A normalizing system of natural deduction for intuitionistic linear logic

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Abstract

The main result of this paper is a normalizing system of natural deduction for the full language of intuitionistic linear logic. No explicit weakening or contraction rules for !-formulas are needed. By the systematic use of general elimination rules a correspondence between normal derivations and cut-free derivations in sequent calculus is obtained. Normalization and the subformula property for normal derivations follow through translation to sequent calculus and cut-elimination.

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1 Introduction

It is well known that a natural deduction formulation for the full language of intuitionistic linear logic presents difficulties.

In the first place, the coexistence in linear logic of multiplicative and additive connectives, with corresponding context-independent and context-sharing rules, impairs the modularity of the logical calculus. Consequently partial solutions, for the multiplicative fragment only, have been given.

Another difficulty in the formulation of a natural deduction system for linear logic was pointed out in Wadler (1992). The direct translation of the context-dependent rule $R!$ of sequent calculus produces a context-dependent rule of !-introduction, with the undesirable feature that the resulting system is not closed under substitution.

In the sequent calculus formulation of linear logic the structural rules of weakening and contraction are permitted only for special formulas and correspondingly, restricted rules of weakening and contraction are given. For intuitionistic logic the sequent calculus rules of weakening and contraction are implicit in a natural deduction formulation, in the form of vacuous and multiple discharge of assumptions. The same implicit treatment of weakening and contraction cannot be easily extended to linear logic, and therefore explicit weakening and contraction rules are usually given, even in what are taken to be natural deduction formulations.

There is no "official" definition of what a natural deduction system should be, but some guidelines in designing such a system can be inferred both from the classic literature (Gentzen 1934-35, Prawitz 1965) and from the desiderata expressed in the various attempts regarding linear logic. The main features can be summarized in the following seven requirements:

1. The rules are motivated through a constructive semantics, giving the meaning of the connectives in terms of derivability. Thus each logical constant has an introduction rule, given by the meaning explanation, and an elimination rule, justified by the introduction rule through an inversion principle.

In comparison to sequent calculus there are no explicit structural rules:

2. The substitution rule, corresponding to cut, is admissible and thus not needed, because the derivability relation of natural deduction has to be compositional;

3. Also weakening and contraction are not needed, but the same deductive strength is achieved by permitting vacuous and multiple discharge of assumptions.

4. The syntax should be as simple and natural as possible.

5. The system should be normalizing.

6. Normal derivations should satisfy the subformula property.

7. Normal derivations in natural deduction should correspond to cut-free derivations in sequent calculus.

The last point has been open for a long time also for intuitionistic logic (Zucker 1974, Pottinger 1977). Recently it has been shown by von Plato that such a correspondence can be achieved for intuitionistic logic by the use of general elimination rules.

All these questions have been addressed and answered in different systems proposed in the literature, sometimes at the cost of a complex syntax. However, there has been no "best" solution with respect to all of the above requirements and it is generally believed that these are opposing goals that cannot all be achieved in the same system.

The main result of this work is a system **N-ILL** of natural deduction for the full language of intuitionistic linear logic. Each logical constant has an introduction and an elimination rule, expressed with the standard notation of natural deduction, closed under substitution and without explicit structural rules, normalizing and with the subformula property for normal derivations, with normal derivations corresponding to cut-free derivations in sequent calculus. Thus our system satisfies all the above seven requirements. These essential properties are contained in the results 3.2, 3.3, and 4.2–4.5.

The introduction rules for the system are the standard ones (except for !) and the elimination rules are found by the use of an inversion principle. A suitable formulation of the rule of !-introduction gives a system with implicit weakening and contraction, closed under substitution. Closure under substitution is obtained following in part the natural deduction formulation of the necessitation rule of modal logic, as made explicit for linear logic by several authors. The structural rules of weakening and contraction for !-formulas follow as derived rules thanks to the possibility of vacuous and multiple discharge of !-formulas in the rule of introduction for the !-modality.

The heterogeneous character of elimination rules in standard natural deduction, with $\vee E$ and $\exists E$ differing from the other elimination rules, leads to a difficult definition of normal derivation. By using general elimination rules the definition is uniform: *A derivation is normal when all major premisses of elimination rules are assumptions.*

We define local translations between the sequent calculus **ILL** and the natural deduction system **N-ILL** for intuitionistic linear logic and show that cut-free derivations are mapped into normal derivations, thus obtaining an indirect normalization proof. The translation has the important consequence that normal derivations satisfy the subformula property.

In Section 2 we recall the standard sequent calculus formulation **ILL** of intuitionistic linear logic. In Section 3 we review the use of general elimination rules in intuitionistic logic and discuss its relevance for linear logic. The rules for the constants 1, $\top$ and 0 are shown to be uniquely determined starting from the rules for $\otimes$, & and $\oplus$. The rules for the modality ! are discussed and related to previous choices in the literature. The natural deduction system **N-ILL** is presented both in "sequent calculus style" and in a natural deduction formulation and is shown to be closed under substitution. The usual rules of weakening and contraction for !-formulas are shown derivable; thus completeness of our system follows. We recall the counterexample by Troelstra of a normal derivation not satisfying the subformula property and show that this problem is solved by our approach. In Section 4 we define a translation between sequent calculus and natural deduction, showing that cut-free derivations in the sequent calculus **ILL** correspond to normal derivations in **N-ILL**. We apply the translation for obtaining an indirect proof of normalization and the subformula property for normal derivation.

In the final section we relate our system to other systems in the literature.

2 Preliminaries

We recall a sequent calculus system for Intuitionistic Linear Logic. We use Γ, Γ', Δ, Δ' ... for finite multisets of formulas and A, B, C, ... for arbitrary formulas. In a sequent the antecedent is a multiset, so that no exchange rule is needed. As for the linear connectives we conform to Girard's notation, but use two-sided sequents. For rules $R\forall$ and $L\exists$ the usual side conditions on the variable are assumed.

Sequent Calculus for Intuitionistic Linear Logic **ILL**

Axioms:

$$A \Rightarrow A \qquad \Rightarrow 1$$

$$0, \Gamma \Rightarrow C \qquad \Gamma \Rightarrow \top$$

Logical rules:

$$\frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}\,R\otimes \qquad \frac{A, B, \Gamma \Rightarrow C}{A \otimes B, \Gamma \Rightarrow C}\,L\otimes$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B}\,R\& \qquad \frac{A, \Gamma \Rightarrow C}{A \& B, \Gamma \Rightarrow C}\,L\&_1 \qquad \frac{B, \Gamma \Rightarrow C}{A \& B, \Gamma \Rightarrow C}\,L\&_2$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \oplus B}\,R\oplus_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \oplus B}\,R\oplus_2 \qquad \frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \oplus B, \Gamma \Rightarrow C}\,L\oplus$$

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \multimap B}\,R\multimap \qquad \frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow C}{A \multimap B, \Gamma, \Delta \Rightarrow C}\,L\multimap$$

$$\frac{!\Gamma \Rightarrow C}{!\Gamma \Rightarrow !C}\,R! \qquad \frac{A, \Gamma \Rightarrow C}{!A, \Gamma \Rightarrow C}\,L!$$

$$\frac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \forall x A}\,R\forall \qquad \frac{A(t/x), \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\,L\forall$$

$$\frac{\Gamma \Rightarrow A(t/x)}{\Gamma \Rightarrow \exists x A}\,R\exists \qquad \frac{A(y/x), \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\,L\exists$$

Structural rules:

$$\frac{\Gamma \Rightarrow C}{1, \Gamma \Rightarrow C}\,1w \qquad \frac{\Gamma \Rightarrow C}{!A, \Gamma \Rightarrow C}\,!w \qquad \frac{!A, !A, \Gamma \Rightarrow C}{!A, \Gamma \Rightarrow C}\,!c$$

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\,cut$$

The cut rule can be dispensed with, since we have:

Theorem 2.1 *The cut rule is admissible in the system* **ILL**.

The proof essentially uses the method of Gentzen's original proof of admissibility of cut for intuitionistic and classical logic, by showing how the cuts can be transformed to cuts on smaller formulas or cuts of less height. The use of multicut is needed when the right premiss of cut is derived by $!c$. Details are given in Section 3.14 of Troelstra (1992).

3 Natural deduction for intuitionistic linear logic

We present the system **N-ILL** of natural deduction for intuitionistic linear logic. For convenience of notation, we first use a sequent-style presentation for the rules of natural deduction, and later give it in the usual natural deduction style. We write $\Gamma \vdash C$ for the formal derivability relation in natural deduction, meaning that there is a derivation of C from the multiset of open assumptions Γ. Each axiom or logical rule of **ILL** corresponds to an axiom or logical rule of **N-ILL**, except the structural rules, which need not be taken as primitive in **N-ILL**. Observe that in our system each connective as well as the modality ! has an introduction and an elimination rule. No explicit weakening or contraction for !-formulas need be assumed: $!w$ and $!c$ are absorbed into $!I$ and are shown to be derivable.

3.1 General elimination rules

The main novelty of the system is the use of general elimination rules for all the connectives and for the universal quantifier. We shall give here some background on the use of general elimination rules in natural deduction. General elimination rules for intuitionistic logic were introduced in von Plato (2001, see also von Plato 2000). A thorough exposition of natural deduction with general elimination rules can be found in Negri and von Plato (2001).

The standard way of justifying introduction rules in proof-theoretical terms is through the BHK-interpretation of logical constants, which gives the *sufficient grounds* for deriving a formula. General elimination rules can then be obtained from the introduction rules through an **inversion principle**: *Whatever follows from the sufficient grounds for deriving a formula must follow from that formula.*

In contrast to Prawitz' inversion principle that only justifies the elimination rules, this stronger inversion principle uniquely determines the elimination rules, once the introduction rules are given. For instance, given the introduction rule for conjunction

$$\frac{\begin{matrix}\vdots & \vdots \\ A & B\end{matrix}}{A \wedge B}\ {\wedge I}$$

the elimination rule is determined through the inversion principle. The grounds for asserting $A \wedge B$ are (derivations of) A and B, thus the rule

$$\frac{\begin{array}{cc} & [A,B] \\ \vdots & \vdots \\ A \wedge B & C \end{array}}{C} \wedge E$$

is determined. By writing out the multisets of open assumptions and replacing the vertical dots with the symbol for the formal derivability relation $\vdash$, the rules read as

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \wedge I \qquad \frac{\Gamma \vdash A \wedge B \quad A, B, \Delta \vdash C}{\Gamma, \Delta \vdash C} \wedge E$$

The two standard elimination rules for conjunction

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge E_1 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge E_2$$

are obtained as special cases from the above when C is A or B, respectively, with the second premiss derivable. The general elimination rule for $\wedge$ was first given for a system of intuitionistic logic in Schroeder-Heister (1984).

We shall here extend the treatment of natural deduction based on general elimination rules to linear logic.

To some extent, the use of general elimination rules has already been undertaken in linear logic. General elimination rules have been given for $\oplus$, analogously to the elimination rule for disjunction, and for $\otimes$: These choices were wellnigh inevitable, the first due to the meaning of disjunction, the second due to the fact that one cannot project from $A \otimes B$ as from $A \& B$.

The sufficient grounds for deriving an implication $A \supset B$ are given by a derivation of B from A. In the formulation of the general elimination rule for implication we should have to express that something follows from a derivation, but there is no way to do this except with the use of higher-level rules, as in Schroeder-Heister (1984). On the other hand a satisfactory solution in first-order logic is obtained by noting that if $A \supset B$ holds, arbitrary consequences of B are already consequences of A. With this proviso, the general implication elimination rule for intuitionistic logic is formulated as

$$\frac{\Gamma \vdash A \supset B \quad \Delta \vdash A \quad B, \Pi \vdash C}{\Gamma, \Delta, \Pi \vdash C} \supset E$$

which replaces the special implication elimination rule of *modus ponens*

$$\frac{\Gamma \vdash A \supset B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \supset E$$

For linear implication we adopt the same elimination rule as for intuitionistic implication:

$$\frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A \quad B, \Pi \vdash C}{\Gamma, \Delta, \Pi \vdash C} \multimap E$$

In linear logic there are difficulties for obtaining the rules for the context-dependent connectives, or additives: The grounds for obtaining $A\&B$ are derivations of A and B from *the same* multiset of open assumptions. Again, this would lead to a higher-level condition in the elimination rules, not expressible in first-order logic. The two, instead of one, elimination rules for the additive conjunction are a way to overcome this problem:

$$\frac{\Gamma \vdash A\&B \quad A, \Delta \vdash C}{\Gamma, \Delta \vdash C} \&E_1 \qquad \frac{\Gamma \vdash A\&B \quad B, \Delta \vdash C}{\Gamma, \Delta \vdash C} \&E_2$$

The special elimination rules, i.e., the standard elimination rules of natural deduction, are special cases of the general elimination rules. For instance, the first special elimination rule for & follows from the general elimination rule when the right premiss is the axiom $A \vdash A$. Modus ponens is similarly an instance of the general implication elimination rule.

With special elimination rules there is no obvious correspondence between normal derivations in natural deduction and cut-free derivations in sequent calculus. A step of the form

$$\frac{\Gamma \vdash A\&B}{\Gamma \vdash A} \&E_1$$

is translated as

$$\frac{\Gamma \Rightarrow A\&B \quad \dfrac{A \Rightarrow A}{A\&B \Rightarrow A} L\&_1}{\Gamma \Rightarrow A} cut$$

thus requiring the insertion of a cut. In the converse translation from sequent calculus to natural deduction a step of the form

$$\frac{A, \Gamma \Rightarrow C}{A\&B, \Gamma \Rightarrow C} L\&_1$$

is translated into

$$\frac{\dfrac{A\&B \vdash A\&B}{A\&B \vdash A} \&E_1 \quad A, \Gamma \vdash C}{A\&B, \Gamma \vdash C} subst$$

where substitution can produce a non-normal derivation.

With general elimination rules no cut is introduced in the translation from normal natural deduction to sequent calculus, and no non-normalities arise in a translation of cut-free derivations.

Consider the case, in the process of cut-elimination, where the cut formula is principal in both premisses of cut, for instance

$$\dfrac{\dfrac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}\, R\otimes \qquad \dfrac{A, B, \Pi \Rightarrow C}{A \otimes B, \Pi \Rightarrow C}\, L\otimes}{\Gamma, \Delta, \Pi \Rightarrow C}\, cut$$

By taking the left premiss of cut and the premiss of $L\otimes$ (and renaming the contexts) we obtain the general elimination rule for $\otimes$

$$\dfrac{\Gamma \vdash A \otimes B \quad A, B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\, \otimes E$$

By taking the right premiss of cut and the premisses of $R\otimes$, we obtain

$$\dfrac{A \otimes B, \Gamma \vdash C \quad \Delta \vdash A \quad \Pi \vdash B}{\Gamma, \Delta, \Pi \vdash C}\, \otimes I$$

the *general introduction* rule for $\otimes$. General introduction rules are dual to general elimination rules: The principal formula appears in the antecedent of one of the premisses (referred to as major premiss), whereas in general elimination rules the principal formula appears in the succedent of the major premiss. The semantical justification for general introduction rules can be given in terms of a **dual inversion principle**: *Whatever follows from a formula must follow from the sufficient grounds for deriving the formula.*

General introduction rules for all connectives of linear logic and for the modality ? are given in Negri (2000); here we shall use, among general introduction rules, only the one for !, to be presented in Section 3.3.

3.2 Rules for 1, $\top$, 0

Once we have given the rules for $\otimes$, & and $\oplus$, the rules (axioms) for the constants 1, $\top$ and 0 follow as special cases: since the constants 1, $\top$ and 0 are the units of $\otimes$, & and $\oplus$ respectively, their rules are obtained as "nullary" cases of the rules for the connectives.

For the constant 1, first we formulate rule $\otimes I$ for a finite set of conjuncts indexed by I

$$\dfrac{\{\Gamma_i \vdash A_i : i \in I\}}{\bigcup_i \Gamma_i \vdash \otimes_i A_i}$$

where $\bigcup_i \Gamma_i$ denotes the multiset union of the Γ_i. By identifying 1 with $\otimes_i A_i$, where $I = \emptyset$, we single out a special case of the rule where I is the empty set: the premiss is empty, and the conclusion gives

$$\vdash 1$$

The elimination rule for 1 is obtained similarly from the general $\otimes$-elimination rule for a finite set of conjuncts

$$\frac{\Gamma \vdash \otimes_i A_i \quad \{A_i : i \in I\}, \Delta \vdash C}{\Gamma, \Delta \vdash C}$$

which gives, for $I = \emptyset$,

$$\frac{\Gamma \vdash 1 \quad \Delta \vdash C}{\Gamma, \Delta \vdash C}$$

For the additive unit $\top$, rule $\& I$ extended to a finite set

$$\frac{\{\Gamma \vdash A_i : i \in I\}}{\Gamma \vdash \&_i A_i}$$

and specialized to $I = \emptyset$ gives

$$\Gamma \vdash \top$$

for any multiset Γ since the premiss of the rule is vacuously satisfied. Rule $\& E$ has no vacuous instance and therefore no elimination rule for $\top$ exists.

The rule of falsity elimination is obtained as a special case from the rule

$$\frac{\Gamma \vdash \oplus_i A_i \quad \{A_i, \Delta \vdash C : i \in I\}}{\Gamma, \Delta \vdash C}$$

which gives, in the 0-ary case

$$\frac{\Gamma \vdash 0}{\Gamma, \Delta \vdash C}$$

There is no vacuous instance for $\otimes I$, thus no 0-introduction rule. Rule $0E$ is quite analogous to the rule of falsity elimination suggested by Prawitz (1965) for intuitionistic logic

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash C}$$

giving as special cases the two forms of axioms of *ex falso quodlibet* $0, \Delta \vdash C$ and $\bot \vdash C$, for linear and intuitionistic logic, respectively. The use of the rule of falsity elimination in place of the axiom of *ex falso quodlibet* allows to infer anything from falsity at any place of the derivation, not just on top of it. However, our notion of normal derivation and procedure of normalization will produce derivations where the premiss of $0E$ is of the form $0 \vdash 0$, and therefore the rule is only applied immediately below assumptions.

3.3 Rules for !

The sequent calculus rules for the modality ! include two logical rules, introduction and elimination, and two structural rules, weakening and contraction. In the natural deduction system **N-ILL** an introduction and an elimination rule suffice.

Rule $R!$ of **ILL** is context-dependent, allowing to infer $!C$ if C can be inferred from a multiset of exclamative formulas. Its natural deduction correspondent (often called *promotion* rule) could be given by

$$\frac{!\Gamma \vdash C}{!\Gamma \vdash !C}\ !I$$

or, in natural deduction style, by

$$\frac{\begin{array}{c} !\Gamma \\ \vdots \\ C \end{array}}{!C}\ !I$$

However, as was first observed by Wadler (1992), the above rule does not have the desirable property of being closed under substitution, i.e., compositionality for natural deduction derivations is lost with such a rule. Several alternatives have been proposed in the literature in order to overcome this problem (see Benton *et al.* 1992, Troelstra 1995, Mints 1995).

The introduction rule for ! can be formulated as a general introduction rule, by following the dual inversion principle: The grounds for deriving $!A$ are given by a derivation of A from a multiset of exclamative formulas; thus the rule takes the form.

$$\frac{!A, \Gamma \vdash C \quad !\Delta \vdash A}{\Gamma, !\Delta \vdash C}\ !I$$

The same rule has been used in Wadler (1992), although with a different motivation. In order to close the above rule under substitution we modify its premiss $!\Delta \vdash A$ along the lines suggested for another form of the introduction rule for ! in Troelstra (1995), and for the necessitation rule of the modal logic **S4** in Bierman and de Paiva (1996), following Prawitz (1965), and obtain

$$\frac{!A, \Gamma \vdash C \quad \Delta_1 \vdash !B_1 \quad \ldots \quad \Delta_n \vdash !B_n \quad !B_1, \ldots, !B_n \vdash A}{\Gamma, \Delta_1, \ldots, \Delta_n \vdash C}\ !I$$

or, in natural deduction style

$$\frac{\begin{array}{c} [!A], \Gamma \\ \vdots \\ C \end{array} \quad \begin{array}{c} \Delta_1 \\ \vdots \\ !B_1 \end{array} \quad \ldots \quad \begin{array}{c} \Delta_n \\ \vdots \\ !B_n \end{array} \quad \begin{array}{c} [!B_1, \ldots, !B_n] \\ \vdots \\ A \end{array}}{C}$$

where the formulas $!A$ and $!B_1, \ldots, !B_n$ in brackets are discharged at the rule.

Together with the above rule one has to add explicit structural rules which allow to discard and duplicate !-formulas. In their general forms, rules $!w$ and $!c$ are

$$\frac{\Gamma \vdash !A \quad \Delta \vdash C}{\Gamma, \Delta \vdash C}\ !w \qquad \frac{\Gamma \vdash !A \quad !A, !A, \Delta \vdash C}{\Gamma, \Delta \vdash C}\ !c$$

Sequent calculus versions follow when $\Gamma = !A$. However, one of the main features of natural deduction, that is, the absence of explicit structural rules, is lost by such addition. In natural deduction systems for intuitionistic and classical logic the effect of the structural rules of weakening and contraction is achieved by means of vacuous and multiple discharge of assumptions. As shown in von Plato (2001), this works to perfection (only) with general elimination rules. The obvious adaptation of the method to linear logic would consist in allowing vacuous and multiple discharge of assumptions consisting of !-formulas. However, this method only works with context-independent rules, and thus is not suited for extension to linear logic beyond the multiplicative fragment. Instead, as shown by Proposition 3.3 below, we can obtain a system with implicit weakening and contraction by allowing vacuous and multiple discharge in the generalized rule of !-introduction. We thus introduce and use the following

$$\frac{!A^m, \Gamma \vdash C \quad \Delta_1 \vdash !B_1 \quad \ldots \quad \Delta_n \vdash !B_n \quad !B_1, \ldots, !B_n \vdash A}{\Gamma, \Delta_1, \ldots, \Delta_n \vdash C}\ !I$$

where $!A^m$ denotes m occurrences of the formula $!A$ (zero included).

Rule $!E$ follows the pattern of the other elimination rules.

In the table that follows some axioms and rules are grouped differently with respect to the table for **ILL**: In fact, $0E$ is an elimination rule and $1I$ as well as $\top I$ are introduction rules with zero premisses. Instances of the axiom $A \vdash A$ are called *assumptions*. Rule $\forall I$ has the condition that y is not free in Γ and $y = x$ or y is not free in A. Rule $\exists E$ has the condition that y is not free in Δ, C and $y = x$ or y is not free in A.

Natural Deduction for Intuitionistic Linear Logic **N-ILL**

(Sequent calculus style)

Axioms:

$A \vdash A$

Logical rules:

$$\vdash 1 \; {\scriptstyle 1I} \qquad \frac{\Gamma \vdash 1 \quad \Delta \vdash C}{\Gamma, \Delta \vdash C}\, {\scriptstyle 1E}$$

$$\Gamma \vdash \top \; {\scriptstyle \top I} \qquad \frac{\Gamma \vdash 0}{\Gamma, \Delta \vdash C}\, {\scriptstyle 0E}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\, {\scriptstyle \otimes I} \qquad \frac{\Gamma \vdash A \otimes B \quad A, B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\, {\scriptstyle \otimes E}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}\, {\scriptstyle \& I} \qquad \frac{\Gamma \vdash A \& B \quad A, \Delta \vdash C}{\Gamma, \Delta \vdash C}\, {\scriptstyle \& E_1} \qquad \frac{\Gamma \vdash A \& B \quad B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\, {\scriptstyle \& E_2}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\, {\scriptstyle \oplus I_1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\, {\scriptstyle \oplus I_2} \qquad \frac{\Gamma \vdash A \oplus B \quad A, \Delta \vdash C \quad B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\, {\scriptstyle \oplus E}$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}\, {\scriptstyle \multimap I} \qquad \frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A \quad B, \Pi \vdash C}{\Gamma, \Delta, \Pi \vdash C}\, {\scriptstyle \multimap E}$$

$$\frac{!A^m, \Gamma \vdash C \quad \Delta_1 \vdash !B_1 \quad \ldots \quad \Delta_n \vdash !B_n \quad !B_1, \ldots !B_n \vdash A}{\Gamma, \Delta_1, \ldots, \Delta_n \vdash C}\, {\scriptstyle !I} \qquad \frac{\Gamma \vdash !A \quad A, \Delta \vdash C}{\Gamma, \Delta \vdash C}\, {\scriptstyle !E}$$

$$\frac{\Gamma \vdash A(y/x)}{\Gamma \vdash \forall x A}\, {\scriptstyle \forall I} \qquad \frac{\Gamma \vdash \forall x A \quad A(t/x), \Delta \vdash C}{\Gamma, \Delta \vdash C}\, {\scriptstyle \forall E}$$

$$\frac{\Gamma \vdash A(t/x)}{\Gamma \vdash \exists x A}\, {\scriptstyle \exists I} \qquad \frac{\Gamma \vdash \exists x A \quad A(y/x), \Delta \vdash C}{\Gamma, \Delta \vdash C}\, {\scriptstyle \exists E}$$

Definition 3.1 *The* height *of a derivation in* **N-ILL** *is its height as a tree.*

We have:

Theorem 3.2 (Closure under substitution) *If $\Gamma \vdash A$ is derivable in* **N-ILL** *with derivation of height $\leqslant n$ and $A, \Delta \vdash B$ is derivable with derivation of height $\leqslant m$, then $\Gamma, \Delta \vdash B$ is derivable with derivation of height $\leqslant max(n, m)$.*

Proof: By induction on the height of the derivation of $A, \Delta \vdash B$. If the height is 0, then there are two cases: In the first case, Δ is empty and B is A, thus the first premiss gives the conclusion without substitution. In the second case, $B = \top$ and the conclusion is an instance of the axiom of $\top I$.

If the height is greater than 0, we consider the last rule applied in the derivation of $A, \Delta \vdash B$. In all the cases substitution is inductively applied to the premisses, and the rule is applied. We show only one case, all the others being dealt with similarly. Let d be the derivation of $\Gamma \vdash A$, and suppose $A, \Delta \vdash B$ is $A, \Delta', \Delta'' \vdash B_1 \otimes B_2$, derived by $\otimes I$ from the premisses $A, \Delta' \vdash B_1$, $\Delta'' \vdash B_2$, with derivations d_1, d_2 of height $h(d_1), h(d_2)$, respectively. By induction hypothesis $\Gamma, \Delta' \vdash B_1$ is derivable with derivation of height $\leqslant max(h(d), h(d_1))$, and by applying $\otimes I$, $\Gamma, \Delta, \Delta' \vdash B_1 \otimes B_2$ is derivable with derivation of height $\leqslant max(max(h(d), h(d_1)), h(d_2)) + 1$ and the conclusion follows since $max(max(h(d), h(d_1)), h(d_2))+1 \leqslant max(h(d), max(h(d_1), h(d_2))+1)$. □

We observe that closure under substitution is the property that guarantees that pasting together two proof trees, one with A at the root, another with A in a leaf, produces another proof tree. This property is usually assumed without mention, but can fail to hold.

As announced, we have:

Proposition 3.3 *The rules* $!w$ *and* $!c$ *are derivable in* **N-ILL**.

Proof: Consider the following derivations:

$$\dfrac{\Delta \vdash C \quad \Gamma \vdash !A \quad \dfrac{!A \vdash !A \quad A \vdash A}{!A \vdash A}\,!E}{\Gamma, \Delta \vdash C}\,!I\ (m{=}0)$$

$$\dfrac{!A, !A, \Delta \vdash C \quad \Gamma \vdash !A \quad \dfrac{!A \vdash !A \quad A \vdash A}{!A \vdash A}\,!E}{\Gamma, \Delta \vdash C}\,!I\ (m{=}2)$$

□

Whereas a sequent calculus style of natural deduction is suitable for studying the metamathematical properties of the system, it is often more convenient to use the tree-style presentation for actually drawing inferences. We include a presentation of **N-ILL** in the usual natural deduction style. The axiom is known as the *rule of assumption* of natural deduction, that allows to start up inferences.

Observe that the assumptions in context-sharing rules get a label. Formulas with the same label are treated as if they were the same formula: For instance, when a formula with a label is discharged, all formulas with the same label are discharged at the same time; when an open assumption with a label is substituted by a derivation concluding the formula, the same substitution is performed for all formulas with the same label, and the open assumptions of these identical derivations all get the same label. This is just a notational convention since the natural deduction style presentation would not otherwise allow to identify formulas as in the sequent calculus style presentation.

When formulas with the same label are discharged in a derivation (for instance in an implication introduction step) we have a form of contraction, hidden in the additive treatment of the context, which differs in character from explicit contraction for !-formulas.

In elimination rules, the *major premiss* is the premiss containing the connective or the constant of the rule in question. The other premisses are called *minor premisses.* The assumptions in brackets are *discharged assumptions.* Discharged assumptions in elimination rules are called *auxiliary assumptions.*

Natural Deduction for Intuitionistic Linear Logic **N-ILL**

(Natural deduction style)

Axioms:

A

Logical rules:

$$\frac{}{1}\ 1I \qquad\qquad \frac{\begin{array}{c}\Gamma\\ \vdots\\ 1\end{array}\quad \begin{array}{c}\Delta\\ \vdots\\ C\end{array}}{C}\ 1E$$

$$\frac{\Gamma}{\top}\ \top I \qquad\qquad \frac{\begin{array}{c}\Gamma\\ \vdots\\ 0\end{array}\quad \Delta}{C}\ 0E$$

$$\frac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}\quad \begin{array}{c}\Delta\\ \vdots\\ B\end{array}}{A \otimes B}\ \otimes I \qquad\qquad \frac{\begin{array}{c}\Gamma\\ \vdots\\ A \otimes B\end{array}\quad \begin{array}{c}[A, B], \Delta\\ \vdots\\ C\end{array}}{C}\ \otimes E$$

$$\frac{\begin{array}{c}\Gamma^{\gamma}\\ \vdots\\ A\end{array}\quad \begin{array}{c}\Gamma^{\gamma}\\ \vdots\\ B\end{array}}{A \& B}\ \& I \qquad\qquad \frac{\begin{array}{c}\Gamma\\ \vdots\\ A \& B\end{array}\quad \begin{array}{c}[A], \Delta\\ \vdots\\ C\end{array}}{C}\ \& E_1 \qquad \frac{\begin{array}{c}\Gamma\\ \vdots\\ A \& B\end{array}\quad \begin{array}{c}[B], \Delta\\ \vdots\\ C\end{array}}{C}\ \& E_2$$

$$\frac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{A \oplus B}\ \oplus I_1 \qquad \frac{\begin{array}{c}\Gamma\\ \vdots\\ B\end{array}}{A \oplus B}\ \oplus I_2 \qquad\qquad \frac{\begin{array}{c}\Gamma\\ \vdots\\ A \oplus B\end{array}\quad \begin{array}{c}[A], \Delta^{\delta}\\ \vdots\\ C\end{array}\quad \begin{array}{c}[B], \Delta^{\delta}\\ \vdots\\ C\end{array}}{C}\ \oplus E$$

$$\frac{\begin{array}{c}[A], \Gamma\\ \vdots\\ B\end{array}}{A \multimap B}\ \multimap I \qquad\qquad \frac{\begin{array}{c}\Gamma\\ \vdots\\ A \multimap B\end{array}\quad \begin{array}{c}\Delta\\ \vdots\\ A\end{array}\quad \begin{array}{c}[B], \Pi\\ \vdots\\ C\end{array}}{C}\ \multimap E$$

$$\dfrac{\begin{array}{c}[!A^m],\Gamma\\ \vdots\\ C\end{array} \quad \begin{array}{c}\Delta_1\\ \vdots\\ !B_1\end{array} \quad \dots \begin{array}{c}\Delta_n\\ \vdots\\ !B_n\end{array} \quad \begin{array}{c}[!B_1,\dots,!B_n]\\ \vdots\\ A\end{array}}{C}\,!I \qquad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ !A\end{array} \quad \begin{array}{c}[A],\Delta\\ \vdots\\ C\end{array}}{C}\,!E$$

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A(y/x)\end{array}}{\forall xA}\,\forall I \qquad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \forall xA\end{array} \quad \begin{array}{c}[A(t/x)],\Delta\\ \vdots\\ C\end{array}}{C}\,\forall E$$

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A(t/x)\end{array}}{\exists xA}\,\exists I \qquad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \exists xA\end{array} \quad \begin{array}{c}[A(y/x)],\Delta\\ \vdots\\ C\end{array}}{C}\,\exists E$$

We observe that the rule

$$\dfrac{\begin{array}{c}!\Gamma\\ \vdots\\ C\end{array}}{!C}$$

is derivable in our system as follows (where $\Gamma = B_1, \dots, B_n$)

$$\dfrac{[!C] \quad !B_1 \quad \dots \quad !B_n \qquad \begin{array}{c}[!B_1,\dots,!B_n]\\ \vdots\\ C\end{array}}{!C}\,!I$$

and we shall sometimes use the former as a shortcut for the latter.

Definition 3.4 *An instance of* $!I$ *is* simple *if it is of one of the following forms:*

$$\dfrac{!A \vdash !A \quad !B_1 \vdash !B_1 \quad \dots !B_n \vdash !B_n \quad !B_1, \dots, B_n \vdash A}{!B_1, \dots, !B_n \vdash !A}$$

$$\dfrac{!A^m, \Gamma \vdash C \quad !A \vdash !A \quad !A \vdash A}{!A, \Gamma \vdash C}$$

where $n, m \geqslant 0$.

Definition 3.5 *A derivation in* **N-ILL** *is in* normal form *if all major premisses of elimination rules are assumptions and all occurrences of* $!I$ *are simple.*

There are several other definitions of normal form in the literature. In Troelstra (1995) an example of a normal derivation (with respect to detour and permutation conversions) is given which does not satisfy the subformula property. The rules used in the derivation are like ours except for $\multimap E$ which is the usual modus ponens rule, instead of our generalized $\multimap E$, and the rules for !, which are

$$\dfrac{!A}{A}\;{\scriptstyle !E} \qquad\qquad \dfrac{!A_1,\ldots !A_n \qquad \begin{matrix}\overset{1}{[!A_1]},\ldots,\overset{n}{[!A_n]}\\ \vdots \\ C\end{matrix}}{!C}\;{\scriptstyle !I,1,\ldots,n}$$

The derivation is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{7}{[!(!A \multimap (B \multimap C))]} \quad \overset{6}{[!A]} \quad \dfrac{\dfrac{\overset{1}{[!(!A \multimap (B \multimap C))]}}{!A \multimap (B \multimap C)}\;{\scriptstyle !E} \quad \overset{2}{[!A]}}{B \multimap C}\;{\scriptstyle \multimap E}}{!(B \multimap C)}\;{\scriptstyle !I,1,2} \quad \overset{5}{[!B]} \quad \dfrac{\dfrac{\overset{3}{[!(B \multimap C)]}}{B \multimap C}\;{\scriptstyle !E} \quad \dfrac{\overset{4}{[!B]}}{B}\;!E}{C}\;{\scriptstyle \multimap E}}{\dfrac{!C}{!B \multimap !C}\;{\scriptstyle \multimap I,5}}\;{\scriptstyle !I,3,4}}{!A \multimap (!B \multimap !C)}\;{\scriptstyle \multimap I,6}}{!(!A \multimap (B \multimap C)) \multimap (!A \multimap (!B \multimap !C))}\;{\scriptstyle \multimap I,7}$$

The subformula property is violated because the formula $!(B \multimap C)$ does not appear in the conclusion. In order to avoid this problem Troelstra adds a conversion to contract the sequence promotion/dereliction rule $(!I/!c)$. In our approach instead we have a normal derivation with the subformula property, as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{7}{[!(!A \multimap (B \multimap C))]} \quad \dfrac{\overset{4}{[!A \multimap (B \multimap C)]} \quad \overset{6}{[!A]} \quad \dfrac{\overset{3}{[B \multimap C]} \quad \dfrac{\overset{5}{[!B]} \quad \overset{1}{[B]}}{B}\;{\scriptstyle !E,1} \quad \overset{2}{[C]}}{C}\;{\scriptstyle \multimap E,2}}{C}\;{\scriptstyle \multimap E,3}}{\dfrac{C}{!C}\;{\scriptstyle !I}}\;{\scriptstyle !E,4}}{!B \multimap !C}\;{\scriptstyle \multimap I,5}}{\dfrac{!A \multimap (!B \multimap !C)}{!(!A \multimap (B \multimap C)) \multimap (!A \multimap (!B \multimap !C))}\;{\scriptstyle \multimap I,7}}\;{\scriptstyle \multimap I,6}$$

It seems that the problem lies in non-normalities hidden in the special elimination rule of modus ponens, rather than in the new rules for the modality ! of linear logic, because also other choices for the modality rules produce a normal derivation with the desired subformula property.

We shall prove in the next section that every derivation in **N-ILL** can be transformed into a derivation in normal form and that derivations in normal form satisfy the subformula property.

4 From sequent calculus to natural deduction and back: A proof of normalization

We define a translation **N** from sequent calculus to natural deduction and a translation **S** from natural deduction to sequent calculus that gives through cut elimination a proof of normalization for the system **N-ILL**. We use for **N-ILL** the sequent calculus style notation, so that no discharge labels are needed.

The translation **N** is defined inductively, starting with the translation of the axioms and proceeding with the translation of the sequent calculus rules, as follows:

$$A \Rightarrow A \quad \rightsquigarrow \quad A \vdash A$$

$$0, \Gamma \Rightarrow C \quad \rightsquigarrow \quad \dfrac{0 \vdash 0}{0, \Gamma \vdash C}\ {\scriptstyle 0E}$$

$$\Rightarrow 1 \quad \rightsquigarrow \quad \vdash 1$$

$$\Gamma \Rightarrow \top \quad \rightsquigarrow \quad \Gamma \vdash \top$$

$$\dfrac{\overset{d_1}{\Gamma \Rightarrow A} \quad \overset{d_2}{\Delta \Rightarrow B}}{\Gamma, \Delta \Rightarrow A \otimes B}\ {\scriptstyle R\otimes} \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{N}(d_1)}{\Gamma \vdash A} \quad \overset{\mathbf{N}(d_2)}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \otimes B}\ {\scriptstyle \otimes I}$$

$$\dfrac{\overset{d}{A, B, \Gamma \Rightarrow C}}{A \otimes B, \Gamma \Rightarrow C}\ {\scriptstyle L\otimes} \quad \rightsquigarrow \quad \dfrac{A \otimes B \vdash A \otimes B \quad \overset{\mathbf{N}(d)}{A, B, \Gamma \vdash C}}{A \otimes B, \Gamma \vdash C}\ {\scriptstyle \otimes E}$$

$$\dfrac{\overset{d_1}{\Gamma \Rightarrow A} \quad \overset{d_2}{\Gamma \Rightarrow B}}{\Gamma \Rightarrow A\&B}\ {\scriptstyle R\&} \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{N}(d_1)}{\Gamma \vdash A} \quad \overset{\mathbf{N}(d_2)}{\Gamma \vdash B}}{\Gamma \vdash A\&B}\ {\scriptstyle \&I}$$

$$\dfrac{\overset{d}{A, \Gamma \Rightarrow C}}{A\&B, \Gamma \Rightarrow C}\ {\scriptstyle L\&_1} \quad \rightsquigarrow \quad \dfrac{A\&B \vdash A\&B \quad \overset{\mathbf{N}(d)}{A, \Gamma \vdash C}}{A\&B, \Gamma \vdash C}\ {\scriptstyle \&E_1}$$

A derivation ending with $L\&_2$ is translated similarly to the above one.

$$\dfrac{\overset{d}{\Gamma \Rightarrow A}}{\Gamma \Rightarrow A \oplus B}\ {\scriptstyle R\oplus_1} \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{N}(d)}{\Gamma \vdash A}}{\Gamma \vdash A \oplus B}\ {\scriptstyle \oplus I_1}$$

A derivation ending with an application of the rule $R\oplus_2$ is translated similarly.

$$\dfrac{\overset{d_1}{A,\Gamma \Rightarrow C} \quad \overset{d_2}{B,\Gamma \Rightarrow C}}{A\oplus B,\Gamma \Rightarrow C}\ L\oplus \quad \rightsquigarrow \quad \dfrac{A\oplus B \vdash A\oplus B \quad \overset{\mathbf{N}(d_1)}{A,\Gamma \vdash C} \quad \overset{\mathbf{N}(d_2)}{B,\Gamma \vdash C}}{A\oplus B,\Gamma \vdash C}\ \oplus E$$

$$\dfrac{\overset{d}{A,\Gamma \Rightarrow B}}{\Gamma \Rightarrow A \multimap B}\ R\multimap \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{N}(d)}{A,\Gamma \vdash B}}{\Gamma \vdash A \multimap B}\ \multimap I$$

$$\dfrac{\overset{d_1}{\Gamma \Rightarrow A} \quad \overset{d_2}{B,\Delta \Rightarrow C}}{A\multimap B,\Gamma,\Delta \Rightarrow C}\ L\multimap \quad \rightsquigarrow \quad \dfrac{A\multimap B \vdash A\multimap B \quad \overset{\mathbf{N}(d_1)}{\Gamma \vdash A} \quad \overset{\mathbf{N}(d_2)}{B,\Delta \vdash C}}{A\multimap B,\Gamma,\Delta \vdash C}\ \multimap E$$

$$\dfrac{\overset{d}{!\Gamma \Rightarrow C}}{!\Gamma \Rightarrow !C}\ R! \quad \rightsquigarrow \quad \dfrac{!C \vdash !C \quad !B_1 \vdash !B_1 \quad \dots \quad !B_n \vdash !B_n \quad \overset{\mathbf{N}(d)}{!B_1,\dots,!B_n \vdash C}}{!\Gamma \vdash !C}\ !I$$

where $!\Gamma$ is $!B_1,\dots,!B_n$.

$$\dfrac{\overset{d}{A,\Gamma \Rightarrow C}}{!A,\Gamma \Rightarrow C}\ L! \quad \rightsquigarrow \quad \dfrac{!A \vdash !A \quad \overset{\mathbf{N}(d)}{A,\Gamma \vdash C}}{!A,\Gamma \vdash C}\ !E$$

For the quantifiers we have the translations

$$\dfrac{\overset{d}{\Gamma \Rightarrow A(y/x)}}{\Gamma \Rightarrow \forall x A}\ R\forall \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{N}(d)}{\Gamma \vdash A(y/x)}}{\Gamma \vdash \forall x A}\ \forall I$$

$$\dfrac{\overset{d}{A(t/x),\Gamma \Rightarrow C}}{\forall x A \Gamma \Rightarrow C}\ L\forall \quad \rightsquigarrow \quad \dfrac{\forall x A \vdash \forall x A \quad \overset{\mathbf{N}(d)}{A(t/x),\Gamma \vdash C}}{\forall x A,\Gamma \vdash C}\ \forall E$$

$$\dfrac{\overset{d}{\Gamma \Rightarrow A(t/x)}}{\Gamma \Rightarrow \exists x A}\ R\exists \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{N}(d)}{\Gamma \vdash A(t/x)}}{\Gamma \vdash \exists x A}\ \exists I$$

$$\dfrac{\overset{d}{A(y/x),\Gamma \Rightarrow C}}{\exists x A,\Gamma \Rightarrow C}\ L\exists \quad \rightsquigarrow \quad \dfrac{\exists x A \vdash \exists x A \quad \overset{\mathbf{N}(d)}{A(y/x),\Gamma \vdash C}}{\exists x A,\Gamma \vdash C}\ \exists E$$

Finally, the structural rules of weakening and contraction are translated through $1E$ and introduction and elimination rules for !, and cut through substitution:

$$\dfrac{\overset{d}{\Gamma \Rightarrow C}}{1, \Gamma \Rightarrow C}\ 1w \quad \rightsquigarrow \quad \dfrac{1 \vdash 1 \quad \overset{\mathbf{N}(d)}{\Gamma \vdash C}}{1, \Gamma \vdash C}\ 1E$$

$$\dfrac{\overset{d}{\Gamma \Rightarrow C}}{!A, \Gamma \Rightarrow C}\ !w \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{N}(d)}{\Gamma \vdash C} \quad !A \vdash !A \quad \dfrac{!A \vdash !A \quad A \vdash A}{!A \vdash A}\ !E}{!A, \Gamma \vdash C}\ !I\ (m{=}0)$$

$$\dfrac{\overset{d}{!A, !A, \Gamma \Rightarrow C}}{!A, \Gamma \Rightarrow C}\ !c \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{N}(d)}{!A, !A, \Gamma \vdash C} \quad !A \vdash !A \quad \dfrac{!A \vdash !A \quad A \vdash A}{!A \vdash A}\ !E}{!A, \Gamma \vdash C}\ !I\ (m{=}2)$$

$$\dfrac{\overset{d_1}{\Gamma \Rightarrow A} \quad \overset{d_2}{A, \Delta \Rightarrow C}}{\Gamma, \Delta \Rightarrow C}\ cut \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{N}(d_1)}{\Gamma \vdash A} \quad \overset{\mathbf{N}(d_2)}{A, \Delta \vdash C}}{\Gamma, \Delta \vdash C}\ subst$$

We have thus proved:

Theorem 4.1 *If d is a derivation of a sequent $\Gamma \Rightarrow C$ in* **ILL**, $\mathbf{N}(d)$ *is a derivation of* $\Gamma \vdash C$ *in* **N-ILL**.

In defining the translation we have also given the translation of the cut rule. This is not strictly necessary, since cut is an admissible rule of the system **ILL**. However, as we shall see in the following section, the translation of a derivation in **N-ILL** which is not in normal form produces cuts, so it is convenient to translate cut as well for having a direct back and forth translation. On the other hand, the translation of a cut-free derivation in **ILL** produces a derivation in **N-ILL** in which no use of substitution is made. This is made possible by the full use of the general elimination rules (like the implication elimination in place of modus ponens, or the general elimination rules for & instead of the special elimination rules). Other translations in the literature (cf. Benton *et al.*) use special elimination rules and substitution has to be used. By inspection of all the cases considered, we note that in the translation of all the left rules, the major premisses of the corresponding elimination rules are assumptions and all occurrences of $!I$ are simple. Occurrences of $R!$ are mapped into simple instances of $!I$ of the first form, whereas occurrences of $!w$ and $!c$ are mapped into simple instances of the second form. Therefore we have:

Corollary 4.2 *If d is a cut-free derivation of a sequent $\Gamma \Rightarrow C$ in* **ILL**, *then* $\mathbf{N}(d)$ *is a derivation of* $\Gamma \vdash C$ *in* **N-ILL** *in normal form.*

The translation $\mathbf{S}$ from natural deduction to sequent calculus is also defined by induction on the derivation tree. We start with the axioms:

$$A \vdash A \quad \rightsquigarrow \quad A \Rightarrow A$$

$$\vdash 1 \quad \rightsquigarrow \quad \Rightarrow 1$$

$$\Gamma \vdash \top \quad \rightsquigarrow \quad \Gamma \Rightarrow \top$$

$$\frac{\overset{\pi}{\Gamma \vdash 0}}{\Gamma, \Delta \vdash C}\ 0E \quad \rightsquigarrow \quad \frac{\overset{\mathbf{S}(\pi)}{\Gamma \Rightarrow 0} \quad 0, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\ cut$$

A derivation ending with an introduction rule is translated by the corresponding right rule of sequent calculus and one ending with an elimination rule by the corresponding left rule and cut.

$$\frac{\overset{\pi_1}{\Gamma \vdash A} \quad \overset{\pi_2}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \otimes B}\ \otimes I \quad \rightsquigarrow \quad \frac{\overset{\mathbf{S}(\pi_1)}{\Gamma \Rightarrow A} \quad \overset{\mathbf{S}(\pi_2)}{\Delta \Rightarrow B}}{\Gamma, \Delta \Rightarrow A \otimes B}\ R\otimes$$

$$\frac{\overset{\pi_1}{\Gamma \vdash A \otimes B} \quad \overset{\pi_2}{A, B, \Delta \vdash C}}{\Gamma, \Delta \vdash C}\ \otimes E \quad \rightsquigarrow \quad \frac{\overset{\mathbf{S}(\pi_1)}{\Gamma \Rightarrow A \otimes B} \quad \dfrac{\overset{\mathbf{S}(\pi_2)}{A, B, \Delta \Rightarrow C}}{A \otimes B, \Delta \Rightarrow C}\ L\otimes}{\Gamma, \Delta \Rightarrow C}\ cut$$

$$\frac{\overset{\pi_1}{\Gamma \vdash A} \quad \overset{\pi_2}{\Gamma \vdash B}}{\Gamma \vdash A \& B}\ \& I \quad \rightsquigarrow \quad \frac{\overset{\mathbf{S}(\pi_1)}{\Gamma \Rightarrow A} \quad \overset{\mathbf{S}(\pi_2)}{\Gamma \Rightarrow B}}{\Gamma \Rightarrow A \& B}\ R\&$$

$$\frac{\overset{\pi_1}{\Gamma \vdash A \& B} \quad \overset{\pi_2}{A, \Delta \vdash C}}{\Gamma, \Delta \vdash C}\ \& E_1 \quad \rightsquigarrow \quad \frac{\overset{\mathbf{S}(\pi_1)}{\Gamma \Rightarrow A \& B} \quad \dfrac{\overset{\mathbf{S}(\pi_2)}{A, \Delta \Rightarrow C}}{A \& B, \Delta \Rightarrow C}\ L\&_1}{\Gamma, \Delta \Rightarrow C}\ cut$$

A derivation ending with $\& E_2$ is translated similarly by $L\&_2$ and cut.

$$\frac{\overset{\pi}{\Gamma \vdash A}}{\Gamma \vdash A \oplus B}\ \oplus I_1 \quad \rightsquigarrow \quad \frac{\overset{\mathbf{S}(\pi_2)}{\Gamma \Rightarrow A}}{\Gamma \Rightarrow A \oplus B}\ R\oplus_1$$

A derivation ending with $\oplus I_2$ is translated similarly by $R\oplus_2$.

$$\dfrac{\Gamma \vdash \overset{\pi_1}{A} \oplus B \quad A,\overset{\pi_2}{\Delta} \vdash C \quad B,\overset{\pi_3}{\Delta} \vdash C}{\Gamma,\Delta \vdash C}\,\oplus E \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{S}(\pi_1)}{\Gamma \Rightarrow A\oplus B} \quad \dfrac{\overset{\mathbf{S}(\pi_2)}{A,\Delta\Rightarrow C} \quad \overset{\mathbf{S}(\pi_3)}{B,\Delta\Rightarrow C}}{A\oplus B,\Delta\Rightarrow C}\,L\oplus}{\Gamma,\Delta\Rightarrow C}\,\mathit{cut}$$

$$\dfrac{A,\overset{\pi}{\Gamma} \vdash B}{\Gamma \vdash A \multimap B}\,\multimap I \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{S}(\pi)}{A,\Gamma \Rightarrow B}}{\Gamma \Rightarrow A \multimap B}\,R\multimap$$

$$\dfrac{\Gamma \vdash \overset{\pi_1}{A} \multimap B \quad \Delta \overset{\pi_2}{\vdash} A \quad B,\overset{\pi_3}{\Pi} \vdash C}{\Gamma,\Delta,\Pi \vdash C}\,\multimap E \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{S}(\pi_1)}{\Gamma \Rightarrow A \multimap B} \quad \dfrac{\overset{\mathbf{S}(\pi_2)}{\Delta \Rightarrow A} \quad \overset{\mathbf{S}(\pi_3)}{B,\Pi \Rightarrow C}}{A \multimap B,\Delta,\Pi \Rightarrow C}\,L\multimap}{\Gamma,\Delta,\Pi \Rightarrow C}\,\mathit{cut}$$

$$\dfrac{!A^m,\overset{\pi}{\Gamma} \vdash C \quad \Gamma_1 \overset{\pi_1}{\vdash} !B_1 \quad \dots \quad \Gamma_n \overset{\pi_n}{\vdash} !B_n \quad !B_1,\dots,\overset{\pi}{!}B_n \vdash A}{\Gamma_1,\dots,\Gamma_n,\Gamma \vdash C}\,!I \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\overset{\mathbf{S}(\pi)}{!A^m,\Gamma \Rightarrow C}}{!A,\Gamma \Rightarrow C}\,\mathit{str} \quad \dfrac{\overset{\mathbf{S}(\pi_n)}{\Gamma_n \Rightarrow !B_n} \quad \begin{matrix} \dfrac{\overset{\mathbf{S}(\pi_1)}{\Gamma_1 \Rightarrow !B_1} \quad \dfrac{\overset{\mathbf{S}(\pi)}{!B_1,\dots,!B_n \Rightarrow A}}{!B_1,\dots,!B_n \Rightarrow !A}\,R!}{!B_2,\dots,!B_n,\Gamma_1 \Rightarrow !A}\,\mathit{cut} \\ \vdots \\ !B_n,\Gamma_1,\dots,\Gamma_{n-1} \Rightarrow !A \end{matrix}}{\Gamma_1,\dots,\Gamma_n \Rightarrow !A}\,\mathit{cut}}{\Gamma_1,\dots,\Gamma_n,\Gamma \Rightarrow C}\,\mathit{cut}$$

where *str* denotes applications of $!w$ or $!c$, depending on m.

$$\dfrac{\Gamma \overset{\pi_1}{\vdash} !A \quad A,\overset{\pi_2}{\Delta} \vdash C}{\Gamma,\Delta \vdash C}\,!E \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{S}(\pi_1)}{\Gamma \Rightarrow !A} \quad \dfrac{\overset{\mathbf{S}(\pi_2)}{A,\Delta \Rightarrow C}}{!A,\Delta \Rightarrow C}\,L!}{\Gamma,\Delta \Rightarrow C}\,\mathit{cut}$$

$$\dfrac{\Gamma \vdash \overset{\pi}{A}(y/x)}{\Gamma \vdash \forall x A}\,\forall I \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{S}(\pi)}{\Gamma \Rightarrow A(y/x)}}{\Gamma \Rightarrow \forall x A}\,R\forall$$

$$\dfrac{\Gamma \vdash \overset{\pi_1}{\forall} x A \quad A(t/x),\overset{\pi_2}{\Delta} \vdash C}{\Gamma,\Delta \vdash C}\,\forall E \quad \rightsquigarrow \quad \dfrac{\overset{\mathbf{S}(\pi_1)}{\Gamma \Rightarrow \forall x A} \quad \dfrac{\overset{\mathbf{S}(\pi_2)}{A(t/x),\Delta \Rightarrow C}}{\forall x A,\Delta \Rightarrow C}\,L\forall}{\Gamma,\Delta \Rightarrow C}\,\mathit{cut}$$

$$\frac{\overset{\pi}{\Gamma \vdash A(t/x)}}{\Gamma \vdash \exists x A}\,{\scriptstyle \exists I} \quad\rightsquigarrow\quad \frac{\overset{\mathbf{S}(\pi)}{\Gamma \Rightarrow A(t/x)}}{\Gamma \Rightarrow \exists x A}\,{\scriptstyle R\exists}$$

$$\frac{\overset{\pi_1}{\Gamma \vdash \exists x A} \quad \overset{\pi_2}{A(y/x), \Delta \vdash C}}{\Gamma, \Delta \vdash C}\,{\scriptstyle \exists E} \quad\rightsquigarrow\quad \frac{\overset{\mathbf{S}(\pi_1)}{\Gamma \Rightarrow \exists x A} \quad \dfrac{\overset{\mathbf{S}(\pi_2)}{A(y/x), \Delta \Rightarrow C}}{\exists x A, \Delta \Rightarrow C}\,{\scriptstyle L\exists}}{\Gamma, \Delta \Rightarrow C}\,{\scriptstyle cut}$$

$$\frac{\overset{\pi_1}{\Gamma \vdash 1} \quad \overset{\pi_2}{\Delta \vdash C}}{\Gamma, \Delta \vdash C}\,{\scriptstyle 1E} \quad\rightsquigarrow\quad \frac{\overset{\mathbf{S}(\pi_1)}{\Gamma \Rightarrow 1} \quad \dfrac{\overset{\mathbf{S}(\pi_2)}{\Delta \Rightarrow C}}{1, \Delta \Rightarrow C}\,{\scriptstyle 1w}}{\Gamma, \Delta \Rightarrow C}\,{\scriptstyle cut}$$

When translating elimination rules the major premisses of which are assumptions, the translation produces *redundant cuts*, of the form

$$\frac{A \Rightarrow A \quad A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}$$

In all such cases the conclusion of cut is equal to the right premiss of cut, and the cut can simply be removed by deleting the left premiss and the conclusion, and the derivation continued as before. It is just a matter of choice whether the removal of redundant cuts should be made part of the translation or not. Henceforth we shall not distinguish between a cut-free derivation and a derivation with redundant cuts.

We have:

Theorem 4.3 *If π is a derivation of $\Gamma \vdash A$ in* **N-ILL**, *then* $\mathbf{S}(\pi)$ *is a derivation of $\Gamma \Rightarrow A$ in* **ILL**. *If in addition π is in normal form, then* $\mathbf{S}(\pi)$ *is cut free.*

Proof: The first assertion holds by the definition of the translation given above. As for the second part, we observe that when translating a derivation in normal form only redundant cuts are introduced by the translation. For instance, the translation of a step of $\&E_1$ in a derivation in normal form is:

$$\frac{A\&B \vdash A\&B \quad \overset{\pi}{A, \Gamma \vdash C}}{A\&B, \Gamma \vdash C}\,{\scriptstyle \&E_1} \quad\rightsquigarrow\quad \frac{A\&B \Rightarrow A\&B \quad \overset{\mathbf{S}(\pi)}{A, \Gamma \Rightarrow C}}{A\&B, \Gamma \Rightarrow C}\,{\scriptstyle L\&_1}$$

It is routine to check that all the other elimination rules, the translation of which in general requires cuts, are translated without nonredundant cuts when major premisses are assumptions. The translation **S** of a simple instance of $!I$ of the first form produces redundant cuts and a step of $R!$. For the second form, $!w$ or steps of $!c$ are used, depending on m. □

As applications of the translation between sequent calculus and natural deduction we have:

Corollary 4.4 *Every derivation in* **N-ILL** *can be transformed into a derivation in normal form.*

Proof: Given a derivation π of $\Gamma \vdash A$ in **N-ILL**, consider its translation $\mathbf{S}(\pi)$, which is by Theorem **??** a derivation of $\Gamma \Rightarrow A$ in **ILL**. By Theorem **??** we obtain a cut-free derivation $\mathbf{S}(\pi)'$ of $\Gamma \Rightarrow A$, hence by Corollary **??**, $\mathbf{N}(\mathbf{S}(\pi)')$ is a normal derivation of $\Gamma \vdash A$ in **N-ILL**. □

Corollary 4.5 *In a normal derivation in* **N-ILL** *all formulas are subformulas of open assumptions or of the conclusion.*

Proof: All formulas occurring in a derivation π of $\Gamma \vdash C$ are also in its translation $\mathbf{S}(\pi)$, which by **??** is cut free under the hypotheses. □

By the results proved in Thm. **??**, Prop. **??**, Cor. **??**, Thm. **??**, Cor. **??**, and Cor. **??**, we have shown that the system **N-ILL** fulfils all the seven criteria for a satisfactory system of natural deduction for intuitionistic linear logic that have been listed in the Introduction.

Concluding remarks and related work

There is a vast literature on normalization for systems of natural deduction in intuitionistic linear logic: Valentini (1992) gives a system with proof terms with a context-dependent !-introduction rule and explicit weakening and contraction. The elimination rules for $\otimes$, & and $\oplus$ are the same as ours, whereas $\multimap E$ is the usual modus ponens. Also Ronchi della Rocca and Roversi (1997) give a system with context-dependent $!I$ thus satisfying only a partial substitution property. Weakening is in-built through a generalized identity axiom and contraction through a shared treatment of the !-part of the context in the rules for the multiplicatives. There are no rules for 0, $\top$, 1. Pfenning and Polakow (1999) give a system with implicit structural rules: The method for absorbing contraction in the logical rules is the additive treatment of the context as in the sequent system **G3**. In the case of linear logic the additive treatment is for the part of the context made of !-formulas in the rules for the multiplicative connectives, which leads to a syntax where the antecedents of sequents consist of two parts, a linear and an intuitionistic one. A context-free rule of !-introduction permitting closure under substitution has been given by several authors following the idea from Prawitz' treatment of **S4** (1965): Benton *et al.* (1993) give a system with proof terms for the $1, \otimes, \multimap, !$-fragment of intuitionistic linear logic, with explicit weakening and contraction. The same fragment is considered by Troelstra (1995), in which contraction is absorbed into $!I$ through a suitable management of multiple occurrences of labels. The implicit treatment of contraction cannot however be extended to the additives. Mints (1998) gives a system for 2^{nd}-order intuitionistic linear logic, where rules have premisses and conclusions of the form $\langle \Gamma \rangle_{!A}, \Delta \vdash C$ meaning that C is derivable from $!A$ and Δ and $!A$ is derivable from Γ.

We have here brought natural deduction and sequent calculus closer by the use of general elimination rules. At the same time the redundancies caused by sequent proofs are avoided with natural deduction with general elimination rules: principal cuts correspond to non-normal instances of elimination rules and non-principal cuts (i.e., cuts where the cut formula is not principal in the right premiss), which are the cause of non-determinism in the cut-elimination process, are absent in natural deduction.

We mention another trend in the literature, that, instead of modifying natural deduction, looks for identifications in sequent calculus derivations for applications in proof search, as in Andreoli (1992) and Howe (1998).

An attempt to identify all possible sequent calculus proofs for classical linear logic is pursued with proof nets, introduced by Girard (1987), but the extension beyond the multiplicative fragment has remained unwieldy.

The main novelty of our system **N-ILL** with respect to the previous in the literature is the implicit treatment of weakening and contraction working for the whole system, not just for a fragment, and with no complication of the syntax, and the use of general elimination rules for all logical constants. At the same time the system satisfies the main desideratum of a natural deduction system, namely the subformula property for normal derivations.

Another possibility of absorbing the structural rules in an implicit way in a system of natural deduction for linear logic consists in allowing multiple and vacuous discharge of !-formulas in rules that discharge assumptions, thus following more closely what is done for intuitionistic logic. The system thus obtained satisfies the same properties of **N-ILL** but at the cost of multiplication of open assumptions in a derivation in the normalization process, so we could not regard the system as completely satisfactory. However such a phenomenon would justify in the resource semantics a slogan such as: The simpler a proof, the more expensive.

General elimination rules for all the intuitionistic connectives and quantifiers have been introduced by von Plato (2001). General elimination rules are systematically used in Negri and von Plato (2001) for establishing a direct isomorphism between derivations in natural deduction and in sequent calculus.

The introduction rule for ! has the form of a general introduction rule.

General introduction rules are employed in Negri and von Plato (2001) in a uniform calculus for intuitionistic logic that gives both natural deduction and sequent calculus as special cases. A uniform calculus for linear logic is presented in Negri (2000).

A proof of strong normalization for natural deduction with general elimination rules for intuitionistic logic has been given in Joachimski and Matthes (2001) by the use of proof terms. A term assignment for our system **N-ILL** is left to future work.

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