Sequent calculus proof theory of intuitionistic apartness and order relations

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Abstract

Contraction-free sequent calculi for intuitionistic theories of apartness and order are given and cut-elimination for the calculi proved. Among the consequences of the result is the disjunction property for these theories. Through methods of proof analysis and permutation of rules, we establish conservativity of the theory of apartness over the theory of equality defined as the negation of apartness, for sequents in which all atomic formulas appear negated. The proof extends to conservativity results for the theories of constructive order over the usual theories of order.

1 Introduction

In constructive theories a notion of apartness is often taken as basic, and equality appears as a defined concept, the negation of apartness. The same can be done with order relations, and weak linear order can be defined as the negation of a strict linear order. The expression of order relations in positive terms can be pushed further so as to include partial order, lattices and Heyting algebras (as in [vP]). Here we take up the task of investigating the proof theory of these apartness and positive order relations.

The extension of sequent calculi to axiomatic theories presents a well known problem, namely, that cut elimination is not necessarily maintained in such extensions. We establish a way of adding to the logical calculus for intuitionistic propositional logic **G3ip** the (free variable) axioms of the theories under consideration in the form of sequent calculus rules. These are put in a form that guarantees admissibility of contraction and cut to be maintained also for these extensions of the calculus. This will work for axioms of the form $\neg P$, $\neg(P\&Q)$, $P \supset Q$, $P \supset Q \vee R$, where P, Q, R are atomic formulas of the theory under consideration.

Although the calculi do not have the subformula property, typical consequences of the subformula property like the disjunction property follow from a weaker subformula property, namely, in a derivation only subformulas of the endsequent or atomic formulas occur.

We pose the question of whether the positive theories, based on apartness or positive order relations, are conservative over those based on equality or weak order when these are defined as the negations of the positive relations. This is not just a question of application of proof theory to the study of mathematical structures but was originally prompted by the application of positive Heyting algebras for a semantics of refutation for intuitionistic logic (cf. [vP, NvP]).

With sequent calculi in which cut is admissible we can prove conservativity of positive theories over the usual theories in an elementary and direct way, using only induction on derivations. By techniques of analysis and manipulation of proofs, we show how to transform derivations of a sequent (with all atoms negated) in theories based on apartness or positive order into derivations of the same sequent in the corresponding theories based on defined equality or partial order. The proof of conservativity thus obtained is also modular, fitting all the theories considered.

The paper is organized as follows: In Section 2 we review from [vP] the basic definitions for the intuitionistic theories of apartness and order. After recalling in Section 3 the sequent calculus **G3ip** for intuitionistic propositional logic, we give in Section 4 sequent calculi for the theories of apartness and order. In Section 5 we show that in these calculi all the structural rules are admissible. In Section 6 we prove conservativity of apartness over equality defined as the negation of apartness. In the final section the conservativity result is extended to a proof of conservativity of positive theories of order over the theories based on partial order defined as the negation of positive order.

2 Intuitionistic theories of apartness and order

We recall here the axioms of the intuitionistic theories of apartness and order. For a general discussion of these axioms we refer to [vP] where the positive axiomatizations, based on apartness and excess, are introduced and contrasted to the usual ones based on equality and weak partial order. Our notation for these theories follows [S] and [vP], with the symbol $\neq$ in place of $\#$ for denoting apartness.

The intuitionistic theory of *apartness* (Heyting) poses the following two axioms for a relation $a \neq b$, where $\neg\, a \neq b$ abbreviates $a \neq b \supset \bot$:

AP1 $\quad \neg\, a \neq a$, $\qquad\qquad$ AP2 $\quad a \neq b \supset a \neq c \vee b \neq c$.

By setting a for c in AP2, we get $a \neq b \supset b \neq a$, that is, symmetry is derivable.

The negation of apartness,

$$a = b =_{df} \neg\, a \neq b$$

is then easily seen to be an equivalence relation.

The intuitionistic theory of *(positive) linear order* (Scott) has a relation $a < b$ with the axioms

PLO1 $\quad \neg\,(a < b \,\&\, b < a)$, $\qquad\qquad$ PLO2 $\quad a < b \supset a < c \vee c < b$.

It follows that $a < b$ is irreflexive, asymmetric and transitive.

The negation of a positive linear order

$$a \leqslant b =_{df} \neg\, b < a$$

defines a *(weak) linear order*, which is reflexive and transitive.

Apartness can be defined by taking the symmetrization of a positive linear order

$$a \neq b =_{df} a < b \vee b < a \ .$$

A positive linear order with *minimum* and *maximum* is obtained by adding to PLO1 and PLO2 the operations $min(a,b)$ and $max(a,b)$ axiomatized by:

MNL $\neg\, a < min(a,b)$, MXL $\neg\, max(a,b) < a$,

MNR $\neg\, b < min(a,b)$, MXR $\neg\, max(a,b) < b$,

MNU $min(a,b) < c \supset a < c \vee b < c$, MXU $c < max(a,b) \supset c < a \vee c < b$.

The classical uniqueness axioms for *min* and *max* in a weak linear order,

$c \leqslant a \,\&\, c \leqslant b \supset c \leqslant min(a,b)$, $a \leqslant c \,\&\, b \leqslant c \supset max(a,b) \leqslant c$

are obtained by taking the contrapositions of MNU and MXU.

Just as a positive linear order is the constructive counterpart of a (weak) linear order, an excess relation is the positive, or constructive, counterpart of a (weak) partial order. An *excess relation* is a relation $a \nleqslant b$, read *a exceeds b*, such that

PPO1 $\neg\, a \nleqslant a$, PPO2 $a \nleqslant b \supset a \nleqslant c \vee c \nleqslant b$.

Since such a relation is the positive counterpart of a partial order, we may loosely call it a *positive partial order*; It is not however a partial order, for transitivity does not in general hold, but a relation the negation of which is a partial order.

By putting

$$a \leqslant b =_{df} \neg\, a \nleqslant b$$

a (weak) partial order, i.e., a reflexive and transitive relation, is obtained.

A positive linear order can be obtained from a positive partial order by adding the axiom

$$\neg\, (a \nleqslant b \,\&\, b \nleqslant a) .$$

We can define apartness from an excess relation by posing

$$a \neq b =_{df} a \nleqslant b \vee b \nleqslant a$$

so the negation of apartness satisfies

$$\neg\, a \neq b \supset\subset a \leqslant b \,\&\, b \leqslant a .$$

Therefore, if equality is defined, antisymmetry holds by definition, i.e., the relation $\leqslant$ is a partial order in the usual sense.

A *positive lattice* is obtained by adding to PPO1 and PPO2 the lattice operations $a \wedge b$ and $a \vee b$, axiomatized by:

MTL $\neg\, a \wedge b \nleqslant a$, JNL $\neg\, a \nleqslant a \vee b$,

MTR $\neg\, a \wedge b \nleqslant b$, JNR $\neg\, b \nleqslant a \vee b$,

MTU $c \nleqslant a \wedge b \supset c \nleqslant a \vee c \nleqslant b$, JNU $a \vee b \nleqslant c \supset a \nleqslant c \vee b \nleqslant c$.

A *positive Heyting algebra* is obtained by adding to a positive lattice the positive Heyting arrow operation $a \to b$. The axioms are:

PHI $\neg\, a \wedge (a \to b) \nleqslant b$, PHU $c \nleqslant a \to b \supset c \wedge a \nleqslant b$.

For top and bottom elements, 1 and 0, we add the axiom

PHB $\neg\, 0 \nleqslant a$,

and define the top element 1 by $1 =_{df} 0 \to 0$.

3 Sequent calculus for intuitionistic propositional logic

We recall here the sequent calculus for intuitionistic propositional logic **G3ip** (see [TS]).

We use Greek upper case Γ, Δ ... for finite multisets of formulas, $P, Q, R, \ldots$ for atomic formulas, $A, B, C, \ldots$ for arbitrary formulas.

G3ip

Axioms:

$$\frac{}{P, \Gamma \Rightarrow P}\ Ax \qquad \frac{}{\bot, \Gamma \Rightarrow A}\ Efq$$

Rules:

$$\frac{A, B, \Gamma \Rightarrow C}{A\&B, \Gamma \Rightarrow C}\ L\& \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A\&B}\ R\&$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ L\vee \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ R\vee_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ R\vee_2$$

$$\frac{A \supset B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C}\ L\supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ R\supset$$

As the notation suggests, we regard axioms as zero premise rules. However, we shall sometimes omit the inference line for such rules and refer to them as axioms. Observe that it is not restrictive to consider only atomic formulas in the first axiom since it can be proved, by induction on the complexity of an arbitrary formula A, that $A, \Gamma \Rightarrow A$ is provable. This feature of the calculus is essential in the proofs of admissibility of cut of Section 5 and in the proofs of conservativity of Section 6. Also, it is necessary in order to prove admissibility of cut that the rule of *ex falso quodlibet* is not restricted to atomic formulas. Moreover $\bot$ has to be regarded as a logical constant, not as an atomic formula (cf. [D], p. 11).

In the above system the structural rules are built in: This is guaranteed by the fact that in the rule $L\supset$ the principal formula is repeated in the left premise, as in Dragalin's **GHPC** (cf. [D]), of which the above calculus is a single succedent version. In fact, the system is a simplified variant of Kleene's **G3** (see [K1], Section 80) where the repetition of the principal formulas also occurs in $L\&$ and $L\vee$.

4 Sequent calculi for intuitionistic theories of apartness and order

In this section we give sequent calculus formulations for intuitionistic theories of apartness and order. These sequent calculi are obtained by adding to **G3ip** the axioms of the theories in the form of rules. These rules are put in such a way that the structural rules stay admissible also in these extensions of the purely logical calculus. As we shall see in Section 5 the proof of cut elimination for these extensions is routine, once the correct way of formulating the axioms as rules has been found. The general idea is to absorb the logical content of these axioms into the geometry of the sequent calculus rules in such a way that only atomic formulas will occur as principal in non-logical rules of the extended calculi. We begin with some guidelines for passing from a Hilbert style axiomatization, with axioms added to the calculus, to a sequent calculus where the axioms are built in, as inference rules.

Let P, Q, R, ... denote atomic formulas of the theory (for instance, of the form $a \neq b$ for the theory of apartness, or of the form $a \not\leq b$ for the theories of positive order). Axioms of the form $\neg(P \,\&\, Q)$ are translated into the zero premise rule (non-logical axiom)

$$\overline{P, Q, \Gamma \Rightarrow C}$$

Notice that the alternative way of translating such axioms, namely as the rule

$$\overline{P, Q, \Gamma \Rightarrow \bot}$$

does not allow a cut free calculus, as, for instance, the following derivation

$$\frac{\overline{P, Q, \Gamma \Rightarrow \bot} \quad \overline{\bot \Rightarrow C}\ {\scriptstyle Efq}}{P, Q, \Gamma \Rightarrow C}\ {\scriptstyle cut}$$

cannot be transformed into a cut free one.

Axioms of the form $P \supset Q \vee R$ are translated into sequent calculus rules of the form

$$\frac{P, Q, \Gamma \Rightarrow C \qquad P, R, \Gamma \Rightarrow C}{P, \Gamma \Rightarrow C}$$

where the repetition of P in the premises is needed for admissibility of contraction.

Axioms of the form $\neg P$ and $P \supset Q$ also follow this pattern, since they can be seen as degenerate forms of the previous types (with one conjunct and one disjunct only, respectively) and thus translate into sequent calculus rules of the form

$$\overline{P, \Gamma \Rightarrow C} \qquad \frac{P, Q, \Gamma \Rightarrow C}{P, \Gamma \Rightarrow C}\ .$$

4.1 Apartness

The sequent calculus for the theory of apartness is obtained by adding to the purely logical rules of **G3ip** the axioms AP1 and AP2 of an apartness relation in the form of the sequent calculus rules *irref* and *split*, respectively,

$$\overline{a \neq a, \Gamma \Rightarrow C}\ {\scriptstyle irref} \qquad \frac{a \neq b, a \neq c, \Gamma \Rightarrow C \qquad a \neq b, b \neq c, \Gamma \Rightarrow C}{a \neq b, \Gamma \Rightarrow C}\ {\scriptstyle split}$$

The resulting calculus will be denoted by **G3AP**.

In the rule *split* the principal formula $a \neq b$ has to be repeated in the premises in order to get a contraction-free calculus, similarly to the rule $L\supset$ of **G3ip** (cf. the discussion in [D, Dy, TS]).

4.2 Positive partial and linear order

The sequent calculus **G3PPO** for the theory of positive partial order is obtained by adding to **G3ip** the rules

$$\overline{a \not\leq a, \Gamma \Rightarrow C}\ {\scriptstyle irref} \qquad \frac{a \not\leq b, a \not\leq c, \Gamma \Rightarrow C \quad a \not\leq b, c \not\leq b, \Gamma \Rightarrow C}{a \not\leq b, \Gamma \Rightarrow C}\ {\scriptstyle split}$$

The two rules are of the same logical form as the corresponding rules of the calculus of apartness and therefore the proof-theoretic properties of **G3AP** will hold for **G3PPO** as well. This is also the reason why we call the axioms and rules by the same names in both calculi.

The calculus for positive linear order **G3PLO** is obtained by adding to **G3PPO** the rule

$$\frac{}{a \not\leq b, b \not\leq a, \Gamma \Rightarrow C}\ \textit{asym}\ .$$

Observe that, in the presence of *split*, *irref* can be derived from *asym*, so we can regard **G3PLO** as **G3ip** plus *split* plus *asym*, and use freely *irref*.

4.3 Positive lattices

The calculus for positive lattices **G3PLT** is obtained by adding to **G3PPO** appropriate rules for join and meet

$$\frac{}{a \wedge b \not\leq a, \Gamma \Rightarrow C}\ \textit{mtl} \qquad \frac{}{a \not\leq a \vee b, \Gamma \Rightarrow C}\ \textit{jnl}$$

$$\frac{}{a \wedge b \not\leq b, \Gamma \Rightarrow C}\ \textit{mtr} \qquad \frac{}{b \not\leq a \vee b, \Gamma \Rightarrow C}\ \textit{jnr}$$

$$\frac{c \not\leq a \wedge b, c \not\leq a, \Gamma \Rightarrow C \qquad c \not\leq a \wedge b, c \not\leq b, \Gamma \Rightarrow C}{c \not\leq a \wedge b, \Gamma \Rightarrow C}\ \textit{mtu}$$

$$\frac{a \vee b \not\leq c, a \not\leq c, \Gamma \Rightarrow C \qquad a \vee b \not\leq c, b \not\leq c, \Gamma \Rightarrow C}{a \vee b \not\leq c, \Gamma \Rightarrow C}\ \textit{jnu}$$

These reduce to the rules for *min* and *max* in the case of linear order.

4.4 Positive Heyting algebras

The calculus **G3PHA** for positive Heyting algebras is obtained by adding to **G3PLT** the following sequent calculus rules corresponding to the axioms PHI and PHU for positive Heyting arrow

$$\frac{}{a \wedge (a \to b) \not\leq b, \Gamma \Rightarrow C}\ \textit{phi} \qquad \frac{c \not\leq a \to b, c \wedge a \not\leq b, \Gamma \Rightarrow C}{c \not\leq a \to b, \Gamma \Rightarrow C}\ \textit{phu}$$

The rule corresponding to PHB is

$$\frac{}{0 \not\leq a, \Gamma \Rightarrow C}\ \textit{phb}\ .$$

5 Cut elimination

We recall that a (schematic) rule

$$\frac{S}{S'}$$

is *admissible* in a logical calculus **G** if for every derivation of an instance of S there is one of the corresponding instance of S'. Similarly for rules with several premises.

In this section we shall prove that the cut rule

$$\frac{\Gamma \Rightarrow A \qquad A, \Gamma' \Rightarrow C}{\Gamma, \Gamma' \Rightarrow C}$$

is an admissible rule in each of the sequent calculi for the theories of apartness and positive order introduced in the previous section. The proof of admissibility of cut for these systems follows quite closely the proof done for **G3** in [TS], which, in turn, follows the proof presented in [D] for **GHPC** (see [Dy1] for a more elaborate exposition). The starting point is the proof of admissibility of the structural rules of weakening and contraction.

Lemma 5.1 *The rule of* weakening

$$\frac{\Gamma \Rightarrow B}{A, \Gamma \Rightarrow B}\, w$$

is admissible in the systems **G3AP**, **G3PPO**, **G3PLO**, **G3PLT**, **G3PHA**.

Proof: For each system, the proof is by induction on the derivation of $\Gamma \Rightarrow B$. If $\Gamma \Rightarrow B$ is an axiom, then also $A, \Gamma \Rightarrow B$ is an axiom. Otherwise weakening is permuted upwards with the last rule used in a derivation of $\Gamma \Rightarrow B$ and the inductive hypothesis is used. □

The proof of admissibility of contraction for **G3** requires an *inversion lemma* (cf. [TS], p. 66). This result can be extended as follows to each of the systems **G3AP**, **G3PPO**, **G3PLO**, **G3PLT**, **G3PHA**. We write $\vdash_n \Gamma \Rightarrow A$ if the sequent $\Gamma \Rightarrow A$ has a derivation of height $\leq n$ in the system considered.

Lemma 5.2 *In each of the systems* **G3AP**, **G3PPO**, **G3PLO**, **G3PLT**, **G3PHA** *the following hold:*

(i) *If* $\vdash_n A\&B, \Gamma \Rightarrow C$, *then* $\vdash_n A, B, \Gamma \Rightarrow C$.

(ii) *If* $\vdash_n A \vee B, \Gamma \Rightarrow C$, *then* $\vdash_n A, \Gamma \Rightarrow C$ *and* $\vdash_n B, \Gamma \Rightarrow C$.

(iii) *If* $\vdash_n A \supset B, \Gamma \Rightarrow C$, *then* $\vdash_n B, \Gamma \Rightarrow C$.

Proof: Since all the statements have similar proofs we shall show only the first. We proceed by induction on n. If $n = 0$ then $A\&B, \Gamma \Rightarrow C$ is an axiom. By the restriction to atomic formulas for the first logical axiom and the fact that only atomic formulas occur as principal in non-logical axioms, we have that also $A, B, \Gamma \Rightarrow C$ is an axiom, and the conclusion follows. Suppose now that the statement has been proved for n and we establish it for $n+1$. Suppose $\vdash_{n+1} A\&B, \Gamma \Rightarrow C$. If $A\&B$ is principal in the last rule used in a derivation of $A\&B, \Gamma \Rightarrow C$, then the premise of the rule is $A, B, \Gamma \Rightarrow C$ and therefore we have $\vdash_n A, B, \Gamma \Rightarrow C$ and a fortiori $\vdash_{n+1} A, B, \Gamma \Rightarrow C$. If $A\&B$ is not principal, then we apply the inductive hypothesis to the premise(s) of the last rule applied in a derivation of $A\&B, \Gamma \Rightarrow C$ and then apply the same rule. Observe that all the rules added to **G3ip** for the theories of apartness and order fall under this case since their principal formulas are atomic. □

In the sequel we shall use the notation $A \equiv B$ for literal identity of the expressions A and B.

Lemma 5.3 *In each of the systems* **G3AP**, **G3PPO**, **G3PLO**, **G3PLT**, **G3PHA** *we have:*

If $\vdash_n A, A, \Gamma \Rightarrow C$, *then* $\vdash_n A, \Gamma \Rightarrow C$.

Proof: By induction on n. If $n = 0$ then $A, A, \Gamma \Rightarrow C$ is an axiom, and also $A, \Gamma \Rightarrow C$ is an axiom. We remark here that in the case we are dealing with **G3PLO** and the axiom is *asym*, with $A \equiv a \not\leq a$, then we have $A, \Gamma \Rightarrow C$ by *irref.*

Suppose now the statement holds for n and we prove it for $n+1$.

If A is not principal in the last rule of the derivation of $A, A, \Gamma \Rightarrow C$, apply the inductive hypothesis to the premise(s) of the last rule and the rule so that a derivation of height $\leq n+1$ of $A, \Gamma \Rightarrow C$ is obtained.

If A is principal, say $D\&B$, the premise is $D, B, D\&B, \Gamma \Rightarrow C$, and by Lemma 5.2 (i) we get a derivation of $D, B, D, B, \Gamma \Rightarrow C$ with height $\leq n$, so by inductive hypothesis applied twice we get $\vdash_n D, B, \Gamma \Rightarrow C$. By applying $L\&$ the conclusion follows. The case with $A \equiv D \vee B$ is treated similarly. If $A \equiv D \supset B$, the derivation ends with

$$\frac{D \supset B, D \supset B, \Gamma \Rightarrow D \qquad B, D \supset B, \Gamma \Rightarrow C}{D \supset B, D \supset B, \Gamma \Rightarrow C}$$

From the left premise, by the induction hypothesis, we get $\vdash_n D \supset B, \Gamma \Rightarrow D$; from the right premise, by the inversion lemma, we get $\vdash_n B, B, \Gamma \Rightarrow C$ and by inductive hypothesis $\vdash_n B, \Gamma \Rightarrow C$. The conclusion follows by applying $L\supset$.

The only cases left are those with A principal and atomic, that is, those specific to the theories under consideration.

If we are considering **G3AP** and $A \equiv a \neq b$, the derivation ends with

$$\frac{a \neq b, a \neq b, a \neq c, \Gamma \Rightarrow C \qquad a \neq b, a \neq b, b \neq c, \Gamma \Rightarrow C}{a \neq b, a \neq b, \Gamma \Rightarrow C}\ \textit{split}$$

This case is simply dealt with by applying the inductive hypothesis to the premises and then *split*. The non-logical rules for all the other systems are treated similarly. We observe that the proof goes through owing to the form of the rules, with the repetition of the principal formula in the premises. □

Corollary 5.4 *The rule of* contraction

$$\frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ c$$

is admissible in the systems **G3AP**, **G3PPO**, **G3PLO**, **G3PLT**, **G3PHA**.

We observe that we could have proved Corollary 5.4 directly, without the stronger Lemma 5.3 that we shall not need. However, the direct proof uses induction on the complexity of the contracted formula, with subinduction on the height of the derivation of the premise of contraction, so it would be no real simplification.

The argument used by Dragalin in [D] in the proof of cut-elimination for **GHPC** also works for the sequent calculus systems for apartness and positive order:

Theorem 5.5 *The cut rule*

$$\frac{\Gamma \Rightarrow A \qquad A, \Gamma' \Rightarrow C}{\Gamma, \Gamma' \Rightarrow C}$$

is admissible in **G3AP**, **G3PPO**, **G3PLO**, **G3PLT**, **G3PHA**.

Proof: Let $\mathcal{D}$ and $\mathcal{D}'$ be the derivations of $\Gamma \Rightarrow A$ and $A, \Gamma' \Rightarrow C$, respectively. The proof is by induction on the complexity of the cut formula with subinduction on the sum of the heights of $\mathcal{D}$ and $\mathcal{D}'$. We distinguish three cases:

1. At least one of the premises is an axiom.
2. The formula A is not principal in at least one of the premises and none of them is an axiom.
3. The formula A is principal in both premises.

The first case, when logical axioms are involved, and second case are dealt with in a straightforward way as in [D, TS]. If $\Gamma \Rightarrow A$ is a non-logical axiom of the theory, then also $\Gamma, \Gamma' \Rightarrow C$ is. If $A, \Gamma' \Rightarrow C$ is a non-logical axiom in which A is not principal, then also $\Gamma, \Gamma' \Rightarrow C$ is a non-logical axiom. If A is principal in the axiom then the cut has the form (in **G3AP**)

$$\frac{\overset{\mathcal{D}}{\Gamma \Rightarrow a \neq a} \qquad a \neq a, \Gamma' \Rightarrow C}{\Gamma, \Gamma' \Rightarrow C}$$

We analyze $\mathcal{D}$ to show how this cut can be permuted upward in the derivation. The atomic formula $a \neq a$ is not principal in the last rule of $\mathcal{D}$ so if this rule is a one-premise rule we transform

$$\frac{\dfrac{\Gamma'' \Rightarrow a \neq a}{\Gamma \Rightarrow a \neq a}\,\textit{rule} \qquad a \neq a, \Gamma' \Rightarrow C}{\Gamma, \Gamma' \Rightarrow C}\,\textit{cut}$$

into

$$\dfrac{\dfrac{\Gamma'' \Rightarrow a \neq a \qquad a \neq a, \Gamma' \Rightarrow C}{\Gamma'', \Gamma' \Rightarrow C}\,\textit{cut}}{\Gamma, \Gamma' \Rightarrow C}\,\textit{rule}$$

If the last rule of $\mathcal{D}$ is a two-premise rule (different from $L\supset$) we trasform

$$\frac{\dfrac{\Gamma'' \Rightarrow a \neq a \qquad \Gamma''' \Rightarrow a \neq a}{\Gamma \Rightarrow a \neq a}\,\textit{rule} \qquad a \neq a, \Gamma' \Rightarrow C}{\Gamma, \Gamma' \Rightarrow C}\,\textit{cut}$$

into

$$\frac{\dfrac{\Gamma'' \Rightarrow a \neq a \qquad a \neq a, \Gamma' \Rightarrow C}{\Gamma'', \Gamma' \Rightarrow C}\,\textit{cut} \qquad \dfrac{\Gamma''' \Rightarrow a \neq a \qquad a \neq a, \Gamma' \Rightarrow C}{\Gamma''', \Gamma' \Rightarrow C}\,\textit{cut}}{\Gamma, \Gamma' \Rightarrow C}\,\textit{rule}$$

If the last rule of $\mathcal{D}$ is $L\supset$, with $\Gamma \equiv A \supset B, \Gamma''$, we transform

$$\frac{\dfrac{A \supset B, \Gamma'' \Rightarrow A \quad B, \Gamma'' \Rightarrow a \neq a}{A \supset B, \Gamma'' \Rightarrow a \neq a}\,\mathrm{L}\supset \qquad a \neq a, \Gamma' \Rightarrow C}{A \supset B, \Gamma'', \Gamma' \Rightarrow C}\,\textit{cut}$$

into

$$\frac{\dfrac{A \supset B, \Gamma'' \Rightarrow A}{A \supset B, \Gamma'', \Gamma' \Rightarrow A}\,\vdots \qquad \dfrac{B, \Gamma'' \Rightarrow a \neq a \quad a \neq a, \Gamma' \Rightarrow C}{B, \Gamma'', \Gamma' \Rightarrow C}\,\textit{cut}}{A \supset B, \Gamma'', \Gamma' \Rightarrow C}\,\mathrm{L}\supset$$

where the dots denote possibly repeated applications of weakening.

Also in the third case we proceed as in [D, TS] for non-atomic A. This exhausts all possible cases since the formula A cannot be atomic and principal in both premises, for atomic and principal formulas only occur on the left of the sequent arrow in the systems **G3AP**, **G3PPO**, **G3PLO**, **G3PLT**, **G3PHA**. □

We observe that some authors, beginning with Gentzen in [G], have considered extensions of sequent calculus with "basic mathematical sequents" of form $P_1, \ldots, P_n \Rightarrow Q$ as extra axioms. For these extensions the cut elimination theorem becomes a reduction of cuts to cuts on axioms (cf. Theorem 4.4.1 in [TS]). In our extensions we deal both with non-logical axioms and with non-logical rules. From this perspective our result can be summarized as follows: Thanks to the form of these rules, contraction stays admissible in these extension. Since cut commutes with non-logical rules, the cuts in a derivation can be reduced to cuts on axioms. Further, our non-logical axioms have an arbitrary formula as succedent that makes them closed under cut, and therefore cut can be eliminated.

Corollary 5.6 *The systems* **G3AP**, **G3PPO**, **G3PLO**, **G3PLT**, **G3PHA** *enjoy the* disjunction property*, that is, if* $\Rightarrow A \vee B$*, then* $\Rightarrow A$ *or* $\Rightarrow B$*.*

Proof: For each of the systems, by inspection of the sequent calculus rules, $\Rightarrow A \vee B$ can only derive by $\vee R$ from a derivation of $\Rightarrow A$ or of $\Rightarrow B$. □

The calculi for the theories of apartness and positive order do not enjoy the subformula property owing to the form of the rules added. Nevertheless a *weak subformula property* holds, namely:

In a derivation only subformulas of the conclusion plus atomic formulas can occur.

The weak subformula property will be crucial in the proof of conservativity of Section 6.

The systems defined by means of rules are equivalent to the systems obtained by adding to the logic the axioms of apartness and order in the form of sequents with empty antecedent. In these latter systems, which we call **H**-systems, the structural rules are not built in, but have to be added.

Definition 5.7 *Let* **H-G3AP** *be the system obtained by adding to* **G3ip+w+c+cut** *the axioms*

$$\Rightarrow \neg\, a \neq a \qquad (i),$$
$$\Rightarrow a \neq b \supset a \neq c \vee b \neq c \qquad (s).$$

Then we have:

Proposition 5.8 *The systems* **H-G3AP** *and* **G3AP** *are equivalent, that is, the same sequents are provable.*

Proof: If $\Gamma \Rightarrow A$ is provable in **H-G3AP** then it is provable in **G3AP** since the axioms i and s are provable in **G3AP**

$$\dfrac{\dfrac{}{a \neq a \Rightarrow \bot}\ \textit{irref}}{\Rightarrow a \neq a \supset \bot}\ R\supset \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{a \neq b, a \neq c \Rightarrow a \neq c}\ Ax}{a \neq b, a \neq c \Rightarrow a \neq c \vee b \neq c}\ R\vee \quad \dfrac{\dfrac{}{a \neq b, b \neq c \Rightarrow b \neq c}\ Ax}{a \neq b, b \neq c \Rightarrow a \neq c \vee b \neq c}\ R\vee}{a \neq b \Rightarrow a \neq c \vee b \neq c}\ \textit{split}}{\Rightarrow a \neq b \supset a \neq c \vee b \neq c}\ R\supset$$

For the converse, it is enough to prove that the rules *irref* and *split* are derivable in **H-G3AP**. The derivations are as follows:

$$
\dfrac{\dfrac{\Rightarrow a \neq a \supset \bot \qquad \Gamma, a\neq a, a\neq a \supset \bot \Rightarrow \bot}{\Gamma, a \neq a \Rightarrow \bot}\,\mathit{cut} \qquad \dfrac{}{\bot \Rightarrow C}\,\mathit{Efq}}{\Gamma, a\neq a \Rightarrow C}\,\mathit{cut}
$$

$$
\dfrac{\dfrac{\Rightarrow a\neq b \supset a\neq c \vee b\neq c \qquad \dfrac{a\neq b, a\neq b \supset a\neq c \vee b\neq c \Rightarrow a\neq c \vee b\neq c \qquad \dfrac{\Gamma, a\neq b, a\neq c \Rightarrow C \quad \Gamma, a\neq b, b\neq c \Rightarrow C}{\Gamma, a\neq b, a\neq c \vee b\neq c \Rightarrow C}\,L\vee}{\Gamma, a\neq b, a\neq b, a\neq b \supset a\neq c \vee b\neq c \Rightarrow C}\,\mathit{cut}}{\Gamma, a\neq b, a\neq b \Rightarrow C}\,\mathit{cut}}{\Gamma, a\neq b \Rightarrow C}\,c
$$

where we have omitted the derivation of the sequents of the form $\Gamma, A, A \supset B \Rightarrow B$. □

The systems **H-G3PPO**, **H-G3PLT**, etc., are defined in a similar way by adding to **G3ip+w+c+cut** the axioms for positive partial order, positive lattices, etc.. We have:

Proposition 5.9 *Each of the systems* **G3AP**, **G3PPO**, **G3PLO**, **G3PLT**, **G3PHA** *is equivalent to the corresponding* **H***-system.*

6 Conservativity of apartness over equality

Classically equality is given as a primitive notion, by means of a relation $a = b$ satisfying the axioms of reflexivity and transitivity

$$
\begin{array}{l}
a = a \\
a = c \;\&\; b = c \supset a = b
\end{array}
$$

Intuitionistically equality is defined as the negation of apartness

$$
a = b =_{df} \neg\, a \neq b \ .
$$

In this way a constructively stronger theory is obtained: For example, defined equality is stable, that is $\neg\neg\,(a = b) \supset a = b$ holds, whereas primitive equality is not.

The usual axioms of reflexivity and transitivity for equality defined as the negation of apartness assume the form

$$
\begin{array}{ll}
\neg\, a \neq a & (\mathit{refl}) \\
\neg\, a \neq c \;\&\; \neg\, b \neq c \supset \neg\, a \neq b & (\mathit{trans})
\end{array}
$$

By *theory of equality* **G3EQ** we refer to the theory with the above two axioms (with no further properties assumed on $\neq$) added to **G3ip** plus the structural rules.

Within the theory of apartness we can single out a suitable class of formulas, in which all occurrences of apartness are negated:

Definition 6.1 *We say that A is a* negatomic formula *if all its atoms are negated and that $\Gamma \Rightarrow A$ is a* negatomic sequent *if all the formulas in the sequent are negatomic.*

As we noted earlier, $\perp$ is not to be considered an atomic formula. Thus, according to the above definition, $\perp$ is negatomic, for all its atoms are vacuously negated.

We remark that the theory of equality defined as the negation of apartness is equivalent to the theory of stable equality, i.e., the theory with $=$ as primitive and reflexivity, symmetry, transitivity and stability as axioms. More precisely, if A is a negatomic formula provable in the theory of defined equality, then the formula A^* obtained by translating $\neg a \neq b$ into $a = b$ is provable in the theory of stable equality. Conversely, if A is an equality formula provable in the theory of stable equality, then the formula A° obtained by the symmetric translation is provable in the theory of defined equality, as stability for negations is logically derivable.

In this section we shall prove that the theory of apartness is conservative over the theory of equality, that is:

> *If a negatomic sequent $\Gamma \Rightarrow A$ is derived in the theory of apartness, then it can also be derived in the theory of equality.*

Unlike for the theory of apartness, we do not have for the theory of equality a calculus with structural rules admissible (and we conjecture that this is not possible due to the form of the axiom of transitivity), but this suffices for our purposes: When defining the transformation establishing conservativity we only need full control on derivations for the source theory (apartness) whereas for the target theory (equality) any derivation goes, structural rules or not.

The proof of conservativity is performed by analyzing and manipulating derivations: First, by the weak subformula property, since subformulas of a negatomic formulas are either negatomic or atoms, we have the following crucial fact:

(∗) *A derivation of a negatomic sequent $\Gamma \Rightarrow A$ in* **G3AP** *can only contain atomic or negatomic formulas.*

The manipulation of derivations uses the technique of permutation of rules dating back to [K] in 1952. This technique is used here for "pulling down" applications of the rule *split*, by permuting them downwards in all possible cases. In turn, these permutations are used to prove (Proposition 6.3) that all occurrences of split in a derivation of a negatomic sequent in **G3AP** can be brought to the form

$$\frac{a \neq b, a \neq c, \Gamma \Rightarrow \perp \qquad a \neq b, b \neq c, \Gamma \Rightarrow \perp}{a \neq b, \Gamma \Rightarrow \perp},$$

that is, a form in which they can be replaced by transitivity of equality, as we shall see.

In the sequel we shall call *split formula* the principal formula of the split rule. We have:

Lemma 6.2 *In a derivation of a negatomic sequent $\Gamma \Rightarrow A$ in* **G3AP***, the rule* split *permutes with $L\&$, $R\&$, $L\vee$, $R\vee$, $L\supset$ and with $R\supset$ in case the split formula is not active in $R\supset$.*

Proof: The proof consists of a series of conversions. We remark once and for all that we freely use weakening since it is an admissible rule of the system, furthermore it does not affect the property (∗) of sequents since its use is here limited to atomic formulas. We also observe that in all the occurrences of split preceding a left rule, the split formula cannot be an active formula of the rule, for otherwise the conclusion of the rule would contain a sequent with a formula of the form $A \circ a \neq b$ (where $\circ$ is $\&$, $\vee$ or $\supset$) which is ruled out by the property (∗).

If *split* occurs above $L\&$ we transform

$$\dfrac{\dfrac{a\neq b, a\neq c, A, B, \Gamma' \Rightarrow C \qquad a\neq b, b\neq c, A, B, \Gamma' \Rightarrow C}{a\neq b, A, B, \Gamma' \Rightarrow C}\,split}{a\neq b, A\&B, \Gamma' \Rightarrow C}\,L\&$$

into

$$\dfrac{\dfrac{a\neq b, a\neq c, A, B, \Gamma' \Rightarrow C}{a\neq b, a\neq c, A\&B, \Gamma' \Rightarrow C}\,L\& \qquad \dfrac{a\neq b, b\neq c, A, B, \Gamma' \Rightarrow C}{a\neq b, b\neq c, A\&B, \Gamma' \Rightarrow C}\,L\&}{a\neq b, A\&B, \Gamma' \Rightarrow C}\,split$$

If *split* occurs above $R\&$ we transform

$$\dfrac{\dfrac{a\neq b, a\neq c, \Gamma' \Rightarrow A \qquad a\neq b, b\neq c, \Gamma' \Rightarrow A}{a\neq b, \Gamma' \Rightarrow A}\,split \qquad a\neq b, \Gamma' \Rightarrow B}{a\neq b, \Gamma' \Rightarrow A\&B}\,R\&$$

into

$$\dfrac{\dfrac{a\neq b, a\neq c, \Gamma' \Rightarrow A \qquad \dfrac{a\neq b, \Gamma' \Rightarrow B}{a\neq b, a\neq c, \Gamma' \Rightarrow B}\,w}{a\neq b, a\neq c, \Gamma' \Rightarrow A\&B}\,R\& \qquad \dfrac{a\neq b, b\neq c, \Gamma' \Rightarrow A \qquad \dfrac{a\neq b, \Gamma' \Rightarrow B}{a\neq b, b\neq c, \Gamma' \Rightarrow B}\,w}{a\neq b, b\neq c, \Gamma' \Rightarrow A\&B}\,R\&}{a\neq b, \Gamma' \Rightarrow A\&B}\,split$$

If *split* occurs above $L\vee$ we transform

$$\dfrac{\dfrac{a\neq b, a\neq c, A, \Gamma' \Rightarrow C \quad a\neq b, b\neq c, A, \Gamma' \Rightarrow C}{a\neq b, A, \Gamma' \Rightarrow C}\,split \qquad a\neq b, B, \Gamma' \Rightarrow C}{a\neq b, A\vee B, \Gamma' \Rightarrow C}\,L\vee$$

into

$$\dfrac{\dfrac{a\neq b, a\neq c, A, \Gamma' \Rightarrow C \quad \dfrac{a\neq b, B, \Gamma' \Rightarrow C}{a\neq b, a\neq c, B, \Gamma' \Rightarrow C}\,w}{a\neq b, a\neq c, A\vee B, \Gamma' \Rightarrow C}\,L\vee \qquad \dfrac{a\neq b, b\neq c, A, \Gamma' \Rightarrow C \quad \dfrac{a\neq b, B, \Gamma' \Rightarrow C}{a\neq b, b\neq c, B, \Gamma' \Rightarrow C}\,w}{a\neq b, b\neq c, A\vee B, \Gamma' \Rightarrow C}\,L\vee}{a\neq b, A\vee B, \Gamma' \Rightarrow C}\,split$$

If *split* occurs above $R\vee_1$ we transform

$$\dfrac{\dfrac{a\neq b, a\neq c, \Gamma' \Rightarrow A \qquad a\neq b, b\neq c, \Gamma' \Rightarrow A}{a\neq b, \Gamma' \Rightarrow A}\,split}{a\neq b, \Gamma' \Rightarrow A\vee B}\,R\vee$$

into

$$\dfrac{\dfrac{a\neq b, a\neq c, \Gamma' \Rightarrow A}{a\neq b, a\neq c, \Gamma' \Rightarrow A\vee B}\,R\vee_1 \qquad \dfrac{a\neq b, b\neq c, \Gamma' \Rightarrow A}{a\neq b, b\neq c, \Gamma' \Rightarrow A\vee B}\,R\vee_1}{a\neq b, \Gamma' \Rightarrow A\vee B}\,split$$

If *split* occurs above $L\supset$, since this rule is not symmetric, we have to distinguish two cases according to whether it occurs above the left or right premise. We also observe that, in this latter case, B is negatomic so it cannot be the split formula. In the first case we transform

$$\dfrac{\dfrac{A\supset B, a\neq b, a\neq c, \Gamma' \Rightarrow A \quad A\supset B, a\neq b, b\neq c, \Gamma' \Rightarrow A}{A\supset B, a\neq b, \Gamma' \Rightarrow A}\,split \qquad B, a\neq b, \Gamma' \Rightarrow C}{A\supset B, a\neq b, \Gamma' \Rightarrow C}\,L\supset$$

into

$$\dfrac{\dfrac{A\supset B, a\neq b, a\neq c, \Gamma' \Rightarrow A \quad \dfrac{B, a\neq b, \Gamma' \Rightarrow C}{B, a\neq b, a\neq c, \Gamma' \Rightarrow C}\,w}{A\supset B, a\neq b, a\neq c, \Gamma' \Rightarrow C}\,L\supset \quad \dfrac{A\supset B, a\neq b, b\neq c, \Gamma' \Rightarrow A \quad \dfrac{B, a\neq b, \Gamma' \Rightarrow C}{B, a\neq b, b\neq c, \Gamma' \Rightarrow C}\,w}{A\supset B, a\neq b, b\neq c, \Gamma' \Rightarrow C}\,L\supset}{A\supset B, a\neq b, \Gamma' \Rightarrow C}\,split$$

In the second case we transform

$$\dfrac{A\supset B, a\neq b, \Gamma' \Rightarrow A \qquad \dfrac{B, a\neq b, a\neq c, \Gamma' \Rightarrow C \quad B, a\neq b, b\neq c, \Gamma' \Rightarrow C}{B, a\neq b, \Gamma' \Rightarrow C}\,split}{A\supset B, a\neq b, \Gamma' \Rightarrow C}\,L\supset$$

into

$$\dfrac{\dfrac{\dfrac{A\supset B, a\neq b, \Gamma' \Rightarrow A}{A\supset B, a\neq b, a\neq c, \Gamma' \Rightarrow A}\,w \quad B, a\neq b, a\neq c, \Gamma' \Rightarrow C}{A\supset B, a\neq b, a\neq c, \Gamma' \Rightarrow C}\,L\supset \quad \dfrac{\dfrac{A\supset B, a\neq b, \Gamma' \Rightarrow A}{A\supset B, a\neq b, b\neq c, \Gamma' \Rightarrow A} \quad B, a\neq b, b\neq c, \Gamma' \Rightarrow C}{A\supset B, a\neq b, b\neq c, \Gamma' \Rightarrow C}\,L\supset}{A\supset B, a\neq b, \Gamma' \Rightarrow C}\,split$$

If *split* occurs above $R\supset$ with split formula not active, we trasform

$$\dfrac{\dfrac{A, a\neq b, a\neq c, \Gamma' \Rightarrow B \qquad A, a\neq b, b\neq c, \Gamma' \Rightarrow B}{A, a\neq b, \Gamma' \Rightarrow B}\,split}{a\neq b, \Gamma' \Rightarrow A\supset B}\,R\supset$$

into

$$\dfrac{\dfrac{A, a\neq b, a\neq c, \Gamma' \Rightarrow B}{a\neq b, a\neq c, \Gamma' \Rightarrow A\supset B}\,R\supset \qquad \dfrac{A, a\neq b, b\neq c, \Gamma' \Rightarrow B}{a\neq b, b\neq c, \Gamma' \Rightarrow A\supset B}\,R\supset}{a\neq b, \Gamma' \Rightarrow A\supset B}\,split \qquad \Box$$

We shall call a derivation in which all occurrences of split have $\bot$ on the right of the sequent arrow a derivation with *split reduction*.

We have:

Proposition 6.3 *A negatomic sequent* $\Gamma \Rightarrow A$ *derivable in* **G3AP** *has a derivation with split reduction.*

Proof: Let $\mathcal{D}$ be a given derivation of $\Gamma \Rightarrow A$ and suppose that $\mathcal{D}$ contains n occurrences of the split rule with succedent different from $\bot$. We show by induction on n that $\mathcal{D}$ can be transformed into a derivation $\mathcal{D}'$ with split reduction. If $n = 0$, $\mathcal{D}$ is already a derivation with split reduction. If $n > 0$ we consider among the n splits with succedent different from $\bot$ one with none of these splits below it. By permuting this split downwards, the succedent formula of the split stays the same or becomes more complex, except when the split is left premise of L$\supset$, because then the split after permutation inherits the succedent from the second premise. If the succedent of second premise is $\bot$, then we are done, the offending instance of split has disappeared. If not, we continue permuting downwards. Ultimately we must arrive at a split with $\bot$ in the succededent; for its conclusion eventually becomes either the premise of a $R\supset$ rule with the split formula active in it, or the premise of a split with $\bot$ as succedent. (Observe that it cannot become the last rule in the derivation, since the endsequent is negatomic). In the former case the principal formula of $R\supset$ is of form $a \neq b \supset B$ and, by property $(*)$, B has to be $\bot$, therefore the split thus transformed has $\bot$ as succedent. Also in the latter case the conclusion of split has $\bot$ as succedent. The derivation thus obtained has $n-1$ splits with succedent different form $\bot$ and by inductive hypothesis it can be transformed into a derivation with split reduction. $\Box$

The following lemmas constitute the core of the proof of conservativity (Theorem 6.7). They are necessary for dealing with those rules (implication rules) in which the premises of negatomic conclusions are not necessarily negatomic.

Lemma 6.4 *Let Γ be negatomic, and assume to have a derivation with split reduction of height m in* **G3AP** *of*

$$\Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow a \neq b \ .$$

Then:

$a_i \equiv b_i$ *for some i, or*

$a \neq b \equiv a_i \neq b_i$ *for some i, or*

$\Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow \neg\neg\, a \neq b$ *has a derivation of height $\leq m$ in* **G3AP**.

Proof: By induction on m. If $m = 0$, $\Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow a \neq b$ is an axiom. If it is *ex falso quodlibet*, then $\Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow \neg\neg\, a \neq b$ also is. If it is *irref* then $a_i \equiv b_i$ for some i. If it is $\Gamma', P \Rightarrow P$ then $a \neq b \equiv a_i \neq b_i$ for some i.

If $m > 0$, suppose the statement true for m and we prove it for $m+1$. Consider the last rule applied in the derivation of the given sequent. Since we are considering a derivation with split reduction, the last rule has to be a logical rule. Moreover, the last rule has to be a left rule.

If the last rule is $L\&$ or $L\vee$ then apply the inductive hypothesis to the premise(s), which are of the same form of the conclusion, and possibly the same rule.

If the last rule is $L\supset$, then $\Gamma = \Gamma', A \supset B$ and the derivation ends with

$$\frac{\Gamma', A\supset B, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow A \quad \Gamma', B, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow a \neq b}{\Gamma', A\supset B, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow a \neq b}$$

Since $A\supset B$ is negatomic, B is negatomic, so the inductive hypothesis applies to the right premise. Thus $a_i \equiv b_i$ for some i, or $a \neq b \equiv a_i \neq b_i$ for some i, or $\Gamma, B, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow \neg\neg\, a \neq b$

has a derivation of height $\leq m$. In this last case the conclusion is obtained by applying $L\supset$ to this latter sequent and the left premise. □

It thus follows that in a derivation in **G3AP** of a sequent with negatomic antecedent and atomic succedent the third conclusion of the lemma obtains, and we can replace the atom by its double negation while continuing to have a correct derivation and preserving the height bound:

Corollary 6.5 *Let Γ be negatomic, and assume to have a derivation with split reduction of height $\leq n$ in* **G3AP** *of $\Gamma \Rightarrow a \neq b$. Then $\Gamma \Rightarrow \neg\neg a \neq b$ has a derivation of height $\leq n$ in* **G3AP**.

Lemma 6.6 *Let Γ and A be negatomic, and assume to have a derivation with split reduction in* **G3AP** *of*

$$\Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow A \ .$$

Then either $a_i \equiv b_i$ for some i, or there is a derivation in **G3EQ** *of*

$$\Gamma, \neg\neg a_1 \neq b_1, \ldots, \neg\neg a_n \neq b_n \Rightarrow A \ .$$

Proof: By induction on the derivation of the sequent $\Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow A$.

If the sequent is an axiom, then it cannot be an axiom of the form $\Gamma, P \Rightarrow P$ since A is negatomic, therefore either Γ contains $\bot$, and thus

$$\Gamma, \neg\neg a_1 \neq b_1, \ldots, \neg\neg a_n \neq b_n \Rightarrow A$$

is an axiom in **G3EQ**, or it is an instance of the axiom *irref*, thus $a_i \equiv b_i$ for some i, and we have proved the claim.

If the sequent comes from & rules or $\vee$ rules, we distinguish two cases, according to whether the rule has one or two premises. In the first case, we apply the induction hypothesis to the premise. If it gives $a_i \equiv b_i$ for some i, we are done. Otherwise we obtain a derivation with double negations in **G3EQ**, to which we apply the same rule. If the rule has two premises and for at least one the inductive hypothesis gives $a_i \equiv b_i$ for some i, we are done. Else we obtain two derivation with double negations in **G3EQ**, to which we apply the same rule.

If the sequent comes from a $L\supset$ rule with both active formulas negatomic we proceed as above. Otherwise the principal formula is of the form $a \neq b \supset \bot$ and the last step of the inference is

$$\frac{\Gamma', a \neq b \supset \bot, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow a \neq b \quad \Gamma', \bot, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow A}{\Gamma', a \neq b \supset \bot, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow A}$$

where $\Gamma = \Gamma', a \neq b \supset \bot$. By Lemma 6.4 applied to the left premise, we either have $a_i \equiv b_i$ for some i, and we are done, or $a \neq b \equiv a_i \neq b_i$ for some i, and therefore the sequent

$$\Gamma', a \neq b \supset \bot, \neg\neg a_1 \neq b_1, \ldots, \neg\neg a_n \neq b_n \Rightarrow A$$

is provable in **G3EQ**, or we have a derivation with height bounded by the height of the derivation of the left premise of

$$\Gamma', a \neq b \supset \bot, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow \neg\neg a \neq b \ .$$

Now the succedent is negatomic and we can thus apply the inductive hypothesis to this derivation and the one of the right premise. If for at least one the inductive hypothesis gives $a_i \equiv b_i$ for some i, we are done. Otherwise we have a derivation in **G3EQ** of

$$\Gamma', a \neq b \supset \bot, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n \Rightarrow \neg\neg\, a \neq b \tag{1}$$

and of

$$\Gamma', \bot, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n \Rightarrow A\ . \tag{2}$$

By means of logical steps (1) gives

$$\Gamma', a \neq b \supset \bot, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n \Rightarrow \bot$$

and thus by cut, together with (2), we obtain a derivation in **G3EQ** of

$$\Gamma, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n \Rightarrow A\ .$$

If the sequent comes from $R\supset$ with both active formulas negatomic, then we proceed as we did for & and $\vee$ rules. Otherwise the inference step has the form

$$\frac{\Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n, c \neq d \Rightarrow \bot}{\Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow c \neq d \supset \bot}$$

(which, as an aside, explains why the statement of this lemma cannot be restricted to one atomic formula). Then either $a_i \equiv b_i$ for some i, or $c \equiv d$, or we have a derivation in **G3EQ** of

$$\Gamma, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n, \neg\neg\, c \neq d \Rightarrow \bot\ .$$

In the first case we have finished. If $c \equiv d$, then by *refl* we have a proof in **G3EQ** of $\Rightarrow \neg\, c \neq d$, and thus the conclusion follows by (repeated applications of) weakening. In the last case, we obtain, by applying $R\supset$, a derivation in **G3EQ** of

$$\Gamma, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n \Rightarrow \neg\neg\neg\, c \neq d$$

hence of

$$\Gamma, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n \Rightarrow \neg\, c \neq d\ .$$

If the sequent comes from a split rule, since this is a derivation with split reduction, it is of the form

$$\frac{\Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n, a_1 \neq c_1 \Rightarrow \bot \qquad \Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n, b_1 \neq c_1 \Rightarrow \bot}{\Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow \bot}$$

By induction hypothesis applied to the left premise we have threee cases: either $a_i \equiv b_i$ for some i, or $a_1 \equiv c_1$, or we have a derivation **G3EQ** of

$$\Gamma, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n, \neg\neg\, a_1 \neq c_1 \Rightarrow \bot\ . \tag{3}$$

In the first case we are done. In the second case, in the right premise we have by $a_1 \equiv c_1$

$$\Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n, b_1 \neq a_1 \Rightarrow \bot$$

to which we apply the inductive hypothesis that gives, via symmetry of equality and via contraction, the conclusion. In the third case, we have to analyze what is known from the inductive hypothesis applied to the right premise. Except for cases symmetrical with those already considered, we are left with a derivation in **G3EQ** of

$$\Gamma, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n \neg\neg\, b_1 \neq c_1, \;\Rightarrow\; \perp \;. \tag{4}$$

By applying to (3) and (4) the rule $R\supset$ we obtain

$$\Gamma, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n \Rightarrow \neg\neg\neg\, a_1 \neq c_1$$

and

$$\Gamma, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n \Rightarrow \neg\neg\neg\, b_1 \neq c_1$$

which by logical steps and transitivity give in **G3EQ**

$$\Gamma, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n \Rightarrow \neg\, a_1 \neq b_1$$

hence

$$\Gamma, \neg\neg\, a_1 \neq b_1, \ldots, \neg\neg\, a_n \neq b_n \Rightarrow \;\perp\; .$$

□

We observe a connection here: The above lemma is structurally similar to a central lemma of another proof of conservativity, namely the syntactic proof in formal topology of the localic Hahn-Banach theorem [CCN].

We are now ready to give the syntactic proof of conservativity:

Theorem 6.7 G3AP *is conservative over* **G3EQ** *for negatomic sequents.*

Proof: Let $\Gamma \Rightarrow A$ be a negatomic sequent derivable in **G3AP**. Then by Proposition 6.3 it has a derivation with split reduction. We prove by induction on this derivation that the sequent is also derivable in **G3EQ**.

If $\Gamma \Rightarrow A$ is an axiom it can only be a logical axiom since Γ is negatomic. So the conclusion holds.

If $\Gamma \Rightarrow A$ is derived by a & rule, or by a $\vee$ rule, or by a $L\supset$ rule with negatomic active formulas, then the premise(s) is (are) negatomic if the conclusion is. We can thus apply the inductive hypothesis to the premise(s) and the same rule.

If $\Gamma \Rightarrow A$ is derived by a $L\supset$ rule with an atomic active formula, i.e.,

$$\frac{\Gamma, a \neq b \supset \perp \;\Rightarrow a \neq b \quad \Gamma, \perp \;\Rightarrow A}{\Gamma, a \neq b \supset \perp \;\Rightarrow A}$$

then by Corollary 6.5 we get a derivation in **G3AP**, with the same height of the derivation of the left premise, of

$$\Gamma, a \neq b \supset \perp \;\Rightarrow \neg\neg\, a \neq b \;.$$

By inductive hypothesis applied to this latter negatomic sequent we obtain a derivation in **G3EQ** of the same sequent, and therefore, by logical steps, $\Gamma, a \neq b \supset \perp \;\Rightarrow\; \perp$ in **G3EQ**. The right premise is negatomic so by inductive hypothesis we obtain a derivation of it in **G3EQ**. By cut of $\Gamma, a \neq b \supset \perp \;\Rightarrow\; \perp$ with $\Gamma, \perp \;\Rightarrow A$ we obtain the conclusion.

The same reasoning used for & and $\vee$ rules applies to a $R\supset$ rule in case the premise is a negatomic sequent. If it is not, the last step of the derivation has the form

$$\frac{\Gamma, a \neq b \Rightarrow \;\perp}{\Gamma \Rightarrow \neg\, a \neq b} \;.$$

By the previous lemma, either $a \equiv b$, and therefore $\Gamma \Rightarrow \neg\, a \neq b$ in **G3EQ**, or we have a derivation in **G3EQ** of $\Gamma, \neg\neg\, a \neq b \Rightarrow\ \bot$. From the latter we obtain the conclusion by means of logical steps.

The last rule cannot be split since Γ is negatomic, and therefore the proof is finished. □

7 Conservativity results for positive order

The proof of conservativity of apartness over equality can be extended to a proof of conservativity of theories of excess over theories of order. In these latter theories the partial order they are based upon is defined through the negation of excess

$$a \leqslant b =_{df} \neg\, a \nleqslant b$$

and the axioms are obtained by taking the negative axioms for excess and the contraposition of the positive ones. For instance, the axioms for defined partial order are

$$\begin{array}{ll} \neg\, a \nleqslant a & \textit{(refl)} \\ \neg\, a \nleqslant c \;\&\; \neg\, b \nleqslant c \supset \neg\, a \nleqslant b & \textit{(trans)} \end{array}$$

We call **G3LO**, **G3PO**, **G3LT**, **G3HA** the theories based on a partial order thus obtained from **G3PLO**, **G3PPO**, **G3PLT**, **G3PHA**, respectively.

The analogue to conservativity of apartness over equality for the theories of order can be stated as follows:

> *If a negatomic sequent $\Gamma \Rightarrow A$ is derived in a theory of positive order, then it can also be derived in the corresponding theory of partial order.*

Indeed, the proof of this general result follows the pattern of the proof of conservativity of equality over apartness: First we observe that property $(*)$ of derivations of negatomic sequents also holds for derivations in the calculi of positive order **G3PLO**, **G3PPO**, etc.. Then all the non-logical rules are shown to be permutable with the logical rules as in Lemma 6.2 so that it is guaranteed that in a derivation of a negatomic sequent they all can be brought to the form with $\bot$ on the right of the sequent arrow. We call such a derivation a *reduced derivation*. At this point the derivation can be transformed into a derivation in the corresponding theory of partial order.

Since the proofs all follow the same pattern, we shall just limit here to giving explicit statements of the intermediate steps and proofs in outline for conservativity of the theory of positive Heyting algebras over the usual theory of Heyting algebras with partial order defined as negation of excess.

By permuting downward the non-logical rules of **G3PHA** as in Lemma 6.2 we obtain, by an easy adaptation of the proof of Proposition 6.3:

Proposition 7.1 *A negatomic sequent $\Gamma \Rightarrow A$ derivable in* **G3PHA** *has a reduced derivation.*

The following lemma can be proved along the lines of the proof of Lemma 6.4:

Lemma 7.2 *Let Γ be negatomic, and assume to have a reduced derivation of height m in* **G3PHA** *of*

$$\Gamma, a_1 \nleqslant b_1, \ldots, a_n \nleqslant b_n \Rightarrow a \nleqslant b\ .$$

Then:

$a_i \equiv b_i$ *for some* i*, or*

$a \not\leq b \equiv a_i \not\leq b_i$ *for some* i*, or*

$\Gamma, a_1 \not\leq b_1, \ldots, a_n \not\leq b_n \Rightarrow \neg\neg\, a \not\leq b$ *has a derivation of height* $\leq m$ *in* **G3PHA**.

Corollary 7.3 *Let* Γ *be negatomic, and assume* $\vdash_n \Gamma \Rightarrow a \neq b$ *in* **G3PHA**. *Then* $\vdash_n \Gamma \Rightarrow \neg\neg\, a \neq b$ *in* **G3PHA**.

Then we have

Lemma 7.4 *Let* Γ *and* A *be negatomic, and assume to have a reduced derivation in* **G3PHA** *of*

$$\Gamma, a_1 \not\leq b_1, \ldots, a_n \not\leq b_n \Rightarrow A\ .$$

Then either $a_i \equiv b_i$ *for some* i *or we have a derivation in* **G3HA** *of*

$$\Gamma, \neg\neg\, a_1 \not\leq b_1, \ldots, \neg\neg\, a_n \not\leq b_n \Rightarrow A\ .$$

Proof: By induction on the derivation of the sequent $\Gamma, a_1 \not\leq b_1, \ldots, a_n \not\leq b_n \Rightarrow A$.

If the sequent is an axiom, we proceed as in Lemma 6.6 in case it is a logical axiom. If the sequent is a non-logical axiom, then we distinguish all possible cases. If it is *irref*, then $a_i \equiv b_i$ for some i, and therefore we have proved the claim. If it is a lattice axiom, for instance

$$\Gamma, a\wedge b \not\leq a, a_2 \not\leq b_2, \ldots, a_n \not\leq b_n \Rightarrow A$$

we observe that in **G3HA** we have $\Rightarrow \neg\, a\wedge b \not\leq a$ and thus by logical steps the conclusion follows.

If the sequent is an instance of the axiom *phi*, e.g.

$$\Gamma, (a\to b)\wedge a \not\leq b, a_2 \not\leq b_2, \ldots, a_n \not\leq b_n \Rightarrow A$$

then we use the fact that in **G3HA** we have $\Rightarrow \neg\, (a\to b)\wedge a \not\leq b$ and therefore

$$\Gamma, \neg\neg\, (a\to b)\wedge a \not\leq b \Rightarrow A$$

is provable in **G3HA**. The conclusion then follows by weakening.

If the sequent is an instance of the axiom *phb*, we argue in a similar way, using that in Heyting algebras we have $\Rightarrow \neg\, 0 \not\leq a$.

If the last rule of the derivation is a & rule, or a $\vee$ rule, or a $R\supset$ rule with negatomic active formulas we proceed as in Lemma 6.6.

If the last rule is $L\supset$ with an atomic active formula then we proceed as we did for Lemma 6.6 by using here Lemma 7.2.

If the last rule is a $R\supset$ rule where the implication has atomic antecedent, i.e.,

$$\frac{\Gamma, a_1 \not\leq b_1, \ldots, a_n \not\leq b_n, c \not\leq d \Rightarrow \bot}{\Gamma, a_1 \not\leq b_1, \ldots, a_n \not\leq b_n \Rightarrow \neg\, c \not\leq d}$$

then, by the inductive hypothesis, either $a_i \equiv b_i$ for some i, and we are done, or $c \equiv d$, and we conclude by using reflexivity of partial order plus weakening, or we have a derivation of $\Gamma, \neg\neg\, a_1 \not\leq b_1, \ldots, \neg\neg\, a_n \not\leq b_n, \neg\neg\, c \not\leq d \Rightarrow \bot$ in **G3HA**. From this latter derivation we get the conclusion by $R\supset$.

If the last rule is a non-logical rule, then it must have $\perp$ on the right of the sequent arrow since we are considering a reduced derivation. If it is a lattice rule, for instance

$$\frac{\Gamma, a{\vee}b \nleq c, a \nleq c, a_1 \nleq b_1, \ldots, a_n \nleq b_n \Rightarrow \perp \quad \Gamma, a{\vee}b \nleq c, b \nleq c, a_1 \nleq b_1, \ldots, a_n \nleq b_n \Rightarrow \perp}{\Gamma, a{\vee}b \nleq c, a_1 \nleq b_1, \ldots, a_n \nleq b_n \Rightarrow \perp}$$

then we apply the inductive hypothesis to the premises. Then either we have $a_i \equiv b_i$ for some i, or $a{\vee}b \equiv c$, or $a \equiv c$, or $b \equiv c$ (and we conclude easily) or we have derivations in **G3HA** of

$$\Gamma, \neg\neg\, a{\vee}b \nleq c, \neg\neg\, a \nleq c, \neg\neg\, a_1 \nleq b_1, \ldots, \neg\neg\, a_n \nleq b_n \Rightarrow \perp$$

$$\Gamma, \neg\neg\, a{\vee}b \nleq c, \neg\neg\, b \nleq c, \neg\neg\, a_1 \nleq b_1, \ldots, \neg\neg\, a_n \nleq b_n \Rightarrow \perp$$

From these we obtain

$$\Gamma, \neg\neg\, a{\vee}b \nleq c, \neg\neg\, a_1 \nleq b_1, \ldots, \neg\neg\, a_n \nleq b_n \Rightarrow \neg\, a \nleq c$$

$$\Gamma, \neg\neg\, a{\vee}b \nleq c, \neg\neg\, a_1 \nleq b_1, \ldots, \neg\neg\, a_n \nleq b_n \Rightarrow \neg\, b \nleq c$$

and thus, by the lattice laws of **G3HA**,

$$\Gamma, \neg\neg\, a{\vee}b \nleq c, \neg\neg\, a_1 \nleq b_1, \ldots, \neg\neg\, a_n \nleq b_n \Rightarrow \neg\, a{\vee}b \nleq c$$

and the conclusion follows by propositional logic.

If the last rule is a Heyting arrow rule we argue in a way similar to the above, by using the axiom of **G3HA** $\neg\, c{\wedge}a \nleq b \supset \neg\, c \nleq a{\to}b$. □

Theorem 7.5 **G3PHA** *is conservative over* **G3HA** *for negatomic sequents.*

Proof: By induction on a reduced derivation of the given negatomic sequent. Since the conclusion is negatomic, the last rule cannot be a non-logical rule. The proof proceeds as the proof of Theorem 6.7, and Corollary 7.3 and Lemma 7.4 are applied in case the last rule is a $L\supset$ or a $R\supset$ with atomic antecedent. □

Concluding remarks

We have given sequent calculus formulations for the theories of apartness and positive order and shown admissibility of all the structural rules for these calculi. As cut does not have to be taken as a primitive rule, proof analysis is made possible and applied to a syntactic proof of conservativity.

The disjunction property for the theory of apartness, obtained here by direct syntactic methods, can also be obtained, by indirect classical reasoning, with the method of gluing of Kripke models (Dirk van Dalen, personal communication).

A characterization of the equality fragment for the first order theory of apartness has been given by van Dalen and Statman in [vDS]. The equality fragment of the theory of apartness is characterized as follows: let

$$x \,\#^{0} y =_{df} \neg x = y, \quad x \,\#^{n+1} y =_{df} \forall z (x \,\#^{n} z \vee y \,\#^{n} z).$$

Then the equality fragment of apartness is the theory $\mathbf{SEQ}^{\omega}$ obtained by adding to the pure theory of equality the generalized stability axioms:

$$\forall xy(\neg x \# ^{n} y \rightarrow x = y)$$

for all natural numbers n. The method is an analysis of normal natural deductions in the theory of apartness. Afterwards, Smorynski showed in [Sm] how to obtain this result by means of Kripke semantics.

In our proof the weak subformula property is crucial. Also in [vDS] this property is essential, although it is not explicitly stated. It is needed there for justifying the restriction to atomic occurrences only of formulas containing apartnesses in normal derivations of equality formulas (p. 106).

Our conservativity theorem relates to the result by van Dalen and Statman as follows: If A is an equality formula derivable in the theory with the axioms of apartness and stable equality, then, by the remark at the beginning of Section 6, A° is a negatomic formula derivable in the theory of apartness. By our conservativity result, A° is derivable in the theory of defined equality, hence $(A^{\circ})^{*} = A$ is derivable in the theory of stable equality. Summing up, this shows that for the propositional part, stability suffices as an extra property of equality to characterize the equality fragment of apartness.

We have also given a conservativity result for systems with several rules of split form. It would have to be studied separately how to normalize derivations in a corresponding natural deduction system with several non-logical elimination rules.

Originally our proof of conservativity was done using as a logical part of the calculus the system **G4ip**, introduced independently by Dyckhoff and Hudelmaier in [Dy] and [H]. The characteristic feature of this calculus, namely the refinement of the $L\supset$ rule according to the form of the antecedent, allowed a better control on negatomic formulas, and some difficulties here occurring in the case of the implication rules were not present. However, the proof relied on the admissibility of cut for the extension of **G4ip** with the rules for apartness. Such an extension of cut admissibility could not be proved using the technique in [Dy], because the proof there is by induction on the weight of sequents; therefore it is only suitable for systems in which the premises have a smaller weight than the conclusion, whereas the rules added for the theory of apartness have premises that are greater in weight than the conclusion. This problem was one of the motivations for an alternative direct syntactic proof of admissibility of cut for **G4ip**. A proof of admissibility of structural rules for **G4ip** using induction on weight of formulae and on height of derivations rather than on weight of sequents, and avoiding use of metatheorems about calculi based on **G3i**, together with an extension to the first-order case **G4i**, is given in [DN]. Proofs for various extensions of **G4ip**, including the theories of apartness and positive order, will be given in [DN1].

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