Decidability of positivity under Stone compactification

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Summary

The original definition of formal topology contains a positivity predicate that is a constructive expression of non-emptiness. We show that keeping the same positivity predicate for the Stone compactification of a formal topology implies its decidability.

We recall here the original definition of formal topology, as given in [S]. For references, further details and motivations the reader is referred to [S] (where sometimes a different notation is used), [NV], [SVV].

Definition A *formal topology* is a structure $\mathcal{A} = \langle S, \cdot, 1, \lhd, Pos \rangle$ where

- $(S, \cdot, 1)$ is a *(formal) base*, i.e. a commutative monoid with unit;
- $\lhd$ is a relation between elements and subsets of S, called *(formal) cover*, satisfying:

  $$\frac{a \in U}{a \lhd U} \ \textit{reflexivity,}$$

  $$\frac{a \lhd U \quad U \lhd V}{a \lhd V} \ \textit{transitivity,}$$

  $$\frac{a \lhd U \quad a \lhd V}{a \lhd U \cdot V} \ \textit{dot-right,}$$

  $$\frac{a \lhd U}{a \cdot b \lhd U} \ \textit{dot-left,}$$

  where $U \lhd V \equiv (\forall u \in U)\ u \lhd V$ and $U \cdot V \equiv \{u \cdot v \mid u \in U, v \in V\}$;

- Pos is a predicate, called *positivity predicate* satisfying:

$$\frac{Pos(a) \qquad a \lhd U}{Pos(U)} \ \textit{monotonicity,}$$

$$\frac{Pos(a) \rightarrow a \lhd U}{a \lhd U} \ \textit{positivity,}$$

where $Pos(U) \equiv (\exists b \in U)Pos(b)$.

There are several equivalent formulations of *positivity* (cf. [S1]). The one we will use here is the following:

$$a \lhd a^+$$

where a^+ denotes the subset of $\{a\}$ given by $\{b \in S \mid b \in \{a\} \ \& \ Pos(b)\}$.

The notion of Stone compactification of a formal topology has been introduced in [S] and further explored in [N] in its application to a constructive version of Stone representation for distributive lattices. We recall here the basic definitions:

Definition A formal cover $\lhd$ on a base S is said to be a *Stone cover* if for all $a \in S$ and U subset of S, $a \lhd U$ implies that there exists a finite subset U_0 of U such that $a \lhd U_0$.

Given a cover on a base S one can define a Stone cover $\lhd_\omega$ from this by

$$a \lhd_\omega U \equiv (\exists U_0 \in \mathcal{P}_\omega(S))(U_0 \subseteq U \ \& \ a \lhd U_0)$$

where $\mathcal{P}_\omega(S)$ is the set of finite subsets of S. The set of finite subsets of a given set can be formally defined in type theory as in [NV], but for our purposes it will be enough to think of it as the set of finite lists (modulo exchange and contraction) over S.

The cover $\lhd_\omega$ thus obtained is called the *Stone compactification* of the cover $\lhd$.

Definition Given a formal topology $\mathcal{A} = \langle S, \cdot, 1, \lhd, Pos \rangle$, its *Stone compactification* is defined by $\mathcal{A}_\omega \equiv \langle S, \cdot, 1, \lhd_\omega, Pos \rangle$

Whereas it is immediate to verify that $\lhd_\omega$ is a cover and that monotonicity of Pos w.r.t. $\lhd_\omega$ is satisfied, it is not at all clear that *positivity* is satisfied. In fact, we have:

Theorem Let $\mathcal{A} = \langle S, \cdot, 1, \lhd, Pos \rangle$ be a formal topology. If Pos is a positivity predicate for the Stone compactification of $\mathcal{A}$ then it is decidable.

Proof: Suppose that Pos satisfies *positivity* with respect to the cover $\lhd_\omega$. Then we have, by the equivalence recalled above,

$$a \lhd_\omega a^+ \ .$$

Since $\lhd_\omega$ is a Stone cover, there exists a finite subset U_0 of the base which is a subset of a^+ and such that

$$a \lhd_\omega U_0 \ .$$

Consider such a finite subset U_0. Whatever definition for the set $\mathcal{P}_\omega(S)$ we choose, we know, from the fact that U_0 belongs to this set, whether it is empty or it contains at least one element. In the first case we have

$$a \lhd_\omega \emptyset$$

and therefore, by monotonicity of Pos, also $\neg Pos(a)$, for, if we assume $Pos(a)$, we obtain that there exists $b \in \emptyset$ such that $Pos(b)$, a contradiction. In the second case, since U_0 is a subset of a^+, there exists at least one element in a^+, that is, $Pos(a)$. Therefore for all $a \in S$, $Pos(a) \vee \neg Pos(a)$ holds, that is, Pos is decidable. $\square$

Conclusion

We have shown that positivity for the Stone compactification of a cover is not satisfied, unless we assume the positivity predicate to be decidable. In particular, for a given formal topology its Stone compactification is not in general a formal topology, contrary to [S], proposition 4.8.

Subsequent literature can be adjusted on this point. The notion of Stone base defined in [N] must be restricted to the cover part and the positivity predicate has to be left out. The representation theorem for distributive lattices continues to hold, and no corresponding notion of apartness from zero has to be introduced on the algebraic side to match the equivalence between Stone bases and distributive lattices.

However, in cases studied so far in applications to constructive analysis, where the Stone compactification of a cover plays an important role, namely in the formal topologies of formal reals, formal intervals, linear functionals (see [CCN, CN, N1, NS]), the positivity predicate is either not needed or it is decidable.

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