(\mathcal{C}, T) \arrow[r, "K"] \arrow[d, "P"'] & K(\mathcal{C}) \arrow[d, "P"'] \\
P(\mathcal{C}) \arrow[r, "\mu_T"] & (P(\mathcal{C}))_{\mu_T} \cong P(K(\mathcal{C}))
\end{tikzcd} \tag{17}$$

Appendix

Appendix A. Formal topologies and formal pretopologies with stable interior operator

The relationship between frames and quantales with modality described in the first section can equivalently be expressed by means of formal topologies and pretopologies. For this purpose we first extend the representation theorems of quantales and frames (cf. Battilotti and Sambin) to quantales with modalities. We show that if $\mathcal{F}$ is the formal pretopology representing the quantale Q, i.e. $Q \cong Sat(\mathcal{F})$, then modalities μ on Q correspond to stable interior operators R on $\mathcal{F}$. Furthemore any formal pretopology $\mathcal{F}$ with R gives rise in a natural way to a formal topology $\mathcal{A}$ such that the frame of its saturated subsets is isomorphic to the frame Q_μ.

We briefly recall some definitions from Sambin (1989).

Definition Appendix A.1. Let $\mathcal{F} \equiv (S, \cdot, 1, \lhd_{\mathcal{F}})$ be a formal pretopology. A map $R : \mathcal{P}S \longrightarrow \mathcal{P}S$ is called an *interior operator* on $\mathcal{F}$ if the following conditions (a)-(d) hold, it is a *stable interior operator* if it satisfies also (e)-(g), where U, V are arbitrary subsets of $\mathcal{P}S$:

(a) $RU \in Sat(\mathcal{F})$;

(b) $RU \lhd U$;

(c) $U \lhd V \rightarrow RU \lhd RV$;
(d) $RU \lhd RRU$;
(e) $RU \cdot RU =_{\mathcal{F}} RU$;
(f) $RU \cdot RV \lhd R(RU \cdot RV)$;
(g) $RS = \mathcal{F}1$.

Observe that an interior operator on $\mathcal{F}$ is just a coclosure operator on $Sat(\mathcal{F})$. Moreover it is determined completely by its restriction to $Sat(\mathcal{F})$, since it directly follows from definitions that for all $U \subseteq S$, $RU = R\mathcal{F}U$. Indeed, when R is a stable interior operator, this restriction is a modality on the quantale $Sat(\mathcal{F})$, since its definition is equivalent to the definition of modality given in the previous section.

We can make the analogy between modalities and stable interior operators more explicit through an extension of the representation theorem for quantales, which we recall now. For further details the reader is referred to Battilotti and Sambin (199?).

Let S be a *base* for a quantale Q, i.e. S is a submonoid of Q such that for all $a \in Q$, $a = \vee\{b \in S : b \leq a\}$. We can define two maps $\phi : \mathcal{P}S \longrightarrow Q$, $\phi(U) \equiv \vee U$ and $\psi : Q \longrightarrow \mathcal{P}S$, $\psi(a) \equiv \downarrow_S a \equiv \{b \in S : b \leq a\}$. It is straightforward, by definition of base, that $\phi(\psi(a)) = a$, so that ϕ is surjective. In general ϕ is not injective, and imposing injectivity actually amounts to consider the quotient set determined on $\mathcal{P}S$ by the equivalence relation $U =_{\mathcal{F}} V \equiv \downarrow_S \vee U = \downarrow_S \vee V$. It is not difficult to prove that $=_{\mathcal{F}}$ is indeed a congruence relation w.r.t. the quantale structure of $\mathcal{P}S$ defined by set theoretic operations of (arbitrary) union and pointwise multiplication, and the quotient $\mathcal{P}S/_{=_{\mathcal{F}}}$ can equivalently be viewed as the quantale $Sat(\mathcal{F})$, where the base of the pretopology $\mathcal{F}$ is S itself and for all $U \subseteq S$, $\mathcal{F}U \equiv \downarrow_S \vee U$. The (well defined !) map induced by ϕ on the quotient is an isomorphism of quantales (for abuse of notation we call it ϕ again). We can extend this isomorphism to modalities and stable interior operators considered as unary operations on Q and $Sat(\mathcal{F})$ respectively. In order to have a morphism w.r.t. the additional structure, the following conditions must be satisfied:

(a) $\psi(\mu a) = R_\mu \psi a$
(b) $\phi(RU) = \mu_R \phi U$

where R_μ and μ_R must be suitably defined in such a way that if μ is a modality then R_μ is a stable interior operator, and conversely if R is a stable interior operator then μ_R is a modality, and the correspondence is biunivocal, i.e. $R_{\mu_R} = R$ and $\mu_{R_\mu} = \mu$. Observe that $R_\mu U = R_\mu \downarrow_S \vee U \equiv R_\mu \psi(\vee U) = \psi\mu(\vee U) \equiv \downarrow_S \mu(\vee U)$, where the first equality holds because $RU = R\mathcal{F}U$ and the second because ψ is a morphism from (Q, μ) to $(Sat(\mathcal{F}), R)$. On the other way round, $\mu_R a = \mu_R \vee \downarrow_S a \equiv \mu_R \phi \downarrow_S a = \phi(R \downarrow_S a) \equiv \vee R \downarrow_S a$. This justifies the following definition.

Definition Appendix A.2. With the above notation we put $R_\mu U \equiv \downarrow_S \mu(\vee U)$ and $\mu_R a \equiv \vee R \downarrow_S a$.

By routine checking of definitions we can prove the following result:

Proposition Appendix A.3. If μ is a modality on Q, then R_μ is a stable interior operator on $\mathcal{F}$. Conversely, if R is a stable interior operator on $\mathcal{F}$, then μ_R is a modality on $Sat(\mathcal{F})$. This correspondence between modalities and stable interior operator is biunivocal, i.e. $R_{\mu_R} = R$ and $\mu_{R_\mu} = \mu$.

Proof. Let μ be a modality on Q; we prove that R_μ satisfies conditions (a)-(g) of definition Appendix A.1. For all $U \in \mathcal{P}S$, $\mathcal{F}R_\mu U \equiv \downarrow_S \vee \downarrow_S \mu \vee U = \downarrow_S \mu \vee U \equiv R_\mu U$, thus (a) is satisfied. As for (b), observe that $\mu \vee U \leq \vee U$ gives $\downarrow_S \mu \vee U \subseteq \downarrow_S \vee U$, that is $R_\mu U \lhd U$. Suppose $U \lhd V$, that is $U \subseteq \downarrow_S \vee V$; then $\vee U \leq \vee V$, thus by monotonicity $\mu \vee U \leq \mu \vee V$ and therefore $\downarrow_S \mu \vee U \subseteq \downarrow_S \mu \vee V$, hence, by reflexivity, $R\mu U \lhd R\mu V$, thus (c) is proved. Condition (d) holds iff $\downarrow_S \mu \vee U \subseteq \downarrow_S \vee \downarrow_S \mu(\vee \downarrow_S \mu \vee U)$. By definition of base, the right hand side is equal to $\downarrow_S \vee \downarrow_S \mu(\mu \vee U)$ and therefore, by idempotence of μ, it is equal to $\downarrow_S \vee \downarrow_S \mu \vee U$; again by definition of base equality follows. Condition (e) amounts to $\downarrow_S \mu \vee U = \downarrow_S \vee(\downarrow_S \mu \vee U \cdot \downarrow_S \mu \vee U)$; the two members are equal by distributivity of joins w. r. t. product, definition of base and idempotence of the product of μ-elements. By (a), condition (f) holds iff $\downarrow_S \mu \vee U \cdot \downarrow_S \mu \vee V \subseteq \downarrow_S \mu \vee (\downarrow_S \mu \vee U \cdot \downarrow_S \mu \vee V)$. By distributivity and definition of base the right hand side is equal to $\downarrow_S \mu(\mu \vee U \cdot \mu \vee V)$; therefore inclusion follows from (2) of proposition 2.5 and stability. As for (g), observe that $R_\mu S \equiv \downarrow_S \mu\top$, thus by 2.4, (e), $R_\mu S = \mathcal{F}1$.

Now, let R be a stable interior operator; we prove that μ_R is a modality via the characterization given in proposition 2.5. Since $R \downarrow_S a \lhd \downarrow_S a$, then by definition of precover and base, $R\downarrow_S a \subseteq \downarrow_S a$, hence $\vee R\downarrow_S a \leq a$, i.e. $\mu_R a \leq a$. Suppose $a \leq b$, then $\downarrow_S a \subseteq \downarrow_S b$; by reflexivity of the precover and monotonicity of R, we get $R\downarrow_S a \lhd R\downarrow_S b$, i.e. $R\downarrow_S a \subseteq R\downarrow_S b$ and therefore $\vee R\downarrow_S a \leq \vee R\downarrow_S b$, that is $\mu_R a \leq \mu_R b$. By idempotence of R and the fact that $RU = R\mathcal{F}U$, we get $R \downarrow_S a =_{\mathcal{F}} R \downarrow_S \vee R \downarrow_S a$, and therefore $\vee R \downarrow_S a =_{\mathcal{F}} \vee R \downarrow_S \vee R \downarrow_S a$ which by definition means $\mu_R a = \mu_R \mu_R a$. We have thus proved that μ_R is a coclosure operator. The product of μ_R-elements is idempotent, since by (e) of definition Appendix A.1, we have $\downarrow_S \vee(R\downarrow_S a \cdot R\downarrow_S a) = R\downarrow_S a$ and therefore $\vee\downarrow_S \vee(R\downarrow_S a \cdot R\downarrow_S a) = \vee R\downarrow_S a$ which by distributivity and definition of base gives the claim. To prove that $\mu_R a \cdot \mu_R b \leq \mu_R(\mu_R a \cdot \mu_R b)$, i.e. $\vee R\downarrow_S a \cdot \vee R\downarrow_S b \leq \vee R\downarrow_S (\vee R\downarrow_S a \cdot \vee R\downarrow_S b)$, observe that, by distributivity and the property of R of depending only on the saturation of its argument, the claim amounts to $\vee(R\downarrow_S a \cdot R\downarrow_S b) \leq \vee R(R\downarrow_S a \cdot R\downarrow_S b)$ which directly follows from definition of R. The last condition $\mu_R \top = 1$ or equivalently $\vee R\downarrow_S \top = 1$ holds since $\downarrow_S \top = S$ and $RS = \mathcal{F}1 \equiv \downarrow_S \vee 1$.

The correspondence is biunivocal, in fact $R_{\mu_R} U \equiv \downarrow_S \vee R \downarrow_S \vee U = RU$ since $RU = R\mathcal{F}U$ and RU is $\mathcal{F}$-saturated and $\mu_{R_\mu} a \equiv \vee\downarrow_S \mu \vee \downarrow_S a = \mu a$ by definition of base. □

Therefore we have proved that any quantale with modality is isomorphic to the class of saturated subsets of a suitable pretopology, endowed with a stable interior operator. Our next problem is to define the formal topology $\mathcal{A}$ representing the frame Q_μ directly in terms of $\mathcal{F}$ and R. The idea is to use the representation theorem for frames applied to Q_μ, where μ is defined in terms of R. By now, we take the whole frame as a base of itself, that is we define $S_{\mathcal{A}} \equiv \{a \in Q : \vee R \downarrow a = a\}$. This set inherits from Q the structure of a commutative monoid with unity. The cover relation is, as usual, $a \lhd_{\mathcal{A}} U \equiv a \leq \vee U$, and this shows, in particular, that the restriction of the precover $\lhd_{\mathcal{F}}$ to $S_{\mathcal{A}} \times \mathcal{P}(S_{\mathcal{A}})$ is indeed a cover.

We obtain the following diagram, in which the vertical arrows are the isomorphisms given by the representation theorems:

$$\begin{tikzcd} (Q,\mu) \arrow[r, "\mu"] \arrow[d, "\wr"'] & Q_\mu \arrow[d, "\wr"'] \\ (Sat\mathcal{F}, R) & Sat\mathcal{A} \end{tikzcd} \tag{18}$$

The bottom arrow is absent because the codomain of R is not exactly $Sat\mathcal{A}$, but the frame associated to the quantale $Sat(\mathcal{F})$ and the modality R, $L \equiv Sat(\mathcal{F})_R \equiv \{U \in Sat(\mathcal{F}) : RU =_{\mathcal{F}} U\}$ [††]; this set is a frame with the following operations (cf. Lemma 2.14, and the definition of the quantale structure of $Sat(\mathcal{F})$ in Sambin (1989)):

$U \wedge^L V \equiv R(U \cap V)$

$\vee^L_{i\in I} U_i \equiv \vee^Q_{i\in I} U_i \equiv \mathcal{F}(\cup_{i\in I} U_i)$.

It is easy to prove that these operations indeed define infima and suprema for the partial order induced by the precover, in fact set theoretic inclusion, on L. Moreover the supremum can be defined without involving $\mathcal{F}$, since one can prove that if the U_i are R-stable, then $\mathcal{F}(\cup_{i\in I} U_i) = R(\cup_{i\in I} U_i)$.

The following proposition shows that this frame is in fact isomorphic to Q_μ:

Proposition Appendix A.4. The map

$$\begin{array}{rcl} \psi : Q_\mu & \longrightarrow & L \equiv \{U \in Sat(\mathcal{F}) : U = R_\mu U\} \\ a & \longmapsto & \downarrow_S a \end{array}$$

defines an isomorphism of frames with inverse $\phi : U \longmapsto \vee U$.

Proof. Let $a \in Q_\mu$; by the representation theorem we already know that $\downarrow_S a \in Sat\mathcal{F}$, moreover $R \downarrow_S a =\downarrow_S (\mu_R a) =\downarrow_S a$ because a is μ-stable, hence $\downarrow_S a \in L$. Conversely, if $U \in L$ then $\mu(\vee U) = \vee(R_\mu U) = \vee U$, so that the given maps are well defined. We now prove that ψ is a morphism of frames. Let $a, b \in Q_\mu$. Then $\psi(a \wedge^\mu b) \equiv \psi(\mu(a \wedge b)) = R_\mu \psi(a \wedge b)$ since ψ is a morphism (cf. (a) above), moreover $\psi(a \wedge b) = \psi a \cap \psi b$, thus by definition of the frame structure of L, $\psi(a \wedge^\mu b) = R_\mu \downarrow_S a \wedge^L R_\mu \downarrow_S b$; then by (a) again and the fact that a and b are μ-stable we obtain $\psi(a \wedge^\mu b) = \psi a \wedge^L \psi b$. The map ψ also preserves arbitrary sups. Let $\{a_i\}_{i\in I}$ be an indexed family of elements of Q_μ. Then $\psi(\vee^\mu_{i\in I} a_i) \equiv \psi(\mu(\vee_{i\in I} a_i)) = R_\mu \psi(\vee_{i\in I} a_i)$. On the other side $\vee^L_{i\in I} \psi(a_i) \equiv R(\cup_{i\in I} \psi(a_i)) = R\mathcal{F}(\cup_{i\in I} \psi(a_i))$ and it is easy to prove that $R\mathcal{F}(\cup_{i\in I} \psi(a_i)) \equiv R(\downarrow_S \vee \cup_{i\in I} \psi(a_i)) = R(\downarrow_S \vee_{i\in I} (\vee \psi(a_i))$. Finally $R(\downarrow_S \vee_{i\in I} (\vee \psi(a_i)) = R(\downarrow_S \vee_{i\in I} a_i) \equiv R(\psi \vee_{i\in I} a_i)$. The verification that ϕ is the inverse of ψ is immediate. $\square$

[††] Observe that by condition (a) we can equivalently write $L = \{U \in \mathcal{P}S : RU = U\}$.

We thus obtain the following commutative diagram:

$$
\begin{tikzcd}
(Q,\mu) \arrow[r, "\mu"] \arrow[d, "\wr"'] & Q_\mu \arrow[d, "\wr"'] \\
(Sat\mathcal{F}, R) \arrow[r, "R"] & Sat\mathcal{F}_R \cong Sat\mathcal{A}
\end{tikzcd}
\tag{19}
$$

Appendix B. Table of sequent calculus rules for IL and ILL

Here follows a sequent calculus version of Intuitionistic Logic and Intuitionistic Linear Logic. A remark on notation is in order: we use greek upper case Γ, Γ', Δ, Δ' ...for finite sequences of formulas, A, B, C, ...for arbitary formulas. As for linear connectives we conform to Girard's notation, but use two sided sequents. For the rules of $\forall R$ and $\exists L$ the usual side conditions on the variable are assumed.

Intuitionistic Logic **IL**

Axioms:

$A \vdash A \qquad \Gamma, \bot \vdash A$

Rules:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge R \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge L_1 \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge L_2$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee R_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee R_2 \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ \vee L$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}\ \rightarrow R \qquad \frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \rightarrow B \vdash C}\ \rightarrow L$$

$$\frac{\Gamma \vdash Ax}{\Gamma \vdash \forall x Ax}\ \forall R \qquad \frac{\Gamma, At \vdash B}{\Gamma, \forall x Ax \vdash B}\ \forall L$$

$$\frac{\Gamma \vdash At}{\Gamma \vdash \exists x Ax}\ \exists R \qquad \frac{\Gamma, Ax \vdash B}{\Gamma, \exists x Ax \vdash B}\ \exists L$$

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}\ w \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ c \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ e$$

$$\frac{\Gamma_1 \vdash A \quad \Gamma_2, A \vdash B}{\Gamma_1, \Gamma_2 \vdash B}\ cut$$

Intuitionistic Linear Logic **ILL**

Axioms:

$A \vdash A$ $\qquad \vdash 1$

$\Gamma, 0 \vdash A$ $\qquad \Gamma \vdash \top$

Rules :

$$\frac{\Gamma \vdash B}{\Gamma, 1 \vdash B}\ 1L$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes L \qquad \frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \otimes R$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A\&B \vdash C}\ \&L_1 \qquad \frac{\Gamma, B \vdash C}{\Gamma, A\&B \vdash C}\ \&L_2 \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A\&B}\ \&R$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ \oplus R_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ \oplus R_2 \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\ \oplus L$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap R \qquad \frac{\Gamma_1 \vdash A \quad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}\ \multimap L$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ !L_1 \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ !L_2 \qquad \frac{!\Gamma \vdash C}{!\Gamma \vdash !C}\ !R \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ !c$$

$$\frac{\Gamma \vdash Ax}{\Gamma \vdash \forall x Ax}\ \forall R \qquad \frac{\Gamma, At \vdash B}{\Gamma, \forall x Ax \vdash B}\ \forall L \qquad \frac{\Gamma \vdash At}{\Gamma \vdash \exists x Ax}\ \exists R \qquad \frac{\Gamma, Ax \vdash B}{\Gamma, \exists x Ax \vdash B}\ \exists L$$

$$\frac{\Gamma_1 \vdash A \quad \Gamma_2, A \vdash B}{\Gamma_1, \Gamma_2 \vdash B}\ cut \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ e$$

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