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Permutability of rules in lattice theory

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Abstract. Lattice theory with the meet and join operations is formulated as a system of rules of inference. The order of application of these rules can be permuted so that a *subterm* property follows: *If an atomic formula is derivable from given atomic formulas by the rules, it has a derivation all terms of which are terms in the given formulas or the conclusion.* A direct decision method for universal formulas in lattice theory with the meet and join operations follows.

1. Introduction

Thoralf Skolem published in 1920 a method for deciding the derivability of an atomic formula from a given set of atomic formulas in lattice theory. He also showed, by the use of conjunctive normal form, that the decidability of universal formulas in lattice theory follows. His axiomatization did not have the meet and join operations, but used instead two ternary relations $M(a, b, c)$ and $J(a, b, c)$ and existential axioms stating that there is always some meet c for any a, b, and the same for join.

Skolem formulated the axioms for lattice theory as **rules** for formal derivations, as he emphasizes, even if derivation trees constructed by the rules are not used in his article. We study the permutability of lattice rules with meet and join operations and show that suitable permutations lead to the following "subterm property": *All terms in a loop-free derivation of an atomic formula $a \leqslant b$ are terms in the open assumptions or the conclusion of the derivation.* This property will give a bound on the size of a sought derivation and a direct decision method for the class of universal formulas in lattice theory with the meet and join operations.

Skolem's paper was forgotten and rediscovered by lattice theorists in the 1990s (see [1] and [2]).

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2. Lattice axioms and rules

There is a binary partial order relation $a \leqslant b$, and equality is defined by

$$a = b \equiv a \leqslant b \ \& \ b \leqslant a.$$

The axioms of lattice theory are

$$\begin{array}{lll} a \leqslant a, \quad (\mathit{Ref}), & a \leqslant b \ \& \ b \leqslant c \supset a \leqslant c, \quad (\mathit{Trans}), & \\ a \wedge b \leqslant a \quad (L\wedge_1), & a \wedge b \leqslant b \quad (L\wedge_2), & c \leqslant a \ \& \ c \leqslant b \supset c \leqslant a \wedge b \quad (R\wedge), \\ a \leqslant a \vee b \quad (R\vee_1), & b \leqslant a \vee b \quad (R\vee_2), & a \leqslant c \ \& \ b \leqslant c \supset a \vee b \leqslant c \quad (L\vee). \end{array}$$

The substitutability of equals in the lattice operations can be proved, because equality is defined through the partial order relation.

The axioms can be written as rules of inference, which makes the logical operations disappear. Some rules have zero premisses:

$$\frac{}{a \leqslant a}\,\mathit{Ref} \qquad \frac{a \leqslant b \quad b \leqslant c}{a \leqslant c}\,\mathit{Trans}$$

$$\frac{}{a \wedge b \leqslant a}\,L\wedge_1 \qquad \frac{}{a \wedge b \leqslant b}\,L\wedge_2 \qquad \frac{c \leqslant a \quad c \leqslant b}{c \leqslant a \wedge b}\,R\wedge$$

$$\frac{}{a \leqslant a \vee b}\,R\vee_1 \qquad \frac{}{b \leqslant a \vee b}\,R\vee_2 \qquad \frac{a \leqslant c \quad b \leqslant c}{a \vee b \leqslant c}\,L\vee$$

Term b in rule *Trans* is a **middle term**. An inspection of the rules shows that middle terms in *Trans* are the only terms in premisses that need not be also terms in a conclusion. We consider only derivations with atomic formulas (**atoms**) as assumptions and conclusion. Logical rules can be set aside because they can be applied only after the lattice rules. Derivation trees have assumptions and instances of zero-premiss rules as leaves.

3. The subterm property

We show that the search for a derivation of an atom from given atoms can be restricted to the terms in these atoms.

Definition 1. A **new term** in a derivation tree is a term that is not a term or a subterm in an assumption or in the conclusion.

Theorem 1 (**Subterm property**). *If an atom is derivable from atomic assumptions in lattice theory, it has a derivation with no new terms.*

Proof. We show how to transform derivations so that they have no new terms. Only rule *Trans* can remove a new term from a derivation. Assume a derivation has new terms, and consider the subderivation down to a first instance of *Trans* that removes a new term b:

$$\dfrac{\begin{matrix}\vdots & \vdots\\ a \leqslant b & b \leqslant c\end{matrix}}{a \leqslant c}\,\mathit{Trans} \tag{1}$$

1. First consider the derivation of the left premiss. If $a \leqslant b$ has been concluded by *Trans*, we permute up the *Trans* removing b:

$$\dfrac{\dfrac{a \leqslant d \quad d \leqslant b}{a \leqslant b}\,{\scriptstyle Trans} \qquad b \leqslant c}{a \leqslant c}\,{\scriptstyle Trans} \quad \rightsquigarrow \quad \dfrac{a \leqslant d \qquad \dfrac{d \leqslant b \quad b \leqslant c}{d \leqslant c}\,{\scriptstyle Trans}}{a \leqslant c}\,{\scriptstyle Trans} \tag{2}$$

Note that, by assumption, d is not a new term.

If $a \leqslant b$ has been concluded by $L\vee$, the term a has a form $a \equiv d\vee e$ and *Trans* is permuted up as follows:

$$\dfrac{\dfrac{d \leqslant b \quad e \leqslant b}{d\vee e \leqslant b}\,{\scriptstyle L\vee} \qquad b \leqslant c}{d\vee e \leqslant c}\,{\scriptstyle Trans} \quad \rightsquigarrow \quad \dfrac{\dfrac{d \leqslant b \quad b \leqslant c}{d \leqslant c}\,{\scriptstyle Trans} \qquad \dfrac{e \leqslant b \quad b \leqslant c}{e \leqslant c}\,{\scriptstyle Trans}}{d\vee e \leqslant c}\,{\scriptstyle L\vee} \tag{3}$$

The permutation of *Trans* removing b as in (2) and (3) is repeated until a left premiss $d' \leqslant b$ has been concluded by some other rules. Observe that d' is not a new term.

1.1. If the rule concluding $d' \leqslant b$ is *Ref*, the right premiss of *Trans* is identical to the conclusion and the term b is not removed from the derivation, against the assumption. The rule cannot be $L\wedge_1$, for then $d' \equiv b\wedge e$ and the new term b would be a subterm of d'. Rule $L\wedge_2$ is excluded similarly.

1.2. If the rule is $R\vee_1$, we have $b \equiv d'\vee b'$ with b' a new term and the step

$$\begin{matrix}\dfrac{\dfrac{}{d' \leqslant d'\vee b'}\,{\scriptstyle R\vee_1} \qquad d'\vee b' \leqslant c}{d' \leqslant c}\,{\scriptstyle Trans}\\ \vdots \\ a \leqslant c\end{matrix} \tag{4}$$

The case of $R\vee_2$ is similar.

1.3. If the rule is $R\wedge$, we have some terms a' and d, e such that $b \equiv d\wedge e$ and

$$\begin{matrix}\dfrac{\dfrac{a' \leqslant d \quad a' \leqslant e}{a' \leqslant d\wedge e}\,{\scriptstyle R\wedge} \qquad d\wedge e \leqslant c}{a' \leqslant c}\,{\scriptstyle Trans}\\ \vdots \\ a \leqslant c\end{matrix} \tag{5}$$

2. Now consider the right premiss $b \leqslant c$ of (4) and (5). If it is concluded by rules *Trans* or $R\wedge$, these are permuted dually to (2) and (3). Rules $R\vee_1, R\vee_2$ are excluded dually to the excluded rules $L\wedge_1, L\wedge_2$ in the left branch of (1). This leaves two cases for (4) and also for (5):

2.1. In (4), the right premiss after permutation becomes $d'\vee b' \leqslant c'$ for some term c'. The right premiss does not match rules $L\wedge_1$ or $L\wedge_2$. If $d'\vee b' \leqslant c'$ is an instance of *Ref*, then $b \equiv c'$ and b is a subterm of c. The other case in (4) is rule $L\vee$. We have, after the permutation of *Trans*,

$$\begin{matrix} \dfrac{\dfrac{}{d' \leqslant d'\vee b'}\,{\scriptstyle R\vee_1} \qquad \dfrac{\begin{matrix}\vdots\\ d' \leqslant c'\end{matrix} \quad b' \leqslant c'}{d'\vee b' \leqslant c'}\,{\scriptstyle L\vee}}{d' \leqslant c'}\,{\scriptstyle Trans} \\ \vdots \\ d' \leqslant c \end{matrix}$$

and the derivation is transformed into

$$\begin{matrix} \vdots \\ d' \leqslant c' \\ \vdots \\ d' \leqslant c \end{matrix}$$

with the transitivity step removed.

2.2 In the derivation of the right premiss in (5), rule $L\vee$ does not match. This leaves $L\wedge_1$ and $L\wedge_2$. We have, after permutations, some term c' and, say, rule $L\wedge_1$:

$$\begin{matrix} \dfrac{\dfrac{\begin{matrix}\vdots\\ a' \leqslant d\end{matrix} \quad a' \leqslant e}{a' \leqslant d\wedge e}\,{\scriptstyle R\wedge} \qquad \dfrac{}{d\wedge e \leqslant c'}\,{\scriptstyle L\wedge_1}}{a' \leqslant c'}\,{\scriptstyle Trans} \\ \vdots \\ a \leqslant c \end{matrix} \tag{6}$$

Now $c' \equiv d$ so the derivation is transformed into

$$\begin{matrix} \vdots \\ a' \leqslant c \\ \vdots \\ a \leqslant c \end{matrix}$$

with the transitivity step removed. Rule $L\wedge_2$ is treated similarly. □

Corollary 2. *Lattice theory is conservative over partial order for universal formulas.*

This result was proved in [3], (Theorem 6.6.5). Theorem 1 is a generalization of that result.

Corollary 3 (**Decidability of universal formulas).** *The derivability of universal formulas in lattice theory is decidable.*

Proof. Consider a universal formula in prenex form $\forall x \cdots \forall z A$ with A in conjunctive normal form. Each conjunct A_k is of the form $P_1 \& \cdots \& P_m \supset Q_1 \vee \cdots \vee Q_n$, with P_i, Q_j atoms. The lattice axioms have no disjunctions in positive parts and therefore (by Harrop's theorem, see, e.g., [3]) A_k is derivable if and only if $P_1 \& \cdots \& P_m \supset Q_j$ is derivable for some j. Apply theorem 2 to each of the Q_j. $\square$

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