Under consideration for publication in Math. Struct. in Comp. Science

Continuous domains as formal spaces

Sara Negri

University of Helsinki, Finland

e-mail: negri@helsinki.fi

Received 12 August 1999

1. Introduction

Continuous domains are partially ordered sets based on an order relation motivated through an ideal notion of computation. A computation is viewed as a sequence of increasingly refined approximations that give, as limit, the result to be computed. More precisely, a computation for x is a directed set with x as supremum. A given element a approximates x if, whenever a directed set D has x as supremum, then some element of D is greater than a, i.e., any computation for x gives, in a finite number of steps, a value that is closer to x than a.

Complete lattices with the property that the set of approximations of any element x form a directed set having x as least upper bound, are called continuous lattices.

Continuous lattices were first introduced by Dana Scott in 1972 and used as models for untyped λ-calculus (Scott 1973). They were later extensively studied (cf. Gierz *et al.* 1980, Banaschewski and Hoffmann 1981) and used as a unifying viewpoint for domain theory (as in Abramsky and Jung 1994).

By requiring closure under suprema only for directed subsets, rather than for arbitrary subsets, a computational model of useful generality is obtained. These partially ordered sets are called continuous directed complete partial orders, and they generalize both continuous lattices and algebraic domains. These latter are domains with a base of compact elements, and they have received particular attention in the literature for the reason that Scott domains are algebraic.

What is known as formal topology was presented in a series of lectures by Per Martin-Löf in 1985 with the question whether it is possible to generalize the concept of Scott domain so as to comprise all thinkable topological spaces. Thus the connection to Scott domains is one of the motivating reasons at the origin of formal topology.

In formal topology the order-theoretic properties of a topological space are assumed as primitive by means of a relation, called formal cover, between elements and subsets of a set. The properties that a cover relation has to satisfy correspond to the properties of the set-theoretic relation of inclusion between an open and the union of a set of opens of a topological space. Formal topology is pointfree, as the points are not primitive objects, and constructive, as the underlying set theory is Martin-Löf's constructive type theory.

Predicativity mainly differentiates formal topology from the development of pointfree topology called locale theory.

Points are not among the primitive notions of formal topology, but are defined as particular, well-behaved collections of opens. Formal spaces are the spaces arising as spaces of formal points of a suitable formal topology.

The interplay between topology and order, perhaps the most typical feature of domain theory, appears in this work in the form of a constructive Stone representation. We show how the relevant structures of domain theory can be represented as formal spaces, via the lattices of formal opens and formal points of suitable formal topologies.

We motivate our analysis of the connections between domain theory and formal topology by showing that continuous domains are central in the applications of formal topology to constructive analysis. Important examples are given by formal reals, formal intervals and formal linear functionals (cf. Negri and Soravia 1999, Cederquist and Negri 1996, Cederquist Coquand and Negri 1998). In all these cases the cover is defined by means of a finitary inductive definition which, as we shall show, has a precise link to the notion of continuity in domains.

We start by setting down all the basic notions of formal topology needed. Some definitions differ from the ones given for instance in Sambin (1987), mainly because we avoid using the positivity predicate. Also, we define a formal topology on a preordered set, as in Coquand (1996), rather than on a commutative monoid or semilattice.

A relation more general than the cover relation is needed when facing the problem of representing nondistributive structures. To this end we recall that complete lattices (also called sup-lattices in the literature) can be presented by means of closure operators, as in Joyal and Tierney (1984). A representation for complete lattices can be given in terms of formal topology by using a generalization of the cover relation (cf. Battilotti and Sambin 1993).

The basic examples of formal topologies recalled in section 3 all have a cover relation with approximation features that are summarized in the definition of *locally Stone* formal topology. The further example of partial reals, i.e., formal reals allowing unsharp points, is analyzed here, and motivates the definition of *locally Scott* formal topology.

In section 4 we show that these two classes of formal topologies give rise, by considering the corresponding lattices of saturated subsets, to continuous and prime-continuous lattices, respectively, and that every continuous or prime-continuous lattice can be represented in this way via formal topology. The representation for continuous and prime-continuous lattices is part the equivalence between lattice-theoretic structures and the corresponding structures in formal topology.

Continuous directed complete partial orders are represented via the formal points of a suitable class of locally Scott formal topologies. The representation of Scott domains via formal topologies is then obtained as a special case in section 4.5.

The idea of representing lattices by means of lattices of opens of a topological space goes back to Stone and was often taken over in the literature for various lattice-theoretic structures. In section 4.6 we show how the spectral theory of continuous lattices (cf. Hoffmann and Lawson 1978 and Gierz *et al.* 1980, ch. V) can be rephrased in terms of formal spaces in a direct and general way.

In the last section we apply our results to get a representation theorem for sober locally compact spaces via formal topology. The axiom of choice is here needed in order to prove spatiality of locally Stone formal topologies.

Our proofs of the representation theorems for the various domain-theoretic structures in this work are all elementary, although some quantifications over powersets appear. Thus our results show that the predicative notion of locally Stone (resp. Scott) formal topology can replace the classically equivalent but impredicative and nonconstructive notion of continuous (resp. prime continuous) domain.

2. Formal topology and pointfree representations

In this section we present the background on formal topology needed for our developments and the basic results on pointfree representations (theorems 2.5 and 2.7) upon which we build in sections 4 and 5. We shall also discuss the role of the positivity predicate.

We remark that the definition of formal topology given here differs from the definition in Sambin (1987): a cover is defined here on a preordered set rather than on a commutative monoid with unit, or on a semilattice with top element, so that a more flexible definition is obtained. The positivity predicate is left out for reasons that will be explained in 2.4. Consequently the definitions of points on a formal topology and of morphisms between formal topologies have to be restated accordingly. Also the definition of Scott formal topology has to be modified in the absence of the positivity predicate.

For general motivations on formal topology as a predicative approach to pointfree topology we refer to Sambin (1987) and the introduction of Negri and Soravia (1999) (and pointers therein to the more recent literature).

2.1. *Basic definitions*

Formal topology is a constructive approach to pointfree topology in the tradition of *locale theory* (see Isbell 1972, Johnstone 1982, 1983, Fourman and Grayson 1982), that uses Martin-Löf's constructive type theory instead of set theory.

Formal topology starts from an axiomatization of the inclusion relation between opens of a topological space $\langle X, \Omega(X)\rangle$, with no reference to the points. Since a point-set topology can always be presented using one of its bases, the abstract structure that we consider is a partially ordered set $\langle S, \sqsubseteq \rangle$ where the set S corresponds to a base of the point-set topology $\Omega(X)$ and $\sqsubseteq$ corresponds to the relation of inclusion between sets belonging to the base.

In point-set topology any open set is obtained as a union of elements of the base; without points we do not have such a union and we regard an open set as a subset of the base S. Set-theoretic inclusion between basic opens and opens is replaced by a relation $\lhd$, called *cover*, between elements and subsets of the base S. This justifies the following:

Definition 2.1. A *formal topology* over a set S consists of a reflexive and transitive relation $\sqsubseteq$ on S and a relation $\lhd$ between elements and subsets of S, called *(formal) cover*, such that, for any $a, b \in S$ and $U, V \subseteq S$, the following conditions hold:

reflexivity	$\dfrac{a \in U}{a \lhd U}$	
transitivity	$\dfrac{a \lhd U \qquad U \lhd V}{a \lhd V}$	where $U \lhd V \equiv (\forall u \in U)\ u \lhd V$
left	$\dfrac{b \sqsubseteq a \quad a \lhd U}{b \lhd U}$	
right	$\dfrac{a \lhd U \qquad a \lhd V}{a \lhd U \sqcap V}$	where $U \sqcap V \equiv \{c \in S \mid (\exists u \in U)(\exists v \in V)(c \sqsubseteq u \,\&\, c \sqsubseteq v)\}$

An *quasi formal topology* on a set S is given by a relation $\lhd$ called *quasi cover* † between elements and subsets of S satisfying only reflexivity and transitivity.

We shall sometimes indicate a formal topology $\mathcal{A}$ on S with preoder $\sqsubseteq$ and cover $\lhd$ with the notation $\langle S, \sqsubseteq, \lhd \rangle$. A similar notation will apply to quasi formal topologies.

If the preordered set S is also a $\wedge$-semilattice, the above definition is equivalent to the definition of cover on a semilattice (cf. Sambin 1987), with the rules of left and right replaced by

$\wedge$ - *left*	$\dfrac{a \lhd U}{a \wedge b \lhd U}$	
$\wedge$ - *right*	$\dfrac{a \lhd U \qquad a \lhd V}{a \lhd U \wedge V}$	where $U \wedge V \equiv \{u \wedge v \mid u \in U, v \in V\}$

Given a quasi formal topology $\mathcal{A}$, we obtain a closure operator $\mathcal{C}_{\mathcal{A}}$ on $\mathcal{P}(S)$ defined by

$$\mathcal{C}_{\mathcal{A}}(U) \equiv \{a \in S \mid a \lhd U\}$$

and shortly denoted by $\mathcal{A}U$. Conversely, any closure operator $\mathcal{C}$ on $\mathcal{P}(S)$ gives rise to a quasi cover $\lhd_c$ on S defined by

$$a \lhd_c U \equiv a \in \mathcal{C}U$$

and the correspondence thus established is biunivocal.

The collection of subsets U of S which are closed with respect to the closure operator induced by the quasi formal topology $\mathcal{A}$, i.e., which satisfy $\mathcal{A}U = U$, is denoted by $Sat(\mathcal{A})$ and called the collection of *saturated subsets* of $\mathcal{A}$. Clearly, for all $U \subseteq S$, $\mathcal{A}U$ is in $Sat(\mathcal{A})$.

For a quasi formal topology $\mathcal{A}$, $Sat(\mathcal{A})$ is partially ordered by inclusion and for U, V in $Sat(\mathcal{A})$ we have $U \subseteq V$ if and only if $U \subseteq \mathcal{A}V$, if and only if $U \lhd V$. Two saturated subsets U, V are equal if $U \lhd V$ and $V \lhd U$. $Sat(\mathcal{A})$ is a *complete lattice*, i.e., it is closed under the formation of arbitrary joins, which are obtained as follows: If $\{U_i\}_{i \in I}$ is a family of saturated subsets of $\mathcal{A}$, then for all $i \in I$, $U_i \lhd \bigcup_{i \in I} U_i$, so for all $i \in I$, $U_i \leqslant \mathcal{A}(\bigcup_{i \in I} U_i)$.

† In Battilotti and Sambin (1993) a quasi cover is called an *infinitary preorder*. Observe that an infinitary preorder is not a preorder.

If V is a saturated subset such that $U_i \leqslant V$ for all $i \in I$, then $\bigcup_{i \in I} U_i \lhd V$, and therefore $\mathcal{A}(\bigcup_{i \in I} U_i) \lhd V$. We thus have

$$\bigvee_{i \in I} U_i \equiv \mathcal{A}(\bigcup_{i \in I} U_i) \ .$$

If $\mathcal{A}$ is a formal topology, by the rules of left and right we have $\mathcal{A}U \cap \mathcal{A}V = \mathcal{A}(U \sqcap V)$, and therefore $Sat(\mathcal{A})$ is closed under intersection. Thus the meet in $Sat(\mathcal{A})$ is given by

$$\mathcal{A}U \wedge \mathcal{A}V \equiv \mathcal{A}U \cap \mathcal{A}V = \mathcal{A}(U \sqcap V)$$

and it is easily seen to be distributive over arbitrary joins, hence making $Sat(\mathcal{A})$ into a *frame*, that is, a complete lattice with finite meets that distribute over arbitrary joins.

Insted of considering for a formal topology $\mathcal{A}$ the frame of saturated subsets $Sat(\mathcal{A})$ we can consider the quotient of $\mathcal{P}(S)$ modulo the equivalence relation

$$U =_{\mathcal{A}} V \equiv U \lhd V \mathbin{\&} V \lhd U \ .$$

Clearly $U =_{\mathcal{A}} V$ if and only if $\mathcal{A}U = \mathcal{A}V$. Such a quotient is denoted by $Open(\mathcal{A})$ and called the collection of *formal opens* of $\mathcal{A}$. $Open(\mathcal{A})$ is a frame isomorphic to $Sat(\mathcal{A})$ by the assignment that selects, for each equivalence class $[U]$, the canonical representative $\mathcal{A}U$.

Frames constitute the basic structure of traditional pointfree topology: They are obtained by abstraction from the order-theoretic properties of the lattice of opens of a topological space. As we shall see in section 2.2, *Sat* extends to a functor giving a categorical equivalence between the category of formal topologies and the category of frames.

Summing up, we can say that the idea of formal topology is to present a topological space as a quotient frame of $\mathcal{P}(S)$ for a chosen set of basic neighbourhoods S in terms of a closure operator $\mathcal{A}$ on $\mathcal{P}(S)$ such that the opens of the topology of discourse are the $\mathcal{A}$-saturated subsets. The closure operator can be axiomatized by a quasi cover. Using a quasi cover one can usually avoid quantification over powersets.

In topological spaces (with a sufficient separation property[‡]) a point can be obtained as the intersection of the opens containing it. Starting from this as the basic intuition, in pointfree topology the definition of point is obtained through the properties of the family of opens of a topological space containing a point: A point is contained in some open set; if two opens contain a point, their intersection does; if a point is in an open set, then it is in any bigger open set; finally, if a point is in an union of opens, it is in some open of the union. A subset of a frame with these properties is called a *completely prime filter*. Formally, it is defined as follows:

Definition 2.2. A subset F of a frame A is a *filter* if the following conditions hold:

1. $(\exists a \in A)(a \in F)$;
2. $a, b \in F$ implies $a \wedge b \in F$;
3. $a \in F$, $b \in A$ and $a \leqslant b$ imply $b \in F$.

‡ This property is the T_1 separation axiom, stating that given two distinct points x and y, there is an open set which contains y but not x.

A filter is *completely prime* if for any subset $\{a_i : i \in I\}$ of A, $\bigvee_{i \in I} a_i \in F$ implies $a_i \in F$ for some $i \in I$.

The definition of *formal point* of formal topology is obtained by a reformulation of the definition of completely prime filter.

Definition 2.3. A *formal point* on a formal topology on S is a subset of S such that, for all $a, b \in S$, $U \subseteq S$, the following conditions hold:

1. $(\exists a \in S)(a \in \alpha)$;
2. $\dfrac{a \in \alpha \quad b \in \alpha}{(\exists c \in S)(c \in \alpha \,\&\, c \sqsubseteq a \,\&\, c \sqsubseteq b)}$;
3. $\dfrac{a \in \alpha \quad a \lhd U}{(\exists b \in U)(b \in \alpha)}$.

The third of the above conditions will be referred to as *monotonicity*. If a is a basic neighbourhood in α we also say that α *forces* a, or α *is a point in* a.

2.2. *The category of formal topologies and the representation of complete lattices, frames and sober topological spaces*

The inverse of a continuous function between two topological spaces X and Y gives a map between ΩY and ΩX that preserves finite intersections and arbitrary unions, that is, finite meets and arbitrary joins of the frames of opens of the topological spaces. These properties are taken as the defining properties of a *frame morphism*. We shall denote by **Frm** the category of frames.

Likewise frames, formal topologies are made into a category starting from a similar intuition. The three conditions below express the fact that the map preserves the top element (nullary meet), binary meets and arbitrary joins. For a subset U of S, $f(U)$ denotes $\bigcup_{a \in U} f(a)$.

Definition 2.4. Let $\mathcal{A} = \langle S, \sqsubseteq_{\mathcal{A}}, \lhd_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle T, \sqsubseteq_{\mathcal{B}}, \lhd_{\mathcal{B}} \rangle$ be two formal topologies. A *morphism* from $\mathcal{A}$ to $\mathcal{B}$ is a map f, from elements of S to subsets of T, such that the following conditions are satisfied:

1. $f(S) =_{\mathcal{B}} T$;
2. $f(a) \sqcap f(b) \lhd_{\mathcal{B}} f(a \sqcap b)$;
3. $\dfrac{a \lhd_{\mathcal{A}} U}{f(a) \lhd_{\mathcal{B}} f(U)}$.

Two morphisms f and g from $\mathcal{A}$ to $\mathcal{B}$ are *equal* if, for all $a \in S$, $f(a) =_{\mathcal{B}} g(a)$.

It is easy to prove that formal topologies and morphisms of formal topologies, with composition given by $(f \circ g)(a) \equiv f(g(a))$ and identity defined by $1(a) \equiv \{a\}$, form a category, called the category of formal topologies and denoted by **FTop**.

If S and T are $\wedge$-semilattices, the above definition can be reformulated with the second condition replaced by

2′. $f(a) \wedge f(b) \lhd_{\mathcal{B}} f(a \wedge b)$.

Morphisms of quasi formal topologies are defined similarly, omitting condition 2. We shall denote by **QFTop** the category of quasi formal topologies.

In the presence of the unit (or top element, if we consider $\wedge$-semilattices rather than monoids), the first condition of definition 2.4 is equivalent to

$$f(1_{\mathcal{A}}) =_{\mathcal{B}} 1_{\mathcal{B}} \ ,$$

and ensures that the corresponding frame morphism (defined in the proof of theorem 2.5) preserves finite meets, hence, in particular, the top element, that can be given as the meet of the empty set.

The *initial object* in the category of frames will be denoted by $\mathbf{2}$. By definition, for every frame A there exists a unique frame morphism $!_A : \mathbf{2} \to A$. In traditional pointfree topology (cf. Johnstone 1982, II, 1.3) points on a frame A are shown to correspond to *prime elements* of A, or equivalently, to frame morphisms from A to the frame $\mathbf{2}$.

In formal topology the role of the frame $\mathbf{2}$ is taken over by the formal topology

$$\mathcal{P}(1) \equiv \langle \{1\}, \sqsubseteq, \lhd \rangle$$

with basic preordered set consisting of a singleton set with the identity relation, and cover relation given by

$$a \lhd U \equiv a \in U \ .$$

For every formal topology $\mathcal{A}$, there exists a unique morphism $!_{\mathcal{A}} : \mathcal{P}(1) \to \mathcal{A}$; it maps 1 in the whole set S, thus $\mathcal{P}(1)$ is initial object in the category of formal topologies.

Formal points of $\mathcal{A}$ can be characterized as morphisms from $\mathcal{A}$ to the initial object. Given a formal point α of $\mathcal{A}$, we define $f_\alpha(a) = \{1\}$ if $a \in \alpha$, otherwise we put $f_\alpha(a) = \emptyset$. Clearly f_α is a morphism of formal topologies from $\mathcal{A}$ to $\mathcal{P}(1)$. Conversely, if f is a morphism from $\mathcal{A}$ to the initial object, by defining

$$\alpha_f \equiv \{a \in S : f(a) =_{\mathcal{P}(1)} \{1\}\}.$$

a formal point of $\mathcal{A}$ is obtained, and the correspondence thus established is biunivocal.

By viewing points as morphisms, we can define a contravariant functor Pt, from formal topologies to spaces of formal points, acting on morphisms as follows: If f is a morphism from $\mathcal{A}$ to $\mathcal{B}$, then $Pt(f)$ is the map from $Pt(\mathcal{B})$ to $Pt(\mathcal{A})$ defined by

$$Pt(f)(\alpha) \equiv \alpha \circ f$$

where $\circ$ denotes composition of morphisms.

We can also extend Sat from objects to morphisms as follows: For $U \in Sat(\mathcal{A})$ and f a morphism from $\mathcal{A}$ to $\mathcal{B}$, we put

$$Sat(f)(U) \equiv \mathcal{B}f(U) \ .$$

It is well known (see MacLane 1971) that a functor F between two categories $\mathcal{C}$ and $\mathcal{D}$ gives an equivalence if and only if it is *full*, *faithful* and *dense*, that is, it is bijective on morphisms and surjective, up to isomorphism, on objects.

We shall denote by **SL** the category of complete lattices with join preserving maps as morphism. Due to the morphisms, this category is sometimes called the category of sup-lattices (e.g. in Joyal and Tierney 1984). We have:

Theorem 2.5. The functor *Sat* gives an equivalence between the category **QFTop** of quasi formal topologies and the category **SL** of complete lattices.

Proof: (Sketch. See also Battilotti and Sambin 1993) As shown in section 2.1, for any quasi formal topology $\mathcal{A}$, $Sat(\mathcal{A})$ is a complete lattice. Conversely, given a complete lattice L with base S, we obtain a quasi formal topology $\mathcal{A}_L$ on S by defining, for $a \in S$ and $U \subseteq S$

$$a \lhd U \equiv a \leqslant \bigvee U$$

The map

$$\begin{array}{rcl} Sat(\mathcal{A}_L) & \to & L \\ U & \mapsto & \bigvee U \end{array}$$

gives a bijection between $Sat(\mathcal{A}_L)$ and L with inverse

$$\begin{array}{rcl} L & \to & Sat(\mathcal{A}_L) \\ a & \mapsto & \{b \in S : b \leqslant a\} \equiv \downarrow_S a \end{array}$$

Clearly we have $\bigvee \downarrow_S a = a$. For $\downarrow_S \bigvee U = U$, suppose $a \leqslant \bigvee U$. Since U is a saturated subset, $a \in U$ follows, thus $\downarrow_S \bigvee U \subseteq U$. The other inclusion is obvious. The bijection preserves arbitrary suprema, and therefore it is an isomorphism of complete lattices, thus showing that *Sat* is a dense functor.

Next we prove that given the quasi formal topologies $\mathcal{A}$ and $\mathcal{B}$ the functor *Sat* gives a bijection between morphisms

$$\begin{array}{rcl} Sat : \mathbf{QFTop}(\mathcal{A}, \mathcal{B}) & \to & \mathbf{SL}(Sat(\mathcal{A}), Sat(\mathcal{B})) \\ f & \mapsto & Sat(f) \end{array}$$

Clearly, the mapping is 1-1 by definition of equality of morphisms. As for surjectivity, given a morphism g in $\mathbf{SL}(Sat(\mathcal{A}), Sat(\mathcal{B}))$, define f by putting, for a in the base S of $\mathcal{A}$, $f(a) \equiv g(\mathcal{A}a)$. It is routine to verify that f is a morphism of quasi formal topologies between $\mathcal{A}$ and $\mathcal{B}$ and that $Sat(f) = g$. $\Box$

All the equivalences throughout the paper will be proved by restriction of the basic equivalences given in theorem 2.5 and theorem 2.7 below, between the category of quasi formal topologies and the category of complete lattices and between the category of spatial formal topologies and the category of sober topological spaces. In these proofs we shall make use of the following general proof method: First we recall that a subcategory $\mathcal{C}'$ of $\mathcal{C}$ is *full* if the morphisms of $\mathcal{C}'$ between two objects are the same as the morphism in $\mathcal{C}$. Suppose we are given an equivalence between $\mathcal{C}$ and $\mathcal{D}$ through a functor F which is dense, full and faithful and suppose we are given two full subcategories $\mathcal{C}'$ and $\mathcal{D}'$ of $\mathcal{C}$ and $\mathcal{D}$, respectively. Suppose that F restricts to a dense functor between $\mathcal{C}'$ and $\mathcal{D}'$, that is, for all A in $\mathcal{C}'$, FA is in $\mathcal{D}'$ and for all B in $\mathcal{D}'$ there exists A in $\mathcal{C}'$ such that FA is isomorphic to B. Then F is full, faithful and dense also as a functor between $\mathcal{C}'$ and $\mathcal{D}'$ and therefore gives an equivalence between $\mathcal{C}'$ and $\mathcal{D}'$. In all these cases we shall say that the equivalence between $\mathcal{C}'$ and $\mathcal{D}'$ is proved *by restriction* of the equivalence between $\mathcal{C}$ and $\mathcal{D}$.

As a first application of this method, by restriction of the equivalence between the category quasi formal topologies and the category of complete lattices, we obtain:

Corollary 2.6. The category of formal topologies is equivalent to the category of frames.

Proof: As indicated in section 2.1, if $\mathcal{A}$ is a formal topology, $Sat(\mathcal{A})$ is a frame. Moreover, if the complete lattice is a frame, $\mathcal{A}_L$ defined as in the proof of theorem 2.5 is a formal topology. □

We summarize these equivalences in the diagram

$$\begin{tikzcd} \mathbf{QFTop} \arrow[r, "\sim"] & \mathbf{SL} \\ \mathbf{FTop} \arrow[u, hook] \arrow[r, "\sim"] & \mathbf{Frm} \arrow[u, hook] \end{tikzcd} \tag{1}$$

where horizontal arrows are equivalences and vertical arrows inclusions of categories.

For any formal topology $\mathcal{A}$, the *spatial topology* on $Pt(\mathcal{A})$ is the topology with base given by the family $\{ext(a)\}_{a \in S}$ where $a \in S$ and $ext(a)$ is the collection of formal points forcing a. In this way $Pt(\mathcal{A})$ becomes a topological space thus justifying the word "space" in the expression "formal space".

By the condition of monotonicity for formal points, if $a \lhd U$, then for any formal point α such that $a \in \alpha$, there exists b in U with $b \in \alpha$. The converse does not necessarily hold, and indeed is the defining property of *spatial* formal topologies (sometimes also called *extensional* formal topologies).

The following result is the counterpart of a well known result for spatial locales; its details can be easily filled in from the proof of the latter (cf. Johnstone 1982, ch. II, 1.4–1.7):

Theorem 2.7. The category **EFTop** of spatial formal topologies is equivalent to the category **STop** of sober topological spaces with continuous maps.

Proof: (Sketch) Given a formal topology $\mathcal{A}$, the space $Pt(\mathcal{A})$ with the spatial topology is a sober topological space and for a morphism of formal topologies f from $\mathcal{A}$ to $\mathcal{B}$, $Pt(f)$ is a continuous map from $Pt(\mathcal{B})$ to $Pt(\mathcal{A})$. Given a sober topological space X, by sobriety we have $X \cong Pt(\Omega X)$. We consider, for a base S of the topological space X, the formal cover on S defined by

$$a \lhd U \ \equiv \ a \subseteq \bigcup U$$

where a is a basic open of X, U is a collection of basic opens, and $\subseteq$ is set-theoretic inclusion. This gives a spatial formal topology $\mathcal{A}$ and we have $X \cong Pt(\mathcal{A})$. We conclude with observing that Pt is a dense, full and faithful functor between the two categories. □

To be more precise, what we have proved above is a *duality* between the two categories, that is, an equivalence between **STop** and $\mathbf{EFTop}^{op}$, the opposite category of **EFTop**, since Pt is a contravariant functor, i.e., a functor that reverses the direction of morphisms.

In section 5 we specialize this result to an equivalence between a particular subcategory of the category of formal topologies and the category of locally compact sober topological spaces with continuous functions as morphisms.

2.3. *Stone and Scott formal topologies and the representation of algebraic frames*

We recall from Sambin (1987) that a cover $\lhd$ on a base S is called a *Stone cover* if, for all $a \in S$ and $U \subseteq S$, $a \lhd U$ implies $a \lhd U_0$ for some finite subset U_0 of U (written $U_0 \subseteq_\omega U$). We remark that the notion of being Stone can apply not just to a cover, but, with more generality, to any quasi cover.

Before proceeding we recall some definitions from lattice theory.

Definition 2.8. An element a of a complete lattice L is *compact* (*finite*) if for every $U \subseteq L$ such that $a \leqslant \bigvee U$ there exists $K \subseteq_\omega U$ such that $a \leqslant \bigvee K$. A complete lattice is *algebraic* if it has a base of compact element, that is, every element can be obtained as the supremum of a set of compact elements.

Definition 2.9. Given a subset U of a lattice L, the *ideal generated by* U, denoted $\mathcal{I}(U)$, is the subset of L consisting of finite suprema $u_1 \vee \ldots \vee u_n$ of elements from U.

If $\mathcal{A}$ is a Stone quasi formal topology, then by definition all elements of $Sat(\mathcal{A})$ of the form $\mathcal{A}a$ for a in the base S are compact. Furthermore it is an immediate consequence of the definition of compact element that finite suprema of compact elements are compact. Therefore saturations of finite sets of the base are compact. Conversely, if U is compact in $Sat(\mathcal{A})$ then there exists a finite subset K of U such that $U \lhd K$. Summing up, we have:

Lemma 2.10. Let $\mathcal{A}$ be a Stone quasi formal topology. Then an element U of $Sat(\mathcal{A})$ is compact if and only if there exists a finite subset K of U such that $U = \mathcal{A}K$.

We have:

Proposition 2.11. If $\mathcal{A}$ is a Stone quasi formal topology, then $Sat(\mathcal{A})$ is an algebraic lattice.

Proof: The complete lattice $Sat(\mathcal{A})$ is generated by the base of compact elements $\{\mathcal{A}a\}_{a \in S}$ since, for any element U we have $U = \bigvee_{a \in U} \mathcal{A}a$, and for all $a \in S$, $\mathcal{A}a$ is compact by definition of Stone quasi formal topology. □

Corollary 2.12. If $\mathcal{A}$ is a Stone formal topology, then $Sat(\mathcal{A})$ is an algebraic frame.

Conversely we have:

Theorem 2.13. Every algebraic lattice L is isomorphic to a lattice of the form $Sat(\mathcal{A})$ for a Stone quasi formal topology $\mathcal{A}$. If L is an algebraic frame, $\mathcal{A}$ is a Stone formal topology.

Proof: Given an algebraic lattice L, the relation between elements and subsets of L defined by

$$a \lhd U \equiv a \in \mathcal{I}(U),$$

gives a Stone quasi formal topology with a lattice of saturated subsets isomorphic to L. If L is an algebraic frame, it gives a Stone formal topology with a frame of saturated subsets isomorphic to L. □

We denote by **SQFTop** and by **SFTop** the categories of Stone quasi formal topologies and Stone formal topologies, respectively. Let moreover **AL** and **AFrm** be the categories of algebraic lattices and algebraic frames, which are full subcategories of **SL** and **Frm**, respectively. We have the diagram

$$\begin{tikzcd} \mathbf{SQFTop} \arrow[r, "\sim"] & \mathbf{AL} \\ \mathbf{SFTop} \arrow[r, "\sim"] \arrow[u, hook] & \mathbf{AFrm} \arrow[u, hook] \end{tikzcd} \tag{2}$$

where each category is a full subcategory of the corresponding category in diagram 1. By proposition 2.11, corollary 2.12 and theorem 2.13 and equivalence by restriction, horizontal arrows are equivalences. In section 4 we shall exhibit an "intermediate" diagram between diagram 1 and diagram 2: each vertex is labelled with a category including the corresponding category of diagram 2 and inlcuded in the corresponding category of diagram 1.

As observed above, compact elements in a lattice are closed under finite joins. They are not however closed under finite meets. An algebraic frame with the property that compact elements are closed under finite meets is called a *coherent* frame.

We remark that the slightly different definition of formal topology given in Sambin 1987 gives, as a consequence of the properties of $\wedge$-*left* and $\wedge$-*right*, $\mathcal{A}U \cap \mathcal{A}V = \mathcal{A}(U \wedge V)$. In particular, if $\mathcal{A}U$ and $\mathcal{A}V$ are compact elements of $Sat(\mathcal{A})$, by using the characterization given in lemma 2.10, we obtain that their meet is compact (This is no longer true with the more general definition of formal topology adopted here). The representation of algebraic frames given above thus becomes in Sambin (1987) a representation of coherent frames. An alternative way of representing coherent frames is given in Negri (1996) by using, instead of Stone formal topologies the structures called *Stone bases*. In these structures the covers, which are relations between elements and subsets of a set, are replaced by their finite traces, which are relations between elements and *finite* subsets of a set, thus allowing for a predicative treatment within type theory.

We say that a cover $\lhd$ on S is a *Scott cover* (or alternatively, as in Sigstam 1990, that it has the *Scott property*) if for all $a \in S$ and $U \subseteq S$,

$$a \lhd U \text{ implies } a \lhd b \text{ for some } b \in U \tag{3}$$

We remark that this definition of Scott cover differs from the one given in Sambin (1987), and used in Sambin, Valentini and Virgili (1996), where condition (3) is only required for positive elements of the base.

Scott quasi formal topologies are defined similarly.

The lattice-theoretic structures corresponding to Scott quasi formal topologies are complete lattices in which every element a is generated by *supercompact* elements, i.e., elements satisfying, for arbitrary subsets T

$$a \leqslant \bigvee T \text{ implies } a \leqslant b \text{ for some } b \in T\ .$$

We could give here a diagram similar to (2) above for Scott quasi formal topologies and Scott formal topologies, but we avoid doing so not to introduce more terminology than necessary, as there is no established name for the corresponding lattice-theoretic categories.

In section 4 we shall study a generalization of these structures, the *locally supercompact lattices*, also known in the literature as *prime-continuous lattices*, and shall relate them to a generalization of Scott quasi formal topologies.

Given two covers $\lhd_1$ and $\lhd_2$ on the same base S, we say that $\lhd_2$ is a *quotient* of $\lhd_1$ (or is *greater* than $\lhd_1$) if for all $a \in S$ and $U \subseteq S$,

$$a \lhd_1 U \Rightarrow a \lhd_2 U \ .$$

The *Stone* (resp. *Scott*) *compactification* of a cover is defined as the greatest Stone (resp. Scott) cover of which the given cover is a quotient. They are defined, respectively, by

$$\begin{aligned} a \lhd_f U &\equiv (\exists U_0 \subseteq_\omega U)(a \lhd U_0) \\ a \lhd_s U &\equiv (\exists b \in U)(a \lhd b) \ . \end{aligned}$$

The Stone and Scott compactifications for quasi covers are defined in the same way.

The Stone (resp. Scott) compactification of a formal topology $\mathcal{A} \equiv \langle S, \sqsubseteq, \lhd \rangle$ is the formal topology $\mathcal{A}_f \equiv \langle S, \sqsubseteq, \lhd_f \rangle$ (resp. $\mathcal{A}_s \equiv \langle S, \sqsubseteq, \lhd_s \rangle$).

2.4. *Positivity and open locales*

The definition of formal topology given in Sambin (1987) differs from definition 2.1 in two respects: The first is that in Sambin (1987) a base is a commutative monoid with unit, rather than a partially ordered set. This is inessential, for, as we have shown, all the basic concepts can be reformulated starting from such a more general structure. The second, not inessential, difference, is that in Sambin (1987) a formal topology comes equipped with a *positivity predicate*, that is, a predicate $Pos(a)$ on elements a of the base S that satisfies

$$\textit{monotonicity} \quad \frac{Pos(a) \quad a \lhd U}{Pos(U)} \ , \qquad \text{where} \qquad Pos(U) \equiv (\exists b \in U)\ Pos(b)$$

$$\textit{positivity} \quad \frac{a \lhd U}{a \lhd U^+} \ , \qquad \text{where} \qquad U^+ \equiv \{b \in S \mid b \in U \,\&\, Pos(b)\}$$

A condition equivalent to *positivity* is

$$\textit{openness} \quad a \lhd a^+ \ .$$

The choice of terminology is not casual: *Positivity* in formal topology corresponds to the condition of *openness* in locale theory. We recall from Johnstone (1984, section 2), the following definition and characterizing property:

Definition 2.14. A locale A is *open* if the unique frame map $!_A : \mathbf{2} \to A$ has a left adjoint `Pos`.

Lemma 2.15. A locale A is open iff there exists a map $\mathtt{Pos} : A \to \mathbf{2}$ such that, whenever $a \in A$, $S \subseteq A$ and $a \leqslant \bigvee S$, then we also have

$$a \leqslant \bigvee \{s \in S \mid \mathtt{Pos}(s)\} \ .$$

We have:

Proposition 2.16. If a formal topology has a positivity predicate then $Sat(\mathcal{A})$ is an open locale and conversely, if A is an open locale, then any formal topology representing A via the functor Sat has a positivity predicate.

Proof: Given a formal topology $\mathcal{A}$ with positivity predicate Pos, define, for U in $Sat(\mathcal{A})$, $\mathtt{Pos}(U) \equiv (\exists b \in U) Pos(b)$. $\mathtt{Pos}$ is a frame morphism from $Sat(\mathcal{A})$ to $\mathbf{2}$ and by *positivity* the condition of lemma 2.15 is satisfied, thus $Sat(\mathcal{A})$ is an open locale. Conversely, given an open locale A, let $\mathcal{A}$ be a formal topology representing A via the funtor Sat, and define, for a in the base of $\mathcal{A}$, $Pos(a) \equiv \mathtt{Pos}(\mathcal{A}a)$. Monotonicity for Pos holds since a frame morphism is order preserving and positivity follows from lemma 2.15 □

Definition 2.3 of formal points reduces to the definition given in Sambin (1987) if the formal topology $\mathcal{A}$ is equipped with a unit 1 and a positivity predicate Pos. Indeed, $1 \in \alpha$ follows from condition 1 in 2.3 and the fact that if 1 is the unit, we have $a \lhd 1$ for all a; the implication $a \in \alpha \to Pos(a)$ follows from *positivity* and condition 3 of definition 2.3.

The definition of morphism of formal topologies in Sambin (1987) contains an additional requirement, namely

$$Pos(f(a)) \to Pos(a) \ .$$

Also this follows from the other properties of morphisms in the presence of a positivity predicate: From $a \lhd a^+$, by condition 3 of definition 2.4, we obtain $f(a) \lhd f(a^+)$. If $Pos(f(a))$ holds, by monotonicity we get $Pos(f(a^+))$, that is, $(\exists b \in a^+)(\exists c \in f(b)) Pos(c)$. Thus in particular a^+ contains an element, therefore $Pos(a)$ holds.

The above shows that if the formal topology is equipped with a positivity predicate, our definitions of formal points and of morphisms are equivalent to those in Sambin 1987. However, the following result shows that positivity is too strong a requirement in the presence of compactness.

Proposition 2.17. Let $\mathcal{A} = \langle S, \sqsubseteq, \lhd \rangle$ be a Stone formal topology with positivity predicate Pos. Then Pos is decidable.

Proof: Suppose that Pos satisfies *positivity*, therefore $a \lhd a^+$. Since $\lhd$ is a Stone cover, there exists a finite subset U_0 of the base which is a subset of a^+ for which $a \lhd U_0$. We know, from the fact that such a U_0 is a finite set, whether it is empty or contains at least one element. In the first case we have $a \lhd \emptyset$, and therefore, by monotonicity of Pos, also $\neg Pos(a)$, for, if we assume $Pos(a)$, we obtain that there exists $b \in \emptyset$ such that $Pos(b)$, a contradiction. In the second case, since U_0 is a subset of a^+, there exists at least one element in a^+, that is, $Pos(a)$. Therefore for all $a \in S$, $Pos(a) \vee \neg Pos(a)$ holds, that is, Pos is decidable. □

For a decidable positivity predicate the property of positivity trivializes, as it follows directly from monotonicity that nonpositive elements are covered by the empty set. In particular, proposition 2.17 shows that for a given formal topology a genuine, undecidable positivity predicate does not extend to its Stone compactification. This limitation and the fact that the positivity predicate is not needed for the results of this work have motivated the choice of leaving it out in our definition of formal topology.

3. Reals, partial reals and intervals

In this section we show examples of formal topologies in which the cover is presented via a Stone or Scott cover. They are all given by means of a *finitary inductive definition* (cf. Aczel 1977), where each rule involved has only finitely many premises.

We start with the topology of formal reals (cf. Negri and Soravia 1999, Cederquist and Negri 1996, Cederquist, Coquand and Negri 1998). This is our motivating example, and we recall the presentation of real numbers as formal points in detail. In addition, we introduce the topology of partial reals.

In the approach to constructive reals via formal topology, real numbers are obtained as increasingly refined rational approximations. *Formal reals* are the formal points of the formal topology on the rationals, defined as follows:

Definition 3.1. The *formal topology of formal reals* is the formal topology on the set $Q\times Q$ of pairs of rationals with preorder defined by $(p,q)\sqsubseteq(r,s)\equiv r\leqslant p\,\&\,q\leqslant s$ and cover $\lhd_{\mathcal{R}}$ defined by

$$(p,q)\lhd_{\mathcal{R}}U\equiv(\forall p',q')(p<p'<q'<q\rightarrow(p',q')\lhd_{\mathcal{R}_f}U),$$

where the relation $\lhd_{\mathcal{R}_f}$ is inductively defined by

1. $\dfrac{q\leqslant p}{(p,q)\lhd_{\mathcal{R}_f}U}$;
2. $\dfrac{(p,q)\in U}{(p,q)\lhd_{\mathcal{R}_f}U}$;
3. $\dfrac{(p,s)\lhd_{\mathcal{R}_f}U\quad(r,q)\lhd_{\mathcal{R}_f}U\quad p\leqslant r<s\leqslant q}{(p,q)\lhd_{\mathcal{R}_f}U}$;
4. $\dfrac{(p,q)\sqsubseteq(p',q')\quad(p',q')\lhd_{\mathcal{R}_f}U}{(p,q)\lhd_{\mathcal{R}_f}U}$.

There are other equivalent definitions for the topology of formal reals. We have chosen the above definition in order to single out the feature of interest for our analysis, namely the fact that the cover $\lhd_{\mathcal{R}_f}$ is the Stone compactification of the cover $\lhd_{\mathcal{R}}$, as proved in Cederquist and Negri (1996).

The proof that $\lhd_{\mathcal{R}}$ is a cover makes essential use of a lemma we recall here from Cederquist and Negri (1996):

Lemma 3.2. Suppose $(p,q)\lhd_{\mathcal{R}_f}U, U\lhd_{\mathcal{R}}V$, and let $p<p'<q'<q$. Then $(p',q')\lhd_{\mathcal{R}_f}V$.

The definition of formal reals can be given more explicitly by unfolding the condition of monotonicity in the general definition of formal point and specializing it to the inductive clauses of definition 3.1:

Definition 3.3. A *formal real* is a subset α of $Q\times Q$ such that:

1. $(\exists(p,q)\in Q\times Q)((p,q)\in\alpha)$;

2. $(p,q) \in \alpha \,\&\, (r,s) \in \alpha \;\leftrightarrow\; (max(p,r), min(q,s)) \in \alpha$;
3. $(p,q) \in \alpha \,\&\, r < s \;\rightarrow\; (p,s) \in \alpha \vee (r,q) \in \alpha$;
4. $(p,q) \in \alpha \;\rightarrow\; (\exists(p',q'))(p < p' < q' < q \,\&\, (p',q') \in \alpha)$.

The collection of formal reals is extended by allowing also "unsharp" elements that we call *partial reals.* The definition of partial reals is like the definition of formal reals, with the locatedness condition of clause 3 omitted The presentation of the formal topology having these as formal points is obtained by omitting the first and third axiom from the definition of $\lhd_{\mathcal{R}_f}$:

Definition 3.4. The *formal topology of partial reals* is the formal topology on $Q \times Q$ with preorder defined as in definition 3.1. The cover relation $\lhd_{\mathcal{PR}}$ is defined as in 3.1 from a relation $\lhd_{\mathcal{PR}_s}$ fulfilling conditions 2 and 4 of the definition of $\lhd_{\mathcal{R}_f}$.

It is easy to prove the following result, showing that the topology of partial reals constitutes an example of the class of *locally Scott formal topologies*, to be introduced in section 4.2:

Proposition 3.5.

1. The relations $\lhd_{\mathcal{PR}}$ and $\lhd_{\mathcal{PR}_s}$ are covers.
2. For all $(p,q) \in Q \times Q$ and $U \subseteq Q \times Q$, $(p,q) \lhd_{\mathcal{PR}_s} U$ implies $(p,q) \lhd_{\mathcal{PR}} U$.
3. For all $(p,q) \in Q \times Q$ and $U \subseteq Q \times Q$, $(p,q) \lhd_{\mathcal{PR}_s} U$ implies that there exists $(r,s) \in U$ such that $(p,q) \lhd_{\mathcal{PR}_s} (r,s)$.

We shall denote by $Pt(\mathcal{PR})$ the formal points of $\mathcal{PR}$.

The relation on formal reals defined by

$$\alpha < \beta \equiv (\exists(p,q) \in \alpha)(\exists(r,s) \in \beta)(q < r)$$

is a relation of *strict linear order*, that is, it satisfies, for all $\alpha, \beta, \gamma \in Pt(\mathcal{PR})$,

1. $\sim (\alpha < \beta \,\&\, \beta < \alpha)$
2. $\alpha < \beta \rightarrow \alpha < \gamma \vee \gamma < \beta$

It follows that the relation defined by

$$\alpha \leqslant \beta \equiv \sim \beta < \alpha$$

is a partial order, and that the relation

$$\alpha \# \beta \equiv \alpha < \beta \vee \beta < \alpha$$

is an apartness relation, i.e., a relation satisfying

1. $\sim \alpha \# \alpha$
2. $\alpha \# \beta \rightarrow \alpha \# \gamma \vee \beta \# \gamma$

By representing rationals p via the embedding $\bar{p} \equiv \{(r,s) : r < p < s\}$, formal reals can be characterized as partial reals that are well located with respect to the rationals:

Proposition 3.6. Let $\alpha \in Pt(\mathcal{PR})$. Then the following are equivalent:

1. $\alpha \in Pt(\mathcal{R})$.
2. $(\forall p,q)(p < q \rightarrow \bar{p} < \alpha \vee \alpha < \bar{q})$.
3. $(\forall k \in Q^+)(\exists(p,q))(q - p < k \,\&\, (p,q) \in \alpha)$.

A global version of the above characterization is given by the following:

Corollary 3.7. $Pt(\mathcal{PR}) = Pt(\mathcal{R})$ iff the relation $\#$ is an apartness relation on $Pt(\mathcal{PR})$.

We now turn to another example of inductively generated formal topology. Similarly to the definition of formal reals, we can define the formal space $[a, b]$ that corresponds to the closed interval of the real line with rational endpoints a and b. The formal points of this space are exactly the formal reals α with $a \leqslant \alpha \leqslant b$. The following definition is equivalent to the one used in Cederquist and Negri (1996), but here an explicit presentation of its Stone compactification is given. This is achieved by adding to the axioms for the finitary cover $\lhd_{\mathcal{R}_f}$ of formal reals two axioms expressing that intervals not overlapping with $[a, b]$ are covered by anything.

Definition 3.8. Let a, b be rationals such that $a < b$. The *formal topology of the closed interval* $[a, b]$ is the formal topology on $Q \times Q$ preordered as in 3.1, with cover relation $\lhd$ defined by

$$(p,q) \lhd U \equiv (\forall p', q')(p < p' < q' < q \to (p', q') \lhd_f U),$$

where the relation $\lhd_f$ is defined inductively by

1. $\dfrac{(p,q) \lhd_{\mathcal{R}_f} U}{(p,q) \lhd_f U}$;
2. $\dfrac{q \leqslant a}{(p,q) \lhd_f U}$;
3. $\dfrac{b \leqslant p}{(p,q) \lhd_f U}$.

Then we have

Lemma 3.9. $(p,q) \lhd_f U$ if and only if $(p,q) \lhd_{\mathcal{R}_f} U \cup \{(p,a), (b,q)\}$

Proof: Routine induction on derivation of covers. □

Proposition 3.10. The relation $\lhd$ is a cover, with Stone compactification given by $\lhd_f$.

Proof: We start with proving that $\lhd_f$ is a cover: Since we already know that $\lhd_{\mathcal{R}_f}$ is a cover, to get to the conclusion we only need to prove that transitivity, left and right hold when $(p,q) \lhd_f U$ is derived from the new axioms 2 and 3 in definition 3.8, which is straightforward. In order to prove that $\lhd$ is a cover, we observe that lemma 3.2 also holds when $\lhd_{\mathcal{R}}$ and $\lhd_{\mathcal{R}_f}$ are replaced by $\lhd$ and $\lhd_f$, respectively: This is seen by the trivial inspection of the cases in which $(p,q) \lhd_f U$ is obtained by axiom 2 or 3. The above directly implies that $\lhd$ satisfies transitivity. The verification that it satisfies reflexivity, left and right is easy.

The proof that $\lhd_f$ is a Stone cover is done by induction on the derivation: If $(p,q) \lhd_f U$ is obtained from $(p,q) \lhd_{\mathcal{R}_f} U$ then the information that $\lhd_{\mathcal{R}_f}$ is a Stone cover provides us with a finite subcover. If it is obtained from $q \leqslant a$ or $b \leqslant p$ then the empty set is a finite subcover of (p,q).

The verification that $\lhd$ is a quotient of $\lhd_f$ is straightforward.

If $(p,q) \lhd U$, with U finite, then, by an immediate consequence of the definitions and the lemma, we have $(p,q) \lhd_{\mathcal{R}} U \cup \{(p,a), (b,q)\}$. Since $\lhd_{\mathcal{R}_f}$ is the Stone compactification of $\lhd_{\mathcal{R}}$, we have that $(p,q) \lhd_{\mathcal{R}_f} U \cup \{(p,a),(b,q)\}$. By the lemma we obtain $(p,q) \lhd_f U$. □

Another example of inductively generated formal topology is the topology of linear and continuous functionals of norm $\leqslant 1$ from a seminormed linear space to the reals (cf. Cederquist, Coquand and Negri 1998). We shall not discuss it here, but just observe that also in this case, as for the topology of formal reals and closed intervals, the cover is given by a quotient of a Stone cover.

The topology of formal reals, of closed intervals, and of linear and continuous functionals have been completely formalized in type theory and implemented in the type-theoretical proof editor Half in Cederquist (1997,1998).

We shall see in section 4.2 how the above examples of formal topologies arise as instances of two special classes of formal topologies.

4. Representation of continuous domains in formal topology

In this section we shall give representation theorems for various types of continuous domains using formal topology. The representations will in particular cover continuous directed complete partial orders, algebraic directed complete partial orders, Scott domains, continuous lattices and prime-continuous lattices.

4.1. *Background on continuous domains*

We proceed with recalling some standard definitions and basic facts from domain theory (see Gierz *et al.* 1980, Abramsky and Jung 1994). We refer to the bibliographical notes of Johnstone (1982) for an account of the origins of these notions and to Smyth (1992) and Vickers (1989) for a general background on topology in theoretical computer science. The relation $\lll$ appears in Abramsky and Jung (1994).

Definition 4.1. A partially ordered set $(L, \leqslant)$ is *directed* if it is nonempty and if for all $x, y \in L$ there exists $z \in L$ such that $x \leqslant z$ and $y \leqslant z$.
A *directed complete partial order* (dcpo for short) is a partially ordered set in which every directed subset has a least upper bound.

Definition 4.2. Let L be a dcpo and let $x, y \in L$. We say that x *approximates* y, or x is *way-below* y, and write

$$x \ll y \ ,$$

if, for all directed subsets A of L, $y \leqslant \bigvee A$ implies that there exists $a \in A$ such that $x \leqslant a$.
We say that x is *compact* if it approximates itself.
We say that x *prime-approximates* y, and write

$$x \lll y \ ,$$

if, for all subsets A of L such that $\bigvee A$ exists in L, $y \leqslant \bigvee A$ implies that there exists $a \in A$ such that $x \leqslant a$. An element is called *super-compact* if it prime-approximates itself.

A dcpo is *continuous* if, for all $x \in L$, the set $\{y \mid y \ll x\}$ is directed and $x = \bigvee\{y \mid y \ll x\}$.
A *continuous lattice* is a continuous complete lattice.
A complete lattice is *prime-continuous* if, for all $x \in L$, $x = \bigvee\{y \mid y \lll x\}$.
A dcpo is called *algebraic* if it has a base of compact elements.
An algebraic dcpo D is a *Scott domain* if the join of every upper bounded pair of compact elements exists in D.

Clearly, every algebraic dcpo is continuous, but not conversely. The condition characterizing Scott domains among algebraic dcpo's is sometimes called *consistent completeness* in the literature.

Observe that the above notion of compactness reduces to the usual definition for spaces, the finite subcover property, since the least upper bound of a set is the same as the least upper bound of the directed set of the least upper bounds of its finite subsets.

We recall some basic facts about the way-below relation:

Proposition 4.3. Let L be a dcpo and let x, y, z, w be elements of L. Then:

1. $x \ll y$ implies $x \leqslant y$;
2. $x \leqslant y \ll z \leqslant w$ implies $x \ll w$;
3. If y is compact, $x \leqslant y$ implies $x \ll y$.

Proof: 1. Choose $\{y\}$ as directed subset. 2. Holds by the definitions. 3. Follows from 2 since for compact y, $y \ll y$. □

Definition 4.4. A subset S of a continuous dcpo L is a *base* of L if, for all $a \in L$, the collection of $x \in S$ with $x \ll a$ is a directed subset (denoted by $\twoheaddownarrow_S a$) with supremum a.
A subset S of a prime-continuous lattice L is a *base* of L if, for all $a \in L$, the collection of $x \in S$ with $x \lll a$ is a subset (denoted by $\Downarrow_S a$) with supremum a.

In the sequel we will refer only to set-based domains and sometimes omit the subscript denoting the base.

Definition 4.5. A subset O of a partially ordered set L is *Scott open* if it is upward closed (i.e., $x \in O$ and $x \leqslant y$ imply $y \in O$) and it splits least upper bounds of directed subsets (i.e., if A is directed with a least upper bound, then $\bigvee A \in O$ implies $(\exists x \in A)(x \in O)$).

In any continuous dcpo, subsets of the form $\twoheaduparrow b = \{x \mid b \ll x\}$, where b belongs to a base of L, give a base of the Scott topology.

For a subset U of a continuous dcpo L, we shall sometimes write $\twoheaddownarrow U$ for $\bigcup_{a \in U} \twoheaddownarrow a$ and $\twoheaduparrow U$ for $\bigcup_{a \in U} \twoheaduparrow a$

Proposition 4.6. (Interpolation property) In a continuous dcpo L, if $a \ll b$, there exists c such that $a \ll c \ll b$. In a prime-continuous lattice, if $a \lll b$, there exists c such that $a \lll c \lll b$. If S is a base of L, c may be chosen from S.

Proof: Cf. Abramsky and Jung (1994, pp. 14 and 69). □

The following two results for continuous lattices are instances of theorems 7.1.1 and 7.1.3 in Abramsky and Jung (1994) (for the equational characterization of the class of continuous lattices, see also theorem I.2.3 in Gierz *et al.* 1980). The proofs of these stronger statements require the axiom of choice, whereas those we need have constructive proofs:

Lemma 4.7. A continuous lattice L satisfies the *directed infinite distributive law*: For any $a \in L$ and any directed subset $\{b_i : i \in I\}$ of L,

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} a \wedge b_i \ .$$

Proof: The inequality from right to left holds in any lattice, so the claim amounts to proving that $a \wedge \bigvee_{i \in I} b_i \leqslant \bigvee_{i \in I} a \wedge b_i$. Let $x \ll a \wedge \bigvee_{i \in I} b_i$. Then $x \ll \bigvee_{i \in I} b_i$ and therefore there exists $i \in I$ such that $x \leqslant b_i$ and indeed such that $x \leqslant a \wedge b_i$ since $x \leqslant a$. Thus $x \leqslant \bigvee_{i \in I} a \wedge b_i$. Since L is continuous this proves the claim. □

Lemma 4.8. A distributive continuous lattice L satisfies the *infinite distributive law*: For any $a \in L$ and any subset $\{b_i : i \in I\}$ of L

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} a \wedge b_i \ .$$

Proof: Just observe that $\bigvee_{i \in I} b_i = \bigvee_{I_0 \subseteq_\omega I} \{\vee_{i \in I_0} b_i\}$ and that the set of finite subsets of I is directed and so are the suprema indexed on these. □

The above lemma explains why distributive continuous lattices are also called *continuous Heyting algebras.*

Lemma 4.9. A prime-continuous lattice L satisfies the infinite distributive law.

Proof: As for 4.7, using prime-continuity instead of continuity. □

4.2. *Locally Stone and locally Scott formal topologies*

The examples of the topologies of formal reals and formal intervals of section 3 motivate the introduction of two particular classes of formal topologies. These two classes permit, as we shall see, a representation theorem for continuous and prime-continuous lattices. The definition of locally Stone formal topology is obtained by restating the properties of the way-below relation in terms of formal topology. The definition of locally Scott formal topology is obtained analogously by translating the notion of prime-approximation.

Definition 4.10. A formal topology $\mathcal{A} \equiv \langle S, \sqsubseteq, \lhd \rangle$ is called *locally Stone* if there exists a map i from elements to subsets of S such that, for all $a \in S$ and $U \subseteq S$,

1. $a =_{\mathcal{A}} i(a)$;
2. $a \lhd U$ implies $i(a) \lhd_f i(U)$, where $i(U) \equiv \bigcup_{b \in U} i(b)$.

A formal topology is called *locally Scott* if there exists a map i as above such that, for all $a \in S$ and $U \subseteq S$,

1. $a =_{\mathcal{A}} i(a)$,
2. $a \lhd U$ implies $i(a) \lhd_s i(U)$.

The definition of locally Stone formal topology can be explained in terms of local compactness as it states that every basic neighbourhood is covered by relatively compact neighbourhoods (the elements of $i(a)$). This informal motivation will be made precise in the representation theorem of section 5. The aim of such a definition is to express local compactness in formal topology without making use of quantification over subsets, which is instead the evident impredicative feature of the definition of the way-below relation for continuous lattices.

It follows from the results in the previous section that the topologies of formal reals and of formal closed intervals are locally Stone and that the topology of partial reals is locally Scott. In these examples we have

$$i((p,q)) = \{(p',q') : p < p' < q' < q\} .$$

For instance, in order to prove that the topology of formal reals satisfies condition 2 for locally Stone covers, suppose $(p,q) \lhd_{\mathcal{R}} U$ and let (p',q') in $i(p,q)$. Then there exists (p'',q'') with $p < p'' < p' < q' < q'' < q$. By definition of $\lhd$ we have $(p'',q'') \lhd_{\mathcal{R}_f} U$ and $U \lhd_{\mathcal{R}} i(U)$, thus by lemma 3.2, $(p'',q'') \lhd_{\mathcal{R}_f} i(U)$ and therefore $(p',q') \lhd_{\mathcal{R}_f} i(U)$.

The topology of linear functionals is locally Stone. Using the notation of Cederquist, Coquand and Negri (1998), we have

$$i(\langle x_1 \in I_1, \ldots, x_n \in I_n\rangle) = \{\langle x_1 \in J_1, \ldots, x_n \in J_n\rangle \mid J_1 < I_1, \ldots, J_n < I_n\} .$$

The definitions of *locally Stone* and *locally Scott quasi formal topologies* are obtained from the above in the obvious way, simply by replacing "formal topology" with " quasi formal topology".

We observe that a Stone (resp. Scott) formal topology is locally Stone (resp. Scott) with $i(a) = \{a\}$. The same holds for quasi formal topologies.

4.3. *Representation of continuous and prime-continuous lattices*

We proceed by showing that the frames of saturated subsets of locally Stone and locally Scott quasi formal topologies give continuous and prime-continuous lattices, respectively.

Theorem 4.11. If $\mathcal{A}$ is a locally Stone quasi formal topology, then $Sat(\mathcal{A})$ is a continuous lattice.

Proof: Let $U \in Sat(\mathcal{A})$. We have to prove that $U = \bigvee\{V \in Sat(\mathcal{A}) : V \ll U\}$, i.e., $U = \mathcal{A}(\bigcup\{V \in Sat(\mathcal{A}) : V \ll U\})$, that is, $U \lhd \bigcup\{V \in Sat(\mathcal{A}) : V \ll U\}$ and $\bigcup\{V \in Sat(\mathcal{A}) : V \ll U\} \lhd U$. Since $V \ll U$ implies $V \leqslant U$, which in $Sat(\mathcal{A})$ means $V \lhd U$, the latter holds, so it is enough to prove the former. Observe that $b \in i(a)$ implies $\mathcal{A}b \ll \mathcal{A}a$: If $\mathcal{A}a \leqslant \bigvee_{i\in I} U_i$ where $\{U_i : i \in I\}$ is a directed subset of $Sat(\mathcal{A})$, then $a \lhd \bigcup_{i\in I} U_i$; since $b \in i(a)$, $b \lhd U_0$ with $U_0 \subseteq_\omega \bigcup_{i\in I} U_i$, and since the family is directed there exists $i \in I$ such that $b \lhd U_i$, so $\mathcal{A}b \leqslant \mathcal{A}U_i$. For all $a \in U$ and for all $b \in i(a)$, we have $\mathcal{A}b \ll \mathcal{A}a$, $\mathcal{A}a \leqslant U$ and therefore by proposition 4.3, $\mathcal{A}b \ll U$, thus, since $U \lhd \bigcup_{a\in U}\{b : b \in i(a)\}$, we have the claim. □

Corollary 4.12. If $\mathcal{A}$ is a locally Stone formal topology, then $Sat(\mathcal{A})$ is a distributive continuous lattice.

In order to prove the converses of the above results, we need a couple of lemmas:

Lemma 4.13. Let L be a continuous lattice. If $b \ll u_1 \vee \ldots \vee u_n$, there exist $b_1 \ll u_1$, ... , $b_n \ll u_n$ such that $b \leqslant b_1 \vee \ldots \vee b_n$.

Proof: By continuity we have

$$u_1 \vee \ldots \vee u_n = \bigvee_{b_1 \ll u_1} b_1 \vee \ldots \vee \bigvee_{b_n \ll u_n} b_n$$

and therefore

$$u_1 \vee \ldots \vee u_n = \bigvee_{b_1 \ll u_1, \ldots, b_n \ll u_n} b_1 \vee \ldots \vee b_n$$

where the right-hand side is a directed join. The conclusion follows by definition of the way-below relation. □

The following lemma is the general lattice-theoretic formulation of lemma 3.2, with $c \in \mathcal{I}(U)$ in place of the finitary cover, $\Downarrow U \subseteq \mathcal{I}(V))$ in place of the cover $U \lhd V$, and the relation $\ll$ in place of strict inclusion of formal intervals.

Lemma 4.14. Let L be a continuous lattice, b, c elements of L and U, V subsets of L. Suppose $c \in \mathcal{I}(U)$, for all $u \in U$, $\Downarrow U \subseteq \mathcal{I}(V))$ and $b \ll c$. Then $b \in \mathcal{I}(V)$.

Proof: If $b \ll c$ and $c \leqslant u_1 \vee \ldots \vee u_n$, where $u_i \in U$ for $i = 1, \ldots, n$, then $b \ll u_1 \vee \ldots \vee u_n$. By lemma 4.13, there exist $b_1 \ll u_1, \ldots b_n \ll u_n$ such that $b \leqslant b_1 \vee \ldots \vee b_n$. By the assumption $\Downarrow U \subseteq \mathcal{I}(V))$, we have that $b_i \in \mathcal{I}(V)$ for $i = 1, \ldots, n$, and therefore $b \in \mathcal{I}(V)$ as well. □

Theorem 4.15. Every continuous lattice is isomorphic to the lattice of saturated subsets of a locally Stone quasi formal topology. If it is distributive it is isomorphic to the lattice of saturated subsets of a locally Stone formal topology.

Proof: Let L be a continuous lattice with base S. For $a \in S$ and $U \subseteq S$, let $i(a) \equiv \Downarrow_S a$, $\sqsubseteq \; \equiv \; \leqslant$ and

$$a \lhd U \equiv i(a) \subseteq \mathcal{I}(U) \; .$$

The relation $\lhd$ is a quasi cover (satisfying in addition left):

Reflexivity holds since $b \ll a$ implies $b \leqslant a$, so if $a \in U$, then $b \in \mathcal{I}(U)$.

Transitivity: Suppose $a \lhd U$ and $U \lhd V$ and let $b \ll a$. By the interpolation property, there exists $c \in L$ such that $b \ll c \ll a$ and therefore $c \in \mathcal{I}(U)$. By lemma 4.14, $b \in \mathcal{I}(V)$, so $a \lhd V$.

Left: If $a \lhd U$, $b \sqsubseteq a$ and $c \ll b$, then $c \ll a$, so $c \in \mathcal{I}(U)$, therefore $b \lhd U$.

If L is distributive, then $\lhd$ also satisfies right since in this case $\mathcal{I}(U) \cap \mathcal{I}(V) \subseteq \mathcal{I}(U \sqcap V)$.

Conditions 1 and 2 of the definition of locally Stone (quasi) formal topology are immediate consequences of the definition of $\lhd$.

Finally, the isomorphism between L and $\mathit{Sat}(\mathcal{A})$ is given as in theorem 2.5. □

Theorems 4.11 and 4.15 show that the basic equivalence between quasi formal topologies and complete lattices restricts to equivalences between the category of locally Stone quasi formal topologies **LSQFTop** and the category of continuous lattices **CL** and between the category of locally Stone formal topologies **LSFTop** and the category of continuous distributive lattices (or continuous frames) **CFrm**.

We can summarize the situation with the diagram

$$
\begin{tikzcd}
\mathbf{QFTop} \arrow[rrr, "\sim"] & & & \mathbf{SL} & & \\
& \mathbf{LSQFTop} \arrow[ul] \arrow[rrr, "\sim"] & & & \mathbf{CL} \arrow[ul] & \\
& & \mathbf{SQFTop} \arrow[ul] \arrow[rrr, "\sim"] & & & \mathbf{AL} \arrow[ul] \\
\mathbf{FTop} \arrow[uuu, hook] \arrow[rrr, dotted, "\sim"] & & & \mathbf{Frm} \arrow[uuu, dotted, hook] & & \\
& \mathbf{LSFTop} \arrow[ul] \arrow[uuu, hook] \arrow[rrr, dotted, "\sim"] & & & \mathbf{CFrm} \arrow[ul, dotted] \arrow[uuu, dotted, hook] & \\
& & \mathbf{SFTop} \arrow[ul] \arrow[uuu, hook] \arrow[rrr, "\sim"] & & & \mathbf{AFrm} \arrow[ul, dotted] \arrow[uuu, hook]
\end{tikzcd}
\tag{4}
$$

where all horizontal arrows are equivalences and vertical and diagonal arrows are inclusions of categories

A similar representation theorem holds for prime-continuous lattices:

Theorem 4.16. If $\mathcal{A}$ is a locally Scott quasi formal topology, then $Sat(\mathcal{A})$ is a prime-continuous lattice.

Proof: As in the proof of theorem 4.11, we obtain $U = \bigvee\{V \in Sat(\mathcal{A}) : V \lll U\}$ since $b \in i(a)$ implies $\mathcal{A}b \lll \mathcal{A}a$ by definition of locally Scott quasi formal topology. □

Theorem 4.17. Every prime-continuous lattice is isomorphic to the frame of saturated subsets of a locally Scott formal topology.

Proof: Given a prime-continuous lattice L with base S, for $a \in S$ and $U \subseteq S$, let $i(a) \equiv \not\downarrow_S a$ and

$$a \lhd U \equiv i(a) \subseteq \downarrow U \ ,$$

where $\downarrow U \equiv \bigcup_{u \in U} \downarrow u$.

The verification that $\lhd$ is a locally Scott cover is straightforward. The bijection between L and $Sat(\mathcal{A})$ is obtained as in the proof of theorem 2.5. □

By the representation theorem for continuous lattices via locally Stone quasi formal topologies we also obtain an alternative proof a well known result (cf. Scott 1972):

Theorem 4.18. Every continuous lattice is the retract of an algebraic lattice (via a continuous s-r pair).

Proof: Let L be a continuous lattice. By theorem 4.15, there exists a locally Stone quasi formal topology $\mathcal{A} = \langle S, \lhd \rangle$ such that L is isomorphic to $Sat(\mathcal{A})$.

Let $\mathcal{A}_f$ be $\langle S, \sqsubseteq, \lhd_f \rangle$, with $\sqsubseteq$ defined as in the proof of 4.15 and $\lhd_f$ being the Stone compactification of $\lhd$. We have the following diagram, where $s \equiv i$ and r is the morphism of $\mathbf{FTop}(\mathcal{A}_f, \mathcal{A})$ induced by the identity map $a \mapsto \{a\}$

$$\begin{tikzcd} & \mathcal{A}_f \arrow[dr, "r"] & \\ \mathcal{A} \arrow[ur, "s"] \arrow[rr, "id_{\mathcal{A}}"] & & \mathcal{A} \end{tikzcd} \tag{5}$$

Since $\mathcal{A}$ is locally Stone, every basic neighbourhood a is equicovered with $i(a)$, and therefore r and s factorize the identity arrow on $\mathcal{A}$ so that the diagram is commutative. By applying the functor *Sat* we obtain another commutative diagram (in the category of complete lattices)

$$\begin{tikzcd} & Sat(\mathcal{A}_f) \arrow[dr, "Sat(r)"] & \\ Sat(\mathcal{A}) \arrow[ur, "Sat(s)"] \arrow[rr, "id_{Sat(\mathcal{A})}"] & & Sat(\mathcal{A}) \end{tikzcd} \tag{6}$$

that yields the conclusion since $Sat(\mathcal{A}_f)$ is an algebraic lattice by proposition 2.11. □

Then, simply by adding distributivity we get:

Corollary 4.19. Every distributive continuous lattice is the retract of an algebraic frame (via a continuous s-r pair).

4.4. *Representation of continuous dcpo's*

Continuous dcpo's cannot be represented as frames of opens of a formal topology, as these latter are complete lattices. However, continuous dcpo's arise in a natural way as formal spaces by considering the posets of formal points of a suitable class of of locally Scott formal topologies, defined below.

We start with recalling the following (cf. theorem 3.1.7 in Sigstam 1990, where a different terminology is used):

Proposition 4.20. If $\mathcal{S} \equiv \langle S, \sqsubseteq, \lhd \rangle$ is a Scott formal topology, then $Pt(\mathcal{S})$ with order given by set inclusion is an algebraic dcpo.

Proof: If $\{\alpha_i \mid i \in I\}$ is a directed subset of $Pt(\mathcal{S})$, then $\bigcup_{i \in I} \alpha_i$ is in $Pt(\mathcal{S})$. For all $a \in S$, $\Uparrow a \equiv \{b \in S \mid a \lhd b\}$ is in $Pt(\mathcal{S})$ since $\mathcal{S}$ is a Scott formal topology, and it is compact. The set consisting of elements of the form $\Uparrow a$, with a ranging in a formal point α is directed (by definition of formal point, there exists c in α with $c \sqsubseteq a$ and $c \sqsubseteq b$, therefore $\Uparrow a$ and $\Uparrow b$ are both subsets of $\Uparrow c$). The compact elements of $Pt(\mathcal{S})$ generate $Pt(\mathcal{S})$ since for all α, by definition 2.3(3), $\alpha = \bigcup_{a \in \alpha} \Uparrow a$. □

Definition 4.21. We say that a locally Scott formal topology $\mathcal{A}$ is *stable* if the map i, given as in definition 4.10, is a morphism of formal topologies from $\mathcal{A}$ to its Scott compactification $\mathcal{A}_s$, i.e., it satisfies in addition the following conditions:

(a) $i(S) =_{\mathcal{A}_s} S$;
(b) $i(a) \sqcap i(b) \lhd_s i(a \sqcap b)$.

The adjective *stable* for denoting this class of locally Scott formal topologies is chosen by analogy to its use in *stably locally compact* (cf. Johnstone 1982, p. 313; see also Smyth 1992a), where it means that the map $a \mapsto \Downarrow a$ preserves finite meets.

If $\mathcal{A}$ is a stable locally Scott formal topology we have the retraction

$$\begin{tikzcd} & \mathcal{A}_s \arrow[dr, "r"] & \\ \mathcal{A} \arrow[ur, "s"] \arrow[rr, "id_{\mathcal{A}}"] & & \mathcal{A} \end{tikzcd} \tag{7}$$

where r and s are the maps as defined above, and $\mathcal{A}_s$ is the Scott compactification of $\mathcal{A}$. By applying the functor Pt we obtain the retraction

$$\begin{tikzcd} & Pt(\mathcal{A}_s) \arrow[dr, "Pt(s)"] & \\ Pt(\mathcal{A}) \arrow[ur, "Pt(r)"] \arrow[rr, "id_{Pt(\mathcal{A})}"] & & Pt(\mathcal{A}) \end{tikzcd} \tag{8}$$

By proposition 4.20, $Pt(\mathcal{A})$ is a retract of an algebraic dcpo. It is well known (see section 3.1.1 in Abramsky and Jung 1994) that retracts of algebraic dcpo's are continuous dcpo's, and thus we obtain:

Theorem 4.22. If $\mathcal{A}$ is a stable locally Scott formal topology, then $Pt(\mathcal{A})$ with order given by set inclusion is a continuous dcpo.

Indeed every continuous dcpo arises in this way, that is, for every continuous dcpo L there is a corresponding locally Scott formal topology $\mathcal{A}$ such that L is isomorphic to $Pt(\mathcal{A})$.

Definition 4.23. Given a continuous dcpo $\langle L, \leqslant \rangle$ with base S, let $\mathcal{A}^L$ be the formal topology on S with $\sqsubseteq \;\equiv\; \leqslant^{op}$ and cover relation between elements and subsets of S that reflects the inclusion between the corresponding Scott opens,

$$a \lhd U \equiv \Uparrow a \subseteq \Uparrow U \ . \tag{9}$$

We have:

Lemma 4.24. If L is a continuous dcpo, then $\mathcal{A}^L$ is a stable locally Scott formal topology, with $i(a) \equiv \Uparrow a$.

Proof: It is straightforward to see that $\lhd$ defined in 9 satisfies property 1 of locally Scott covers. As for the second property, suppose $a \lhd U$ and let $b \in i(a)$. By the interpolation property, there esists c such that $a \ll c \ll b$. Since $c \in i(a)$ and $i(a) \subseteq i(U)$, $c \in i(U)$. Since $c \ll b$, $b \lhd c$, so by definition of $\lhd_s$, $b \lhd_s i(U)$. The morphism i is stable: $i(S) \lhd_s S$ holds since, for all $a \in S$, $b \in i(a)$ implies $b \lhd a$. For the converse, let $a \in S$. By definition of continuous dcpo, there exists $b \in S$ with $b \ll a$, that is, $b \in i(S)$ and $a \lhd b$, so condition

(a) is proved. For condition (b), let $x \in \twoheaduparrow a \sqcap \twoheaduparrow b$. Then there exists a', b' with $a \ll a' \leqslant x$ and $b \ll b' \leqslant x$. Since L is a dcpo, $a \vee b$ exists and $a \vee b \ll x$. By the interpolation property, there exists y in S with $a \vee b \ll y \ll x$. Such a y is thus in $\twoheaduparrow(a \sqcap b)$ and $x \lhd y$. □

Lemma 4.25. If L is an algebraic dcpo with a base of compact elements S, the formal topology $\mathcal{A}^L$ with base S is a Scott formal topology.

Proof: For all $a \in S$, $\twoheaduparrow a = \uparrow a$, thus $a \lhd U$ iff $(\exists b \in U)(b \leqslant a)$ and therefore $\lhd$ is a Scott cover. □

By definition formal points are lower-directed with respect to $\sqsubseteq$, and thus upper-directed with respect to $\leqslant$. Since L is a dcpo, every point of $\mathcal{A}^L$ has supremum in L. We can define a map

$$\begin{array}{rcl} \phi : Pt(\mathcal{A}^L) & \to & L \\ \alpha & \mapsto & \bigvee \alpha \end{array}$$

To show that ϕ is an isomorphism we need a couple of lemmas.

Lemma 4.26. For all $a \in S$, $\twoheaddownarrow_S a \in Pt(\mathcal{A}^L)$.

Proof: Observe that by definition of base of a continuous dcpo, $\twoheaddownarrow_S a$ is directed, thus in particular not empty. The rest is routine. □

Lemma 4.27. For all α in $Pt(\mathcal{A}^L)$, $\alpha = \bigcup_{b \in \alpha} \twoheaddownarrow b$, where the union is directed.

Proof: If $a \in \alpha$, since $a \lhd \twoheaduparrow a$, by monotonicity there exists $b \in \alpha$ such that $a \ll b$, therefore $a \in \bigcup_{b \in \alpha} \twoheaddownarrow b$. Conversely, if $a \ll b$ for some $b \in \alpha$, then $b \lhd a$, so by monotonicity $a \in \alpha$. The union is directed since formal points of $\mathcal{A}^L$ are directed. □

By lemma 4.26 we can define a map

$$\begin{array}{rcl} \psi : L & \to & Pt(\mathcal{A}^L) \\ a & \mapsto & \twoheaddownarrow_S a \end{array}$$

The maps ϕ and ψ are monotone and inverses of each other: $\phi\psi(a) = a$ holds by definition of continuous domain. Further, since formal points are directed, we have $\twoheaddownarrow_S(\bigvee \alpha) = \bigcup_{b \in \alpha} \twoheaddownarrow b$ and therefore, using lemma 4.27, we have $\psi\phi(\alpha) = \alpha$.

We have thus proved:

Theorem 4.28. Every continuous dcpo can be represented as the formal space of points of a stable locally Scott formal topology.

We remark that this result solves the problem left open in Negri (1998). Furthermore, the representation of algebraic dcpo's via Scott formal topologies can be obtained as a special case, using lemma 4.25:

Corollary 4.29. Every algebraic dcpo can be represented as the formal space of points of a Scott formal topology.

We observe the following characterization of the way-below relation in $Pt(\mathcal{A}^L)$:

Proposition 4.30. If $\alpha, \beta \in Pt(\mathcal{A}^L)$, $\beta \ll \alpha$ iff there exists $a \in S$ such that $\beta \subseteq \downarrow a \subseteq \alpha$.

Proof: If $\beta \ll \alpha$, using that $\alpha = \bigcup_{a \in \alpha} \twoheaddownarrow a$, we obtain that there exists $a \in \alpha$ such that $\beta \subseteq \twoheaddownarrow a$. Since $\twoheaddownarrow a \subseteq \downarrow a \subseteq \alpha$, the conclusion follows. Conversely, suppose $\alpha \leqslant \bigcup_{i \in I} \alpha_i$, where the right hand side is a directed union, and let $\beta \subseteq {\downarrow}a \subseteq \alpha$. Then there exists $i \in I$ such that $a \in \alpha_i$, and therefore ${\downarrow}a \subseteq \alpha_i$, so that $\beta \subseteq \alpha_i$. □

As a consequence we obtain directly, instead of using theorem 4.22:

Corollary 4.31. For all α in $Pt(\mathcal{A}^L)$, $\alpha = \bigcup_{\beta \ll \alpha} \beta$.

4.5. *Representation of Scott domains*

Since Scott domains are a special kind of continuous dcpo's, the representation of continuous dcpo's through formal topologies can be specialized to a representation of Scott domains.

In order to achieve our purpose we have to strengthen the definition of Scott formal topologies by adding a condition of existence of meets of lower bounded pairs.

Definition 4.32. A Scott formal topology $\mathcal{A} \equiv \langle \mathcal{S}, \sqsubseteq, \lhd \rangle$ is *consistently complete* if for all $a, b \in S$ such that there exists $c \in S$ with $c \sqsubseteq a$ and $c \sqsubseteq b$, $a \wedge b$ exists in S.

If one thinks of elements of the base S as fragments of information, and of the partial order $\sqsubseteq$ as an order of refinement of information (with smaller elements being more refined) the above condition expresses the requirement that if two fragments of information can be refined to a common one, then their conjunction is also in the base. Informations with a common refinement can be thought as noncontradictory, consistent, thus justifying the name of the condition.

Theorem 4.33. If $\mathcal{A}$ is a consistently complete Scott formal topology, then $Pt(\mathcal{A})$ with order given by set-theoretic inclusion is a Scott domain.

Proof: By proposition 4.20, $Pt(\mathcal{A})$ is an algebraic dcpo. The compact elements of $Pt(\mathcal{A})$ are of the form ${\Uparrow}a$ for $a \in S$. Suppose that ${\Uparrow}a$ and ${\Uparrow}b$ are two compact elements of $Pt(\mathcal{A})$ bounded above by a point α in $Pt(\mathcal{A})$. Then ${\Uparrow}a \subseteq \alpha$ and ${\Uparrow}b \subseteq \alpha$, thus $a \in \alpha$ and $b \in \alpha$. By condition 2 in definition 2.3, there exists $c \in \alpha$ such that $c \sqsubseteq a$ and $c \sqsubseteq b$. By consistent completeness $a \wedge b$ exists, and we have ${\Uparrow}a \vee {\Uparrow}b = {\Uparrow}(a \wedge b)$ since ${\Uparrow}(a \wedge b)$ satisfies the defining properties of the least upper bound of ${\Uparrow}a$ and ${\Uparrow}b$: ${\Uparrow}(a \wedge b)$ is in $Pt(\mathcal{A})$; ${\Uparrow}a \subseteq {\Uparrow}(a \wedge b)$, ${\Uparrow}b \subseteq {\Uparrow}(a \wedge b)$; if α is a formal point containing ${\Uparrow}a$ and ${\Uparrow}b$, then it contains ${\Uparrow}(a \wedge b)$. □

Conversely, let D be a Scott domain. By definition, the compact elements $K(D)$ of D generate D. Let $\mathcal{A}^D$ be the formal topology with base $K(D)$ partially ordered by $\sqsubseteq \; \equiv \; \leqslant^{op}$ with cover defined as in 9. By the proof of lemma 4.25, the definition of its cover reduces to

$$a \lhd U \equiv (\exists b \in U)(b \leqslant a) \ .$$

We have:

Proposition 4.34. $\mathcal{A}^D$ is a consistently complete Scott formal topology.

Proof: By lemma 4.25, $\mathcal{A}^D$ is a Scott formal topology. As for consistent completeness, suppose that $a, b, c \in K(D)$ and $c \sqsubseteq a$, $c \sqsubseteq b$. By definition of Scott domain, $a \vee b$ exists, and $a \vee b$ is the meet of a and b with respect to the partial order $\sqsubseteq$ given by $\leqslant^{op}$. □

The maps ϕ and ψ defined in the previous section give the isomorphism between D and

$Pt(\mathcal{A}^D)$. We observe that in this case $\psi(a) = \{b \in K(D) \mid b \leqslant a\}$. Summing up we have obtained:

Theorem 4.35. Every Scott domain can be represented as the formal space of points of a consistently complete Scott formal topology.

4.6. *Formal topology and the spectral theory of lattices*

In Hoffmann and Lawson (1978), a topological representation for distributive continuous lattices is obtained by using the *hull-kernel topology* on the *spectrum* of L. We recall that the spectrum $Spec(L)$ of a lattice L is the set of non-top *prime* elements, i.e., of elements p satisfying

$$a \wedge b \leqslant p \text{ implies } a \leqslant p \text{ or } b \leqslant p \ .$$

The hull-kernel topology is generated by the subsets $Spec(L) - \uparrow a$ with a ranging in L. For complete lattices this is the same as the spatial topology on the space of completely prime filters on L, $Pt(L)$: Every completely prime filter α on L is of the form $L - \downarrow p$ for a prime p (the supremum of the complement of α in L). By mapping p to $L - \downarrow p$ an anti-order isomorphism ϕ is obtained between $Spec(L)$ and $Pt(L)$. Moreover, this map induces an isomorphism between the hull-kernel and the spatial topology on these spaces since $\phi(Spec(L) - \uparrow a) = \{\alpha \in Pt(L) \mid a \in \alpha\}$. From this observation it follows that the results concerning the spectral theory of (distributive) continuous lattices are obtained in formal topology with dualities replaced by equivalences. In particular, theorem 4.15 is the poinfree part of the result in Hoffmann and Lawson (1978) stating that distributive continuous lattices are isomorphic to the lattices of opens of sober locally compact topological spaces. This latter result in turn is obtained in our approach from the spatiality of locally Stone formal topologies (cf. proposition 5.1 below).

5. Representation of locally compact spaces

In this section we shall show how locally Stone formal topologies can be identified with sober locally compact topological spaces. The identification takes the form of an equivalence of categories. The equivalence is obtained by restriction from the equivalence between formal topologies and sober topological spaces (theorem 2.7). We shall give complete proofs in order to make the exposition self-contained, although the main ideas follow the usual pointfree approach as in Johnstone (1982).

Proposition 5.1. Locally Stone formal topologies are spatial.

Proof: Suppose that a and U are respectively an element and a subset of the base S of a locally Stone formal topology $\mathcal{A}$ and that $\sim a \lhd U$. Our claim is that there exists a formal point α of $\mathcal{A}$ such that $a \in \alpha$ and $\alpha \cap U = \emptyset$. From the assumption that $\sim a \lhd U$ and that $\mathcal{A}$ is a locally Stone formal topology, it follows that there exists $c \in i(a)$ such that $\sim c \lhd_f i(U)$. In the continuous lattice $Sat(\mathcal{A})$ we have $\mathcal{A}c \ll \mathcal{A}a$, so, by the interpolation property, we can inductively define a sequence D_0, D_1, $\ldots$, D_n, D_{n+1} in $Sat(\mathcal{A})$ such that

$$\mathcal{A}c \ll D_{n+1} \ll D_n \ll \ldots \ll D_1 \ll D_0 \ll \mathcal{A}a \ .$$

Consider now $F \equiv \bigcup\{\uparrow D_n \mid n \geq 0\} = \bigcup\{\Uparrow D_n \mid n \geq 0\}$, where $\uparrow D_n = \{V \in Sat(\mathcal{A}) \mid D_n \lhd V\}$ and $\Uparrow D_n = \{V \in Sat(\mathcal{A}) \mid D_n \ll V\}$. Clearly, F is a filter in $Sat(\mathcal{A})$ and it is Scott open (being a union of basic Scott opens). Moreover, $\mathcal{A}a \in F$ and for all $b \in i(U)$, $\sim \mathcal{A}b \in F$. By Zorn's lemma, F extends to a Scott open filter G maximal amongst those containing $\mathcal{A}a$ and having empty intersection with $\{\mathcal{A}b \mid b \in i(U)\}$, hence with $\{\mathcal{A}b \mid b \in U\}$. By lemma VII.4.3 in Johnstone (1982), G is prime, and since it is Scott open, it is completely prime. By taking $\alpha \equiv \{b \in S \mid \mathcal{A}b \in G\}$ the desired point is obtained. $\Box$

We remark that the proof is nonconstructive for the use of Zorn's lemma. The corresponding result for locally compact locales (alias continuous frames) has been proved similarly in Johnstone (1982, theorem VII.4.3).

We also have:

Proposition 5.2. The spatial topology on $Pt(\mathcal{A})$ of a locally Stone formal topology $\mathcal{A}$ is a distributive continuous lattice.

Proof: Since $\mathcal{A}$ is spatial by 5.1, the frame of opens in $Pt(\mathcal{A})$ given by the spatial topology is isomorphic to $Sat(\mathcal{A})$. The conclusion follows by corollary 4.12. $\Box$

Proposition 5.3. If $\mathcal{A}$ is a locally Stone formal topology, the formal space $Pt(\mathcal{A})$ with the spatial topology is a sober locally compact topological space.

Proof: Let α be a point in $Pt(\mathcal{A})$, and let U be a neighbourhood of α in the spatial topology. It is not restrictive to suppose $U = ext(a)$, for some a in the base of $\mathcal{A}$. Since $a \lhd i(a)$ and $a \in \alpha$, by definition of formal points there exists $c \in i(a)$ such that $c \in \alpha$. In $Sat(\mathcal{A})$ we have $\mathcal{A}c \ll \mathcal{A}a$. Let F be the Scott open filter constructed as in the proof of proposition 5.1. We claim that $ext(F)$ is a compact neighbourhood of α contained in $ext(a)$.

It is clear that $ext(F)$ is a neighbourhood of α contained in $ext(a)$ since $c \lhd F$ implies $ext(c) \subseteq ext(F)$ and $F \lhd a$ implies $ext(F) \subseteq ext(a)$. As for compactness, suppose that $ext(F) \subseteq \bigcup_{i \in I} ext(U_i)$. By spatiality we have $F \lhd \bigcup_{i \in I} U_i$, and in the frame $Sat(\mathcal{A})$,

$$F \leqslant \bigvee_{U_0 \subseteq_\omega \bigcup_{i \in I} U_i} U_0$$

where the right-hand side is a directed join. Since F is a Scott open filter, there exists $U_0 \subseteq_\omega \bigcup_{i \in I} U_i$ such that $F \leqslant U_0$. Therefore, there exists $I_0 \subseteq_\omega I$ such that $F \lhd \bigcup_{i \in I_0} U_i$, that is, such that $ext(F) \subseteq \bigcup_{i \in I_0} ext(U_i)$. $\Box$

Conversely, we have the following:

Proposition 5.4. Let $\langle X, \Omega(X)\rangle$ be a sober locally compact topological space. Then there exists a locally Stone formal topology $\mathcal{A}$ such that $\langle X, \Omega(X)\rangle$ is isomorphic to the formal space $Pt(\mathcal{A})$ with the spatial topology.

Proof: We already know from the equivalence between formal topologies and sober topological spaces that the basic partially ordered set of the formal topology corresponding to the topological space X consists of the basic opens with partial order given by set-theoretic inclusion; the cover is defined by

$$a \lhd U \equiv a \subseteq \bigcup U \ .$$

The proof that this is indeed a locally Stone formal topology is routine once we have observed that in the topological space X, $b \ll a$ iff there exists a compact open k with $b \subseteq k \subseteq a$ and that for any (basic) open set a in a locally compact topological space, $a = \bigcup_{b \ll a} b$. $\Box$

By restriction of the equivalence between spatial formal topologies and sober topological spaces we have:

Theorem 5.5. The category of locally Stone formal topologies is equivalent to the category **SLCTop** of sober locally compact topological spaces.

We observe that we can obtain this equivalence in an alternative indirect way. First, by using the functor *Sat* that to a formal topology associates its frame of saturated subsets, by theorem 4.11 and theorem 4.15 we obtain an equivalence between the category of locally Stone formal topologies and the category of distributive continuous lattices. By composing this equivalence with the equivalence between the category of distributive continuous lattices and the category of sober locally compact topological spaces (see, e.g., theorem 7.2.16 in Abramsky and Jung 1994), we get an equivalence between the category of locally Stone formal topologies and the category of sober locally compact topological spaces.

Summing up, we have the following diagram that represents the equivalences between various locally compact structures, as arising in formal topology, lattice theory and point-set topology, where **SFrm** is the category of spatial frames:

$$\begin{tikzcd}
\mathbf{EFTop} \arrow[r, "\sim"] & \mathbf{SFrm} \arrow[r, "\sim"] & \mathbf{STop} \\
\mathbf{LSFTop} \arrow[r, "\sim"] \arrow[u, hook] & \mathbf{CFrm} \arrow[r, "\sim"] \arrow[u, hook] & \mathbf{SLCTop} \arrow[u, hook]
\end{tikzcd} \tag{10}$$

Concluding remarks and related work

The correspondence between formal spaces and locally compact frames, or equivalently, distributive continuous lattices, has been studied in Sigstam (1990) using neighbourhood systems and generators for cover relations.

Another related work is Sigstam and Stoltenberg-Hansen (1997) where two representations for regular locally compact spaces, one based on domains, another on formal spaces, are compared.

We have extended the representation of ordered structures in two directions: to nondistributive structures using the generalization of the notion of cover relation to that of quasi cover, and to partially ordered sets that are complete only with respect to directed joins, namely directed complete partial orders.

Representations of Scott domains based on formal spaces are given in Sigstam (1995) and in Sambin, Valentini and Virgili (1996). In the latter work it is proved that any Scott domain is isomorphic to the partially ordered structure given by the formal points of a Scott formal topology, and an essential use of the positivity predicate is made.

Different definitions of formal space have been used in relation to the representation of Scott domains: In Martin-Löf (1985) a formal space is axiomatized through a formal intersection and union of basic neighbourhoods and a consistency predicate *Con*. In Sigstam (1990) (see also section 6.2 in Stoltenberg-Hansen, Lindström and Griffor 1994) the definition of a formal space is given through a partial operation $a \wedge b$ which is defined when a and b are consistent ($Con(a, b)$) and a covering relation. The condition of consistent completeness (a, b lower bounded implies $Con(a, b)$) characterizes neighbourhood systems among pre-neighbourhood systems. In Sambin, Valentini and Virgili (1996) a formal space is defined through a total monoid operation, a cover relation and a consistency predicate *Pos*.

Here a formal space is defined without an operation and a consistency predicate, by requiring only a preorder relation. In Coquand (1996) it is shown that a similar definition allows to maintain the representation of frames and naturally connects to the definition of cover used in category theory.

In Sambin, Valentini and Virgili (1996) the representation of Scott domains is obtained by defining the cover relation on particular subsets of the given Scott domain by inducing the monoid operation through set-theoretic intersection. It is not clear how to extend such a procedure to include the case of continuous dcpo's, hence our choice of definition of formal space.

Furthermore, we have seen how the inclusion of a positivity predicate in the general definition of a formal space leads to some limitations in its scope.

We have solved the problem left open in Negri (1998), of representing continuous dcpo's. Lattices of this kind are used in the domain-theoretic approach to integration in locally compact topological spaces (cf. Edalat and Negri 1998). One direction of research is to use the representation here established for a constructive approach to measure and integration within the theory of formal spaces.

In Smyth (1977) the notion of R-structure has been introduced as a basis for the study of effectiveness in domains. R-structures are defined by requiring a suitable interpolation property on a transitive relation, and they suffice to obtain continuous dcpo's by ideal completion. A related direction of research not undertaken here is to give a direct presentation of continuous domains by means of an axiomatization of the way-below relation.

A preliminary report of some of the results of this work appears in Negri (1998).

Finally, I wish to thank Inger Sigstam for having made her notes of Martin-Löf's lectures available to me, and Jan von Plato for having translated them from Swedish. I benefitted from discussions with Per Martin-Löf and Peter Aczel. A special thanks is due to Bernhard Reus for having carefully read the manuscript and given helpful advice.

The anonymous referee has provided detailed suggestions to improve the exposition. Roy Dyckhoff has given some comments on the final manuscript.

References

Abramsky, S. and A. Jung (1994) Domain theory, in *Handbook of Logic in Computer Science*, vol. 3, S. Abramsky *et al.* (eds), Clarendon Press, Oxford, pp. 1–168.

Aczel, P. (1977) An introduction to inductive definitions, in *Handbook of Mathematical Logic*, J. Barwise (ed), North-Holland, pp. 739–782.

Banaschewski, B. and R.-E. Hoffmann (1981) *Continuous Lattices*, Lecture Notes in Mathematics 871, pp. 209–248, Springer.

Battilotti, G. and G. Sambin (1993) A uniform presentation of sup-lattices, quantales and frames by means of infinitary preordered sets, pretopologies and formal topologies, Preprint no. 19, Dept. of Pure and Applied Mathematics, University of Padova.

Cederquist, J. (1998) An implementation of the Heine-Borel covering theorem in type theory, in *Types for Proofs and Programs*, Selected papers, Types '96, E. Gimenez and C. Paulin (eds), LNCS 1512, pp. , Springer.

Cederquist, J. (1997) A machine assisted proof of the Hahn-Banach theorem, Chalmers University of Technology and University of Göteborg.

Cederquist, J., Th. Coquand and S. Negri (1998) The Hahn-Banach theorem in type theory, in *Twenty-Five Years of Constructive Type Theory*, G. Sambin and J. Smith (eds), Oxford University Press, pp. 57–72, 1998.

Cederquist, J. and S. Negri (1996) A constructive proof of the Heine-Borel covering theorem for formal reals, in *Types for Proofs and Programs*, S. Berardi and M. Coppo (eds), Lecture Notes in Computer Science 1158, pp. 62–75, Springer.

Coquand, Th. (1996) Formal topology with posets, manuscript, 10 pages.

Edalat, A. and S. Negri (1998) The generalized Riemann integral on locally compact spaces, *Topology and its Applications*, vol. 89, pp. 121–150, 1998.

Fourman, M.P. and R.J. Grayson (1982) Formal spaces, in *The L. E. J. Brouwer Centenary Symposium*, A. S. Troelstra and D. van Dalen (eds), pp. 107–122, North-Holland, Amsterdam.

Gierz, G. *et al.* (1980) *A Compendium on Continuous Lattices*, Springer.

Hoffmann, K.H. and J.D. Lawson (1978) The spectral theory of distributive continuous lattices, *Transactions of the American Mathematical Society* 246, pp. 285–310.

Hoffmann, K.H. and M.W. Mislove (1981) Local compactness and continuous lattices, in *Continuous Lattices*, Banaschewski and Hoffmann (eds), *op. cit.*.

Isbell, J.R. (1972) Atomless parts of spaces, *Mathematica Scandinavica* 31, pp. 5–32.

Johnstone, P.T. (1982) *Stone Spaces*, Cambridge University Press.

Johnstone, P.T. (1984) Open locales and exponentiation, *Contemporary Mathematics* 30, pp. 84–116.

Johnstone, P.T. (1983) The point of pointless topology, *Bulletin of the American Mathematical Society* 8, pp. 216–233.

Joyal, A. and M. Tierney (1984) An extension of the Galois theory of Grothendieck, *Memoirs of the American Mathematical Society* 51, pp. 1–71.

MacLane, S. (1971) *Categories for the Working Mathematician*, Springer.

Martin-Löf, P. (1970) *Notes on Constructive Mathematics*, Almqvist & Wiksell, Stockholm.

Martin-Löf, P. (1984) *Intuitionistic Type Theory*, Bibliopolis, Napoli.

Martin-Löf, P. (1985) Formell (punktfri) topologi, unpublished seminar notes, Stockholm, spring 1985.

Negri, S. (1996) Stone bases, alias the constructive content of Stone representation, in *Logic and Algebra*, A. Ursini and P. Aglianó (eds), Dekker, New York, pp. 617–636.

Negri, S. (1998) Continuous lattices in formal topology, in *Types for Proofs and Programs*, Selected papers, Types '96, E. Gimenez and C. Paulin (eds), LNCS 1512, pp. 333–353, Springer.

Negri, S. and D. Soravia (1999) The continuum as a formal space, *Archive for Mathematical Logic* (in press).

Negri, S. and S. Valentini (1997) Tychonoff's theorem in the framework of formal topologies, *The Journal of Symbolic Logic* 62, pp. 1315–1332.

Sambin, G. (1987) Intuitionistic formal spaces – a first communication, in *Mathematical Logic and its Applications*, D. Skordev (ed), Plenum Press, New York, pp. 187–204.

Sambin, G., S. Valentini and P. Virgili (1996) Constructive domain theory as a branch of intuitionistic pointfree topology, *Theoretical Computer Science* 159, pp. 319–341.

Scott, D.S. (1972) Continuous lattices, in *Toposes, Algebraic Geometry and Logic*, F.W. Lawvere (ed), Lecture Notes in Mathematics 274, pp. 97–136, Springer.

Scott, D.S. (1973) Models for various type-free calculi, in *Logic, Methodology and Philosophy of Science IV*, P. Suppes *et al.* (eds), North-Holland, pp. 157–187.

Sigstam, I. (1990) *On formal spaces and their effective presentations*, Ph. D. thesis, Report 1990:7, Department of Mathematics, University of Uppsala.

Sigstam, I. (1995) Formal spaces and their effective presentation, *Archive for Mathematical Logic* 34, pp. 211–246, 1995.

Sigstam, I. and V. Stoltenberg-Hansen (1997) Representability of locally compact spaces by domains and formal spaces, *Theoretical Computer Science* 179, pp. 319–331.

Smyth, M.B. (1977) Effectively given domains, *Theoretical Computer Science* 5, pp. 257–274.

Smyth, M.B. (1992) Topology, in *Handbook of Logic in Computer Science*, vol.1, S. Abramsky *et al.* (eds), Clarendon Press, Oxford, pp. 641–761.

Smyth, M.B. (1992a) Stable Compactification I, *Journal of the London Mathematical Society* 45, pp. 321–340.

Stoltenberg-Hansen, V., I. Lindström and E.R. Griffor (1994) *Mathematical Theory of Domains*, Cambridge University Press.

Vickers, S. (1989) *Topology via Logic*, Cambridge University Press.