Cut elimination in provability logic (extended abstract)

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Following the method developed in Negri and von Plato (1998) and in Negri (2003), we present a uniform Gentzen-style approach to the proof theory of a large family of normal modal logics. The method covers all the modal logics characterized by geometric conditions on their Kripke models. Each modal system is obtained by adding in a modular way the rules for the accessibility relation to a basic modal system. The resulting (labelled) sequent calculi have all the structural rules–weakening, contraction, and cut–admissible.

A natural challenge is to extend the method to treat also Gödel-Löb provability logic. After Solovay's landmark paper (1976), that characterized axiomatically the modal logic of arithmetical provability G (later called GL), a great effort was directed to producing an adequate sequent calculus and proving cut elimination for it. Semantic completeness proofs for Gentzen's formulations for GL were provided (Sambin and Valentini 1982, Avron 1984) but syntactic proofs of cut elimination (Leivant 1981, Valentini 1983) turned out to be problematic (Moen 2001).

Gödel-Löb provability logic is characterized by irreflexive, transitive, and Noetherian Kripke frames. The non-first-order frame condition of Noetherianity cannot be encoded in the geometric rule scheme, but it becomes part of the characterization of forcing for modal formulas

$$x \Vdash \Box A \textit{ iff for all } y,\ xRy \textit{ and } y \Vdash \Box A \textit{ implies } y \Vdash A$$

This meaning explanation justifies a left and right rule for $\Box$. The resulting sequent calculus derives the Löb axiom, has all the rules invertible, the necessitation, weakening, and contraction rules admissible. Cut elimination is proved by induction on a triple parameter, given by the size of the cut formula, the range of the label of the cut formula (i.e., the set of worlds accessible from it in the derivation) and the sum of the heights of the premisses of cut.

A full proof is presented in Negri (2005).

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