SEQUENT CALCULUS IN NATURAL DEDUCTION STYLE

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Abstract

A sequent calculus is given in which the management of weakening and contraction is organized as in natural deduction. The latter has no explicit weakening or contraction, but vacuous and multiple discharges in rules that discharge assumptions. A comparison to natural deduction is given through translation of derivations between the two systems. It is proved that if a cut formula is never principal in a derivation leading to the right premiss of cut, it is a subformula of the conclusion. Therefore it is sufficient to eliminate those cuts that correspond to detour and permutation conversions in natural deduction.

§1. Introduction. In natural deduction, what corresponds to weakening and contraction in sequent calculus, is achieved by permitting vacuous and multiple discharges in those rules that discharge assumptions. We shall give a formulation of sequent calculus 'in natural deduction style,' in which weakening and contraction work the same way. Discharge in natural deduction corresponds to the application of a sequent calculus rule that has an active formula in the antecedent of a premiss. These are the left rules and the right implication rule. In sequent calculus, ever since Gentzen, weakening and contraction have been made into steps independent of the application of these rules. Cut elimination is much more complicated than normalization, with numerous cases of permutation of cut that do not have any correspondence in the normalization process. Moreover, in usual sequent calculi, due to mentioned independence, there can be formulas concluded by weakening or contraction that remain inactive through a whole derivation. These steps do not contribute anything, they are totally 'useless' and the formulas can either be pruned out, for useless weakening, or left multiplied, for useless contraction. The calculus we present avoids such useless steps altogether.

The calculus we present is characterized by the following two properties: First, two-premiss rules have independent contexts, corresponding to the independent treatment of assumptions in natural deduction. The structure of a calculus with independent contexts but with explicit rules of weakening and contraction has been studied in von Plato (2001a). Secondly, weakening and contraction are rendered implicit by letting any number $n \geqslant 0$ of repetitions of an active formula be removed in a logical rule. This formulation of the rules was found by the first author in connection with studies on linear logic, Negri (2000).

In the calculus we give, only those cuts need be eliminated that correspond to detour and permutation conversions. These are the cases where the cut formula is principal in at least the right premiss of cut. In addition, the cut formula can be principal somewhere higher up in the derivation of the right premiss of cut, and the cut is permuted up there in one step. For all other cases of cut, we prove that the cut formula is a subformula of the conclusion. Therefore the subformula property, Gentzen's original aim in the 'Hauptsatz,' can be concluded by eliminating only those cuts where the cut formula is principal in the derivation of the right premiss.

§2. The calculus and proof of cut elimination. Contexts will be treated as multisets, denoted by $\Gamma, \Delta, \Theta, \ldots$. Arbitrary formulas are denoted by $A, B, C, \ldots$, and n occurrences of a formula A by A^n. Falsity $\perp$ is the constant value of a zero place logical connective. The intuitionistic single succedent sequent calculus in natural deduction style is denoted by *GN*. We shall also briefly indicate a corresponding multisuccedent classical calculus *GM* for which the proof of cut elimination goes through in a similar way. We shall first consider the propositional part of the single succedent calculus. The logical rules have independent contexts.

GN

Logical axiom:

$A \Rightarrow A$

Logical rules:

$$\frac{A^m, B^n, \Gamma \Rightarrow C}{A\&B, \Gamma \Rightarrow C}\, L\& \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A\&B}\, R\&$$

$$\frac{A^m, \Gamma \Rightarrow C \quad B^n, \Delta \Rightarrow C}{A \vee B, \Gamma, \Delta \Rightarrow C}\, L\vee \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\, R\vee 1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\, R\vee 2$$

$$\frac{\Gamma \Rightarrow A \quad B^n, \Delta \Rightarrow C}{A \supset B, \Gamma, \Delta \Rightarrow C}\, L\supset \qquad \frac{A^m, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\, R\supset$$

$$\frac{}{\perp \Rightarrow C}\, L\perp$$

The formula with the connective is *principal* in a rule, the other shown formulas are *active*. For $\perp$, we write a zero premiss rule instead of an axiom, to emphasize that $\perp$, too, has a left rule with principal formula $\perp$.

The left rules $L\&$, $L\vee$ and $L\supset$ as well as $R\supset$ have instances for any $m, n \geqslant 0$. For example, from $L\&$ with $m = 1, n = 0$ we get the first of Gentzen's original left conjunction rules, with premiss $A, \Gamma \Rightarrow C$. We say that formulas A and B are *used* in these rules. Whenever $m = 0$ or $n = 0$ in an instance, there is a *vacuous use*, corresponding to weakening, and whenever $m > 1$ or $n > 1$, there is a *multiple use*, corresponding to contraction.

The intuitionistic calculus *G0i* of von Plato (2001a) is the same as *GN* with the exponents in the rules removed and with explicit weakening and contraction as primitive rules.

The rule of cut is

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\,{\scriptstyle Cut}$$

DEFINITION 1. *If a multiset Δ is obtained from Γ by multiplying formulas in Γ, where zero multiplicity is also permitted, Δ is a* multiset reduct *of* Γ.

The relation of being a multiset reduct is reflexive and transitive. We also call a sequent a reduct of another if its antecendent is a multiset reduct. These reducts are generated by steps of cut elimination, in the same way as assumptions are multiplied in the conversions to normal form in natural deduction. In usual cut elimination procedures, once the cut has been permuted up, the original antecedent of the conclusion of cut is restored by weakenings and contractions following the permuted cut. In our calculus, weakening and contraction are not explicitly available, but the restriction is not essential:

PROPOSITION 2. *If in the derivation of $\Gamma \Rightarrow C$ in* GN+Cut *the sequent $\Delta \Rightarrow D$ occurs and if the subderivation down to $\Delta \Rightarrow D$ is substituted by a derivation of $\Delta^* \Rightarrow D$ where Δ^* is a multiset reduct of Δ, then the derivation can be continued to conclude $\Gamma^* \Rightarrow C$ with Γ^* a multiset reduct of Γ.*

PROOF: It is sufficient to consider an uppermost cut that we may assume to be the last step of the whole derivation. First consider the part before the cut, having only axioms and logical rules. Starting with the derivation of $\Delta^* \Rightarrow D$, the derivation is continued as with $\Delta \Rightarrow D$, save for the steps that use formulas. It is enough to consider such rules when one premiss is $\Delta^* \Rightarrow D$. If in the original derivation a formula from Δ was used that does not occur in Δ^*, a vacuous use is made, and similarly for formulas that occur multiplied in Δ^*, as compared to Δ, a multiple use is made.

It remains to show that the conclusion of cut can be replaced with a sequent having a multiset reduct as antecedent. Let the original cut concluding $\Gamma \Rightarrow C$ be

$$\frac{\Gamma_1 \Rightarrow A \quad A, \Gamma_2 \Rightarrow C}{\Gamma_1, \Gamma_2 \Rightarrow C}\,{\scriptstyle Cut}$$

where $\Gamma_1, \Gamma_2 = \Gamma$, and let the reduced premisses be ${\Gamma_1}^* \Rightarrow A$ and $A^n, {\Gamma_2}^* \Rightarrow C$. If $n = 1$ a cut with the reduced premisses will give a conclusion with a multiset reduct of Γ as antecedent. If $n = 0$ the conclusion of cut is replaced by the right premiss. If $n > 1$, we make n cuts with left premiss ${\Gamma_1}^* \Rightarrow A$

in succession and the conclusion of the last cut has a multiset reduct of Γ as antecedent. QED.

The proposition shows two things: 1. It is enough to consider derivability in *GN* modulo multiset reducts. 2. It is enough to perform cut eliminations modulo multiset reducts.

DEFINITION 3. *A cut with the premisses* $\Gamma \Rightarrow A$ *and* $A, \Delta \Rightarrow C$ *is* redundant *in the following cases:*

(i) Γ *contains* A,

(ii) Γ *or* Δ *contains* $\bot$,

(iii) $A = C$.

(iv) Δ *contains* C,

(v) *the derivation of* $A, \Delta \Rightarrow C$ *contains a sequent with a multiple occurrence of* A.

THEOREM 4. ELIMINATION OF REDUNDANT CUTS. *Given a derivation of* $\Gamma \Rightarrow C$ *in* GN+Cut *there is a derivation with redundant cuts eliminated.*

PROOF. In case (i) of redundant cut, if Γ contains A, then A, Δ is a multiset reduct of Γ, Δ and by proposition 2, the cut is deleted and the derivation continued with $A, \Delta \Rightarrow C$. In case (ii), the conclusion has $\bot$ in the antecedent and the derivation begins with $\bot \Rightarrow C$. In case (iii), if $A = C$, the cut is deleted and the derivation continued with $\Gamma \Rightarrow C$. In case (iv), if Δ contains C, the derivation begins with $C \Rightarrow C$.

Case (v) of redundant cut can obtain in two ways: 1. It can happen that A has another occurence in the context Δ of the right premiss, and therefore also in the conclusion of cut. But in this case, the antecedent A, Δ of the right premiss is a multiset reduct of Γ, Δ and by proposition 2, the cut can be deleted. 2. It can happen that there was a multiple occurrence of A in some antecedent in the derivation of the right premiss and all but one occurrence were active in earlier cuts or logical rules. In the former case, if the right premiss of a cut is $A, \Delta' \Rightarrow C'$ and Δ' contains another occurrence of A the cut is deleted and the derivation continued from $\Delta' \Rightarrow C'$. In the latter case, using all occurrences of A will give a derivation of $\Delta \Rightarrow C$. Again, since Δ is a multiset reduct of the antecedent of conclusion of cut the cut can be deleted and the derivation continued from $\Delta \Rightarrow C$. QED.

Redundant cuts (i)–(iv) have as one premiss a sequent from which an axiom or conclusion of $L\bot$ is obtainable as a multiset reduct. In particular, if one premiss already is an axiom or conclusion of $L\bot$, a special case of redundant cut (i)–(iv) obtains.

DEFINITION 5: *A cut is* hereditarily principal (nonprincipal) *in a derivation if its cut formula is principal in some rule (is never principal) in the derivation of the right premiss of cut.*

Proposition 6. *The first occurrence of a hereditarily principal cut formula in a derivation without redundant cuts is unique.*

Proof. Assume there are at least two such occurrences. But then there is a sequent with a multiple occurrence of the cut formula and the derivation has a redundant cut as in case (v) of definition 3. QED.

A principal cut is the special case of hereditarily principal cut, with the cut formula principal in the last rule deriving the right premiss. The idea of cut elimination is to consider only hereditarily principal cuts and to permute them up in one step to the first occurrence of a cut formula A hereditarily principal in the derivation of the right premiss.

It can happen that the instance of a rule concluding a hereditarily principal cut formula had vacuous or multiple uses of active formulas. These cuts are *hereditarily vacuous* and *hereditarily multiple*, respectively:

Definition 7. *If a hereditarily principal cut formula is concluded by rule $L\&$ and $m, n = 0$, or by rule $L\lor$ and $m = 0$ or $n = 0$, or by rule $L\supset$ and $n = 0$, the cut is* hereditarily vacuous. *If the formula is concluded by $L\&$ and $m > 1$ or $n > 1$, or by $L\lor$ and $m, n > 1$, or by $L\supset$ and $n > 1$, a* hereditarily multiple cut *obtains.*

We now prove a cut elimination theorem for hereditarily principal cuts. The proof is by induction on the length of cut formula, with a subinduction on height of derivation of the left premiss of cut. Length is defined in the usual way, 0 for $\perp$, 1 for atoms, and sum of lengths of components plus 1 for proper connectives. Height of derivation is the greatest number of consecutive steps of inference in it. In the proof, multiplication of every formula occurrence in Γ to multiplicity n is written Γ^n.

Theorem 8. Elimination of hereditarily principal cuts. *Given a derivation of $\Gamma \Rightarrow C$ with cuts, there is a derivation of $\Gamma^* \Rightarrow C$ with no hereditarily principal cuts, with Γ^* a multiset reduct of Γ.*

Proof. First remove possible redundant cuts. Then consider the first hereditarily principal cut in the derivation which we may assume to be the last step. If the cut formula is not principal in the left premiss, the cut is permuted in the derivation of the left premiss, with its height of derivation diminished.

There remain three cases with the cut formula principal in both premisses. In each case, if a step of cut elimination produces redundant cuts, these are at once eliminated.

1. Cut formula is $A\&B$. If $m > 0$ or $n > 0$ we have the derivation

$$\dfrac{\dfrac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A\&B}{\scriptstyle R\&} \qquad \begin{array}{c} \dfrac{A^m, B^n, \Theta' \Rightarrow C'}{A\&B, \Theta' \Rightarrow C'}{\scriptstyle L\&} \\ \vdots \\ A\&B, \Theta \Rightarrow C \end{array}}{\Gamma, \Delta, \Theta \Rightarrow C}{\scriptstyle Cut}$$

We make m cuts with $\Gamma \Rightarrow A$, starting with the premiss $A^m, B^n, \Theta' \Rightarrow C'$, and up to $B^n, \Gamma^m, \Theta' \Rightarrow C'$, then continue with n cuts with $\Delta \Rightarrow B$, up to the conclusion $\Gamma^m, \Delta^n, \Theta' \Rightarrow C'$. Now the derivation is continued as before from where $A\&B$ was principal, to conclude $\Gamma^m, \Delta^n, \Theta \Rightarrow C$, all cuts in the derivation being on shorter formulas than in the initial derivation.

If $m, n = 0$, we have a hereditarily vacuous cut, with $\Theta' \Rightarrow C'$ the premiss of rule $L\&$. It is not a special case of the previous since there is nothing to cut. Instead the derivation is continued without rule $L\&$ until $\Theta \Rightarrow C$ is concluded.

2. Cut formula is $A \vee B$. With $A \vee B$ principal in the left premiss, assume the rule is $R\vee_1$ with $m > 0$:

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}{\scriptstyle R\vee 1} \qquad \begin{array}{c} \dfrac{A^m, \Delta' \Rightarrow C' \quad B^n, \Theta' \Rightarrow C'}{A \vee B, \Delta', \Theta' \Rightarrow C'}{\scriptstyle L\vee} \\ \vdots \\ A \vee B, \Delta, \Theta \Rightarrow C \end{array}}{\Gamma, \Delta, \Theta \Rightarrow C}{\scriptstyle Cut}$$

We make m cuts with $\Gamma \Rightarrow A$, starting with the premiss $A^m, \Delta' \Rightarrow C'$, obtaining $\Gamma^m, \Delta' \Rightarrow C'$. The derivation is continued as before from where $A \vee B$ was principal; Where a formula from Θ' was used in the original derivation, there will be a vacuous use. The derivation ends with $\Gamma^m, \Lambda \Rightarrow C$ where Λ is a multiset reduct of the context Δ, Θ of the right premiss of the original cut. All cuts are on shorter formulas than the initial cut. If in the left premiss the rule was $R\vee_2$ and $n > 0$, the procedure is similar.

If $m = 0$, assuming still that the rule concluding the left premiss is $R\vee_1$, the cut is hereditarily vacuous and proceeding analogously to case *1* we continue from the premiss $\Delta' \Rightarrow C'$ without cuts to a sequent $\Lambda \Rightarrow C$ where Λ is by proposition 2 a reduct of Γ, Δ, Θ. The other cases of hereditarily vacuous cuts are handled similarly.

3. Cut formula is $A \supset B$. With $n > 0$ the derivation is

$$\dfrac{\dfrac{A^m, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}{\scriptstyle R\supset} \qquad \begin{array}{c} \dfrac{\Delta' \Rightarrow A \quad B^n, \Theta' \Rightarrow C'}{A \supset B, \Delta', \Theta' \Rightarrow C'}{\scriptstyle L\supset} \\ \vdots \\ A \supset B, \Delta, \Theta \Rightarrow C \end{array}}{\Gamma, \Delta, \Theta \Rightarrow C}{\scriptstyle Cut}$$

We first cut m times with $\Delta' \Rightarrow A$, starting with $A^m, \Gamma \Rightarrow B$, and obtain $\Gamma, {\Delta'}^m \Rightarrow B$, then cut with this n times, starting with $B^n, \Theta' \Rightarrow C'$, to obtain $\Gamma^n, {\Delta'}^{mn} \Rightarrow C'$. All cuts are on shorter formulas.

The case of $n = 0$ gives a hereditarily vacuous cut that is handled as in the previous cases. QED.

COROLLARY 9. SUBFORMULA PROPERTY. *If the derivation of* $\Gamma \Rightarrow C$ *has no hereditarily principal cuts, all formulas in the derivation are subformulas of* Γ, C.

PROOF. Consider the uppermost hereditarily nonprincipal cut

$$\frac{\Gamma' \Rightarrow A \quad A, \Delta' \Rightarrow C'}{\Gamma', \Delta' \Rightarrow C'}\, Cut$$

Since A is never active in the derivation of the right premiss, its first occurrence is in an axiom $A \Rightarrow A$. By the same, $A \Rightarrow A$ can be replaced by the derivation of the left premiss of cut, $\Gamma' \Rightarrow A$ and the derivation continued as before, until the sequent $\Gamma', \Delta' \Rightarrow C'$ is reached by the rule originally concluding the right premiss of cut. Therefore the succedent A is a subformula of $\Gamma', \Delta' \Rightarrow C'$. Repeating this for each nonhereditary cut formula in succession, we conclude that they all are subformulas of the endsequent. QED.

Theorems 4 and 8 and the proof of corollary 9 actually give an elimination procedure for all cuts:

COROLLARY 10. *Given a derivation of* $\Gamma \Rightarrow C$ *in* GN+Cut, *there is a derivation of* $\Gamma^* \Rightarrow C$ *in* GN, *with* Γ^* *a multiset reduct of* Γ.

There are sequents derivable in calculi with explicit weakening and contraction rules that have no derivation in the calculus *GN*, for example, $A \Rightarrow A\&A$. The last rule must be $R\&$, but its application in *GN* will only give $A, A \Rightarrow A\&A$. Even if the sequent $A \Rightarrow A\&A$ is not derivable, the sequent $\Rightarrow A \supset A\&A$ is, by a multiple use of A in the $R\supset$ rule.

The completeness of the calculus *GN* is easily proved, for example, by deriving any standard set of axioms of intuitionistic logic as sequents with empty antecedents, and by noting that modus ponens in the form

$$\frac{\Rightarrow A \supset B \quad \Rightarrow A}{\Rightarrow B}$$

is admissible: Application of $L\supset$ to $A \Rightarrow A$ and $B \Rightarrow B$ gives $A \supset B, A \Rightarrow B$, and cuts with the premisses of modus ponens give $\Rightarrow B$. We also have completeness in another sense: Sequent calculi with weakening and contraction modify the derivability relation in an inessential way, for if $\Gamma \Rightarrow C$ is derivable in such calculi, obviously there is a derivation of $\Gamma^* \Rightarrow C$ in

GN, with Γ^* a multiset reduct of Γ. In particular, if $\Rightarrow C$ is derivable in such calculi, it is derivable in *GN*.

To extend the calculus to predicate logic, quantifier rules must have multiplicities in antecedents of premisses, similarly to the propositional case:

Quantifier rules for GN

$$\frac{A(t/x)^m, \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}{\scriptstyle L\forall} \qquad \frac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \forall x A}{\scriptstyle R\forall}$$

$$\frac{A(y/x)^m, \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}{\scriptstyle L\exists} \qquad \frac{\Gamma \Rightarrow A(t/x)}{\Gamma \Rightarrow \exists x A}{\scriptstyle R\exists}$$

The variable restrictions in $R\forall$ and $L\exists$ are that y is not free in the conclusion. The results for propositional logic extend in a straightforward manner to predicate logic and will not be detailed out here.

§3. A multisuccedent calculus. We give a classical multisuccedent version of the calculus *GN*, called *GM*. It is obtained by writing the right rules in perfect symmetry to the left rules.

GM

Logical axiom:

$A \Rightarrow A$

Logical rules:

$$\frac{A^m, B^n, \Gamma \Rightarrow \Delta}{A\&B, \Gamma \Rightarrow \Delta}{\scriptstyle L\&} \qquad \frac{\Gamma \Rightarrow \Delta, A^m \quad \Gamma' \Rightarrow \Delta', B^n}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', A\&B}{\scriptstyle R\&}$$

$$\frac{A^m, \Gamma \Rightarrow \Delta \quad B^n, \Gamma' \Rightarrow \Delta}{A \vee B, \Gamma, \Gamma' \Rightarrow \Delta}{\scriptstyle L\vee} \qquad \frac{\Gamma \Rightarrow \Delta, A^m, B^n}{\Gamma \Rightarrow \Delta, A \vee B}{\scriptstyle R\vee}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B^n, \Gamma' \Rightarrow \Delta}{A \supset B, \Gamma, \Gamma' \Rightarrow \Delta}{\scriptstyle L\supset} \qquad \frac{A^m, \Gamma \Rightarrow \Delta, B^n}{\Gamma \Rightarrow \Delta, A \supset B}{\scriptstyle R\supset}$$

$$\frac{}{\bot \Rightarrow \Delta}{\scriptstyle L\bot}$$

$$\frac{A(t/x)^m, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}{\scriptstyle L\forall} \qquad \frac{\Gamma \Rightarrow \Delta, A(y/x)^m}{\Gamma \Rightarrow \Delta, \forall x A}{\scriptstyle R\forall}$$

$$\frac{A(y/x)^m, \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}{\scriptstyle L\exists} \qquad \frac{\Gamma \Rightarrow \Delta, A(t/x)^m}{\Gamma \Rightarrow \Delta, \exists x A}{\scriptstyle R\exists}$$

In von Plato (2001a), a classical calculus called *G0c* was given that is the same as *GM* but without the exponents in the rules and with explicit left and right weakening and contraction.

Similarly to proposition 2, one proves that if in the derivation of $\Gamma \Rightarrow \Delta$ in *GM*+*Cut* a subderivation of $\Theta \Rightarrow \Lambda$ is substituted by a derivation of $\Theta^* \Rightarrow \Lambda^*$ with both contexts reducts of the original ones, then there is a derivation of $\Gamma^* \Rightarrow \Delta^*$ with contexts similarly reduced. A proof of elimination of hereditarily principal cuts and of the subformula property is obtained similarly to the results for the single succedent intuitionistic calculus.

The above calculus is complete for classical logic, by the derivation:

$$\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A, A \supset \bot}\,R\supset}{\Rightarrow A \vee \sim A}\,R\vee$$

The instance of $R\supset$ has $m = 1$, $n = 0$, with Γ empty, $\Delta = A$ and $B = \bot$. More generally, we obtain the full versions of Gentzen's original left and right negation rules from $L\supset$ and $R\supset$ by suitable choices:

$$\dfrac{\Gamma \Rightarrow \Delta, A \quad \dfrac{}{\bot \Rightarrow}\,L\bot}{A \supset \bot, \Gamma \Rightarrow \Delta}\,L\sim \qquad\qquad \dfrac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A \supset \bot}\,R\sim$$

§4. Natural deduction. To compare sequent calculus and natural deduction, the conjunction and implication elimination rules will be formulated as *general elimination rules*, analogously to the disjunction elimination rule. The three rules are

$$\dfrac{\begin{matrix} \\ \vdots \\ A\&B\end{matrix} \quad \begin{matrix}\overset{1.}{[A^m]}, \overset{2.}{[B^n]} \\ \vdots \\ C\end{matrix}}{C}\,\&E,1.,2. \qquad \dfrac{\begin{matrix} \\ \vdots \\ A\vee B\end{matrix} \quad \begin{matrix}\overset{1.}{[A^m]} \\ \vdots \\ C\end{matrix} \quad \begin{matrix}\overset{2.}{[B^n]} \\ \vdots \\ C\end{matrix}}{C}\,\vee E,1.,2. \qquad \dfrac{\begin{matrix} \\ \vdots \\ A\supset B\end{matrix} \quad \begin{matrix} \\ \vdots \\ A\end{matrix} \quad \begin{matrix}\overset{1.}{[B^n]} \\ \vdots \\ C\end{matrix}}{C}\,\supset E,1.$$

The rules have instances for any $m, n \geqslant 0$. The notation $\overset{1.}{[A^m]}$ stands for m occurrences of A, all discharged, which is indicated by the square brackets and number identifying the rule at which the discharge happens. It may very well happen that there are other occurrences of a formula A among the assumptions than those discharged in an inference. The discharge of assumptions is regulated by the principle that no two instances of rules in a derivation can have a common discharge label (number next to the rule symbol).

The first of the above general elimination rules is suggested in Schroeder-Heister (1984), the third in von Plato (2001b). The usual rules come out as special cases, with $C = A$ and $C = B$, respectively, for conjunction, and $C = B$ for implication. As shown in von Plato (2001b), it is precisely these special rules that are responsible for the lack of isomorphism of derivations in sequent calculus and natural deduction. Translations are given there, between cut-free sequent calculus derivations and normal natural deduction derivations with general elimination rules, such that translation back and forth always is the identity. The translations were also extended to non-normal derivations and derivations with principal cuts.

In order to make transparent the connection between the present sequent calculus *GN* and natural deduction, we write the latter in sequent calculus style: Each formula A in a natural deduction derivation is replaced by the expression $\Gamma \rightarrow A$ where Γ is the multiset of open assumptions on which A depends. This variant of natural deduction was first used by Gentzen (1936). It may resemble sequent calculus proper in appearance but is fundamentally different as there are no left rules with principal formulas in the antecedent of the conclusion. We restrict the treatment to the propositional part; The quantifier rules are handled analogously, with a general elimination rule for the universal quantifier. A single arrow will be used for the derivability relation of natural deduction. The natural deduction calculus in sequent calculus style with general elimination rules is denoted *NG*:

NG

Rule of assumption:

$A \rightarrow A$

Logical rules:

$$\frac{\Gamma \rightarrow A \quad \Delta \rightarrow B}{\Gamma, \Delta \rightarrow A\&B}\,\&I \qquad \frac{\Gamma \rightarrow A\&B \quad A^m, B^n, \Gamma \rightarrow C}{\Gamma, \Delta \rightarrow C}\,\&E$$

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee B}\,\vee I_1 \quad \frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \vee B}\,\vee I_2 \qquad \frac{\Gamma \rightarrow A \vee B \quad A^m, \Delta \rightarrow C \quad B^n, \Theta \rightarrow C}{\Gamma, \Delta, \Theta \rightarrow C}\,\vee E$$

$$\frac{A^m, \Gamma \rightarrow B}{\Gamma \rightarrow A \supset B}\,\supset I \qquad \frac{\Gamma \rightarrow A \supset B \quad \Delta \rightarrow A \quad B^n, \Theta \rightarrow C}{\Gamma, \Delta, \Theta \rightarrow C}\,\supset E$$

$$\frac{\Gamma \rightarrow \bot}{\Gamma \rightarrow C}\,\bot E$$

The multiplicities for m, n are as for the sequent calculus rules, with $m = 0$ or $n = 0$ corresponding to *vacuous* and $m > 1$ or $n > 1$ to *multiple discharge.* The notion of a multiset reduct is the same as in sequent calculus.

Derivations in *GN* differ from those in *NG* in that non-normal derivations are already contained in the *NG* rules. Some cuts correspond to non-normal steps, namely the principal ones, but for a full correspondence of derivations in *GN* and *NG*, a rule of substitution for *NG* is needed:

$$\frac{\Gamma \to A \quad A, \Delta \to C}{\Gamma, \Delta \to C}{\scriptstyle Subst}$$

THEOREM 11. ADMISSIBILITY OF SUBSTITUTION. *If* $\Gamma \to C$ *is derivable in* NG+Subst, *then* $\Gamma^* \to C$ *is derivable in* NG, *where* Γ^* *is a multiset reduct of* Γ.

PROOF: It is sufficient to consider an uppermost substitution, and we may assume it to be the last step in the derivation of $\Gamma \to C$. Admissibility of substitution is proved by induction on the height of derivation of the right premiss $A, \Delta \to C$ of substitution. If it is an assumption, Δ is empty and $A = C$, so the conclusion of substitution is equal to the left premiss. Otherwise the right premiss is concluded by a logical rule. Inspection of the logical rules shows that the substitution formula always appears in at least one context of their premisses, and substitution is permuted up until it reaches an assumption. QED.

The rule of substitution resembles cut but is different in nature. The rules of natural deduction in sequent calculus style list in the antecendent the open assumptions, but these are never principal in a rule. Therefore substitution is like a cut where the cut formula is never principal in the derivation of the right premiss of cut. Indeed, the above proof of admissibility of substitution is just another formulation of the proof of elimination of hereditarily nonprincipal cuts in corollary 9.

In terms of the standard representation of natural deduction, instead of the sequent calculus style representation, admissibility of substitution just states that substitution through the putting together of derivations produces a correct derivation. The rule of cut, on the other hand, is not a rule of the systems of sequent calculi we have considered. From its admissibility in these systems follows *closure under cut*, but only of the class of derivable sequents, not of derivations.

DEFINITION 12: *A derivation in* NG *is* normal *if the major premiss of every elimination rule is an assumption.*

Inspecting the rules, we observe that the subformula property for normal derivations follows.

We shall show that cut-free derivations in *GN* have the same structure as normal derivations in *NG*.

§5. Isomorphic translation between sequent calculus and natural deduction. An isomorphism between cut-free derivations and normal derivations will be established through translations in both directions, where by isomorphism is meant that the *order of logical rules* is the same. We first translate from arbitrary, possibly non-normal, natural deduction derivations to sequent calculus, then from cut-free derivations, and finally also cuts will be translated.

The translation from NG begins top-down with assumptions in which it is enough to change the arrow. In introduction rules arrows and rule symbols are changed. When an elimination rule is encountered, say $\&E$, the step of inference and its translation are

$$\frac{\Gamma \to A\&B \quad A^m, B^n, \Delta \to C}{\Gamma, \Delta \to C}{\scriptstyle \&E} \quad \rightsquigarrow \quad \frac{\Gamma \Rightarrow A\&B \quad \dfrac{A^m, B^n, \Delta \Rightarrow C}{A\&B, \Delta \Rightarrow C}{\scriptstyle L\&}}{\Gamma, \Delta \Rightarrow C}{\scriptstyle Cut}$$

The translation of the other cases is perfectly analogous. It is seen that the order of logical rules is the same in the original and the translation.

If the major premiss of an E-rule is an assumption, the context of the major premiss is the same as the major premiss formula; In the translation to sequent calculus it now appears only as the principal formula in the antecedent of the conclusion of the left sequent rule. (In particular, falsity elimination leads to the $L\bot$ rule.)

Given a derivation in GN with principal cuts only, the above translation in reverse gives a derivation in NG with the order of logical rules again preserved. Translation back and forth, starting from either, gives the identity. We therefore conclude the

THEOREM 13: *Derivations in* GN + principal cuts *are isomorphic to derivations in* NG.

For normal derivations, the translation of an E-rule produces a left rule and a cut where the left premiss is an axiom. The right premiss of cut is equal to its conclusion so that the cut can be deleted. We have

THEOREM 14. *Cut-free derivations in* GN *are isomorphic to normal derivations in* NG.

Cut-free derivations in sequent calculus in natural deduction style turn out to be the same as normal natural deduction derivations in sequent calculus style, except for some notational variation and the writing of an assumption of the form $A \to A$ as an extra premiss in NG. It follows that the calculus GN shares the good and bad aspects of natural deduction, for example, it is not suited for proof search, whereas inductive proofs on height of derivation tend to be easier than in calculi supporting proof search.

We still have to consider the translation of derivations with arbitrary cuts. This is done by cases, depending on whether the cut formula is principal in the right premiss or not. The former obtains in four cases, the left rules. The first one is $L\&$, and the translation was already given above, and similarly for the other left rules. If the cut formula is not principal in the right premiss, the cut is translated into a substitution, as nonprincipal cuts do not have any representation in NG without the substitution rule. This finishes the translation of derivations with cuts into non-normal natural deduction derivations. We conclude the

THEOREM 15. *Derivations in* GN+Cut *are isomorphic to derivations in* NG+Subst.

In Gentzen's original work, a translation of natural deduction derivations, normal in Gentzen's sense, into sequent calculus is described (sec. V. §4). Each formula C is first replaced by a sequent $\Gamma \Rightarrow C$ where Γ is the list of open assumptions C depends on, and then the rules are translated. The rules $\&I$ and $\vee I$ are translated in the obvious way. Translations of $\supset I$ and $\vee E$ involve possible weakenings and contractions, corresponding to vacuous and multiple discharges. Whenever in the natural deduction there are instances of $\&E$ and $\supset E$, the first phase of the translation gives steps such as

$$\frac{\Gamma \Rightarrow A\&B}{\Gamma \Rightarrow A} \qquad \frac{\Gamma \Rightarrow A \supset B \quad \Delta \Rightarrow A}{\Gamma, \Delta \Rightarrow B}$$

These are turned into sequent calculus inferences by the replacements

$$\frac{\Gamma \Rightarrow A\&B \quad \dfrac{A \Rightarrow A}{A\&B \Rightarrow A}{\scriptstyle L\&}}{\Gamma \Rightarrow A}{\scriptstyle Cut} \qquad \frac{\Gamma \Rightarrow A \supset B \quad \dfrac{\Delta \Rightarrow A \quad B \Rightarrow B}{A \supset B, \Delta \Rightarrow B}{\scriptstyle L\supset}}{\Gamma, \Delta \Rightarrow B}{\scriptstyle Cut}$$

A normal natural deduction in Gentzen's sense translates into a sequent calculus derivation with cuts. These 'hidden cuts' are brought about by the special elimination rules for conjunction and implication but Gentzen makes no comment about the cuts that the translation to sequent calculus produces.

Natural deduction in sequent calculus style can be translated into the more usual notation of natural deduction with nonlocal discharge of assumptions in a way analogous to the translation of sequent calculus to natural deduction in von Plato (2001b). The translation begins from the root of a derivation and proceeds step by step until assumptions or $\bot E$ are reached. Conjunction and disjunction introductions are translated by writing the principal formula under the inference line. Implication introduction is translated by

$$\frac{\overset{\vdots}{A^m, \Gamma \to B}}{\Gamma \to A \supset B}{\scriptstyle \supset I} \quad \rightsquigarrow \quad \frac{\overset{1.}{[A^m]}, \overset{\vdots}{\Gamma \to B}}{A \supset B}{\scriptstyle \supset I,1.}$$

The translation now continues from the premiss. Conjunction elimination is translated by

$$\dfrac{\overset{\vdots}{\Gamma \to A\&B} \quad A^m, B^n, \Delta \overset{\vdots}{\to} C}{\Gamma, \Delta \to C}\,\&E \quad \rightsquigarrow \quad \dfrac{\overset{\vdots}{\Gamma \to A\&B} \quad [\overset{1.}{A^m}], [\overset{2.}{B^n}], \Delta \overset{\vdots}{\to} C}{C}\,\&E,1.,2.$$

The other elimination rules are translated similarly. Finally, we have the translation of assumptions and of the $\bot E$ rule:

$$A \to A \quad \rightsquigarrow \quad A \qquad\qquad \dfrac{}{\bot \to C}\,\bot E \quad \rightsquigarrow \quad \dfrac{\bot}{C}\,\bot E$$

If assumptions without numerical labels are reached, A and $\bot$ turn into *open* assumptions of the natural deduction derivation. Whenever a formula is discharged, the translation produces formulas with labels. When assumptions are reached, we can have $[\overset{i.}{A}] \to A$ and $[\overset{i.}{\bot}] \to C$ with a label in the antecedent. These are translated into $[\overset{i.}{A}]$ and $\frac{[\overset{i.}{\bot}]}{C}\bot E$.

It is also possible to define a translation in the other direction, from nonlocal natural deduction to natural deduction in sequent calculus style. The translation is analogous to the translation from natural deduction to sequent calculus given in von Plato (2001b).

It is straightforward to write a multiple conclusion calculus corresponding to the multisuccedent calculus *GM*, in analogy to the way *NG* corresponds to *GN*.

§6. Concluding remarks on normalization. A step of normalization corresponds to a cut with cut formula principal in the right premiss or both premisses. Part of the effectiveness of normalization comes from this being the only situation of convertibility. Instead of transforming, step by step, a derivation with a nonprincipal cut into one with a principal cut, we have achieved the same efficient transformation as in natural deduction. Our method of cut elimination has some analogy to that of Gentzen's, where one inductive parameter is the sum of the "left and right rank numbers," i.e., the numbers of steps of inference from first occurrences of a cut formula to the cut. In Gentzen's procedure, the cut was permuted up, but due to independent contexts in our calculus, and the deletion of redundant cuts where the cut formula has a second occurrence in the premiss, a single-step permutation of cut can be done. Our cut elimination procedure for a hereditarily principal cut gives a unique result, whereas the step-by-step permutation up of nonprincipal cuts can usually be done in different ways, leading to cut-free derivations with varying orders of logical rules.

Our proof of elimination of hereditarily principal cuts gives at once a proof of normalization for natural deduction, where permutations of cuts correspond to either detour conversions, into derivations with shorter conversion formulas, or permutation conversions, with height of derivation of major premiss of an elimination rule diminished by one. In the end, major premisses have become assumptions and the derivation is normal.

Since we use the general elimination rules for conjunction and implication, permutation conversions can be extended from such conversions for disjunctions as found by Prawitz (1965), to all elimination steps.

In Prawitz (1971), a simplification of derivations in natural deduction is suggested, called properly *simplification conversion*. The convertibility arises from disjunction elimination when in at least one of the auxiliary derivations, say the first one, a disjunct was not assumed:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\vee B\end{matrix}\qquad\begin{matrix}\Delta\\ \vdots\\ C\end{matrix}\qquad\begin{matrix}\overset{2.}{[B^n]},\Theta\\ \vdots\\ C\end{matrix}}{C}\,{\scriptstyle \vee E,2.}$$

The elimination step is not needed, for C is already concluded in the first auxiliary derivation. With general elimination rules for conjunction and implication, we analogously have:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\&B\end{matrix}\quad\begin{matrix}\Delta\\ \vdots\\ C\end{matrix}}{C}\,{\scriptstyle \&E}\qquad\qquad\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\supset B\end{matrix}\quad\begin{matrix}\Delta\\ \vdots\\ A\end{matrix}\quad\begin{matrix}\Theta\\ \vdots\\ C\end{matrix}}{C}\,{\scriptstyle \supset E}$$

In both inferences, C is already concluded without the elimination rule, and simplification conversion extends to all elimination rules. In terms of *GN*, there is in each of these inferences a (hereditarily) vacuous cut with cut formula concluded by a left rule in the right premiss. For example, for $\vee E$ we have

$$\dfrac{\Gamma\Rightarrow A\vee B\qquad\dfrac{\Delta\Rightarrow C\quad B^n,\Theta\Rightarrow C}{A\vee B,\Delta,\Theta\Rightarrow C}\,{\scriptstyle L\vee}}{\Gamma,\Delta,\Theta\Rightarrow C}\,{\scriptstyle Cut}$$

which converts to $\Delta\Rightarrow C$, and Δ is a reduct of the antecedent of conclusion of the original cut. The other two elimination rules lead to similar conversions. In the notion of vacuous cut, we find the systematic origin of simplification conversions, extending to all elimination rules.

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