From Kripke Models to Algebraic Counter-valuations

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Abstract. Starting with a derivation in the refutation calculus **CRIP** of Pinto and Dyckhoff, we give a constructive algebraic method for determining the values of formulas of intuitionistic propositional logic in a counter-model. The values of compound formulas are computed pointwise from the values on atoms, in contrast to the non-local determination of forcing relations in a Kripke model based on classical reasoning.

1 Introduction

Systems of terminating sequent calculi for intuitionistic propositional logic were first given in Dyckhoff (1992) and in Hudelmaier (1992). These calculi have the property that bottom-up proof search of provable sequents always terminates, a feature obtained through a refinement of the left implication rule of the usual cut-free sequent calculi for intuitionistic propositional logic (see Troelstra and Schwichtenberg 1996 for standard versions of these calculi).

In Pinto and Dyckhoff (1995), a related *refutation calculus* **CRIP** was given, for showing underivability of a sequent $\Gamma \Rightarrow \Delta$. They proved that for intuitionistic propositional logic, either the sequent $\Gamma \Rightarrow \Delta$ is derivable in Dyckhoff's calculus **LJT***, or the *antisequent* $\Gamma \not\Rightarrow \Delta$ is derivable in **CRIP**. For the latter case, a method was given for constructing a Kripke counter-model. A related method was developed by Stoughton (1996) for producing small Kripke counter-models.

We shall here propose an algebraic method for computing the values of compound formulas in a counter-model. The method is constructive, and can replace the determination of forcing of compound formulas in a Kripke model. In the latter, classical reasoning on the meta-level is used; Our method, instead, uses a direct pointwise computation from values on atomic formulas.

In Kripke trees as well as in Heyting algebras, there is no internal notion for expressing that, say, an element is strictly above another one in the partial order, but this can only be seen by looking "from the outside", if at all. We propose a structure, that of a *positive* Heyting algebra, that internalizes the intuitive situation. This is done by requiring a relation $a \nleq b$, read as "a exceeds b", with properties such that the usual partial order comes out as a negation,

$a \leqslant b \equiv \sim a \nleqslant b$. This is quite analogous to the definition of an equality relation as a negation of apartness. Next, we define a formula A to be *invalid* if there exists a valuation v to a positive Heyting algebra such that $v(\top) \nleqslant v(A)$. If not, A is defined as valid, and we have for all valuations v that $v(\top) \leqslant v(A)$. In von Plato (1997), it is shown that this initially perhaps surprising definition of intuitionistic validity as a negative notion coincides with the usual definition. Further, with positive Heyting algebras we can express and prove *soundness* of rules of refutation, by showing that if there is a counter-valuation for the premises, there is a counter-valuation for the conclusion.

The paper is organized as follows: We introduce the algebraic semantics of refutation, and then present the calculus **CRIP**. In Section 5, we show how to construct an algebraic counter-model parallel to the construction of a Kripke counter-model. The key step is the operation of combining (positive) Heyting algebras that corresponds to the gluing of Kripke models. In Section 6, we show how the valuations in positive Heyting algebras are computed, and in Section 7 we give some examples; These show concretely how the values of compound formulas are computed from values on atoms, instead of the non-local and classical determination of forcing in a Kripke model.

2 Positive partial order, lattices and Heyting algebras

We assume given a set with a primitive relation $a \nleqslant b$, to be read *a exceeds b*, and satisfying the axioms of *irreflexivity* and *splitting:*

PPO1. $\sim a \nleqslant a$, PPO2. $a \nleqslant b \supset a \nleqslant c \vee c \nleqslant b$.

A set with such a relation is called a *positive partial order*. Observe that the relation is not a partial order, for transitivity does not in general hold, but a relation whose negation is a partial order, defined by:

Definition 1. $a \leqslant b \equiv \sim a \nleqslant b$.

This *weak partial order* relation is reflexive by PPO1 and transitive by contraposition of PPO2. As there will be no need for a primitive notion of equality, we define equality by $a = b \equiv a \leqslant b \;\&\; b \leqslant a$. Thus, our weak partial order is what is sometimes called a quasi-ordering.

We can further define an apartness relation by $a \neq b \equiv a \nleqslant b \vee b \nleqslant a$. It has the usual properties, and its negation coincides with equality defined above. Strict partial order can be defined by $a < b \equiv b \nleqslant a \;\&\; \sim a \nleqslant b$ and it is irreflexive and transitive.

A *positive lattice* is obtained by adding meet and join operations and the following axioms to a positive partial order.

MTI $\sim a \wedge b \nleqslant a$, $\sim a \wedge b \nleqslant b$, JNI $\sim a \nleqslant a \vee b$, $\sim b \nleqslant a \vee b$,

MTU $c \nleqslant a \wedge b \supset c \nleqslant a \vee c \nleqslant b$, JNU $a \vee b \nleqslant c \supset a \nleqslant c \vee b \nleqslant c$.

Positive Heyting algebras result from adding to a positive lattice a third construction $a \rightarrow b$, to be called *Heyting arrow*, with the axioms

PHI $\sim (a \to b) \wedge a \nleq b$, PHU $c \nleq a \to b \supset c \wedge a \nleq b$.

The first axiom validates modus ponens, the second, a constructive uniqueness principle, identifies implication as the supremum of anything that together with a gives b.

Here we use positive Heyting algebras with a bottom element 0. This is obtained by the principle

PHB $\sim 0 \nleq a$,

and a top element 1 is now defined by $1 = 0 \to 0$.

Each of the positive structures is constructively stronger than the corresponding usual structure, because of the presence of splitting and the uniqueness axioms. But if we define partial order through the negation of excess, the usual axioms for partial order, lattices and Heyting algebras are obtained by taking the negative axioms for excess and the contrapositions of the positive ones. For instance, the axioms for partial order defined negatively are PPO1 and

$$\sim a \nleq c \;\&\; \sim c \nleq b \supset \sim a \nleq b,$$

and the ones to be added for lattices are MTI, JNI, and

$$\sim c \nleq a \;\&\; \sim c \nleq b \supset \sim c \nleq a \wedge b,$$

$$\sim a \nleq c \;\&\; \sim b \nleq c \supset \sim a \vee b \nleq c,$$

and for Heyting algebras PHB, PHI, and

$$\sim c \wedge a \nleq b \supset \sim c \nleq a \to b.$$

If a formula in which all atoms are negated is proved in the theory of positive Heyting algebras, then it can be proved in the theory with the above axioms. This conservativity result is proved in Negri (1997) by means of a cut-free sequent calculus for the theory of positive Heyting algebras. (The ideas and methods of this proof require too much space to be summarized here).

We say that a map ϕ from a positive Heyting algebra H_1 to a positive Heyting algebra H_2 is a *homomorphism of positive Heyting algebras* if it reflects the excess relation and preserves meet, join, Heyting arrow and bottom, that is, for all $a, b \in H_1$ we have

1. $\phi(a) \nleq \phi(b)$ implies $a \nleq b$,
2. $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$,
3. $\phi(a \vee b) = \phi(a) \vee \phi(b)$,
4. $\phi(a \to b) = \phi(a) \to \phi(b)$,
5. $\phi(0_1) = 0_2$,

where 0_1 and 0_2 are the bottom elements of H_1 and H_2, respectively.

If a map ϕ reflects the excess relation, then by contraposition it is monotone with respect to the partial order defined through negation of excess. As a consequence, the conditions 2–5 can be weakened into the following:

2′. $\sim \phi(a){\wedge}\phi(b) \not\leq \phi(a{\wedge}b)$,

3′. $\sim \phi(a{\vee}b) \not\leq \phi(a){\vee}\phi(b)$,

4′. $\sim \phi(a){\to}\phi(b) \not\leq \phi(a{\to}b)$,

5′. $\sim \phi(0_1) \not\leq 0_2$.

An *isomorphism of positive Heyting algebras* is a bijective homomorphism of positive Heyting algebras.

The following lemma will be used in the proof of proposition 11:

Lemma 2. *If H_1 and H_2 are positive Heyting algebras and $\phi : H_1 \to H_2$ and $\psi : H_1 \to H_2$ are maps that reflect the excess relation and are inverses of each other, then ϕ is an isomorphism of positive Heyting algebras.*

Proof. We prove 2′, the other conditions being dealt with similarly.

By bijectivity, we have $a{\wedge}b = \psi\phi(a{\wedge}b)$, and thus also $\psi\phi(a){\wedge}\psi\phi(b) = \psi\phi(a{\wedge}b)$. By monotonicity of ψ we have $\sim \psi(\phi(a){\wedge}\phi(b)) \not\leq \psi\phi(a){\wedge}\psi\phi(b)$, and therefore $\sim \psi(\phi(a){\wedge}\phi(b)) \not\leq \psi\phi(a{\wedge}b)$. By monotonicity of ϕ and the fact that ϕ is the inverse of ψ, we obtain $\sim \phi(a){\wedge}\phi(b) \not\leq \phi(a{\wedge}b)$.

3 Algebraic semantics of refutation

We shall show that positive Heyting algebras lead to a natural formal semantics of refutation, corresponding precisely to the usual algebraic semantics for derivability. A *valuation* is, as usually, a homomorphism $v : \mathrm{Form} \to H$ from the set of formulas Form (here of intuitionistic propositional logic) to a positive Heyting algebra H, satisfying the equations

$v(A\&B) = v(A){\wedge}v(B)$,

$v(A \vee B) = v(A){\vee}v(B)$,

$v(A \supset B) = v(A){\to}v(B)$,

$v(\bot) = 0$.

Let Γ range over finite sets of formulas. We shall write $v(\Gamma)$ for the meet of the values of formulas in Γ, with $v(\Gamma) = 1$ in case Γ is empty.

Definition 3. *A formula A is* invalid under Γ, *written $\Gamma \not\models A$, if there is a valuation v to a positive Heyting algebra such that $v(\Gamma) \not\leq v(A)$. In this case we say that v is a* counter-valuation *to Γ, A.*

In particular, a formula A is invalid, denoted by $\not\models A$, if there is a valuation v to a positive Heyting algebra such that $1 \not\leq v(A)$. We can also define *consistency* of Γ internally, by requiring that there is a valuation v for which $v(\Gamma) \not\leq 0$. This is most naturally written as $\Gamma \not\models$ (or, equivalently, $\Gamma \not\models \bot$).

Definition 4. *$\Gamma \models A$ if and only if not $\Gamma \not\models A$.*

We shall say that A is valid under Γ, or a logical consequence of Γ. In particular, A is valid if not $\not\vDash A$, and Γ is inconsistent if not $\Gamma \not\vDash$.

We emphasize that this order of concepts is essential for reasoning constructively. If a classical meta-logic is used, validity can equally be taken as the basic notion.

It follows from our definition that $\Gamma \vDash A$ if and only if for all valuations v to positive Heyting algebras, $v(\Gamma) \leqslant v(A)$. In particular, we have that a formula A is valid if and only if $v(A) = 1$ for all valuations. This is just like the standard definition of validity for intuitionistic logic except that it refers to positive Heyting algebras, and as shown in von Plato (1997), the new notion of validity coincides with the old one. To give a brief example of a proof of validity, let us show $\vDash A\&B \supset A$. So assume there is a valuation v such that $1 \nleqslant v(A\&B \supset A)$. Then $1 \nleqslant v(A) \wedge v(B) \rightarrow v(A)$, so by PHU, $v(A) \wedge v(B) \nleqslant v(A)$ which gives a contradiction by MTI. So for all valuations v we have $\sim 1 \nleqslant v(A\&B \supset A)$, that is, $1 \leqslant v(A\&B \supset A)$. Observe that the proof is constructive: no *reductio ad absurdum* is used, but the negative definition of validity.

In von Plato (1997), details of the application of positive Heyting algebras to intuitionistic propositional logic can be found. For example, it is shown that the Lindenbaum algebras of intuitionistic propositional logic have the structure of positive Heyting algebras, from which completeness relative to these algebras follows.

4 Refutation calculi

For us, a refutation calculus is a system of syntactic rules for showing refutability. Refutability is a positive notion, in contrast to the weak negative notion of underivability.

We shall here make use of the calculus **CRIP** of Pinto and Dyckhoff (1995), with the role of falsum in the rules made explicit (Roy Dyckhoff, personal communication November 1997). In the rules below, an *antisequent* is an expression of form $\Gamma \not\Rightarrow \Delta$ where Γ, Δ are finite multisets of formulas. The rules of **CRIP**, from Pinto and Dyckhoff (1995, p. 227), are to be read as follows: We start from an antisequent $\Gamma \not\Rightarrow \Delta$ at the bottom, and infer sufficient conditions upwards. If we reach *axioms* in all leaves of the upward-growing tree, the refutation was successful. We can then read the tree top-down as a derivation of the initial antisequent as a theorem of **CRIP**, and, therefore, as a nontheorem of intuitionistic propositional logic. If not, the sequent $\Gamma \Rightarrow \Delta$ is derivable in the multisuccedent calculus **LJT*** of Dyckhoff (1992).

In the rules of **CRIP** below, we use $P, Q, R, \ldots$ for atomic formulas and $A, B, C, \ldots$ for arbitrary formulas. Two of the rules have conditions, and in them, an *atomic implication* is one with an atom as antecedent.

CRIP:

$$\frac{}{P_1 \supset B_1, \ldots, P_k \supset B_k, \Gamma \not\Rightarrow \Delta} \; axiom$$

$$\frac{A, B, \Gamma \not\Rightarrow \Delta}{A\&B, \Gamma \not\Rightarrow \Delta} \; (1) \qquad \frac{\Gamma \not\Rightarrow \Delta, A}{\Gamma \not\Rightarrow \Delta, A\&B} \; (2) \qquad \frac{\Gamma \not\Rightarrow \Delta, B}{\Gamma \not\Rightarrow \Delta, A\&B} \; (3)$$

$$\frac{A, \Gamma \not\Rightarrow \Delta}{A \vee B, \Gamma \not\Rightarrow \Delta} \; (4) \qquad \frac{B, \Gamma \not\Rightarrow \Delta}{A \vee B, \Gamma \not\Rightarrow \Delta} \; (5) \qquad \frac{\Gamma \not\Rightarrow \Delta, A, B}{\Gamma \not\Rightarrow \Delta, A \vee B} \; (6)$$

$$\frac{P, B, \Gamma \not\Rightarrow \Delta}{P, P \supset B, \Gamma \not\Rightarrow \Delta} \; (7) \qquad \frac{C \supset B, D \supset B, \Gamma \not\Rightarrow \Delta}{(C \vee D) \supset B, \Gamma \not\Rightarrow \Delta} \; (8)$$

$$\frac{C \supset (D \supset B), \Gamma \not\Rightarrow \Delta}{(C\&D) \supset B, \Gamma \not\Rightarrow \Delta} \; (9) \qquad \frac{B, \Gamma \not\Rightarrow \Delta}{(C \supset D) \supset B, \Gamma \not\Rightarrow \Delta} \; (10)$$

$$\frac{C_1, D_1 \supset B_1, \Gamma_1 \not\Rightarrow D_1 \ldots C_n, D_n \supset B_n, \Gamma_n \not\Rightarrow D_n \quad \Gamma', E_1 \not\Rightarrow F_1 \ldots \Gamma', E_m \not\Rightarrow F_m}{\Gamma' \not\Rightarrow E_1 \supset F_1, \ldots, E_m \supset F_m, \Delta} \; (11)$$

where we use the abbreviations:

$\Gamma' = (C_1 \supset D_1) \supset B_1, \ldots, (C_n \supset D_n) \supset B_n, \Gamma,$
$\Gamma_i = \Gamma' - (C_i \supset D_i) \supset B_i.$

$$\frac{\Gamma \not\Rightarrow \Delta}{\bot \supset B, \Gamma \not\Rightarrow \Delta} \; (12)$$

The conditions in *axiom* are that Γ contains only atomic formulas, Δ contains only atomic formulas or $\bot$, Γ and Δ are disjoint, and each P_i is atomic and not in Γ.

The restrictions in rule (11) are: Each formula in Γ is either atomic or an atomic implication, no antecedent of an atomic implication is equal to an atom in Γ, Δ contains only atoms or $\bot$, Γ and Δ are disjoint.

The *axiom*-rule is a special case of rule (11), with $m = n = 0$. The conditions and rule (12) are amendments to Pinto and Dyckhoff (1995). It is possible to avoid adding rule (12) if in rule (11) Γ is permitted to contain implications with $\bot$ as antecedent.

5 Construction of counter-valuations

We show how to construct positive Heyting algebras serving as codomains of counter-valuations for the nontheorems of intuitionistic propositional logic. We use the calculus **CRIP** and the construction of Kripke counter-models from derivations of antisequents in **CRIP**, to obtain the construction of positive Heyting algebras and counter-valuations.

We start by recalling the construction of a Heyting algebra out of a Kripke model (for more details, see Fitting 1969). Let K be a Kripke model, with a reflexive and transitive relation $\leq$ and a forcing relation $\Vdash$ between elements w of K and formulas, with the usual properties.[1] The algebraic model $H(K)$ corresponding to K is the collection of the upward closed subsets[2] of K, with ordering given by subset inclusion. The meet and join operations are intersection and union, respectively. The top element 1 is the whole set K, the bottom is the empty set. The *K-valuation* $v(P)$ of an atomic formula P is the set of nodes of the Kripke model forcing P,

Definition 5. $v(P) \equiv \{w \in K \mid w \Vdash P\}$.

We have (Fitting 1969, p. 24):

Proposition 6. *$H(K)$ is a Heyting algebra, with $v(A) = 1$ iff $K \Vdash A$.*

For propositional logic, finite Kripke models suffice for the construction of counter-models. These are discrete structures, with a decidable partial order.

Whereas finite sets have a decidable membership, subfinite sets, i.e., subsets of a finite set, do not necessarily have a decidable membership. We therefore define the Heyting algebra associated to a finite Kripke tree to consist of *finite subsets* of the Kripke tree. Then the associated Heyting algebra has a decidable order, and is indeed a positive Heyting algebra with the excess relation defined by

$$U \nleq V =_{df} (\exists u \in U)(u \notin V) .$$

We therefore have

Proposition 7. *If K is a finite Kripke tree, $H(K)$ is a positive Heyting algebra.*

The following representation of elements of $H(K)$ will be useful:

Lemma 8. *If K is a finite Kripke tree, then every element of $H(K)$ can be uniquely represented as*

$$\bigcup_{a \in F} \uparrow a$$

where $\uparrow a = \{b \in K \mid a \leq b\}$, F is a finite subset of K, and any two distinct elements of F are incomparable.

[1] By well known results (see Troelstra and van Dalen 1988, ch. 2.6) we can consider Kripke models as represented by trees, and call a Kripke tree the lattice structure of a Kripke model.

[2] Recall that a subset U of S is *upward closed* if, whenever $x \in S$ and $a \leq x$ for some $a \in U$, then $x \in U$.

Proof. Immediate.

In the construction of Kripke models, an essential step is the *gluing* of a finite number of Kripke models $K_1, \ldots, K_n$. The resulting Kripke model has an initial world w_0 with immediate successors given by the initial worlds of the n given Kripke models. The forcing relation can be modified by the forcing of certain atoms in the new root w_0.

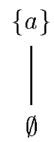
We shall denote by $g(K_1, \ldots, K_n)$ the Kripke tree obtained by gluing of $K_1, \ldots, K_n$. Our next task is to find the operation on (positive) Heyting algebras corresponding to gluing, that is, the operation $\circ$ solving up to positive Heyting algebra isomorphism the equation

$$H(g(K_1, \ldots, K_n)) = \circ(H(K_1), \ldots, H(K_n)).$$

For the sake of simplicity, we consider the case of $n = 2$ only, but what follows generalizes to any finite number in an obvious way.

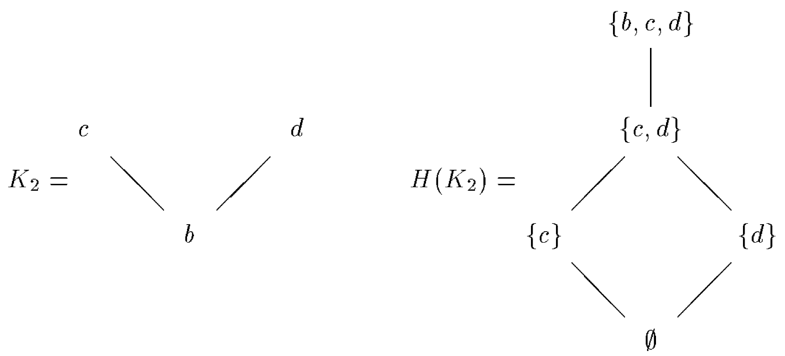
Before giving the general construction, we discuss two examples:

Example 9. Let K_1 be the singleton-set Kripke tree. Then $H(K_1)$ is the (positive) Heyting algebra consisting of two elements



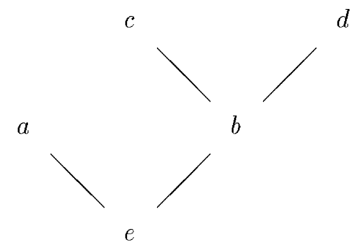
Observe that when one draws diagrams of this kind, one neatly places the points apart, even though in the theories based on partial order there is no internal notion to express the visual effect.

Example 10.

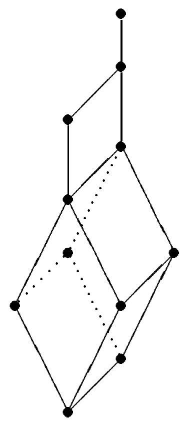


Observe that K_2 is itself the gluing of two Kripke trees of the first kind, so $H(K_2) \cong H(K_1) \circ H(K_1)$ where $\circ$ is the operation to be determined.

If we glue together K_1 and K_2 we obtain

$g(K_1, K_2) =$ 

and the corresponding positive Heyting algebra is (with explicit labels omitted)



The **general construction** behind these examples is as follows: Given two positive Heyting algebras H_1 and H_2, with respective top elements 1_1 and 1_2, let $H_1 \times H_2$ be their Cartesian product with excess relation defined by

$$(a_1, a_2) \nleq (b_1, b_2) \equiv a_1 \nleq b_1 \vee a_2 \nleq b_2$$

and component-wise meet, join and Heyting arrow, and let $\widehat{H_1 \times H_2}$ be the lattice obtained by adding an "extra-top" element 1 and extending the excess relation by posing $\sim (a, b) \nleq 1$ for $a \in H_1,\ b \in H_2$ and $1 \nleq (1_1, 1_2)$. It is clear that in the examples we have

$$H(K_2) = H(g(K_1, K_1)) \cong \widehat{H(K_1) \times H}(K_1),$$

$$H(g(K_1, K_2)) \cong \widehat{H(K_1) \times H}(K_2).$$

Indeed we have, in full generality, that the extra-topped Cartesian product is the operation on Heyting algebras corresponding to the gluing of Kripke models:

Proposition 11. *Let K_1 and K_2 be finite Kripke trees. Then*

$$H(g(K_1, K_2)) \cong \widehat{H(K_1) \times H}(K_2).$$

Proof. By lemma 8, an element in $H(g(K_1, K_2))$ is either $1 = \uparrow w_0$, where w_0 is the root of $g(K_1, K_2)$, or

$$\bigcup_{a \in F_1} \uparrow a \cup \bigcup_{b \in F_2} \uparrow b$$

where F_1 and F_2 are finite subsets of K_1 and K_2. The maps

$$\begin{array}{c} \phi : H(g(K_1, K_1)) \to H(\widehat{K_1) \times H}(K_1) \\ \uparrow w_0 \mapsto 1 \\ \bigcup_{a \in F_1} \uparrow a \cup \bigcup_{b \in F_2} \uparrow b \mapsto (\bigcup_{a \in F_1} \uparrow a, \bigcup_{b \in F_2} \uparrow b) \end{array}$$

$$\begin{array}{c} \psi : H(\widehat{K_1) \times H}(K_1) \to H(g(K_1, K_1)) \\ 1 \mapsto \uparrow w_0 \\ (\bigcup_{a \in F_1} \uparrow a, \bigcup_{b \in F_2} \uparrow b) \mapsto \bigcup_{a \in F_1} \uparrow a \cup \bigcup_{b \in F_2} \uparrow b \end{array}$$

reflect the excess relation and are inverses of each other, therefore by lemma 2 they give an isomorphism between $H(g(K_1, K_2))$ and $H(\widehat{K_1) \times H}(K_2)$.

We adopt from Pinto and Dyckhoff (1995) the following:

Definition 12. *A Kripke tree is a* strong counter-model *to a sequent* $\Gamma \Rightarrow \Delta$ *if in its initial world all the formulas in* Γ *are forced and none of the formulas in* Δ *are forced.*

Our corresponding algebraic notion is:

Definition 13. *A positive Heyting algebra* H *with a valuation* v *is an* algebraic counter-model *to a sequent* $\Gamma \Rightarrow \Delta$ *if for all* A *in* Γ, *we have* $v(A) = 1$ *and* $1 \not\leq \bigvee_{B \in \Delta} v(B)$.

Lemma 14. *If* K *is a strong counter-model to* $\Gamma \Rightarrow \Delta$, *then* $H(K)$, *with the* K*-valuation as defined in 5, is an algebraic counter-model to* $\Gamma \Rightarrow \Delta$.

As an aside, we recall from Pinto and Dyckhoff (1995) that a Kripke tree is a *counter-model* to a sequent $\Gamma \Rightarrow \Delta$ if it has a node in which all formulas in Γ are forced and none of the formulas in Δ are forced. If K is a counter-model to $\Gamma \Rightarrow \Delta$ then the positive Heyting algebra $H(K)$ with the K-valuation has the property that

$$\bigwedge_{A \in \Gamma} v(A) \not\leq \bigvee_{B \in \Delta} v(B)$$

and we call such a positive Heyting algebra with a valuation satisfying the above property a *weak algebraic counter-model.* The relation between algebraic counter-models and weak algebraic counter-models parallels the relation between strong counter-models and counter-models, that is, every algebraic counter-model is a weak algebraic counter-model but not conversely.

Theorem 15. *If $\Gamma \not\Rightarrow \Delta$ is derivable in* **CRIP**, *then there is an algebraic counter-model to the sequent $\Gamma \Rightarrow \Delta$.*

Proof. Consider the derivation in **CRIP** of the antisequent $\Gamma \not\Rightarrow \Delta$. For each step of the construction of the Kripke counter-model as given in proposition 1 of Pinto and Dyckhoff (1995), there is a corresponding step of construction of a positive Heyting algebra and an algebraic counter-valuation, given as follows:

–To the Kripke tree consisting of a single world there corresponds the positive Heyting algebra consisting of two elements. All atoms that are forced are evaluated into the top element, the others into the bottom.

–To the gluing of $n > 1$ Kripke trees K_i there corresponds the extra-topped Cartesian product of Heyting algebras. The atoms forced in the new root are evaluated into the extra top, the other atoms P into $(v_1(P), \ldots, v_n(P))$ where $v_i(P)$ is the K-valuation of $H(K_i)$. In the special case of an application of rule (11) with just one premise, and in all other rules, no gluing of Kripke models is performed, and correspondingly, no extra-topped Cartesian product is taken: an algebraic counter-model for the premise is also a counter-model for the conclusion.

By proposition 11, the Heyting algebra H resulting from this construction is isomorphic to the Heyting algebra $H(K)$ associated to the resulting Kripke tree. Moreover, by lemma 14, H with the K-valuation of $H(K)$ is an algebraic counter-model to $\Gamma \Rightarrow \Delta$.

6 Computation of counter-valuations

The proof of theorem 15 prescribes how to construct an algebraic counter-model starting from a successful **CRIP** refutation. The positive Heyting algebra that serves as codomain of the valuation is defined inductively: The starting points are the two-element Heyting algebras, serving as counter-models for the axioms, and given $n > 1$ positive Heyting algebras that serve as counter-models for the n premises of rule 11, the counter-model for the conclusion is obtained by taking their extra-topped Cartesian product; The construction also gives the valuation for atomic formulas. The evaluation of compound formulas can then be done in a component-wise fashion, but before that a remark on the Cartesian product of positive Heyting algebras is in order:

If $H_1, \ldots, H_n$ are positive Heyting algebras, then the set given by their Cartesian product with excess relation given by

$$(a_1, \ldots, a_n) \nleq (b_1, \ldots, b_n) \equiv a_1 \nleq b_1 \vee \ldots \vee a_n \nleq b_n$$

and meet, join and Heyting arrow defined component-wise, is a positive Heyting algebra.

Let H be the extra-topped Cartesian product of the positive Heyting algebras $H_1, \ldots, H_n$ and let v be a valuation on atoms. Then for all formulas A, $v(A)$ is either 1 or $(a_1, \ldots, a_n)$, where $a_i \in H_i$. For the sake of simplicity we can also denote by a vector $(t_1, \ldots, t_n)$ the extra-top and extend the excess relation and the meet and join operations of H_i by stating that $t_i \nleq a_i$ for all $a_i \in H_i$ and by posing $t_i \wedge a_i = a_i$ and $t_i \vee a_i = t_i$. Then valuations can be computed componentwise, with some care for implication. So assume that $v(B) = (b_1, \ldots, b_n)$ and $v(C) = (c_1, \ldots, c_n)$ have been computed. We then have:

$$v(B \& C) = v(B) \wedge v(C) = (b_1 \wedge c_1, \ldots, b_n \wedge c_n),$$

$$v(B \vee C) = v(B) \vee v(C) = (b_1 \vee c_1, \ldots, b_n \vee c_n).$$

For $v(B \supset C)$ we distinguish three cases:

If $\sim v(B) \nleq v(C)$, then $v(A) = (t_1, \ldots, t_n)$,

if $v(B) \nleq v(C)$ and $v(B) = (t_1, \ldots, t_n)$ then $v(A) = (c_1, \ldots c_n)$,

if $v(B) \nleq v(C)$ and $(t_1, \ldots, t_n) \nleq v(B)$ then $v(A) = (b_1 \rightarrow c_1, \ldots b_n \rightarrow c_n)$.

The evaluation is algorithmic and no use of reasoning on the meta-level is needed, whereas in Kripke models the computation of the values of compound formulas uses classical reasoning on the meta-level.

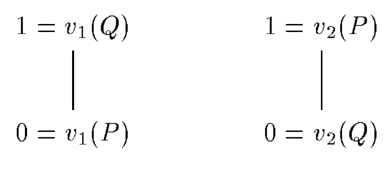
7 Some examples of algebraic counter-models

Example 16. $(P \supset Q) \vee (Q \supset P)$, with P and Q distinct atoms:

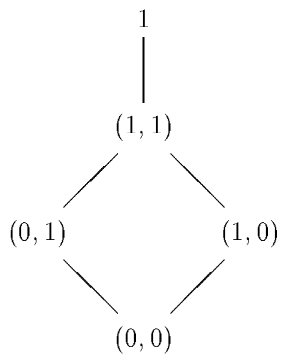
The antisequent $\not\Rightarrow (P \supset Q) \vee (Q \supset P)$ has the following **CRIP** derivation:

$$\cfrac{\cfrac{\cfrac{}{P \not\Rightarrow Q}\,\textit{axiom} \qquad \cfrac{}{Q \not\Rightarrow P}\,\textit{axiom}}{\not\Rightarrow P \supset Q, Q \supset P}\,(11)}{\not\Rightarrow (P \supset Q) \vee (Q \supset P)}\,(6)$$

The Kripke counter-model is obtained by gluing the single-world Kripke models K_1 and K_2, with $K_1 \Vdash P$ and $K_2 \Vdash Q$. The algebraic counter-model is obtained by taking the extra-topped Cartesian product of the corresponding positive Heyting algebras of two elements



that is,



The valuation of the atoms is

$$v(Q) = (v_1(Q), v_2(Q)) = (1, 0), \qquad v(P) = (v_1(P), v_2(P)) = (0, 1)$$

and now we can compute

$$v(P \supset Q) = (1, 0), \quad v(Q \supset P) = (0, 1), \quad v((P \supset Q) \vee (Q \supset P)) = (1, 1),$$

so that

$$1 \nleq v((P \supset Q) \vee (Q \supset P)).$$

Example 17. $(P \supset Q \vee R) \supset (P \supset Q) \vee (P \supset R)$, with P, Q, R distinct atoms: The **CRIP** derivation is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{R, P \not\Rightarrow Q}\,\textit{axiom}}{Q \vee R, P \not\Rightarrow Q}\,(5)}{P \supset Q \vee R, P \not\Rightarrow Q}\,(7) \qquad \dfrac{\dfrac{\dfrac{}{Q, P \not\Rightarrow R}\,\textit{axiom}}{Q \vee R, P \not\Rightarrow R}\,(4)}{P \supset Q \vee R, P \not\Rightarrow R}\,(7)}{\dfrac{P \supset Q \vee R \not\Rightarrow P \supset Q, P \supset R}{P \supset Q \vee R \not\Rightarrow (P \supset Q) \vee (P \supset R)}\,(6)}\,(11)}{\not\Rightarrow (P \supset Q \vee R) \supset (P \supset Q) \vee (P \supset R)}\,(11)$$

We get the Kripke counter-model by gluing the two single-world Kripke models, K_1 forcing R and P, and K_2 forcing Q and P. Observe that the lower instance of rule (11) does not require any gluing. Thus, the corresponding algebraic model is as in example 16, with

$$v(P) = (1, 1) \qquad v(Q) = (0, 1), \qquad v(R) = (1, 0).$$

Therefore

$$v(Q \vee R) = (1, 1), \qquad v(P \supset Q) = (0, 1), \qquad v(P \supset R) = (1, 0),$$

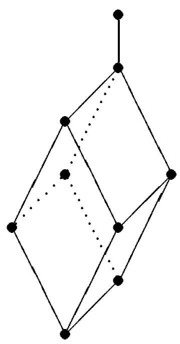
thus

$$v(P \supset Q \vee R) = 1 \nleq v((P \supset Q) \vee (P \supset R)) = (1, 1).$$

Example 18. $(\sim P \supset Q \vee R) \supset (\sim P \supset Q) \vee (\sim P \supset R)$, P, Q, R distinct atoms: The **CRIP** derivation is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \not\Rightarrow \bot}\,\scriptstyle axiom}{\bot \supset Q \vee R, P \not\Rightarrow \bot}\,{\scriptstyle (12)} \qquad \dfrac{\dfrac{\dfrac{}{R, \sim P \not\Rightarrow Q}\,\scriptstyle axiom}{Q \vee R, \sim P \not\Rightarrow Q}\,{\scriptstyle (5)}}{\sim P \supset Q \vee R, \sim P \not\Rightarrow Q}\,{\scriptstyle (10)} \qquad \dfrac{\dfrac{}{Q, \sim P \not\Rightarrow R}\,\scriptstyle axiom}{Q \vee R, \sim P \not\Rightarrow R}\,{\scriptstyle (4)}}{\sim P \supset Q \vee R \not\Rightarrow \sim P \supset Q, \sim P \supset R}\,{\scriptstyle (11)}}{\sim P \supset Q \vee R \not\Rightarrow (\sim P \supset Q) \vee (\sim P \supset R)}\,{\scriptstyle (6)}}{\not\Rightarrow (\sim P \supset Q \vee R) \supset (\sim P \supset Q) \vee (\sim P \supset R)}\,{\scriptstyle (11)}$$

We construct the Kripke counter-model to the end-antisequent by gluing the three Kripke trees forcing, respectively, P, R, and Q. The corresponding positive Heyting algebra is the extra-topped cube



where $v(P) = (1,0,0)$, $v(R) = (0,1,0)$, $v(Q) = (0,0,1)$. We can now illustrate the ease by which the values of compound formulas are determined in the algebraic semantics, by simple computation from values of atomic formulas:

$v(\sim P) = (1,0,0) \rightarrow (0,0,0) = (0,1,1)$
$v(Q \vee R) = (0,1,1)$
$v(\sim P \supset Q \vee R) = (0,1,1) \rightarrow (0,1,1) = 1$
$v(\sim P \supset Q) = (0,1,1) \rightarrow (0,0,1) = (1,0,1)$
$v(\sim P \supset R) = (0,1,1) \rightarrow (0,1,0) = (1,1,0)$
$v((\sim P \supset Q) \vee (\sim P \supset R)) = (1,0,1) \vee (1,1,0) = (1,1,1)$
$v((\sim P \supset Q \vee R) \supset (\sim P \supset Q) \vee (\sim P \supset R)) = 1 \rightarrow (1,1,1) = (1,1,1)$.

8 Concluding remarks

We have given an algebraic semantics of refutation and replaced the determination of forcing of formulas in a Kripke model by a straightforward componentwise computation. Kripke models have been used only for showing the correctness of the construction, that parallels the construction of a Kripke counter-model out of a **CRIP** derivation. In a further work we plan to study the direct construction of counter-valuations avoiding Kripke models altogether.

Positive Heyting algebras and the definition of validity as a negative notion have been here introduced for systematic reasons, even if they could have been avoided in the case of intuitionistic propositional logic because of decidability. We hope to extend the algebraic semantics and counter-valuation construction to intuitionistic predicate logic and expect that the use of positive Heyting algebras will result in a computationally stronger semantics as compared to Kripke models.

Implementation of our algorithm of counter-model construction should present no particular difficulties.

We are indebted to Roy Dyckhoff for his useful comments and advice. Thanks are due to Paul Taylor for his package for drawing diagrams in LaTeX.

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