Permutability of Rules for Linear Lattices[1]

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Abstract: The theory of linear lattices is presented as a system with multiple-conclusion rules. It is shown through the permutability of the rules that the system enjoys a subterm property: all terms in a derivation can be restricted to terms in the conclusion or in the assumptions. Decidability of derivability with the rules for linear lattices follows through the termination of proof-search.
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1 Introduction

In [Negri and von Plato 2002], lattice theory was presented as a system of rules of proof. The rules are used for the construction of formal proofs or *derivations*. Such derivations have a finite number of atomic formulas (atoms) $a_1 \leqslant b_1, \ldots, a_n \leqslant b_n$ as assumptions, and one atom $a \leqslant b$ as a conclusion. The main result was a proof of the decidability of the derivability of an atom from given atoms. Earlier such proofs, starting with [Skolem 1920], seem to have used a relational formulation of lattice theory, whereas our proof used lattice theory with the meet and join operations. The proof was based on permutability properties of the rules of lattice theory. Proof methods for lattice theory, both relational and with operations, are presented in [Negri and von Plato 2004]. In [Negri, von Plato, and Coquand 2004] a proof of decidability for the theory of linear order was presented.

As mentioned in [Negri and von Plato 2002], the proof of decidability can be carried through also for linear lattices. However, it is not possible to have a rule system with just one conclusion, because of the disjunctive alternatives of the linearity axiom. In this sequel to [Negri and von Plato 2002], we carry through the decidability result by a more general permutability argument for a system of rules that permits several alternative consequences from a given set of assumptions. As in our earlier work, decidability of the order relation is not assumed. The presentation is basically self-contained. In particular, no specific knowledge of logic or proof theory is assumed, but a reading of our earlier paper [Negri and von Plato 2002] can be useful.

[1] C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*

2 Linear lattices as systems with rules

The theory of linear lattices has a binary partial order relation $a \leqslant b$, and equality is defined by

$$a = b \equiv a \leqslant b \ \& \ b \leqslant a.$$

The axioms for the order relation are

$$a \leqslant a, \quad \mathit{Ref}, \qquad a \leqslant b \ \& \ b \leqslant c \supset a \leqslant c, \quad \mathit{Trans}, \quad a \leqslant b \vee b \leqslant a, \quad \mathit{Lin},$$

The lattice axioms are standard:

$$a \wedge b \leqslant a \quad (L\wedge_1), \qquad a \wedge b \leqslant b \quad (L\wedge_2), \qquad c \leqslant a \ \& \ c \leqslant b \supset c \leqslant a \wedge b \quad (R\wedge),$$

$$a \leqslant a \vee b \quad (R\vee_1), \qquad b \leqslant a \vee b \quad (R\vee_2), \qquad a \leqslant c \ \& \ b \leqslant c \supset a \vee b \leqslant c \quad (L\vee).$$

The substitutability of equals in the lattice operations can be proved, because equality is defined through the order relation.

In [Negri and von Plato 2002], derivations in lattice theory were represented as trees constructed by rules corresponding to the above axioms, with the conclusion at the root and the assumptions in the leaves. Such rules constitute an extension with mathematical rules of logical systems of natural deduction. Because of the linearity axiom *Lin*, the theory of linear lattices is not what is sometimes called a Harrop theory (that is, it has axioms with unavoidable disjunctions), therefore it cannot be treated as a system of rules that give derivations in tree form. In order to cover non-Harrop theories, one would need a multi-conclusion system of natural deduction; however, natural deduction is inherently a single-conclusion system. Multi-conclusion rules and derivations cannot be written as two-dimensional trees, but the difficulty can be circumvented using *sequent systems*.

The existence of a derivation of a formula C from assumptions Γ can be written on one line as a *sequent* $\Gamma \to C$. If multiple conclusions are permitted, we have sequents of the form $\Gamma \to \Delta$, where both Γ and Δ are finite *multisets* of formulas, i.e., lists with order disregarded. Γ contains the *assumptions* and Δ the *cases* of such sequents. Logical rules modify formulas in the antecedent Γ and in the succedent Δ of sequents. The sequent system we shall use is called **G0c** in [Negri and von Plato 2001]. It is a classical, multi-conclusion sequent calculus system, with explicit *structural rules*. These rules permit the addition of superfluous assumptions and cases, called *weakening*, and the *contraction* of several occurrences of the same formula into one:

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}\, LW \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}\, RW$$

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta}\, LC \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A}\, RC$$

Weakening and contraction can operate on either side of sequents, left or right. These rules are an explicit part of our sequent calculus for lattice theory. A fifth structural rule, called *cut*, permits the combination of two derivations, one having a formula A as a case, the other as an assumption:

$$\frac{\Gamma \rightarrow \Delta, A \quad A, \Gamma' \rightarrow \Delta'}{\Gamma, \Gamma' \rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

This rule is not a part of our calculus, because Gentzen's Hauptsatz (cut elimination theorem) holds for it: Instances of *Cut* can be eliminated from derivations by permuting them up until one premiss of cut is of the form $A \rightarrow A$. It is seen that the other premiss of cut is now identical to the conclusion of cut, so the rule can be removed. We do not need to go into these properties in detail. For our purposes, it suffices to say that cut elimination is maintained when the system is augmented with mathematical rules of the form (with *RRS* standing for *Right Rule Scheme*)

$$\frac{\Gamma_1 \rightarrow \Delta_1, P_1 \quad \dots \quad \Gamma_m \rightarrow \Delta_m, P_m}{\Gamma_1, \dots, \Gamma_m \rightarrow \Delta_1, \dots, \Delta_m, Q_1, \dots, Q_n}\,{\scriptstyle RRS}$$

corresponding to mathematical axioms of the form

$$P_1 \& \dots \& P_m \supset Q_1 \vee \dots \vee Q_n$$

where the P_i's and the Q_j's are atomic formulas. Moreover, we can "forget about logic," since in such extensions all the logical rules permute down with respect to the mathematical rules. In some cases, permutation of logical rules below the mathematical rules produces multiplications of steps of inference. For example, if the rule scheme is preceded by right contraction on a formula P_1

$$\frac{\dfrac{\Gamma_1 \rightarrow \Delta_1, P_1, P_1}{\Gamma_1 \rightarrow \Delta_1, P_1}\,{\scriptstyle RC} \quad \Gamma_2 \rightarrow \Delta_2, P_2 \quad \dots \quad \Gamma_m \rightarrow \Delta_m, P_m}{\Gamma_1, \dots, \Gamma_m \rightarrow \Delta_1, \dots, \Delta_m, Q_1, \dots, Q_n}\,{\scriptstyle RRS}$$

we permute as follows:

$$\frac{\dfrac{\dfrac{\Gamma_1 \rightarrow \Delta_1, P_1, P_1 \quad \Gamma_2 \rightarrow \Delta_2, P_2 \quad \dots \quad \Gamma_m \rightarrow \Delta_m, P_m}{\Gamma_1, \dots, \Gamma_m, P_1 \rightarrow \Delta_1, \dots, \Delta_m, Q_1, \dots, Q_n}\,{\scriptstyle RRS} \quad \Gamma_2 \rightarrow \Delta_2, P_2 \quad \dots \quad \Gamma_m \rightarrow \Delta_m, P_m}{\Gamma_1, \Gamma_2, \Gamma_2, \dots, \Gamma_m, \Gamma_m \rightarrow \Delta_1, \Delta_2, \Delta_2, \dots, \Delta_m, \Delta_m, Q_1, Q_1, \dots, Q_n, Q_n}\,{\scriptstyle RRS}}{\Gamma_1, \Gamma_2 \dots, \Gamma_m \rightarrow \Delta_1, \Delta_2 \dots, \Delta_m, Q_1, \dots, Q_n}\,{\scriptstyle C^*}$$

where C^* denotes repeated steps of left and right contractions.

Whenever we have derivations with more than one formula in the succedent, we say that we have a *derivation with cases.*

For linear lattices, we distinguish between *ground terms* $p, q, r, \dots$, that contain no meet or join operations, and *arbitrary terms* $a, b, c, \dots$. An essential point is that the linearity axiom, as well as the reflexivity of the order relation, can be restricted to ground terms.

The rules for linear lattices are

$$\frac{}{\to p \leqslant p}\,Ref \qquad \frac{}{\to p \leqslant q, q \leqslant p}\,Lin \qquad \frac{\Gamma_1 \to \Delta_1, a \leqslant b \quad \Gamma_2 \to \Delta_2, b \leqslant c}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, a \leqslant c}\,Trans$$

$$\frac{}{\to a \wedge b \leqslant a}\,L\wedge_1 \qquad \frac{}{\to a \wedge b \leqslant b}\,L\wedge_2 \qquad \frac{\Gamma_1 \to \Delta_1, c \leqslant a \quad \Gamma_2 \to \Delta_2, c \leqslant b}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, c \leqslant a \wedge b}\,R\wedge$$

$$\frac{}{\to a \leqslant a \vee b}\,R\vee_1 \qquad \frac{}{\to b \leqslant a \vee b}\,R\vee_2 \qquad \frac{\Gamma_1 \to \Delta_1, a \leqslant c \quad \Gamma_2 \to \Delta_2, b \leqslant c}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, a \vee b \leqslant c}\,L\vee$$

In the rules, the formulas in Γ, Δ form the *contexts*. The atoms in the premisses which are not in a context are called *active*, and those in the conclusion are called *principal*. Derivations start with *initial sequents* of the form $a \leqslant b \to a \leqslant b$ and with instances of the *zero-premiss* rules. The former corresponds to the making of an assumption $a \leqslant b$ in a system of derivation as in [Negri and von Plato 2002]. Of the rules for linear lattics, *Ref* and *Lin* are restricted to ground terms. It is seen that derivations with cases stem from instances of rule *Lin*.

Term b in rule *Trans* is a *middle term*. An inspection of the rules shows that middle terms in *Trans* are the only terms in premisses that need not be also terms in a conclusion. Because of the permutability of logical rules below the mathematical rules observed above, we can consider derivations of sequents with only atomic formulas in antecedents and succedents.

The rules above give a complete system for the theory of linear lattices because reflexivity and linearity are derivable for arbitrary terms:

Lemma 1. *For arbitrary terms a and b, the sequents $\to a \leqslant a$ and $\to a \leqslant b, b \leqslant a$ are derivable in the rule system for linear lattices.*

Proof. By induction on the length of the terms a, b. For ground terms the sequents are conclusions of zero-premiss rules of the system, thus derivable. For a compound term a, for instance $a \equiv a_1 \wedge a_2$, reflexivity follows from the meet rules: $L\wedge_1$ and $L\wedge_2$ give $a_1 \wedge a_2 \leqslant a_1$ and $a_1 \wedge a_2 \leqslant a_2$, and from these, by $R\wedge$, we obtain $a_1 \wedge a_2 \leqslant a_1 \wedge a_2$. If a is a join, the proof uses instead the rules for join.

For linearity, we have to analyze the form of a and b. If a and b are not both ground terms, there are eight cases, reduced to five by symmetry. In all such cases, linearity is reduced to linearity on the components, which latter is derivable by the inductive hypothesis. For example, in the case of $a \equiv a_1 \wedge a_2$ and $b \equiv b_1 \vee b_2$, linearity on the terms a and b is derived by applying $R\wedge$ to the sequents $\to a_1 \wedge a_2 \leqslant b_1 \vee b_2, b_1 \vee b_2 \leqslant a_1$ and $\to a_1 \wedge a_2 \leqslant b_1 \vee b_2, b_1 \vee b_2 \leqslant a_2$. The former is derived by $L\vee$ from the conclusions of

$$\cfrac{\cfrac{\cfrac{}{a_1 \wedge a_2 \leqslant a_1}\,L\wedge_1 \quad \cfrac{}{a_1 \leqslant b_1, b_1 \leqslant a_1}\,Lin}{a_1 \wedge a_2 \leqslant b_1, b_1 \leqslant a_1}\,Trans \quad \cfrac{}{b_1 \leqslant b_1 \vee b_2}\,R\vee_1}{a_1 \wedge a_2 \leqslant b_1 \vee b_2, b_1 \leqslant a_1}\,Trans$$

and

$$\dfrac{\dfrac{}{a_1 \wedge a_2 \leqslant a_1}\,L\wedge_1 \quad \dfrac{}{a_1 \leqslant b_2, b_2 \leqslant a_1}\,Lin}{\dfrac{a_1 \wedge a_2 \leqslant b_2, b_2 \leqslant a_1}{a_1 \wedge a_2 \leqslant b_1 \vee b_2, b_2 \leqslant a_1}}\,Trans \qquad \dfrac{}{b_2 \leqslant b_1 \vee b_2}\,R\vee_2 \quad Trans$$

The latter is derived in a similar way. □

3 The subterm property

The definition of new terms given in [Negri and von Plato 2002] is here extended to the more general setting of derivations with cases:

Definition 1 *A* new term *in a derivation of a sequent* $\Gamma \to \Delta$ *is a term that is not a term or a subterm in* Γ, Δ.

Theorem 2. Subterm property. *If a sequent* $\Gamma \to \Delta$ *with only atoms in* Γ, Δ *is derivable in the theory of linear lattices, it has a derivation with no new terms.*

Before proving the theorem, we need preliminary notions for defining a suitable weight that indicates the presence of new terms and how deep down in a derivation tree they are. The theorem will be proved by giving transformations that reduce such weight, until it becomes zero, with the removal of all new terms from the derivation.

Terms in a derivation are ordered lexicographically. Given any two terms a and b, either a precedes b in the ordering, or b precedes a, or a and b are syntactically identical.

Branches of a derivation tree are sequences of sequents that start with its endsequent and go through one premiss up to conclusions of zero-premiss rules or initial sequents. The *length* of a branch is the number of steps in it. The *height* of a sequent in a derivation is the maximal length of subbranches up from that sequent. The height of a derivation is the height of its endsequent.

Given a derivation $\mathcal{D}$, consider all the occurrences of a new term which is maximal in the lexicographic ordering among the new terms of the derivation, and among such occurrences, consider those which are *downmost* in the derivation, that is, not followed below by other occurrences of the same term. Downmost occurrences of maximal new terms appear in steps of transitivity removing them from the derivation. Each branch of the derivation contains at most one such downmost maximal new term occurrence. Branches $\mathcal{B}_i$ in the derivation are assigned *weight* zero if they do not contain such a term, else have as weight $w(\mathcal{B}_i)$ the length of the subbranch up from the sequent with the last occurrence of the term. The weight of the whole derivation $\mathcal{D}$ is given by the multiset of the weights of its branches $\mathcal{B}_1, \dots, \mathcal{B}_n$

$$w(\mathcal{D}) \equiv \langle w(\mathcal{B}_1), \dots, w(\mathcal{B}_n) \rangle$$

The weight of the derivation is reduced if one or more branches are replaced by one or more branches of smaller weight.

This is a standard, well-founded ordering of finite multisets.

Proof of Theorem 2. We show how to transform derivations so that the weight of the derivation in the multiset ordering gets reduced.

Consider a step of transitivity removing a downmost maximal new term b:

$$\dfrac{\begin{matrix}\vdots \\ \Gamma_1 \to \Delta_1, a \leqslant b\end{matrix} \quad \begin{matrix}\vdots \\ \Gamma_2 \to \Delta_2, b \leqslant c\end{matrix}}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, a \leqslant c}\,\scriptstyle{Trans}$$

Consider the derivations $\mathcal{D}_1$, $\mathcal{D}_2$ of the premisses of *Trans*.

1. If the atoms $a \leqslant b$ and $b \leqslant c$ are not themselves principal in the last rules of $\mathcal{D}_1$ or $\mathcal{D}_2$, they are found in the premisses of that rule, and transitivity can be permuted above the rule. The cases are:

1.1. If the last rule of $\mathcal{D}_1$ is *Trans* with $a \leqslant b$ not principal in it and middle term e different from b, we have, with $\Gamma_1 \equiv \Gamma_{1_1}, \Gamma_{1_2}$ and $\Delta_1 \equiv \Delta_{1_1}, \Delta_{1_2}$

$$\dfrac{\dfrac{\Gamma_{1_1} \to \Delta_{1_1}, a \leqslant b, d \leqslant e \quad \Gamma_{1_2} \to \Delta_{1_2}, e \leqslant f}{\Gamma_1 \to \Delta_1, a \leqslant b, d \leqslant f}\,\scriptstyle{Trans} \qquad \Gamma_2 \to \Delta_2, b \leqslant c}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, a \leqslant c, d \leqslant f}\,\scriptstyle{Trans}$$

We permute as follows:

$$\dfrac{\dfrac{\Gamma_{1_1} \to \Delta_{1_1}, a \leqslant b, d \leqslant e \quad \Gamma_2 \to \Delta_2, b \leqslant c}{\Gamma_{1_1}, \Gamma_2 \to \Delta_{1_1}, \Delta_2, a \leqslant c, d \leqslant e}\,\scriptstyle{Trans} \qquad \Gamma_{1_2} \to \Delta_{1_2}, e \leqslant f}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, a \leqslant c, d \leqslant f}\,\scriptstyle{Trans}$$

Observe that the height of the left premiss of *Trans* removing b is shortened, so that one of the branches of positive weight of the derivation has its weight reduced by the transformation.

If $b \leqslant c$ is not principal in the second premiss of *Trans*, the permutation is analogous.

1.2. If the middle term e is identical to b we have a block of two consecutive transitivities with middle term b, and we postpone the case to point no. 3.

2. If the atom $a \leqslant b$ is principal in *Trans*, we have the following subcases:

2.1. If the last rule of $\mathcal{D}_1$ is *Trans* with $a \leqslant b$ principal, we have

$$\dfrac{\dfrac{\Gamma_{1_1} \to \Delta_{1_1}, a \leqslant d \quad \Gamma_{1_2} \to \Delta_{1_2}, d \leqslant b}{\Gamma_1 \to \Delta_1, a \leqslant b}\,\scriptstyle{Trans} \qquad \Gamma_2 \to \Delta_2, b \leqslant c}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, a \leqslant c}\,\scriptstyle{Trans}$$

Observe a subtlety here (which explains why we have chosen the lexicographic ordering on terms): since b is a downmost maximal new term occurrence, if d is a new term, it is smaller than or equal to b in the lexicographic ordering. In case d is strictly smaller or not a new term, the transformed derivation is

$$\dfrac{\Gamma_{1_1} \to \Delta_{1_1}, a \leqslant d \qquad \dfrac{\Gamma_{1_2} \to \Delta_{1_2}, d \leqslant b \quad \Gamma_2 \to \Delta_2, b \leqslant c}{\Gamma_{1_2}, \Gamma_2 \to \Delta_{1_2}, \Delta_2, d \leqslant c}\,{\scriptstyle Trans}}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, a \leqslant c}\,{\scriptstyle Trans}$$

and the weight is reduced. Else d is identical to b, thus the original derivation has the form

$$\dfrac{\dfrac{\Gamma_{1_1} \to \Delta_{1_1}, a \leqslant b \quad \Gamma_{1_2} \to \Delta_{1_2}, b \leqslant b}{\Gamma_1 \to \Delta_1, a \leqslant b}\,{\scriptstyle Trans} \qquad \Gamma_2 \to \Delta_2, b \leqslant c}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, a \leqslant c}\,{\scriptstyle Trans}$$

and the transformed derivation with reduced weight is

$$\dfrac{\dfrac{\Gamma_{1_1} \to \Delta_{1_1}, a \leqslant b \quad \Gamma_2 \to \Delta_2, b \leqslant c}{\Gamma_{1_1}, \Gamma_2 \to \Delta_{1_1}, \Delta_2, a \leqslant c}\,{\scriptstyle Trans}}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, a \leqslant c}\,{\scriptstyle W^*}$$

Observe that the original endsequent is restored by steps of left and right weakening, denoted by W^*. These do not increase weight because the contexts do not contain the maximal new term b.

A similar permutation is performed in case the right premiss of transitivity is derived by another transitivity.

2.2. If $a \leqslant b$ has been concluded by $L\vee$, the term a has a form $a \equiv d \vee e$ and the derivation

$$\dfrac{\dfrac{\Gamma_{1_1} \to \Delta_{1_1}, d \leqslant b \quad \Gamma_{1_2} \to \Delta_{1_2}, e \leqslant b}{\Gamma_1 \to \Delta_1, d \vee e \leqslant b}\,{\scriptstyle L\vee} \qquad \Gamma_2 \to \Delta_2, b \leqslant c}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, d \vee e \leqslant c}\,{\scriptstyle Trans}$$

is transformed as follows, with *Trans* permuted up to the two premisses of $L\vee$

$$\dfrac{\dfrac{\Gamma_{1_1} \to \Delta_{1_1}, d \leqslant b \quad \Gamma_2 \to \Delta_2, b \leqslant c}{\Gamma_{1_1}, \Gamma_2 \to \Delta_{1_1}, \Delta_2, d \leqslant c}\,{\scriptstyle Trans} \qquad \dfrac{\Gamma_{1_2} \to \Delta_{1_2}, e \leqslant b \quad \Gamma_2 \to \Delta_2, b \leqslant c}{\Gamma_{1_2}, \Gamma_2 \to \Delta_{1_2}, \Delta_2, e \leqslant c}\,{\scriptstyle Trans}}{\Gamma_1, \Gamma_2, \Gamma_2 \to \Delta_1, \Delta_2, \Delta_2, d \vee e \leqslant c}\,{\scriptstyle L\vee}$$

The number of branches with term b is increased, but each has a lesser weight so the multiset ordering is reduced.

A similar permutation is performed if the right premiss of *Trans* is derived by $R\wedge$. Observe that the permutation of *Trans* over $L\vee$ and $R\wedge$ produces duplications in the contexts.

2.3. The premisses of *Trans* are derived by $R\vee_1$ and $L\vee$ and $b \equiv b_1 \vee b_2$. The derivation

$$\dfrac{\dfrac{}{\rightarrow b_1 \leqslant b_1 \vee b_2}\,R\vee_1 \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \rightarrow \Delta_1, b_1 \leqslant c\end{array} \quad \begin{array}{c}\vdots\\ \Gamma_2 \rightarrow \Delta_2, b_2 \leqslant c\end{array}}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, b_1 \vee b_2 \leqslant c}\,L\vee}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, b_1 \leqslant c}\,Trans$$

is transformed into

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \rightarrow \Delta_1, b_1 \leqslant c\end{array}}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, b_1 \leqslant c}\,W^*$$

with the maximal new term occurrence $b_1 \vee b_2$ removed and thus the weight of the derivation reduced.

3. The transformations given can lead to the repetition of case 1.2, with blocks of consecutive instances of *Trans* with the middle term b. We permute such blocks by analyzing the premisses of the rules applied above the topmost transitivities of each block.

Eventually we reach a point in which we have a derivation starting with initial sequents, or zero-premiss rules, or instances of *Trans* with middle term different from b, immediately followed by a block of transitivities with middle term b.

In the latter case, the instance of *Trans* with middle term different from b is permuted below the block of transitivities with middle term b, step by step, as in 1.1.

In the other cases, the term containing b must be principal in the initial sequents or zero-premiss rules, else we can shorten branches with term b because the conclusion of the first rule below them would be again an initial sequent or follow by a zero-premiss rule. We can similarly rule out the possibility of one initial sequent above *Trans* being derived by *Ref*: the conclusion of *Trans* would be identical to the other premiss.

If one premiss of the block is an initial sequent, then b is a term in the antecedent of the conclusion, contrary to the assumption of b being a new term. (Observe that no atom is ever removed from antecedents of sequents.)

If one premiss of the block is the conclusion of a zero-premiss lattice rule, then we have the atoms $b \wedge c \leqslant b$ or $b \leqslant b \vee f$. Now $b \wedge c$, $b \vee f$ would be new terms longer than b, contrary to assumption.

We are thus left with the possibility that b is a ground term q in an atom $p \leqslant q$ in a linearity $\rightarrow p \leqslant q, q \leqslant p$. There is then a second occurrence of the term q, in the atom $q \leqslant p$, that has to disappear from the derivation. Since q is a maximal new term, $q \leqslant p$ is active in an instance of transitivity, not in a lattice rule, or else q would be a subterm of another longer new term. There is therefore an instance of *Trans* with an atom $r \leqslant q$ in the first premiss and the atom $q \leqslant p$ in the second. Now analyze the subderivation down to the first premiss with the atom $r \leqslant q$ in the same way as the original derivation with the atom $p \leqslant q$. The analysis leads to another instance of *Lin*, of the form $\rightarrow s \leqslant q, q \leqslant s$, but this would lead to an infinite derivation. □

The system **G0c** extended with rules for mathematical theories is useful for the analysis of proofs through the permutation of rules. However, the system includes explicit rules of weakening and contraction that make it less suitable for proof search: *a priori*, in a proof search starting from the endsequent to be derived, there is no bound on the number of times formulas can get duplicated through the (roof-first) application of the rule of contraction. There is another system of sequent calculus, called **G3c**, that does not need explicit structural rules. This remains so also when the calculus is extended with mathematical rules [cf. Chapter 6 of Negri and von Plato 2001].

Another problem with the calculus **G0c**, in a root-first proof search, is that one does not know how to divide the contexts in the conclusion between the premisses, thus, proof search is not deterministic even if the conclusion is given. The calculus **G3c** does not have this problem, because its characteristic feature is that rules with more than one premiss have *shared*, i.e., identical contexts. A specific property of extension of the calculus **G3c** is that instances of rules that lead, read root first, into a duplication of a formula in a premiss, can be excluded: Whenever there is such a duplication, the conclusion is equal to the premiss with the duplication contracted. The property of *height-preserving admissibility* of the rule of contraction states that whenever a sequent with a duplication is derivable, the contracted sequent is derivable *with the same height of derivation*. It follows that there is a bound on the length of derivations with the rules of a theory. A drawback of **G3c** is that rules do not usually permute the way they do in **G0c**.

We show that the subterm property of the calculus for linear lattices of theorem 2 carries over to a calculus based on the sequent calculus **G3c**, through a translation of derivations in **G0c** to **G3c** and back. One-premiss rules need no translation. With the two-premiss rules of lattice theory, say *Trans* with $\Gamma_1 \rightarrow \Delta_1, a \leqslant b$ and $\Gamma_2 \rightarrow \Delta_2, b \leqslant c$ as premisses, apply weakening repeatedly to both premisses to get premisses with shared contexts: $\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, a \leqslant b$. The conclusion is $\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, b \leqslant c$. Proceeding in this way, a derivation in the calculus **G3c** with shared contexts is obtained. It is immediate that the terms in both derivations are the same. Thus, if there is a derivation in **G0c**, there is one with the subterm property, consequently also one in **G3c** with the subterm property. Proof search can be done with the calculus **G3c** with terms taken from the endsequent to be derived. Rules that lead to a duplication of an atom in a premiss are excluded. With a bounded number of terms, there is only a bounded number of distinct atoms, so that proof search with the subterm property in

G3c is bounded. Once a derivation has been found, it can be translated back to **G0c** in a routine fashion. We thus conclude with the following theorem:

Theorem 3. *Derivability of given atomic cases Δ from atomic assumptions Γ in the theory of linear lattices is decidable.*

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