The continuum as a formal space

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Abstract

A constructive definition of the continuum based on formal topology is given and its basic properties studied. A natural notion of Cauchy sequence is introduced and Cauchy completeness is proved. Other results include elementary proofs of the Baire and Cantor theorems. From a classical standpoint, formal reals are seen to be equivalent to the usual reals. Lastly, the relation of real numbers as a formal space to other approaches to constructive real numbers is determined.

1 Introduction

The basic idea of pointfree topology, namely, to reverse the traditional conceptual order between points and opens by considering the opens, or approximations, as primitive, and building the points as particular well behaved collections of opens, is here worked out to obtain a constructive definition of the continuum.

The foundational framework used in this work is that of formal topology (cf. [S] for a first introduction). It has the advantage of being formalizable within Martin-Löf's constructive type theory (cf. [ML2]), and the latter in turn can be used directly as a high-level programming language (cf. [ML1], [NPS]). The definition of real numbers in formal topology allows inductive definitions and proofs which are ready to be made algorithmic and machine-checked.

We give an account of all the basic notions of formal topology needed here and introduce the formal topology of intervals with rational endpoints. The formal points of this topology lead, through a natural construction, to our notion of continuum. Arithmetical operations are defined and an order relation giving rise to an apartness relation is studied in detail.

From a constructive point of view formal reals are not order complete: Assuming that every bounded collection of formal reals has a least upper bound would imply the validity of the principle of *testability*, that is $\neg P \vee \neg\neg P$ for every proposition P, which is not intuitionistically acceptable (cf. [BW]). To obtain order completeness we are thus led, as an alternative to using a non-constructive logic, to widen the collection of formal reals, following a procedure typical of intuitionistic mathematics (cf. [T]).

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As a further confirmation of the correctness of our construction, we show how to obtain, by assuming classical logic, extensionality of the topology of formal reals. This means, roughly, that the new pointfree space and the point-set one are classically equivalent.

In order to develop constructive analysis, a natural notion of Cauchy sequence is introduced and Cauchy completeness is proved. Moreover, very elementary and constructive proofs of some classical theorems of analysis, like the Baire theorem and the Cantor theorem are obtained.

Finally, our notion of reals is shown to correspond to Bishop reals and to Dedekind cuts in their usual constructive treatment, and is compared with Martin-Löf's *maximal approximations*.

We emphasize that from the classical point of view, formal reals are equivalent to the usual reals, so that nothing is lost. What is gained is the advantage of using pointfree methods that permit to constructivize classical non-constructive results (cf. [N], [NV], [CN], [CCN] for other examples), and the possibility to pursue the project of machine-implemented formalization of analysis (cf. [Ce], [C]).

2 Preliminaries

We recall here the basic definitions concerning our approach to pointfree topology; further information can be found in [CN], [S], [SVV]. The reader already familiar with formal topology can skip this section.

Formal topology was introduced by Per Martin-Löf and Giovanni Sambin (cf. [S]) as a constructive approach to topology, in the tradition of Johnstone's version of *Grothendieck topologies* [J] and Fourman and Grayson's *formal spaces* [FG], but using simpler technical devices and a constructive set theory based on Martin Löf's type theory.

A point-set topology can always be presented using one of its bases. The abstract structure that we consider is a commutative monoid $\langle S, \cdot_{\mathcal{S}}, 1_{\mathcal{S}}\rangle$ where the set S corresponds to the set of the elements of the base of the point-set topology $\Omega(X)$, $\cdot_{\mathcal{S}}$ corresponds to the operation of intersection between basic elements, and $1_{\mathcal{S}}$ corresponds to the whole collection X.

In a point-set topology any open set is obtained as a union of elements of the base, but without points we don't have such a union; hence we are naturally led to think that an open set may directly correspond to a subset of the set S. Since there may be many different subsets of basic elements whose union is the same open set, we introduce an equivalence relation $\cong_{\mathcal{S}}$ between two subsets U and V of S with the following intended meaning: denoting by c^* the element of the base which corresponds to the formal basic open c, $U \cong_{\mathcal{S}} V$ holds if and only if the opens $U^* \equiv \cup_{a\in U} a^*$ and $V^* \equiv \cup_{b\in V} b^*$ are equal. To this purpose we introduce a relation $\lhd_{\mathcal{S}}$, called *cover*, between a basic element a of S and a subset U of S whose intended meaning is that $a \lhd_{\mathcal{S}} U$ when $a^* \subseteq U^*$ and therefore the equivalence $U \cong_{\mathcal{S}} V$ will amount to $(\forall u \in U)\ u \lhd V\ \&\ (\forall v \in V)\ v \lhd U$. The conditions we require of this relation are a straightforward rephrasing of the similar set-theoretic situation.

We introduce a predicate Pos on elements of S, to express positively (that is without using negation) that a basic open is not empty.

The notion of subset can be formalized within constructive type theory (cf. [NPS]): a subset U of a set S is a function that takes any x in S into a proposition $U(x)$. We will use the informal notation $a \in U$ for $U(a)$ and the following abbreviations for subset inclusion and extensional equality:

$$U \subseteq_S V \equiv (\forall x \in S)(U(x) \to V(x));$$

$$U =_S V \equiv U \subseteq_S V \ \&\ V \subseteq_S U.$$

In the sequel we omit subscripts when clear from the context.

Definition 2.1 (Formal topology) *Let S be a set. A* formal topology *over S is a structure*

$$\mathcal{S} \equiv \langle S, \cdot, 1, \lhd, Pos \rangle$$

where $\langle S, \cdot, 1 \rangle$ is a commutative monoid with unit, $\lhd$ is a relation, called cover, *between elements and subsets of S such that, for any $a, b \in S$ and for any $U, V \subseteq S$ the following conditions hold*

$$\textit{reflexivity} \qquad \frac{a \in U}{a \lhd U}$$

$$\textit{transitivity} \qquad \frac{a \lhd U \qquad (\forall u \in U)\ u \lhd V}{a \lhd V}$$

$$\textit{dot-left} \qquad \frac{a \lhd U}{a \cdot b \lhd U}$$

$$\textit{dot-right} \qquad \frac{a \lhd U \qquad a \lhd V}{a \lhd \{u \cdot v \mid u \in U, v \in V\}}$$

and Pos is a predicate on S, called positivity predicate*, satisfying:*

$$\textit{monotonicity} \qquad \frac{Pos(a) \qquad a \lhd U}{(\exists b \in U)\ Pos(b)}$$

$$\textit{positivity} \qquad a \lhd a^+$$

where $a^+ \equiv \{b \in S \mid a = b \ \&\ Pos(b)\}$

All the conditions are straightforward rephrasings of the preceding intuitive considerations, except positivity. The first reason to introduce positivity is that any non-positive basic open is covered by anything. Indeed, when the predicate Pos is decidable, positivity is equivalent to the rule

$$\frac{\neg Pos(a)}{a \lhd \emptyset} \ .$$

Positivity also allows proof by cases on $Pos(a)$ for deductions involving covers (for a detailed discussion cf. [SVV]).

In a formal topology the following rule is derivable from *dot-left* and *dot-right*:

$$\textit{localization} \qquad \frac{a \lhd U}{a \cdot b \lhd U \cdot b} \ .$$

If the base is a semilattice, *localization* becomes equivalent to *dot-right*.

Given a formal topology $\mathcal{A}$, we denote with $Sat(\mathcal{A})$ the collection of *saturated subsets* of $\mathcal{A}$, that is, of the subsets U of S such that $\mathcal{A}U = U$, where

$$\mathcal{A}U \equiv \{a \in S \mid a \lhd U\} \ .$$

With the following operations

$\mathcal{A}U \wedge \mathcal{A}V \equiv \mathcal{A}U \cap \mathcal{A}V = \mathcal{A}(U \cdot V)$,

$\bigvee_{i \in I} \mathcal{A}U_i \equiv \mathcal{A}(\bigcup_{i \in I} U_i)$,

$Sat(\mathcal{A})$ is a *frame* (cf. [S1]).

In order to connect our pointfree approach to classical point-set topology, the notion of point has to be recovered. Since we reverse the usual conceptual order between points and opens, and take the opens as primitive, points are defined as particular, well behaved, collections of opens. We recall here the definition of formal points of a formal topology:

Definition 2.2 *Let $\mathcal{A}$ be a formal topology. A subset α of S is said to be a* formal point *if for all $a, b \in S$, $U \subseteq S$ the following conditions hold:*

1. $1 \in \alpha$;
2. $\dfrac{a \in \alpha \quad b \in \alpha}{a \cdot b \in \alpha}$;
3. $\dfrac{a \in \alpha \quad a \lhd U}{(\exists b \in U)(b \in \alpha)}$.

We observe that formal points satisfy the rule

$$\frac{a \in \alpha}{Pos(a)}$$

as follows from condition 3 and *positivity.*

In order to maintain the usual intuition on points, in the sequel we often write $\alpha \Vdash a$ (*α forces a*, or *α is a point in a*) in place of $a \in \alpha$. Moreover we interpret 1 as the whole space, product as intersection, $a \lhd U$ as "a is included in the union of U", and $Pos(a)$ as "a is inhabited", that is, a positive way of saying that a is not empty.

For any formal topology $\mathcal{A}$, the formal space $Pt(\mathcal{A})$ of formal points on $\mathcal{A}$ can be endowed with a topology, called the *extensional topology.* Define, for $a \in S$, $ext(a)$ to be the collection of formal points forcing a. The family $\{ext(a)\}_{a \in S}$ is a base for a topology on $Pt(\mathcal{A})$: By the definition of formal points we have $ext(1) = Pt(\mathcal{A})$, thus the whole space is a basic open, and $ext(a) \cap ext(b) = ext(a \cdot b)$, thus the family is closed under intersection. If we denote $\cup_{b \in U} ext(b)$ with $ext(U)$, then the generic open is of the form $ext(U)$ for $U \subseteq S$.

Let $\Omega Pt(\mathcal{A})$ be the topology so obtained. Then the map

$$\begin{array}{rccc} \phi : & Sat(\mathcal{A}) & \to & \Omega Pt(\mathcal{A}) \\ & U & \longmapsto & ext(U) \end{array}$$

is clearly a surjective frame homomorphism, therefore it is a frame isomorphism iff it is injective. Injectivity of ϕ amounts to

if for all α in $Pt(\mathcal{A})$, $\alpha \Vdash a$ implies $\alpha \Vdash U$ then $a \in U$

for arbitrary $a \in S$ and $U \subseteq S$. When the above condition holds, we say that the formal topology is *extensional* or *has enough points.* This condition is classically equivalent to the following one:

if $a \not\lhd U$ then there exists α in $Pt(\mathcal{A})$ such that $\alpha \Vdash a$ and $\alpha \nVdash U$.

Intuitively this says that formal points suffice for "separating" the different opens.

By the representation theorem for frames by means of formal topologies (cf. [S]), extensional formal topologies are "the same" as spatial frames (or locales). We recall a well known result that connects localic and point-set topology (cf. [J]):

Theorem 2.3 *The category of spatial frames is equivalent to the category of sober topological spaces.*

Thus extensionality for a formal topology is a measure of how much the formal space resembles the corresponding classical topological space. Except for very few cases like Scott formal topologies (cf. [SVV] for details), a proof of extensionality requires some non-constructive assumptions. For instance, the prime filter theorem is used in the case of Stone formal topologies (cf. [N]). We give in Section 7 a proof of extensionality for the topology of formal reals that uses classical logic.

3 Basic definitions

Real numbers can be obtained in a pointfree setting starting from intervals with rational endpoints (cf. [J]). Here we show how to obtain real numbers as formal points of a formal topology. Some ideas in this direction have been already hinted at in [S]. The definition we will use here is a slight variant of the definition suggested by Th. Coquand (cf. [CN], [CCN]).

We will assume here the classic construction of the set Q of rationals starting from the integers, since it is unproblematic from the constructive point of view. Moreover, the usual order of rationals is decidable, and this will allow us to prove some results by cases.

Given the set Q of rationals, let Q^+ denote the totally ordered set obtained by adjoining a top element $+\infty$ to Q and let similarly Q^- denote $Q \cup \{-\infty\}$. The set $S = Q^- \times Q^+$ is then partially ordered by the relation of interval inclusion:

$$(p, q) \leq (r, s) \equiv r \leq p \ \& \ q \leq s.$$

The meet operation, intersection of intervals, is defined by

$$(p, q) \cdot (r, s) \equiv (max(p, r), min(q, s))$$

and it makes S a semilattice with top element $(-\infty, +\infty)$.

Definition 3.1 *The* formal topology of formal reals *is the structure*

$$\mathcal{R} \equiv \langle Q^- \times Q^+, \cdot, (-\infty, +\infty), \lhd, Pos\rangle \ ,$$

where the cover $\lhd$ *is defined by*

$$(p, q) \lhd U \equiv (\forall p', q')(p < p' < q' < q \rightarrow (p', q') \lhd_f U) \ ,$$

with the relation $\lhd_f$ *inductively defined by*

1. $\dfrac{q \leq p}{(p, q) \lhd_f U}$;

2. $\dfrac{(p, q) \in U}{(p, q) \lhd_f U}$;

3. $\dfrac{(p, s) \lhd_f U \quad (r, q) \lhd_f U \quad p \leq r < s \leq q}{(p, q) \lhd_f U}$;

4. $\dfrac{(p', q') \lhd_f U \quad p' \leq p < q \leq q'}{(p, q) \lhd_f U}$.

The positivity predicate is defined by

$$Pos(p, q) \equiv p < q \ .$$

We just recall here that it has been proved in [CN] that both $\lhd$ and $\lhd_f$ are covers, the latter being the *Stone compactification* of the former (cf. [N]). This means that the following holds:

Proposition 3.2 *Let $(p,q) \in S$ and $U \subseteq S$. Then*

1. *If $(p,q) \lhd_f U$, then there exists a finite subset U_0 of U such that $(p,q) \lhd_f U_0$.*
2. *If $(p,q) \lhd_f U$, then $(p,q) \lhd U$.*
3. *If $(p,q) \lhd U$ and U is finite, then $(p,q) \lhd_f U$.*

The general definition of formal reals is obtained as an instantiation of Definition 2.2 to the topology of formal reals. It is equivalent to the following more explicit one, where only the order between rationals is involved:

Definition 3.3 *A* formal real *is a subset α of S such that:*

1. $\alpha \Vdash (-\infty, +\infty)$;
2. $\alpha \Vdash (p,q)$ & $\alpha \Vdash (r,s) \quad \Leftrightarrow \quad \alpha \Vdash (p,q) \cdot (r,s)$;
3. $\dfrac{\alpha \Vdash (p,q) \qquad p \leq r < s \leq q}{\alpha \Vdash (p,s) \vee \alpha \Vdash (r,q)}$;
4. $\dfrac{\alpha \Vdash (p,q)}{(\exists (p',q'))(p < p' < q' < q \ \& \ \alpha \Vdash (p',q'))}$.

In the sequel we will denote with $Pt(\mathcal{R})$ the collection of formal points on $\mathcal{R}$. We observe that right to left implication in condition 2 is equivalent to monotonicity of points with respect to the relation $\leq$, i.e., to the rule:

$$\frac{\alpha \Vdash (p,q) \quad (p,q) \leq (t,v)}{\alpha \Vdash (t,v)} \ .$$

Furthermore, condition 3 can be replaced by the equivalent

$$\frac{\alpha \Vdash (p,q) \qquad r < s}{\alpha \Vdash (p,s) \vee \alpha \Vdash (r,q)} \ .$$

In Section 6 we will see that in $Pt(\mathcal{R})$ it is not possible to define constructively a least upper bound and greatest lower bound of a bounded subset. A possible way out is to extend $Pt(\mathcal{R})$ in the following way:

Definition 3.4 *A* weak formal real *is a subset α of S satisfying 1, 2, 4 of Definition 3.3, and the following weak form of 3:*

3′. $\dfrac{\alpha \Vdash (p,q) \qquad p \leq r < s \leq q}{\neg\neg(\alpha \Vdash (p,s) \vee \alpha \Vdash (r,q))}$.

We denote with $Pt(\mathcal{R})^e$ the collection of weak formal reals.

The following results, stating that formal reals can be approximated by rationals up to an arbitrary accuracy, will be useful.

Lemma 3.5 *Let α be a formal real. Then, for any positive rational k, there exists $(p,q) \in \alpha$ such that $q - p < k$.*

Proof: By definition of formal reals, there exists $(r,s) < (-\infty, +\infty)$ such that $\alpha \Vdash (r,s)$. By trisecting the interval (r,s), we have $\alpha \Vdash (r, \frac{r+2s}{3}) \vee \alpha \Vdash (\frac{2r+s}{3}, s)$. Since there exists n such that $(\frac{2}{3})^n \cdot (s-r) < k$, by repeatedly applying this procedure we find the required (p,q). $\square$

4 Arithmetic in $Pt(\mathcal{R})$

We recall that a *continuous function* f between the spaces $(Pt(\mathcal{A}), \mathcal{A})$ and $(Pt(\mathcal{B}), \mathcal{B})$[1] is given by a pair (f^{pt}, f^{Ω}), where f^{pt} maps points of $\mathcal{A}$ into points of $\mathcal{B}$ and f^{Ω} is a *morphism of formal topologies* (defined as in [S]) from $\mathcal{B}$ to $\mathcal{A}$. The two maps are in an adjoint relation, i.e., for all α in $Pt(\mathcal{A})$ and b in the base of $\mathcal{B}$, we have

$$\alpha \Vdash f^{\Omega}(b) \Leftrightarrow f^{pt}(\alpha) \Vdash b.$$

The same holds for continuous maps $h : Pt(\mathcal{R}) \times Pt(\mathcal{R}) \to Pt(\mathcal{R})$ that are in adjoint relation with morphisms of formal topologies from $\mathcal{R}$ to the coproduct $\mathcal{R}+\mathcal{R}$, defined as in [NV].

The arithmetic operations on formal reals can thus be defined as follows (see also [Si]):

- *Successor:* The successor of a point is obtained by shifting its neighbourhoods. So we define

$$\begin{aligned} \alpha + 1 \ \equiv \ f^{pt}(\alpha) \ &\equiv \ \{(r+1, s+1) \mid \alpha \Vdash (r, s)\} \\ f^{\Omega}((p, q)) \ &\equiv \ \{(r, s) \mid r+1 = p, s+1 = q\}. \end{aligned}$$

- *Additive inverse:*

$$\begin{aligned} -\alpha \ \equiv \ f^{pt}(\alpha) \ &\equiv \ \{(-q, -p) \mid (p, q) \in \alpha\} \\ f^{\Omega}((p, q)) \ &\equiv \ \{(-q, -p)\} \ . \end{aligned}$$

- *Sum:* After defining the pointwise sum on S,

$$(p, q) + (r, s) \equiv (p + r, q + s),$$

  we put

$$\begin{aligned} \alpha + \beta \ \equiv \ f^{pt}(\alpha, \beta) \ &\equiv \ \{(p, q) + (r, s) \mid (p, q) \in \alpha, (r, s) \in \beta\} \\ f^{\Omega}((u, v)) \ &\equiv \ \{((p, q), (r, s)) \in S \times S \mid (u, v) = (p, q) + (r, s)\}. \end{aligned}$$

- *Product:* We define the binary operation between basic neighbourhoods,

$$(p, q) * (r, s) \equiv (min(pr, ps, qr, qs), max(pr, ps, qr, qs)) \ .$$

  Then

$$\begin{aligned} \alpha \cdot \beta \ \equiv \ f^{pt}(\alpha, \beta) \ &\equiv \ \{(p, q) * (r, s) \mid (p, q) \in \alpha, (r, s) \in \beta\} \\ f^{\Omega}((u, v)) \ &\equiv \ \{((p, q), (r, s)) \mid u = min(pr, ps, qr, qs), v = max(pr, ps, qr, qs)\} \ . \end{aligned}$$

- *Inverse:* It is defined only for $\alpha < 0 \vee \alpha > 0$:

$$\alpha^{-1} \ \equiv \ f^{pt}(\alpha) \equiv \{(r, s) \mid (\exists (p, q))(r \leq \frac{1}{q} < \frac{1}{p} \leq s \ \& \ \alpha \models (p, q) \ \& \ (p > 0 \vee q < 0))\}$$

- *Difference:*

$$\alpha - \beta \equiv \alpha + (-\beta)$$

[1] Cf. [V] for the analogous notion in the setting of locale theory.

5 Order in $Pt(\mathcal{R})$

We define strict order between formal reals in the following way:

$$\alpha < \beta \equiv (\exists(p,q) \in \alpha)(\exists(r,s) \in \beta)(q < r) \ .$$

Moreover we define:

$$\alpha \leq \beta \equiv \neg\ \beta < \alpha \ .$$

Properties of strict and weak order follow from the following two lemmas (cf. [vP]):

Lemma 5.1 *Let $\alpha,\ \beta \in Pt(\mathcal{R})$. Then*

$$\neg(\alpha < \beta \ \&\ \beta < \alpha)$$

Proof: By definition of order, if $\alpha < \beta$ there exist (p_1, q_1) in α and (r_1, s_1) in β such that $q_1 < r_1$. Moreover, if $\beta < \alpha$ there exist (p_2, q_2) in α and (r_2, s_2) in β such that $s_2 < p_2$. Then we have $\neg Pos((p_1, q_1) \cdot (p_2, q_2))$, hence a contradiction. □

Lemma 5.2 *Let $\alpha, \beta, \gamma \in Pt(\mathcal{R})$. Then $\alpha < \beta$ implies $\alpha < \gamma\ \vee\ \gamma < \beta$.*

Proof: By hypothesis there exist (p,q) in α and (r,s) in β such that $q < r$. Thus, by definition of formal real, there exist (p', q') in α and (r', s') in β with $(p', q') < (p,q)$ and $(r', s') < (r,s)$. Consider now any (u,v) in γ. Since $q < r$, we have $\gamma \models (u,r) \vee \gamma \models (q,v)$. In the first case, $\gamma < \beta$ follows since $r < r'$; otherwise, we have $\alpha < \gamma$ since $q' < q$. □

By Lemmas 5.1 and 5.2 we have:

Proposition 5.3 *The relation $<$ is a* linear order *in $Pt(\mathcal{R})$, i.e., satisfies, for all $\alpha, \beta, \gamma \in Pt(\mathcal{R})$,*

1. *$\alpha < \beta \to \neg(\beta < \alpha)$ (asymmetry) ;*
2. *$\alpha < \beta \ \&\ \beta < \gamma \to \alpha < \gamma$ (transitivity) .*

We observe that Lemma 5.2 does not hold for weak formal reals, i.e., weak formal reals are not located. Anyway, for $Pt(\mathcal{R})^e$, Proposition 5.3 holds as well and it is proved directly form the definitions.

The order on formal reals induces on the rationals their usual order, since we have:

Proposition 5.4 *The map $r \mapsto \bar{r} \equiv \{(p,q) \in S \mid p < r < q\}$ is an injective order homomorphism of Q into $Pt(\mathcal{R})$.*

Proof: The claim amounts to proving that $\bar{r} < \bar{s}$ iff $r < s$: If $\bar{r} < \bar{s}$, by definition of order in $Pt(\mathcal{R})$, there exists (p,q) in $\bar{r}$ and (t,v) in $\bar{s}$ such that $q < t$ and thus $p < r < q < t < s < v$, hence $r < s$; conversely, if $r < s$ then $(-\infty, r + \frac{s-r}{3})$ is in $\bar{r}$ and $(s - \frac{s-r}{3}, +\infty)$ is in $\bar{s}$. Since $r + \frac{s-r}{3} < s - \frac{s-r}{3}$, we have $\bar{r} < \bar{s}$. □

The following lemma will be useful:

Lemma 5.5 *Let $p \in Q$ and $\alpha \in Pt(\mathcal{R})^e$. Then*

$$\bar{p} < \alpha \Leftrightarrow \alpha \Vdash (p, +\infty) \ .$$

Proof: Since $\bar{p} < \alpha$, there exist (q,r) in $\bar{p}$ and (s,t) in α such that $r < s$. We have $p < r$ and therefore $(s,t) \lhd (p, +\infty)$. Thus, by monotonicity, $\alpha \Vdash (p, +\infty)$. Conversely, if $\alpha \Vdash (p, +\infty)$, there exists (q,r) in α with $p < q < r < +\infty$ and thus there exists (s,t) in $\bar{p}$ (taking for instance $s = p - 1$ and $t = \frac{p+q}{2}$) with $t < q$, that is, $\bar{p} < \alpha$. □

Symmetrically, we have:

Lemma 5.6 $\alpha < \bar{p} \Leftrightarrow \alpha \Vdash (-\infty, p)$.

It now follows from the definitions that rationals are dense in formal reals:

Lemma 5.7 *Let $\alpha, \beta \in Pt(\mathcal{R})$ with $\alpha < \beta$. Then there exists a rational number t such that $\alpha < \bar{t} < \beta$.*

We will need the following lemma when comparing formal reals with Martin-Löf's maximal approximations (cf. [ML]) in Section 9.3.

Lemma 5.8 *Let $(p, q), (r, s)$ be two basic neighbourhoods with $r < p < q < s$ and let $\alpha \in Pt(\mathcal{R})$. Then*

$$\alpha \Vdash (r, s) \;\vee\; (\exists (t, u))(\alpha \Vdash (t, u) \;\&\; \neg Pos((p, q) \cdot (t, u))).$$

Proof: For any (t, u) such that $\alpha \Vdash (t, u)$, since $r < p < q < s$, we have $(t, u) \lhd \{(t, p), (r, u)\}$ and $(r, u) \lhd \{(r, s), (q, u)\}$. Then by transitivity $(t, u) \lhd \{(t, p), (r, s), (q, u)\}$ and therefore, by monotonicity, $\alpha \Vdash (t, p)$ or $\alpha \Vdash (q, u)$ (and in both cases we have found an element of α disjoint with (p, q)), or $\alpha \Vdash (r, s)$. $\Box$

Then we have:

Proposition 5.9 *Let $\alpha, \beta \in Pt(\mathcal{R})$. Then $\alpha \leq \beta \;\&\; \beta \leq \alpha$ holds if and only if $\alpha =_S \beta$.*

Proof: Let $(p, q) \in \alpha$. Then $(\exists (p', q') \in \alpha)(p < p' < q' < q)$. By the above lemma, we have $(p, q) \in \beta$ or there exists $(r, s) \in \beta$ such that $\neg Pos((p', q') \cdot (r, s))$. If the latter holds, we have $s < p'$ or $q' < r$ and therefore $\alpha < \beta$ or $\beta < \alpha$, against the hypothesis. So $(p, q) \in \beta$. In the same way it is proved that $\beta \subseteq_S \alpha$. Conversely, since a point cannot contain two disjoint neighbourhoods, both $\beta < \alpha$ and $\alpha < \beta$ are impossible and therefore $\alpha \leq \beta \;\&\; \beta \leq \alpha$ holds. $\Box$

Proposition 5.10 *The relation $\leq$ is a partial order between formal reals, i.e., satisfies, for all $\alpha, \beta, \gamma \in Pt(\mathcal{R})$,*

1. $\alpha \leq \alpha$;
2. $\alpha \leq \beta \;\&\; \beta \leq \gamma \rightarrow \alpha \leq \gamma$;
3. $\alpha \leq \beta \;\&\; \beta \leq \alpha \rightarrow \alpha =_S \beta$.

Proof: Use Lemma 5.1, Lemma 5.2 and Proposition 5.9. $\Box$

It will follow from Proposition 9.5 below that the partial order $\leq$ on $Pt(\mathcal{R})$ is not decidable.

5.1 Apartness

We now introduce an *apartness relation* between formal reals, i.e., a relation # satisfying, for all α, β, $\gamma \in Pt(\mathcal{R})$

1. $\neg(\alpha \# \beta) \Leftrightarrow (\alpha =_S \beta)$;
2. $\alpha \# \beta \Leftrightarrow \beta \# \alpha$;
3. $\alpha \# \beta \Rightarrow \alpha \# \gamma \vee \gamma \# \beta$.

Definition 5.11 (apartness) *Let $\alpha, \beta \in Pt(\mathcal{R})$. Then α is* apart from *β, and we write $\alpha \# \beta$, if*

$$\alpha < \beta \vee \beta < \alpha \; .$$

Observe that, by definition of order, we have $\alpha \# \beta$ iff there exist (p, q) in α and (r, s) in β such that $(q < r) \vee (s < p)$. Thus the above definition conveys the intuitive meaning of two points being apart if there are two disjoint (basic) opens containing them.

We have, by Proposition 5.9 and Lemma 5.2:

Proposition 5.12 *The relation $\#$ is an apartness relation in $Pt(\mathcal{R})$.*

Corollary 5.13 *Let $\alpha, \beta \in Pt(\mathcal{R})$. If $\alpha \subseteq_S \beta$ then $\alpha =_S \beta$.*

Proof: If $\alpha \# \beta$, there exist (p, q) in α and (r, s) in β such that $\neg Pos((p, q) \cdot (r, s))$, which is absurd since $\alpha \subseteq_S \beta$ and a point cannot contain two disjoint intervals. Thus $\neg(\alpha \# \beta)$ and therefore $\alpha =_S \beta$. $\square$

5.2 Minimum and maximum

For $\alpha, \beta \in Pt(\mathcal{R})$ we define the minimum and maximum of α and β in the following way:

$$min(\alpha, \beta) \equiv$$
$$\{(p, q) \in S \mid (\alpha \Vdash (p, +\infty) \ \& \ \beta \Vdash (p, +\infty)) \ \& \ (\alpha \Vdash (-\infty, q) \ \vee \ \beta \Vdash (-\infty, q))\} \ ;$$
$$max(\alpha, \beta) \equiv$$
$$\{(p, q) \in S \mid (\alpha \Vdash (p, +\infty) \ \vee \ \beta \Vdash (p, +\infty)) \ \& \ (\alpha \Vdash (-\infty, q) \ \& \ \beta \Vdash (-\infty, q))\} \ .$$

It is easy to prove that $min(\alpha, \beta)$ and $max(\alpha, \beta)$ are formal reals and satisfy the constructive axioms of minimum and maximum in a linear order (cf. [vP]), that is, we have:

Proposition 5.14 *For all α, β in $Pt(\mathcal{R})$, the following hold:*

MM1: $\neg\ \alpha < min(\alpha, \beta) \qquad \neg\ \beta < min(\alpha, \beta)$
$\neg\ max(\alpha, \beta) < \alpha \qquad \neg\ max(\alpha, \beta) < \beta$;

MM2: $\neg(min(\alpha, \beta) < \alpha \ \& \ min(\alpha, \beta) < \beta)$
$\neg(\alpha < max(\alpha, \beta) \ \& \ \beta < max(\alpha, \beta))$.

The distance between two formal reals and the absolute value can thus be defined as usual:

$$|\alpha - \beta| = max(\alpha - \beta, \beta - \alpha);$$

$$|\alpha| = max(\alpha, -\alpha).$$

5.3 Cantor theorem

By the following result formal reals constitute a non-denumerable collection.

Theorem 5.15 (Cantor) *Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of formal reals and let p, q be rational numbers with $p < q$. Then there exists a formal real α such that $\bar{p} < \alpha < \bar{q}$ and, for all n in $\mathbb{N}$, $\alpha \# \alpha_n$.*

Proof: We define by induction two sequences of rationals $(p_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$ such that, for all n, m with $1 \leq n < m$:

i. $p \leq p_n < p_m < q_m < q_n \leq q$;

ii. $\bar{p}_n > \alpha_n$ or $\bar{q}_n < \alpha_n$;

iii. $q_n - p_n < \frac{1}{n}$.

Let $n > 1$ and suppose that $p_1, ..., p_{n-1}, q_1, ..., q_{n-1}$ have been constructed. Since $p_{n-1} < q_{n-1}$, by Lemma 5.2, $(\alpha_n > \bar{p}_{n-1}) \vee (\alpha_n < \bar{q}_{n-1})$. If the first disjunct holds, then $\bar{p}_{n-1} < min(\alpha_n, \bar{q}_{n-1})$, and therefore, by Lemma 5.7, there exists a rational p_n such that $\bar{p}_{n-1} < \bar{p}_n < min(\alpha_n, \bar{q}_{n-1})$; moreover, there exists q_n such that $\bar{p}_n < \bar{q}_n < min(\alpha_n, \bar{q}_{n-1}, \bar{p}_n + \bar{\frac{1}{n}})$. Similarly, if the second disjunct holds then $max(\alpha_n, \bar{p}_{n-1}) < \bar{q}_{n-1}$ and therefore there exists q_n such that $max(\alpha_n, \bar{p}_{n-1}) < \bar{q}_n < \bar{q}_{n-1}$ and consequently there exists p_n such that $max(\alpha_n, \bar{p}_{n-1}, \bar{q}_n - \bar{\frac{1}{n}}) < \bar{p}_n < \bar{q}_n$. In both cases conditions i-iii are satisfied. It is routine to check that $\alpha = \{(p,q) \in S \mid (\exists n)(\exists m)((p \leq p_n) \& (q_m \leq q))\}$ is a formal real. Finally, for all n, we have by construction $(\bar{p}_n > \alpha_n) \vee (\bar{q}_n < \alpha_n)$ and moreover $\alpha \Vdash (p_n, q_n)$ so that $\bar{p}_n < \alpha < \bar{q}_n$. By transitivity of the order we obtain $(\alpha_n < \alpha) \vee (\alpha_n > \alpha)$ and we can thus conclude that $(\forall n)(\alpha \# \alpha_n)$. $\Box$

6 Order completeness of $Pt(\mathcal{R})^e$

The order we have introduced in $Pt(\mathcal{R})$ is not (conditionally) complete, i.e., it is not possible to prove constructively that a bounded non-empty collection of formal reals has a least upper bound (lub) and a greatest lower bound (glb). Indeed, we can prove that order completeness of $Pt(\mathcal{R})$ implies a logical principle which is not intuitionistically valid.

Proposition 6.1 *If $Pt(\mathcal{R})$ is order complete, then for any proposition P, $\neg P \vee \neg\neg P$ holds.*

Proof: Given a proposition P, we consider the following collection of formal reals:

$$A = \{\alpha \in Pt(\mathcal{R}) \mid (\alpha = 0 \wedge P) \vee (\alpha = 1 \wedge \neg P)\}.$$

If $A = \emptyset$, then $0 \notin A$, $1 \notin A$, and therefore $\neg P \wedge \neg\neg P$. Thus $A \neq \emptyset$. Moreover A is bounded since $\alpha \in A$ implies $\alpha = 0 \vee \alpha = 1$. By the hypothesis that the collection $Pt(\mathcal{R})$ is order complete, there exists $\gamma = \text{lub}(A)$ in $Pt(\mathcal{R})$. We observe that

$$\begin{array}{rcl} P & \Rightarrow & \gamma = 0 \\ \neg P & \Rightarrow & \gamma = 1 \end{array}$$

and thus, since $\alpha \# \beta$ implies $\neg(\alpha = \beta)$, by contraposition we obtain

$$\begin{array}{rcl} \gamma \# 0 & \Rightarrow & \neg P \\ \gamma \# 1 & \Rightarrow & \neg\neg P. \end{array}$$

Since $0 \# 1$ holds, by the property of apartness we have $\gamma \# 0 \vee \gamma \# 1$ and therefore $\neg P \vee \neg\neg P$. $\Box$

To obtain order completeness we are thus led, as an alternative to using a non-constructive logic, to widen the collection of formal reals, by considering $Pt(\mathcal{R})^e$, defined in 3.4. In fact we have:

Proposition 6.2 *Let $F = \{\alpha_i \mid i \in I\}$ be a collection in $Pt(\mathcal{R})^e$ bounded above. Then F has a lub.*

Proof: First we observe that we can suppose that F is bounded by a rational number, since if $(\forall i \in I)(\alpha_i < \beta)$ for some $\beta \in Pt(\mathcal{R})^e$, then there exists $k \in Q$ such that $\beta < \bar{k}$, simply by taking a neighbourhood $(p,k) \in \beta$ with $k < +\infty$.

We are now going to prove that the weak formal real

$$\alpha \equiv \{(p,q) \in S \mid (\exists i \in I)((p, +\infty) \in \alpha_i) \ \& \ (\exists q' < q)(\forall i \in I)((-\infty, q') \in \alpha_i)\}$$

is the lub of F.

In order to prove $(\forall i \in I)(\alpha_i \leq \alpha)$, it is enough to prove, by definition of $\leq$, that supposing $(\exists i \in I)(\alpha < \alpha_i)$ leads to a contradiction. So, let $i \in I$ be such that $\alpha < \alpha_i$. Then there exist $(p,q) \in \alpha$ and $(r,s) \in \alpha_i$ with $q < r$. By definition of α, $(\exists q' < q)((-\infty, q') \in \alpha_i)$ and therefore $(r, q') = (r,s) \cdot (-\infty, q') \in \alpha_i$. Since α_i is a weak formal real, for any such q', $r < q'$, and therefore $r < q$, contradicting $q < r$. Finally, we prove that α is the least upper bound of F, i.e., that if $\bar{\alpha}$ is a weak formal real such that $(\forall i \in I)(\alpha_i \leq \bar{\alpha})$, then $\alpha \leq \bar{\alpha}$. Suppose $\bar{\alpha} < \alpha$. Then there exist $(p,q) \in \alpha$ such that $\bar{\alpha} < \bar{p}$ and $i \in I$ such that $\bar{p} < \alpha_i$. From $\alpha_i \leq \bar{\alpha}$ we also have $\alpha_i \leq \bar{p}$, a contradiction. This proves that $\neg(\bar{\alpha} < \alpha)$, i.e., $\alpha \leq \bar{\alpha}$. □

In a symmetric way one proves that any collection in $Pt(\mathcal{R})^e$ bounded below has a glb which is given by

$$\alpha \equiv \{(p,q) \in S \mid (\exists i \in I)((-\infty, q) \in \alpha_i) \ \&\ (\exists p' > p)(\forall i \in I)((p', +\infty) \in \alpha_i)\}.$$

7 Extensionality of $Pt(\mathcal{R})$

In this section we will provide a classical proof of extensionality for formal reals, or, in other words, of injectivity of the map

$$ext : Sat(\mathcal{R}) \to \Omega Pt(\mathcal{R})$$

defined in Section 2.

The proof of extensionality for the formal topology of formal reals follows from the general result stating extensionality of locally compact locales (cf. Theorem 4.3, chapter VII, in [J]). We will give here a direct proof.

Theorem 7.1 *The formal topology of formal reals is extensional.*

Proof: We will prove classically that

if $(p,q) \not\lhd U$ *then there exists* α *in* $Pt(\mathcal{R})$ *such that* $\alpha \Vdash (p,q)$ *and* $\alpha \not\Vdash U$

by an inductive construction of a formal point forcing (p,q) but not U.

We start with defining $\uparrow(p,q) \equiv \{(p',q') \mid (p,q) \leq (p',q')\}$ and we put $\alpha_0 \equiv \uparrow(p,q)$. Since $(p,q) \not\lhd U$, there exists (p_1, q_1) with $p < p_1 < q_1 < q$ such that $(p_1,q_1) \not\lhd U$. Let $\alpha_1 \equiv \uparrow(p,q) \cup \uparrow(p_1,q_1) = \uparrow(p_1,q_1)$. By trisecting (p_1,q_1), we obtain two overlapping intervals, namely $(p_1, q_1 - \frac{q_1-p_1}{3})$ and $(p_1 + \frac{q_1-p_1}{3}, q_1)$. Since $(p_1,q_1) \not\lhd U$, at least one of the two is not covered by U, and we call it (p_2,q_2). Then put $\alpha_2 \equiv \uparrow(p_1,q_1) \cup \uparrow(p_2,q_2) = \uparrow(p_2,q_2)$. Suppose that $\alpha_n = \uparrow(p_n,q_n)$ has been defined. If n is even we define α_{n+1} to be $\uparrow(p_{n+1},q_{n+1})$ where (p_{n+1},q_{n+1}) is suitably chosen such that $p_n < p_{n+1} < q_{n+1} < q_n$ and $(p_{n+1},q_{n+1}) \not\lhd U$. If n is odd, α_{n+1} is defined by trisecting (p_n,q_n) as in step 2. Observe that $\alpha_0 \subseteq \alpha_1 \subseteq \ldots \subseteq \alpha_n \subseteq \alpha_{n+1}$ for all n. Let $\alpha = \bigcup_{n \in \mathbb{N}} \alpha_n$. In order to achieve the conclusion we only have to prove that α is a formal real since it is clear by the construction that $\alpha \Vdash (p,q)$ and $\alpha \not\Vdash U$. All the conditions of Definition 3.3 are easily seen to be satisfied except the third. Suppose $(p',q') \in \alpha$ and let $p' \leq r < s \leq q'$. By construction of α (and the fact that the sequence of rationals $((\frac{2}{3})^n)_n$ converges to 0), there exists $n \in \mathbb{N}$ and $(p_n,q_n) \in \alpha_n$ such that $q_n - p_n < s - r$ and $(p_n,q_n) \leq (p',q')$. Therefore we have $(p_n,q_n) \leq (p',s) \vee (p_n,q_n) \leq (r,q')$ and thus $\alpha \Vdash (p',s) \vee \alpha \Vdash (r,q')$. □

We conclude this section with the remark: In terms of *propositional theories* (cf. [FG]), what we have achieved with the definition of the formal space of formal reals is a *countable presentation* for the theory. Its *models* are the formal reals, and extensionality is completeness of the theory.

8 Constructive analysis

8.1 Convergence in Pt(R)

The notion of convergence and of Cauchy sequence can be expressed simply in terms of formal reals. We will prove that formal reals satisfy Cauchy completeness, that is, a sequence of formal reals converges (to a formal real) iff it is a Cauchy sequence.

As usual, we say that a sequence of reals converges to a given real if, for any neighbourhood of such real number, the sequence is eventually in it, i.e.:

Definition 8.1 *A sequence $(\alpha_n)_n$ of formal reals* converges to α *in $Pt(\mathcal{R})$ (abbr. $(\alpha_n)_n \to \alpha$) if for all (p,q) in α there exists m such that for all $n > m$, $\alpha_n \Vdash (p,q)$.*

We can then prove that if a sequence converges, then it converges to a unique formal point, which can thus be called the *limit* of the sequence.

Proposition 8.2 *Suppose $(\alpha_n)_n \to \alpha$ and $(\alpha_n)_n \to \beta$. Then $\alpha =_S \beta$.*

Proof: If $\alpha \# \beta$, then $(\exists (p,q) \in \alpha)(\exists (r,s) \in \beta)(\neg Pos((p,q) \cdot (r,s))$. So, by the definition of convergence, there exists m_1 such that, for all $n > m_1$, $\alpha_n \Vdash (p,q)$ and there exists m_2 such that, for all $n > m_2$, $\alpha_n \Vdash (r,s)$. Thus there exists m such that, for all $n > m$, $\alpha_n \Vdash (p,q) \cdot (r,s)$, that contradicts $\neg Pos((p,q) \cdot (r,s))$. So $\neg(\alpha \# \beta)$ and therefore, by 5.12, $\alpha =_S \beta$. □

We can prove the following easy facts about convergent sequences:

Proposition 8.3 *Suppose $(\alpha_n)_n \to \alpha$. Then any subsequence $(\beta_{n_k})_k$ of $(\alpha_n)_n$ converges to the same limit.*

Proof: Straightforward. □

Proposition 8.4 *Any convergent sequence $(\alpha_n)_n$ is bounded.*

Proof: Suppose $(\alpha_n)_n \to \alpha$ and let (p,q) be such that $-\infty < p < q < +\infty$ and $\alpha \Vdash (p,q)$. Then there exists m such that, for all $n > m$, $\alpha_n \Vdash (p,q)$.

Take, for all $n \leq m$, a neighbourhood (p_n, q_n) with $-\infty < p_n < q_n < +\infty$ such that $\alpha_n \Vdash (p_n, q_n)$. Then, for all n, we have $min(\bar{p}, \bar{p}_1, ..., \bar{p}_m) < \alpha_n < max(\bar{q}, \bar{q}_1, ..., \bar{q}_m)$, and therefore the sequence is bounded. □

Definition 8.5 *A sequence of formal reals $(\alpha_n)_n$ is called a* Cauchy sequence *if, for any positive rational k, there exist a neighbourhood (p,q) and a natural number m such that*

$$q - p < k \ \& \ (\forall n > m)(\alpha_n \Vdash (p,q)) \ .$$

As announced, we have:

Theorem 8.6 (Cauchy completeness of $Pt(\mathcal{R})$) *A sequence of formal reals $(\alpha_n)_n$ converges iff it is a Cauchy sequence.*

Proof: Suppose that $(\alpha_n)_n \to \alpha$. This, together with Lemma 3.5, implies that for any positive rational k there exist (p,q) with $q - p < k$ and m such that for all $n > m$, $\alpha_n \Vdash (p,q)$.

On the other hand, if $(\alpha_n)_n$ is a Cauchy sequence, then we can prove that

$$\alpha = \{(p,q) \in S \mid (\exists (p',q'))(p < p' < q' < q \ \& \ (\exists m)(\forall n > m)(\alpha_n \Vdash (p',q')))\}$$

is a formal real and the limit of the given sequence. □

Definition 8.7 *A collection F of formal reals is* complete *if every Cauchy sequence in F converges to a point in F. It is* totally bounded *if for any positive $k \in Q$ there exists a finite sequence $(p_1, q_1), \ldots, (p_n, q_n)$ of intervals such that for all $i \leq n$, $q_i - p_i < k$ and, for all α in F, $\alpha \Vdash \{(p_1, q_1), \ldots, (p_n, q_n)\}$.*

We define here the closed interval $[0, 1]$ as the collection of formal reals α with $0 \leq \alpha \leq 1$. Then we have:

Proposition 8.8 *The closed interval $[0, 1]$ is complete and totally bounded.*

Proof: Completeness. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $[0, 1]$ and let α be its limit. If $\alpha > \bar{1}$ we have $\alpha \Vdash (1, +\infty)$ and therefore, by definition of convergence, $(\exists m)(\forall n > m)(\alpha_n \Vdash (1, +\infty))$ that is $(\forall n > m)(\alpha_n > \bar{1})$. By hypothesis we have instead $\alpha_n \leq \bar{1}$ and therefore we get $\alpha \leq \bar{1}$. In the same way we obtain $\bar{0} \leq \alpha$.

Total boundedness. Observe that $(0, 1)$ can be covered by a finite sequence of overlapping intervals $(p_1, q_1), \ldots, (p_n, q_n)$ such that $p_1 < 0 < p_2 < q_1 < \ldots < p_n < q_{n-1} < 1 < q_n$, and for all $i \leq n$, $q_i - p_i = k$. Since $0 \leq \alpha \leq 1$, $\alpha \Vdash (p_1, q_n)$ and thus, since $(p_1, q_n) \lhd \{(p_1, q_1), \ldots, (p_n, q_n)\}$, by monotonicity, $\alpha \Vdash \{(p_1, q_1), \ldots, (p_n, q_n)\}$. □

We observe that classically a subset of the real numbers is complete and totally bounded if and only if it is compact, in the sense that from any open cover a finite subcover can be extracted. In constructive mathematics this is not any longer true (cf. [Bee]) since compactness only implies completeness and total boundedness. A pointfree definition of closed intervals of the real line allows to prove compactness, in the strongest sense, in a completely constructive way (cf. [CN]).

We can prove as usual that a continuous function preserves convergence of sequences.

Proposition 8.9 *Let f be a continuous function from $Pt(\mathcal{R})$ to $Pt(\mathcal{R})$ (let f^{pt} and f^{Ω} be the map on points and the corresponding morphism on opens respectively) and let $(\alpha_n)_n$ be a sequence of formal reals. If $(\alpha_n)_n \to \alpha$ then $(f^{pt}(\alpha_n))_n \to f^{pt}(\alpha)$.*

Proof: Suppose $f^{pt}(\alpha) \Vdash (p, q)$. Then by the adjunction between f^{pt} and f^{Ω} we have $\alpha \Vdash f^{\Omega}((p, q))$ and therefore, by definition of convergence, $(\exists m)(\forall n > m)(\alpha_n \Vdash f^{\Omega}((p, q)))$. Again, by the relation between f^{pt} and f^{Ω}, we obtain

$$(\exists m)(\forall n > m)(f^{pt}(\alpha_n) \Vdash (p, q))$$

that is $(f^{pt}(\alpha_n))_n \to f^{pt}(\alpha)$. □

8.2 Baire theorem

In this section we will prove that the classical Baire theorem holds for formal reals. As a corollary we will get another proof of Cantor's theorem on the non-denumerability of the continuum. These results follow some ideas already in [ML].

Definition 8.10 *A subset U of S is* dense *in $Pt(\mathcal{R})$ if for any positive interval (p, q) there exists (r, s) in U such that $Pos((p, q) \cdot (r, s))$ holds.*

Observe that a subset U of S is dense in $Pt(\mathcal{R})$ iff, for any positive interval (p, q), there exists at least one formal real α with $\alpha \Vdash (p, q)$ and $\alpha \Vdash U$. Therefore, U is dense (as a collection of intervals) iff $ext(U)$ is dense (as a collection of formal reals) in the usual sense.

In the following we will use the notation $l(p, q)$ for $q - p$.

Theorem 8.11 (Baire) *Let $(U_n)_{n\in\mathbb{N}}$ be a denumerable family of subsets of S such that, for all n, U_n is dense in $Pt(\mathcal{R})$. Then $\cap_{n\in\mathbb{N}} ext(U_n)$ is dense in $Pt(\mathcal{R})$.*

Proof: Let (p,q) be a positive interval in S. We start by taking (p_1,q_1) with $p<p_1<q_1<q$ in order to have a finite interval. Since U_1 is dense in $Pt(\mathcal{R})$, there exists a formal real α_1 such that $\alpha_1 \Vdash (p_1,q_1)$ and $\alpha_1 \Vdash U_1$. Thus there exists (a_1,b_1) in U_1 such that $\alpha_1 \Vdash (p_1,q_1)\cdot(a_1,b_1)$. Consider now an interval (p_2,q_2) with $(p_2,q_2) < (p_1,q_1)\cdot(a_1,b_1) \leq (p_1,q_1)$ and such that $l(p_2,q_2) < \frac{l((p_1,q_1)\cdot(a_1,b_1))}{2}$. Again, since U_2 is dense in $Pt(\mathcal{R})$, there exists a formal real α_2 such that $\alpha_2 \Vdash (p_2,q_2)$ and $\alpha_2 \Vdash U_2$. Then there exists (a_2,b_2) in U_2 such that $\alpha_2 \Vdash (p_2,q_2)\cdot(a_2,b_2)$.
In this way, we obtain a sequence of intervals $\{(p_n,q_n)\}_n$ such that, for all $n>1$,

$l(p_n,q_n) < \frac{l(p_{n-1},q_{n-1})}{2} < \ldots < \frac{l(p_1,q_1)}{2^{n-1}}$;

$p < p_1 < \ldots < p_{n-1} < q_{n-1} < \ldots < q_1 < q$;

$ext((p_n,q_n)) \subseteq ext(U_{n-1})$.

The last follows since $a_{n-1} < p_n < q_n < b_{n-1}$ and $(a_{n-1},b_{n-1}) \in U_{n-1}$, and therefore, for all $m<n$, $ext((p_n,q_n)) \subseteq ext(U_m)$.

We also obtain a sequence of points $(\alpha_n)_n$, which is by construction a Cauchy sequence. By Cauchy completeness of $Pt(\mathcal{R})$, the sequence $(\alpha_n)_n$ converges to the limit

$$\alpha = \{(a,b) \mid (\exists (a',b'))(a<a'<b'<b \ \&\ (\exists m)(\forall n>m)(\alpha_n \Vdash (a',b'))\} .$$

Let $m>0$. Then we have $max(p_m,a_m) < p_{m+1} < q_{m+1} < min(q_m,b_m)$ and moreover for all $n>m$, $\alpha_n \Vdash (p_{m+1},q_{m+1})$. Now, by definition of α, $\alpha \Vdash (p_m,q_m)\cdot(a_m,b_m)$ and so, for all $m>0$, $\alpha \Vdash (a_m,b_m)$, that is $\alpha \in ext(U_m)$ and $\alpha \Vdash (p_m,q_m)$. Since $p<p_m<q_m<q$, from the latter we get $\alpha \Vdash (p,q)$. $\square$

Corollary 8.12 *Let $(\alpha_n)_n$ be a sequence in $Pt(\mathcal{R})$. Then the collection of formal reals α such that $(\forall n)(\alpha \# \alpha_n)$ is dense in $Pt(\mathcal{R})$.*

Proof: For all n, let $U_n \equiv \{(p,q)\in S \mid (\bar{q} \leq \alpha_n) \vee (\alpha_n \leq \bar{p})\}$. It follows from the definition of a formal real that U_n is dense. By Baire's theorem, $\cap_{n\in\mathbb{N}} ext(U_n)$ is dense in $Pt(\mathcal{R})$. The conclusion follows since $ext(U_n) = \{\alpha \mid \alpha\#\alpha_n\}$ and $\cap_{n\in\mathbb{N}} ext(U_n) = \{\alpha \mid (\forall n)(\alpha\#\alpha_n)\}$. $\square$

9 Other axiomatizations

9.1 Dedekind cuts

We recall here the definition of a Dedekind cut on the rationals as given in [FH] and prove that there exists a one-one correspondence between formal points and Dedekind cuts.

Definition 9.1 *A* Dedekind cut *on the rationals is a pair (L,U), where L and U are non-empty strict subsets of Q^- and of Q^+ respectively, such that:*

- $(\forall p)(\neg(p\in U \ \&\ p\in L))$ *(disjointness);*
- $(\forall p\in L)(\exists \in L)(p<q)$,
  $(\forall p\in U)(\exists q\in U)(q<p)$ *(openness);*
- $(\forall p)((\exists q\in L)(p<q) \rightarrow p\in L)$,
  $(\forall p)((\exists q\in U)(q<p) \rightarrow p\in U)$ *(monotonicity);*
- $(\forall p)(\forall q)(p<q \rightarrow p\in L \vee q\in U)$ *(locatedness).*

We have:

Proposition 9.2 *There is a one-one correspondence between the collection $Pt(\mathcal{R})$ of formal points on $\mathcal{R}$ and the collection R of Dedekind cuts on the rationals, given by*

$$\phi(\alpha) \equiv (\{p \in Q^- \mid (p, +\infty) \in \alpha\}, \{q \in Q^+ \mid (-\infty, q) \in \alpha\})$$

with inverse given by $\psi(L, U) \equiv \{(p, q) \in Q^- \times Q^+ \mid p \in L \;\&\; q \in U\}$.

Proof: For any formal point α, $(L_\alpha, U_\alpha) \equiv \phi(\alpha)$ is a Dedekind cut: L_α and U_α are non-empty since $(-\infty, +\infty) \in \alpha$ and therefore $-\infty \in L$ and $+\infty \in U$. Disjointness holds too, since if $p \in L_\alpha$ and $p \in U_\alpha$ then, by closure under intersection, we would have $(p, p) \in \alpha$ and therefore $Pos((p, p))$ which is impossible. Openness and monotonicity follow from the fourth and the second condition of the definition of a formal real, respectively. Finally, locatedness is proved by observing that if $p < q$, then $(-\infty, q) \in \alpha$ or $(p, +\infty) \in \alpha$.

We now prove that $\alpha_{(L,U)} \equiv \psi((L, U)) \equiv \{(p, q) \in S \mid p \in L, q \in U\}$ is a formal real. Of course $(-\infty, +\infty) \in \alpha$ since L and U are both not empty and monotone. Closure under product of $\alpha_{(L,U)}$ is obvious, and the converse follows from monotonicity. If $(p, q) \in \alpha_{(L,U)}$ and $r < s$, then by locatedness $(p, s) \in \alpha_{(L,U)}$ or $(r, q) \in \alpha_{(L,U)}$. Furthermore, if $(p, q) \in \alpha_{(L,U)}$, then by openness there exists (p', q') such that $p < p' < q' < q$ and $(p', q') \in \alpha_{(L,U)}$.

Finally, it is straightforward to prove that ψ is indeed the inverse of ϕ. $\Box$

For $p \in Q$, let (L_p, U_p) be the Dedekind cut defined by $L_p \equiv \{x \in Q^- \mid x < p\}$ and $U_p \equiv \{x \in Q^+ \mid x > p\}$. Then a basic open interval of Dedekind cuts is defined as

$$]p, q[\equiv \{(L, U) \mid L_p \subset L \subset L_q \;\&\; U = \{r \in Q^+ \mid L < r\}$$

where $L < r \equiv (\forall x \in L)(x < r)$. We have:

1. $\psi(\,]p, q[\,) = ext((p, q))$,
2. $\phi(ext((p, q))) =]p, q[$.

1. If $\alpha \in \psi(\,]p, q[\,)$ there exists (L, U) such that $L_p \subset L \subset L_q \;\&\; U = \{r \in Q^+ \mid L < r\}$ and $\alpha = \alpha_{(L,U)}$. Thus $p \in L$ and $q \in U$, i.e., $(p, q) \in \alpha_{(L,U)}$ that is $\alpha \in ext((p, q))$. Conversely, in $\alpha \in ext((p, q))$, then, since $\alpha_{(L,U)_\alpha} \equiv \psi((L, U)_\alpha)$, $L_p \subset L \subset L_q$ and $q \in U$, thus (by disjointness and momotonicity) $U = \{r \mid r > L\}$.
2. Just observe that $\phi(ext((p, q))) = \phi(\psi(\,]p, q[\,)) =]p, q[$ by the above proof and bijectiveness.

Thus we have

Proposition 9.3 *Dedekind reals with their usual topology are homeomorphic to formal reals with the extensional topology.*

9.2 Cauchy sequences à la Bishop

We recall the definition of real numbers à la Bishop (cf. [Bi]):

Definition 9.4 *A* real number *is a sequence of rationals $(x_n)_n$ such that*

$$|x_m - x_n| \leq m^{-1} + n^{-1} \qquad (m, n \in \mathbb{N}^+) .$$

The real numbers $(x_n)_n$ and $(y_n)_n$ are equal if

$$|x_n - y_n| \leq 2n^{-1} \qquad (n \in \mathbb{N}^+) .$$

Then we have (cf. [CN]):

Proposition 9.5 *There is a one-one correspondence between formal reals and real numbers à la Bishop.*

Proof: Let α be a formal real. By Lemma 3.5, there exists $(p,q) \in \alpha$ such that $q - p \leq 2/3$. We can generate inductively a sequence of intervals $((x_n, y_n))_n$ by defining $(x_1, y_1) = (p,q)$ and

$$(x_{i+1}, y_{i+1}) \equiv \begin{cases} (x_i, y_i - \frac{y_i - x_i}{3}) & \text{if } \alpha \Vdash (x_i, y_i - \frac{y_i - x_i}{3}) \\ (x_i + \frac{y_i - x_i}{3}, y_i) & \text{if } \alpha \Vdash (x_i + \frac{y_i - x_i}{3}, y_i) \end{cases}$$

We observe that the definition is sound since $\alpha \Vdash (x_i, y_i)$ and $x_i + \frac{y_i - x_i}{3} < y_i - \frac{y_i - x_i}{3}$ imply $\alpha \Vdash (x_i, y_i - \frac{y_i - x_i}{3}) \vee \alpha \Vdash (x_i + \frac{y_i - x_i}{3}, y_i)$. It can be verified that the sequences $(x_n)_n$ and $(y_n)_n$ are real numbers according to Definition 9.4.

Conversely, if $(x_n)_n$ is a real number à la Bishop, then the set defined by

$$\alpha \equiv \bigcup_{n \in \mathbb{N}^+} \{(p,q) \in S \mid p < x_n - 2/n < x_n + 2/n < q\}$$

is a formal real.

Moreover the correspondence thus established is biunivocal. □

Well known properties of constructive reals, such as undecidability of order, now follow for formal reals.

9.3 Martin-Löf's maximal approximations

In [ML], two positive basic neighbourhoods (p,q) and (r,s) are said to *lie apart* if $q < r \vee s < p$. We write $(p,q) < (r,s)$ for $r < p \;\&\; q < s$. The definition of maximal approximation given in [ML] can thus be expressed in the following form:

Definition 9.6 *An* approximation A *is a recursively enumerable subset of* $Q^- \times Q^+$ *such that:*

1. $\dfrac{(p_1,q_1) \in A \quad (p_1,q_1) < (p_2,q_2)}{(p_2,q_2) \in A}$;

2. $\dfrac{(p_1,q_1) \in A \quad (p_2,q_2) \in A}{(\exists (r,s) \in A)((r,s) < (p_1,q_1) \;\&\; (r,s) < (p_2,q_2))}$.

An approximation A is maximal *if moreover*

3. $\dfrac{(p_1,q_1) < (p_2,q_2)}{(\exists (r,s) \in A)(\neg Pos((p_1,q_1) \cdot (r,s))) \vee (p_2,q_2) \in A}$.

Then we have:

Proposition 9.7 *Every recursively enumerable formal real is a maximal approximation.*

Proof: Let α be a formal real and also a recursively enumerable subset of $Q^- \times Q^+$. Then for all $(p,q) \in \alpha$, $Pos((p,q))$ holds, thus α consists of positive neighbourhoods. The defining conditions of maximal approximation are satisfied:

(1) If $(p_1,q_1) \in \alpha$ and $(p_1,q_1) < (p_2,q_2)$ then $(p_1,q_1) \lhd (p_2,q_2)$, and therefore, by monotonicity, $(p_2,q_2) \in \alpha$.

(2) Let $(p_1,q_1) \in \alpha$ and $(p_2,q_2) \in \alpha$. Then $(p_1,q_1) \cdot (p_2,q_2) \in \alpha$ and therefore, by definition of formal reals, $(\exists (r,s) \in \alpha)((r,s) < (p_1,q_1) \cdot (p_2,q_2))$, so $(\exists (r,s) \in \alpha)((r,s) < (p_1,q_1) \;\&\; (r,s) < (p_2,q_2))$.

(3) Cf. Lemma 5.8. $\square$

Conversely we have:

Proposition 9.8 *Every maximal approximation is a recursively enumerable formal real.*

Proof: First observe that a maximal approximation A satisfies the rule of monotonicity

$$\frac{(p_1,q_1) \in A \quad (p_1,q_1) \leq (p_2,q_2)}{(p_2,q_2) \in A} \ .$$

To see this, assume $(p_1,q_1) \in A$. Then by 2 there exists $(r,s) \in A$ such that $(r,s) < (p_1,q_1)$. For such an (r,s) we have $(r,s) < (p_2,q_2)$ and therefore $(p_2,q_2) \in A$.

1. By condition 3 of Definition 9.6, every maximal approximation is non-empty. So, if $(p,q) \in A$ then $(p,q) \leq (-\infty,+\infty)$ and therefore, by the above observation, $(-\infty,+\infty) \in A$.

2. Suppose $(p_1,q_1) \in A$ and $(p_2,q_2) \in A$. Then, by condition 2 of Definition 9.6, $(\exists(r,s) \in A)((r,s) < (p_1,q_1)\cdot(p_2,q_2))$, thus by monotonicity $(p_1,q_1)\cdot(p_2,q_2) \in A$. The converse simply holds by monotonicity.

3. Let $(p,q) \in A$ and $r < s$. If $r \leq p$ then $(p,q) \leq (r,q)$, thus by monotonicity $(r,q) \in A$. Similarly, if $q \leq s$, then $(p,s) \in A$. Otherwise, $p < r < s < q$ and therefore $(r, \frac{r+s}{2}) < (p,s)$. Thus, by definition of maximal approximation, $(p,s) \in A$ or there exists $(u,v) \in A$ such that $\neg Pos((u,v)\cdot(r,\frac{r+s}{2}))$. Then $v \leq r$ or $\frac{r+s}{2} \leq u$. So we get $(u,v)\cdot(p,q) \leq (p,s)$ or $(u,v)\cdot(p,q) \leq (r,q)$ and the conclusion follows by monotonicity since $(u,v)\cdot(p,q) \in A$.

4. If $(p,q) \in A$, then by condition 2 of Definition 9.6 there exists (p',q') such that $p < p' < q' < q$ and $(p',q') \in A$. $\square$

10 Concluding remarks

The presentation of the topology of formal reals by means of a Stone cover, and its use in mathematical proofs, is the main accomplishment of this work. It gives a precise sense to the computational content of formal reals.

This presentation of real numbers has served as a way of constructivizing mathematical analysis, cf. [CN] and [CCN], even to the point of achieving completely formalized and machine-checked proofs, as in [C]. Formal reals can thus be seen as a computational alternative to classical reals.

An interesting question left open here is to characterize formal topologies having a similar inductive generation.

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