Two ways of finitizing linear time

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Abstract

A labelled sequent calculus for the $\mathcal{U}$-free fragment of LTL is defined through the method of internalization of the possible-worlds semantics into the syntax. The calculus enjoys desirable structural properties, but requires an infinitary rule. Two finitary fragments are identified by replacing the infinitary rule with a weaker finitary rule, and by bounding the number of its premisses. Conservativity results for appropriate fragments are proved in both cases. Full LTL is obtained by adding rules for $\mathcal{U}$ that preserve the structural properties of the system.

Keywords: Linear time logic, labelled sequent calculus, structural properties, finitization.

1 Introduction

It is well known that sequent calculi for Linear Time Logic (LTL) either require a rule with an infinite number of premisses or are not truly cut free. Several attempts have been done in order to obtain a finitary cut-free calculus for LTL: For instance, [10] uses the finite model property in order to give an upper bound to the number of the premisses of an analogous infinitary rule for the logic of common knowledge, whereas in [8] a finitary system is obtained by annotating fixpoint formulas with a history. However, in the first case the finitized rule requires a number of premisses which is exponential in the size of the conclusion and, in general, the whole approach appears quite artificial because it relies on model-theoretical rather than proof-theoretical arguments. The second solution, on the other hand, requires a more complex syntax. Furthermore, both of them do allow to prove closure under cut but not syntactic cut elimination.

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In the present work we have a different goal: Instead of trying to finitize the calculus for LTL, we identify two finitary fragments of the system. We use the method of internalization of the possible world semantics into the syntax of sequent calculi, as detailed in [12]. A labelled system G3K$_t$$LTL$ for the Until-free fragment of LTL is formulated by adding to the basic calculus for temporal logic with future operators mathematical rules that correspond to the properties of the intended class of structures: In particular, discreteness is given by an infinitary rule that defines the accessibility relation $\leq$ for $\mathbf{G}$ as the reflexive and transitive closure of the accessibility relation $\prec$ for $\mathbf{T}$. Admissibility of the structural rules of weakening, contraction, and cut can be proved syntactically along the guidelines of the general method introduced in [14] and developed in [15] and [11] for extending sequent calculi with mathematical rules. Ours, however, is not a straightforward application: The calculus requires mathematical rules that act on atomic formulas both in the left- and in the right-hand side of the sequents, with the consequence of genuinely new cases in the proof of admissibility of cut. A refined measure of complexity for relational atomic formulas is therefore used as in [6].

A weaker system G3K$_t$$LTL^W$ is formulated by substituting $\leq$ with an accessibility relation $\mathcal{R}$ that contains, possibly strictly, the reflexive and transitive closure of $\prec$. In particular, the infinitary rule is replaced by a finitary rule that permits the splitting of an interval $[x, y]$ with an immediate successor of x. Through a suitable translation, we prove that every sequent derivable in the finitary system is derivable in the infinitary one. The converse fails, but the inverse translation allows to identify a fragment of G3K$_t$$LTL$ for which conservativity with respect to G3K$_t$$LTL^W$ is proved.

Next, we give a finite bound to the number of premisses in the infinitary rule, and show that the finitized rule is as strong as the infinitary one for derivations of sequents that do not contain $\mathbf{G}$ in the negative part nor $\mathbf{F}$ in the positive part.

Finally, we extend our system with rules for Kamp's Until (henceforth $\mathcal{U}$) and thus obtain a complete calculus for full LTL. The rules for $\mathcal{U}$ call for special attention: They have active formulas in the premisses that are as complex as the principal formulas in the conclusion so that the standard inductive measure based on formula size is inappropriate for proving structural properties. The use of labels, however, permits a refined analysis and a proof of the structural properties through the definition of an appropriate inductive parameter based on the syntactic notion of *range* of a variable x in a derivation (used also in [12]).

The paper is organized as follows: In Section 2.1 we define a labelled sequent calculus, G3K$_t$$LTL$, for $\mathcal{U}$-free fragment of linear time temporal logic and show in Section 2.2 that all the structural rules are admissible. In Section 3 we prove two partial conservativity results: The finitary fragment G3K$_t$$LTL^W$ of G3K$_t$$LTL$ is presented in Section 3.1 and in Section 3.2 we show that the number of premisses of the infinitary rule can be bounded under suitable conditions. In Section 4 we extend our calculus with rules for $\mathcal{U}$ and prove that they preserve the structural properties. We conclude with a discussion of related literature and further work. The two appendices in [7] provide the proofs that have been omitted here.

2 A sequent calculus for LTL

2.1 Logical and mathematical rules

We refer to [12], [13] for general background on the method of formulating systems of sequent calculus for normal modal logics and non-classical logics. Here we extend the method to the treatment of linear time logic.

Linear Time Logic (LTL) is a temporal logic used in computer science for the specification and the verification of reactive systems. For ease of exposition, we consider first the $\mathcal{U}$-free fragment of LTL, characterized by the presence of two temporal operators **T**, 'tomorrow', and **G**, 'it will always be the case that',[3] corresponding to Prior's system 7.2 (see [16], p. 178). Semantically, the tomorrow operator is a necessity modality with respect to the accessibility relation $x \prec y$, 'x immediately precedes y'[4]; The always-in-the-future operator is a necessity modality with respect to the accessibility relation $x \leq y$, 'x precedes or is equal to y', and its dual operator, of possibility in the future, is **F**.

A sequent calculus $\mathrm{G3K}_t LTL$ for the $\mathcal{U}$-free fragment of LTL is obtained through an internalization of the possible worlds semantics into the syntax, resulting in a *labelled* sequent calculus: Every formula in a sequent $\Gamma \Rightarrow \Delta$ is either a relational atom $x \leq y$, $x \prec y$, $x = y$ or a labelled formula $x : A$. Intuitively, relational atoms and labelled formulas are the counterpart of the accessibility relations (or equality) and of the forcing relation $x \Vdash A$ of Kripke models, respectively.

The rules for propositional connectives are analogous to the standard rules, with the active and principal formulas marked by the same label x; For temporal operators, the rules are obtained from the meaning explanation in terms of their relational semantics:

$x \Vdash \mathbf{T}A$ *iff for all* y*,* $x \prec y$ *implies* $y \Vdash A$

$x \Vdash \mathbf{G}A$ *iff for all* y*,* $x \leq y$ *implies* $y \Vdash A$

$x \Vdash \mathbf{F}A$ *iff there exists* y *such that* $x \leq y$ *and* $y \Vdash A$

The left-to-right direction in the explanation above justifies the left rules, the right-to-left direction the right rules. The role of the quantifiers is reflected in the variable conditions for rules $R\mathbf{G}$, $L\mathbf{F}$, and $R\mathbf{T}$ below.

The logical rules for the calculus are given in Table 1. Observe that initial sequents are restricted to atomic formulas P. This feature, common to all G3 systems of sequent calculus, is needed in order to ensure invertibility of the rules (Lemma 2.5) and other structural properties.

In addition to the logical rules of Table 1, mathematical rules corresponding to the frame properties of accessibility relations have to be considered. In [15] and in [11] a general method was presented for extending sequent calculi with rules for axiomatic theories while preserving all the structural properties of the logical calculus. This method works for mathematical axioms expressible by means of universal

[3] In recent literature it is generally denoted by the necessity symbol $\Box$; In order to avoid any confusion with the necessity modality, we prefer to use the traditional temporal operator **G**. Analogously, we denote its dual 'it will be the case that' by the temporal operator **F**, rather than by the possibility operator $\Diamond$.

[4] Indeed, as shown in Section 2.2, **T** is self-dual under the condition of uniqueness of immediate successor.

Initial sequents:

$$x : P, \Gamma \Rightarrow \Delta, x : P \qquad x = y, \Gamma \Rightarrow \Delta, x = y$$

$$x \leq y, \Gamma \Rightarrow \Delta, x \leq y \qquad x \prec y, \Gamma \Rightarrow \Delta, x \prec y$$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A\&B, \Gamma \Rightarrow \Delta}\, L\& \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A\&B}\, R\&$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\, R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\, L\supset \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\, R\supset$$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta}\, L\bot$$

Rules for G

$$\frac{y : A, x : \mathbf{G}A, x \leq y, \Gamma \Rightarrow \Delta}{x : \mathbf{G}A, x \leq y, \Gamma \Rightarrow \Delta}\, L\mathbf{G} \qquad \frac{x \leq y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{G}A}\, R\mathbf{G}$$

Rules for F

$$\frac{x \leq y, y : A, \Gamma \Rightarrow \Delta}{x : \mathbf{F}A, \Gamma \Rightarrow \Delta}\, L\mathbf{F} \qquad \frac{x \leq y, \Gamma \Rightarrow \Delta, x : \mathbf{F}A, y : A}{x \leq y, \Gamma \Rightarrow \Delta, x : \mathbf{F}A}\, R\mathbf{F}$$

Rules for T

$$\frac{y : A, x : \mathbf{T}A, x \prec y, \Gamma \Rightarrow \Delta}{x : \mathbf{T}A, x \prec y, \Gamma \Rightarrow \Delta}\, L\mathbf{T} \qquad \frac{x \prec y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{T}A}\, R\mathbf{T}$$

(Rules $R\mathbf{G}$, $L\mathbf{F}$ and $R\mathbf{T}$ have the condition that y is not in Γ, Δ)

Table 1
Logical rules for the system $\mathrm{G3K}_t LTL$

axioms or geometric implications. In [12], contraction- and cut-free sequent calculi for various modal logics are obtained by adding to the basic system G3K the mathematical rule(s) corresponding to the properties of the accessibility relation characterizing their frames. We refer to the cited papers for the details. In the following we present the mathematical rules for the system $\mathrm{G3K}_t LTL$

Rules for Equality

$$\frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, EqRef$$

$$\frac{At(y), x = y, At(x), \Gamma \Rightarrow \Delta}{x = y, At(x), \Gamma \Rightarrow \Delta}\, EqSubst_{At} \qquad \frac{y : P, x = y, x : P, \Gamma \Rightarrow \Delta}{x = y, x : P, \Gamma \Rightarrow \Delta}\, EqSubst$$

(where $At(x)$ stands for an equality or a relational formula $x \leq z$, $z \leq x$, $x \prec z$ or $z \prec x$)

Rules for Order Relation ($\leq$)

$$\frac{x \leq z, x \leq y, y \leq z, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \Gamma \Rightarrow \Delta}\, Trans \qquad \frac{x \leq x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Ref$$

Rules for Successor Relation ($\prec$)

$$\frac{y = z, x \prec y, x \prec z, \Gamma \Rightarrow \Delta}{x \prec y, x \prec z, \Gamma \Rightarrow \Delta}\, UnSucc$$

$$\frac{x \prec y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, R\text{-}Ser \qquad \frac{x \leq y, x \prec y, \Gamma \Rightarrow \Delta}{x \prec y, \Gamma \Rightarrow \Delta}\, Inc$$

(Rule *R-Ser* has the condition that y is not in Γ, Δ)

The order relation $x \leq y$ is defined as the transitive and reflexive closure of the immediate successor relation $x \prec y$, that is,

$$x \leq y \;\equiv\; \exists n \in \mathbb{N}\, x \prec^n y$$

where $x \prec^n y$ is defined inductively as follows

$x \prec^0 y \;\equiv\; x = y$
$x \prec^1 y \;\equiv\; x \prec y$
$x \prec^{n+1} y \;\equiv\; \exists z (x \prec^n z \,\&\, z \prec y)$, for $n > 0$

Rules for Iterated Successor Relation ($\prec^n$)

The left to right direction of the definition of $x \prec^n y$ gives the following rule

$$\frac{x \prec^n y, y \prec z, \Gamma \Rightarrow \Delta}{x \prec^{n+1} z, \Gamma \Rightarrow \Delta}\, LDef \qquad (y \text{ not in } \Gamma, \Delta)$$

and the right to left direction gives the right rule

$$\frac{\Gamma \Rightarrow \Delta, x \prec^{n+1} z, x \prec^n y \quad \Gamma \Rightarrow \Delta, x \prec^{n+1} z, y \prec z}{\Gamma \Rightarrow \Delta, x \prec^{n+1} z}\, RDef$$

Infinitary Rule

The left to right direction of the definition of $x \leq y$ as transitive closure of $x \prec y$ gives the following infinitary rule

$$\frac{\{x \prec^n y, x \leq y, \Gamma \Rightarrow \Delta\}_{n \in \mathbb{N}}}{x \leq y, \Gamma \Rightarrow \Delta}\, T^{\omega}$$

The right to left direction gives, for every $n \in \mathbb{N}$, the following generalized form of rule *Inc*

$$\frac{x \leq y, x \prec^n y, \Gamma \Rightarrow \Delta}{x \prec^n y, \Gamma \Rightarrow \Delta}\, Inc_n$$

which is admissible in our system, as well as the analogously generalized rules $UnSucc_n$, $EqSubst_n$ and $R\text{-}Ser_n$ (see Appendix A in [7]).

We observe here that no rule, except for $RDef$, removes a relational atom from the right-hand side of sequents in a derivation, and that such atoms are never active in propositional rules. Therefore, initial sequents of the form $x \leq y, \Gamma \Rightarrow \Delta, x \leq y$ or $x \prec y, \Gamma \Rightarrow \Delta, x \prec y$ or $x = y, \Gamma \Rightarrow \Delta, x = y$, and the derivable sequents $x \prec^n y, \Gamma \Rightarrow \Delta, x \prec^n y$ are needed only for deriving properties of accessibility relations, namely the axioms corresponding to the mathematical rules previously considered.

2.2 *Structural properties*

Next we show that the system G3K$_t LTL$ enjoys desirable structural properties.

Lemma 2.1 *Sequents of the form $x : A, \Gamma \Rightarrow \Delta, x : A$, with A an arbitrary modal formula (not just atomic), are derivable in* G3K$_t LTL$*.*

In order to guarantee invertibility of the rules, initial sequents cannot have the form $x \prec^n y, \Gamma \Rightarrow \Delta, x \prec^n y$ for $n > 1$. However, they are easily derivable by induction:

Lemma 2.2 *Sequents of the form $x \prec^n y, \Gamma \Rightarrow \Delta, x \prec^n y$ are derivable in* G3K$_t LTL$ *for all $n \in \mathbb{N}$.*

Substitution of labels is defined in the obvious way as follows for relational atoms and labelled formulas. We use xRy to denote an equality $x = y$ or a relational formula $x \leq y$, $x \prec y$.

$$xRy(z/w) \equiv xRy \text{ if } w \neq x \text{ and } w \neq y$$

$$xRy(z/x) \equiv zRy \text{ if } x \neq y$$

$$xRy(z/y) \equiv xRz \text{ if } x \neq y$$

$$xRx(z/x) \equiv zRz$$

$$x : A(z/y) \equiv x : A \text{ if } y \neq x$$

$$x : A(z/x) \equiv z : A$$

The definition of substitution is extended to multisets componentwise. We have:

Lemma 2.3 *If $\Gamma \Rightarrow \Delta$ is derivable in* G3K$_t LTL$*, then also $\Gamma(y/x) \Rightarrow \Delta(y/x)$ is derivable, with the same derivation height.*

In what follows, Greek lower case is used for denoting labelled or relational formulas.

Theorem 2.4 *The rules of left and right weakening*

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} LWk \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} RWk$$

are height-preserving admissible in $\mathrm{G3K}_t LTL$.

Lemma 2.5 *All the rules of* $\mathrm{G3K}_t LTL$ *are height-preserving invertible.*

Theorem 2.6 *The rules of left and right contraction*

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} LCtr \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} RCtr$$

are height-preserving admissible in $\mathrm{G3K}_t LTL$.

The system $\mathrm{G3K}_t LTL$ has mathematical rules that act both on the left- and on the right-hand side of sequents, and a measure of complexity for relational atoms is needed in the proof of cut elimination, as in [6].

Definition 2.7 The length of a labelled formula $x : A$ is defined as the length of A. The length of relational or equality formulas is defined as follows: $l(x \prec y) \equiv l(x \leq y) \equiv l(x = y) \equiv 1$ and $l(x \prec^n y) \equiv n$ for $n \geq 1$.

Theorem 2.8 *The rule of cut*

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} Cut$$

is admissible in $\mathrm{G3K}_t LTL$.

Note that the interpretation of operator **T** is self-dual in the presence of the rules *R-Ser* and *UnSucc*, that means that its semantic explanation can be reformulated as follows:

$x \Vdash \mathbf{T}A$ *iff there exists an (unique) instant* y *such that* $x \prec y$ *and* $y \Vdash A$

In fact, we have the following result:

Proposition 2.9 *The sequent* $x \prec y, y : A, \Gamma \Rightarrow \Delta, x : \mathbf{T}A$ *is derivable in* $\mathrm{G3K}_t LTL$.

Some formulations of linear time logic, as for example [9] (pp. 73-74), stress the importance of the recursive definition $\mathbf{G}A \equiv (A \ \& \ \mathbf{TG}A)$. The right to left direction of this equivalence corresponds to the frame property

$$x \leq y \supset (x = y \vee \exists z (x \prec z \ \& \ z \leq y))$$

In our system the latter corresponds to the following rule:

$$\frac{x = y, x \leq y, \Gamma \Rightarrow \Delta \quad x \prec z, z \leq y, x \leq y, \Gamma \Rightarrow \Delta}{x \leq y, \Gamma \Rightarrow \Delta} Mix \qquad (z \text{ not in } \Gamma, \Delta)$$

Rule *Mix* is admissible in the presence of T^ω; Else it can be used for defining an alternative, finitary system, as we shall see below.

Proposition 2.10 *Rule* Mix *is admissible in* $\mathrm{G3K}_t LTL$.

The property of right linearity, $\forall x \forall y \forall z (x \leq z \;\&\; x \leq y \supset y \leq z \vee z \leq y)$, is guaranteed by admissibility of the corresponding rule:

Proposition 2.11 *The rule of right linearity*

$$\frac{y \leq z, x \leq z, x \leq y, \Gamma \Rightarrow \Delta \quad z \leq y, x \leq z, x \leq y, \Gamma \Rightarrow \Delta}{x \leq z, x \leq y, \Gamma \Rightarrow \Delta}\, \textit{R-Lin}$$

is admissible in $\mathrm{G3}_t LTL$.

The labelled sequents corresponding to the standard axioms and the modal rules of the $\mathcal{U}$-free fragment of LTL are derivable/admissible in $\mathrm{G3K}_t LTL$:

Proposition 2.12 *The following characteristic sequents*

$x : \mathbf{G}(A \supset B), x : \mathbf{G}A \Rightarrow x : \mathbf{G}B$ $\qquad$ $x : \mathbf{T}(A \supset B), x : \mathbf{T}A \Rightarrow x : \mathbf{T}B$

$x : \mathbf{T}\neg A \Rightarrow x : \neg\mathbf{T}A$ $\qquad$ $x : \neg\mathbf{T}A \Rightarrow x : \mathbf{T}\neg A$

$x : \mathbf{G}A \Rightarrow x : A \,\&\, \mathbf{TG}A$ $\qquad$ $x : A, x : \mathbf{TG}A \Rightarrow x : \mathbf{G}A$

$x : A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A$

are derivable in $\mathrm{G3K}_t LTL$.

Proposition 2.13 *The necessitation rules for* **G** *and* **T**

$$\frac{\Rightarrow x : A}{\Rightarrow x : \mathbf{G}A}\,\mathbf{G}Nec \qquad\qquad \frac{\Rightarrow x : A}{\Rightarrow x : \mathbf{T}A}\,\mathbf{T}Nec$$

are admissible in $\mathrm{G3K}_t LTL$.

Corollary 2.14 *The calculus* $\mathrm{G3K}_t LTL$ *is complete with respect to the* $\mathcal{U}$*-free fragment of* LTL.

3 Finitization

3.1 A non-standard system for LTL

Let us consider a relation $\mathcal{R}$ that contains the transitive and reflexive closure $\leq$ of $\prec$ and corresponds to the operators $\widetilde{\mathbf{G}}$ and $\widetilde{\mathbf{F}}$. We then define the system $\mathrm{G3K}_t LTL^W$ by substituting, in the calculus $\mathrm{G3K}_t LTL$, the relation $\leq$ with $\mathcal{R}$, the operators **G** and **F** with $\widetilde{\mathbf{G}}$ and $\widetilde{\mathbf{F}}$ respectively, by removing the rules T^ω, $LDef$ and $RDef$, and by adding the rules Mix and $R\text{-}Lin$ as primitive.

The system $\mathrm{G3K}_t LTL^W$ admits non-standard models consisting of several (possibly infinite) consecutive copies of the natural numbers, $\mathbb{N} \oplus \ldots \oplus \mathbb{N}$. In the standard model for discrete time logic, every instant greater than the initial point is the unique successor of its unique predecessor and can be reached from the initial instant by finitely many iterations of the immediate successor relation: This condition corresponds to the infinitary rule T^ω of the calculus $\mathrm{G3K}_t LTL$.

On the contrary, because of the absence of rule T^ω, the systems $\mathrm{G3K}_t LTL^W$ allows non-standard models: Although every point has a unique immediate successor, there can be instants different from the initial one that are not immediate successors of any other instant and are instead initial points of successive copies of

$\mathbb{N}$. Thus it is not always true that between any two points x, y such that $x\mathcal{R}y$ there are finitely many other points.

If the rule of right linearity is dropped from the calculus G3K$_t LTL^W$, we obtain a calculus for the so-called branching time gaps, the models of which are constituted by well-founded trees of copies of $\mathbb{N}$. The first-order temporal logic corresponding to branching time gaps has been investigated from a model theoretic point of view in [5]. Its proof-theoretic aspects have been investigated through a G3-style (unlabelled) sequent calculus in [2].

In the following, we show that the system G3K$_t LTL^W$ can be embedded in G3K$_t LTL$: Every sequent derivable in G3K$_t LTL^W$ is derivable in G3K$_t LTL$ under a suitable translation. The converse fails because of the infinitary rule: For instance, every proof search for the characteristic sequent corresponding to the induction principle $x : A, x : \widetilde{\mathbf{G}}(A \supset \mathbf{T}A) \Rightarrow x : \widetilde{\mathbf{G}}A$ would require to apply rule *Mix* infinitely many times. However, we identify a conservative fragment, for which derivability in G3K$_t LTL$ implies derivability in G3K$_t LTL^W$ under the inverse translation.

The translation * from the language of G3K$_t LTL^W$ to the language of G3K$_t LTL$ is defined inductively as follows:

$(x = y)^* \equiv x = y$
$(x \prec y)^* \equiv x \prec y$
$(x\mathcal{R}y)^* \equiv x \leq y$
$(\bot)^* \equiv \bot$
$(P)^* \equiv P$ (P atomic formula)
$(A \circ B)^* \equiv A^* \circ B^*$ ($\circ$ propositional constant)
$(\widetilde{\mathbf{G}}A)^* \equiv \mathbf{G}A^*$
$(\widetilde{\mathbf{F}}A)^* \equiv \mathbf{F}A^*$
$(\mathbf{T}A)^* \equiv \mathbf{T}A^*$
$(x : A)^* \equiv x : A^*$
$\Gamma^* \equiv \{\varphi^* | \varphi$ is a relational atom or a labelled formula in $\Gamma\}$

We have:

Theorem 3.1 *If* $\vdash_{G3K_t LTL^W} \Gamma \Rightarrow \Delta$ *then* $\vdash_{G3K_t LTL} \Gamma^* \Rightarrow \Delta^*$

The inverse translation $^+$ from G3K$_t LTL$ to G3K$_t LTL^W$ is defined as follows:

$(x = y)^+ \equiv x = y$
$(x \prec y)^+ \equiv x \prec y$
$(x \leq y)^+ \equiv x\mathcal{R}y$
$(\bot)^+ \equiv \bot$
$(P)^+ \equiv P$
$(A \circ B)^+ \equiv A^+ \circ B^+$
$(\mathbf{G}A)^+ \equiv \widetilde{\mathbf{G}}A^+$
$(\mathbf{F}A)^+ \equiv \widetilde{\mathbf{F}}A^+$
$(\mathbf{T}A)^+ \equiv \mathbf{T}A^+$
$(x : A)^+ \equiv x : A^+$

$\Gamma^+ \equiv \{\varphi^+ | \varphi$ is a relational atom or a labelled formula in $\Gamma\}$

We have:

Theorem 3.2 *If the sequent* $\Gamma \Rightarrow \Delta$ *is derivable in* G3K$_t$*LTL and the operator* **G** *does not appear in the positive part, nor* **F** *in the negative part of the sequent* $\Gamma \Rightarrow \Delta$ *and* Γ, Δ *do not contain relational atoms, then* $\Gamma \Rightarrow \Delta$ *is derivable without using the infinitary rule.*

Proof. We show that all the applications of the infinitary rule can be dispensed with. Without loss of generality we assume that the given derivation is minimal, in the sense that no shortenings arising from applications of height preserving contraction are possible: This excludes rule instances such as transitivity with a reflexitity atom as principal. Observe that each relational atom $x \leq y$, in particular those concluded by T^ω, has to disappear before the conclusion. We consider one such downmost atom and the rule that makes it disappear: Rules $R\mathbf{G}$ and $L\mathbf{F}$ are excluded because they would introduce **G** in the positive part or **F** in the negative part. Thus it can disappear by means of Inc or Ref.

If the atom concluded by T^ω is removed by Ref, we have

$$\cfrac{\cfrac{\{x \prec^n x, x \leq x, \Gamma' \Rightarrow \Delta'\}_{n\in\mathbb{N}}}{x \leq x, \Gamma' \Rightarrow \Delta'}\, T^\omega}{\cfrac{\vdots}{\cfrac{x \leq x, \Gamma'' \Rightarrow \Delta''}{\Gamma'' \Rightarrow \Delta''}\, Ref}}$$

we take the leftmost premiss of T^ω and transform the derivation into the following

$$\cfrac{\cfrac{x = x, x \leq x, \Gamma' \Rightarrow \Delta'}{x \leq x, \Gamma' \Rightarrow \Delta'}\, EqRef}{\cfrac{\vdots}{\cfrac{x \leq x, \Gamma'' \Rightarrow \Delta''}{\Gamma'' \Rightarrow \Delta''}\, Ref}}$$

with the application of T^ω removed from the derivation.

Because of the assumption of minimality the instance of Ref cannot be preceded by an application of $Trans$ with the reflexivity atom $x \leq x$ or $y \leq y$ and the atom $x \leq y$ concluded by T^ω as principal.

If the atom concluded by T^ω is removed by Inc, we have

$$\cfrac{\cfrac{\{x \prec^n y, x \leq y, \Gamma' \Rightarrow \Delta'\}_{n\in\mathbb{N}}}{x \leq y, \Gamma' \Rightarrow \Delta'}\, T^\omega}{\cfrac{\vdots}{\cfrac{x \leq y, x \prec y, \Gamma'' \Rightarrow \Delta''}{x \prec y, \Gamma'' \Rightarrow \Delta''}\, Inc}}$$

The second premiss from the left of T^ω has the form $x \prec y, x \leq y, x \prec y, \Gamma''' \Rightarrow \Delta'$, with $\Gamma' \equiv x \prec y, \Gamma'''$. By height-preserving contraction we obtain $x \leq y, \Gamma' \Rightarrow \Delta'$ and proceed with the derivation until we reach $x \prec y, x \leq y, \Gamma'' \Rightarrow \Delta''$. Then we conclude $x \prec y, \Gamma'' \Rightarrow \Delta''$ by an application of Inc. Note that the derivation is shortened contrary to the assumption of minimality.

If the atom concluded by T^ω is removed by applications of $Trans$ followed by applications of Inc, we have the derivation

$$\dfrac{\dfrac{\{x \prec^n y, x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'\}_{n\in\mathbb{N}}}{x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'}\, T^\omega}{\vdots}$$

$$\dfrac{x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma'' \Rightarrow \Delta''}{x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma'' \Rightarrow \Delta''}\, Trans{\times}m$$

$$\vdots$$

$$\dfrac{x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma''' \Rightarrow \Delta'''}{x \prec z_1, \ldots, z_m \prec y, \Gamma''' \Rightarrow \Delta'''}\, Inc{\times}(m+1)$$

that can be transformed into the following derivation

$$\dfrac{\dfrac{x \prec^{m+1} y, x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'}{x \prec z_1, \ldots, z_m \prec y, x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'}\, LDef\text{-}Inv{\times}m}{x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'}\, Ctr^*$$

$$\vdots$$

$$\dfrac{x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma'' \Rightarrow \Delta''}{x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma'' \Rightarrow \Delta''}\, Trans{\times}m$$

$$\vdots$$

$$\dfrac{x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma''' \Rightarrow \Delta'''}{x \prec z_1, \ldots, z_m \prec y, \Gamma''' \Rightarrow \Delta'''}\, Inc{\times}(m+1)$$

where $LDef$-Inv stands for height-preserving invertibility of rule $LDef$. Again the derivation is shortened contrary to the assumption.

Note that if the atom concluded by T^ω is removed by an application of $EqSubst_{At}$, we have the following derivation:

$$\dfrac{\{x \prec^n y, z = y, x \leq y, x \leq z, \Gamma' \Rightarrow \Delta'\}_{n\in\mathbb{N}}}{z = y, x \leq y, x \leq z, \Gamma' \Rightarrow \Delta'}\, T^\omega$$

$$\vdots$$

$$\dfrac{z = y, x \leq y, x \leq z, \Gamma'' \Rightarrow \Delta''}{z = y, x \leq z, \Gamma'' \Rightarrow \Delta''}\, EqSubst_{At}$$

It is possible to permute up rule $EqSubst_{At}$ with respect to the rule T^ω. We modify each premiss of T^ω as follows:

$$\dfrac{x \prec^n y, x \leq y, z = y, x \leq z, \Gamma' \Rightarrow \Delta'}{x \prec^n y, x \leq y, x \prec^n z, z = y, x \leq z, \Gamma' \Rightarrow \Delta'}\, LWk$$

$$\vdots$$

$$\dfrac{\dfrac{x \prec^n y, x \leq y, x \prec^n z, z = y, x \leq z, \Gamma'' \Rightarrow \Delta''}{x \prec^n y, x \prec^n z, z = y, x \leq z, \Gamma' \Rightarrow \Delta'}\, Inc_n}{x \prec^n z, z = y, x \leq z, \Gamma'' \Rightarrow \Delta''}\, EqSubst_n$$

We can now apply previous modifications. The case of $EqSubst_{At}$ with active formulas $z = x, x \leq y, z \leq y$ is analogous. □

Corollary 3.3 *If* $\vdash_{G3K_tLTL} \Gamma \Rightarrow \Delta$ *and* $\Gamma \Rightarrow \Delta$ *is as in the previous theorem, then* $\vdash_{G3K_tLTL^W} \Gamma^+ \Rightarrow \Delta^+$.

3.2 A finite bound for the infinitary rule

Let us consider now the calculus $\mathrm{G3K}_t LTL^\delta$ obtained by substituting the infinitary rule T^ω by the following finitary version T^δ

$$\frac{\{x \prec^m y, x \leq y, \Gamma \Rightarrow \Delta\}_{0 \leq m \leq \delta(\Gamma,\Delta)+1}}{x \leq y, \Gamma' \Rightarrow \Delta'}\, T^\delta$$

where, in order to guarantee admissibility of weakening, Γ' and Δ' are arbitrary multisets containing Γ and Δ respectively, and $\delta(\Gamma, \Delta)$ is defined as the number of occurrences of $\mathbf{T}$ in the positive part of the sequent $x \prec^m y, x \leq y, \Gamma \Rightarrow \Delta$.

Lemma 3.4 *Let*

$$x \prec^m z, z \prec y, x \leq y, \Gamma \Rightarrow \Delta$$

be a sequent derivable in $\mathrm{G3K}_t LTL$ *not containing* $\mathbf{G}$ *in the negative part, nor* $\mathbf{F}$ *in the positive part, nor* z *in* Γ, Δ*, nor relational atoms in* Δ*, and let* m *be* $\delta(\Gamma, \Delta)$*. Then also the sequent*

$$x \prec^m z, z \prec^n y, x \leq y, \Gamma \Rightarrow \Delta$$

is derivable in $\mathrm{G3K}_t LTL$ *for all* $n \geq 1$*.*

Proof. We are interested in minimal derivations. Trace up the atom $z \prec y$ along the derivation. If it is never principal, it can be replaced by $z \prec^n y$ all along the derivation. It cannot be principal in an axiomatic sequent because of the condition that no relational atoms are in Δ. Therefore the possibilities are $L\mathbf{T}$, $EqSubst_{At}$, $UnSucc$, and Inc.

If $z \prec y$ were principal in $L\mathbf{T}$ we would have the derivation

$$\frac{z \prec y, y : A, z : \mathbf{T}A, \Gamma' \Rightarrow \Delta'}{z \prec y, z : \mathbf{T}A, \Gamma' \Rightarrow \Delta'}\, L\mathbf{T}$$

but the principal formula $z : \mathbf{T}A$ can disappear only going back along the chain $x \prec^m z$ by means of applications of $L\mathbf{T}$ and/or instances of $L\mathbf{G}$ or $R\mathbf{F}$ (possibly combined with rule Inc). However, $L\mathbf{G}$ and $R\mathbf{F}$ are excluded by the condition that $\mathbf{G}$ is not in the negative part and $\mathbf{F}$ is not in the positive part of the sequent; whereas applications of $L\mathbf{T}$ would introduce in the positive part of the sequent a number of operators $\mathbf{T}$ greater than m, contrary to the hypothesis that $m = \delta(\Gamma, \Delta)$. Note that by the same reason no atom of the form $z \prec t$ can be principal in a left rule for $\mathbf{T}$, whereas the case of $z \prec t$ active in an instance of $R\mathbf{T}$ would introduce a formula $z : \mathbf{T}A$ in the consequent that cannot disappear without violating the variable conditions on temporal rules or the hypothesis that z is not in Γ, Δ.

Neither the atom $z \prec y$ nor any other atom of the form $z \prec t$ can be principal in $EqSubst_{At}$ with equality on z

$$\frac{w \prec t, z = w, z \prec t, \Gamma' \Rightarrow \Delta'}{z = w, z \prec t, \Gamma' \Rightarrow \Delta'}\, EqSubst_{At}$$

In fact, since z is not in Γ, Δ, the atom $z = w$ should disappear from the derivation. It could not be removed by rule $EqRef$, because otherwise we could shorten the derivation by means of height-preserving admissibility of contraction on formulas $w \prec t, z \prec t$ (by $z \equiv w$), contrary to the hypothesis of minimality. The atom $z = w$

could disappear by $UnSucc$ with principal formulas $v \prec z, v \prec w$ or by T^ω with principal formula $z \leq w$ or $w \leq z$, but both formulas cannot disappear without introducing new relational atoms with variable z or logical formulas labelled by z, contrary to the hypothesis that z is not in Γ, Δ.

For the cases with $z \prec y$ principal in $UnSucc$ or Inc, we observe that we can replace all along the derivation relational formulas of the form $z \prec t$ ($z \prec y$ included) with $z \prec^n t$ and apply the admissible rules Inc_n, $UnSucc_n$, $EqSubst_n$ and $R\text{-}Ser_n$ whenever Inc, $UnSucc$, $EqSubst_{At}$ and $R\text{-}Ser$ are applied in the original derivation. □

Theorem 3.5 *If the sequent $\Gamma \Rightarrow \Delta$ does not contain $\mathbf{G}$ in the negative part nor $\mathbf{F}$ in the positive part and Γ, Δ do not contain relational atoms, then $\vdash_{G3K_tLTL^\delta} \Gamma \Rightarrow \Delta$ iff $\vdash_{G3K_tLTL} \Gamma \Rightarrow \Delta$.*

4 Adding Until

We obtain the system G3K$_t LTL+\mathcal{U}$ for full LTL by adding the rules for the binary temporal operator Until. Instead of using the semantic explanation of Until

$x \Vdash A\mathcal{U}B$ iff there exists y such that $x \leq y$ and $y \Vdash B$ and for all z, if $x \leq z$ and $z < y$ then $z \Vdash A$

we justify the rules through the recursive definition

$$A\mathcal{U}B \equiv B \vee (A \,\&\, \mathbf{T}(A\mathcal{U}B) \& \,\mathbf{F}B)$$

The rules for Until are of a peculiar form, since the complexity of the active formulas is not strictly less than the complexity of the principal formulas. A similar situation occurs in the sequent rules for Gödel-Löb logic presented in [12] and here, as well, admissibility of the structural rules is shown through a refined inductive measure based on the notion of range.

Rules for $\mathcal{U}$:

$$\frac{x \prec y, x : B, \Gamma \Rightarrow \Delta \quad x \prec y, x : A, y : A\mathcal{U}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta}{x \prec y, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta} \, L\mathcal{U}$$

$$\frac{\Gamma \Rightarrow \Delta, x : A\mathcal{U}B, x : B}{\Gamma \Rightarrow \Delta, x : A\mathcal{U}B} \, R\mathcal{U}_1$$

$$\frac{\Gamma \Rightarrow \Delta, x : A\mathcal{U}B, x : A \quad x \prec y, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, y : A\mathcal{U}B \quad \Gamma \Rightarrow \Delta, x : A\mathcal{U}B, x : \mathbf{F}B}{\Gamma \Rightarrow \Delta, x : A\mathcal{U}B} \, R\mathcal{U}_2$$

Rule $R\mathcal{U}_2$ has the condition that y is not in Γ, Δ.

We also need to admit initial sequents of the form:

$$x : A\mathcal{U}B, \Gamma \Rightarrow \Delta, x : A\mathcal{U}B$$

The right rules for $\mathcal{U}$ are invertible by admissibility of weakening. By the following lemma, also the left rule is invertible.

Lemma 4.1 *If the sequent* $x \prec y, x : A\mathcal{U}B, \Gamma \Rightarrow \Delta$ *is derivable, then the sequents*

$$x \prec y, x : B, \Gamma \Rightarrow \Delta$$

$$x \prec y, x : A, y : A\mathcal{U}B, x : \mathbf{F}B, \Gamma \Rightarrow \Delta$$

are derivable.

Definition 4.2 The right *range* of x in a derivation $\mathcal{D}$ is the (finite) set of instants y such that either $x \prec y$ or for some $n \geq 1$ and for some $x_1, \ldots, x_n$, the atoms $x \prec x_1, x_1 \prec x_2, \ldots, x_n \prec y$ appear in the sequents of $\mathcal{D}$. The left range of x is defined analogously as the set of instants y such that either $y \prec x$ or for some $n \geq 1$ and for some $y_1, \ldots, y_n$, the atoms $y \prec y_1, y_1 \prec y_2, \ldots, y_n \prec x$ appear in the sequents of $\mathcal{D}$. Ranges of variables are ordered by set inclusion.

Theorem 4.3 *The rules of contraction are right range-preserving admissible.*

Theorem 4.4 *The rule of cut is admissible in the calculus* $\mathbf{G3}\mathrm{K}_t LTL + \mathcal{U}$*.*

Proposition 4.5 *The following characteristic sequents*

$$x : A\mathcal{U}B \Rightarrow x : \mathbf{F}B \qquad x : \mathbf{F}B \Rightarrow x : \top\mathcal{U}B$$

$$x : A\mathcal{U}B \Rightarrow x : B, x : A \,\&\, \mathbf{T}(A\mathcal{U}B) \qquad x : B \vee (A \,\&\, \mathbf{T}(A\mathcal{U}B)) \Rightarrow x : A\mathcal{U}B$$

are derivable in $\mathbf{G3}\mathrm{K}_t LTL + \mathcal{U}$*.*

Corollary 4.6 *The calculus* $\mathbf{G3}\mathrm{K}_t LTL + \mathcal{U}$ *is complete with respect to* LTL*.*

Note that the peculiar form of the rules for $\mathcal{U}$ does not influence the proof of Lemma 3.2, therefore the conservativity result of Theorem 3.3 holds for the systems $\mathrm{G3K}_t LTL + \mathcal{U}$ and $\mathrm{G3K}_t LTL^W + \mathcal{U}$. On the contrary, the fact that $z \prec y$ can be principal in $L\mathcal{U}$ prevents from proving Lemma 3.4, henceforth Theorem 3.5, for the extended calculus.

5 Discussion of related work and further developments

In [5] a comparison between first-order temporal logics, with future operators $\Box$ and $\bigcirc$,[5] for linear discrete time and for linear time with branching gaps is put forward: Whereas in the former an infinitary rule is needed, the latter is formulated as a cut-free extension of Gentzen's system for classical predicate logic with finitary rules for temporal operators. A conservativity result is then obtained for the $\Box$-free fragment of the system.

As we observed in Section 3, if we drop the rule of right linearity from $\mathrm{G3K}_t LTL^W$ we obtain a labelled sequent calculus that corresponds to the propositional fragment of the system of [5]. However, we can prove a stronger conservativity result: Our theorem has only the condition that endsequents do not contain $\mathbf{G}$ in the positive part (nor $\mathbf{F}$ in the negative part), whereas in [5] $\Box$ cannot appear at all in the formula to be derived. In addition, our calculus contains the rules for $\mathbf{F}$

[5] Corresponding to our $\mathbf{G}$ and $\mathbf{T}$, respectively.

and can easily be extended with rules for $\mathcal{U}$, whereas [5] has only the operators $\Box$ and $\bigcirc$.

In our work we have identified two finitary fragments of LTL: The first is obtained by substituting the rule that corresponds to reflexive and transitive closure with a weaker finitary counterpart. The second, somehow complementary fragment, is obtained through a finite bound on the number of premisses of the infinitary rule. Related results were obtained in a different, but qualitatively similar case, in the logic of common knowledge. A conservativity result for the logic of common knowledge, parallel to the one for our first fragment, is presented in [3,4]. Results similar to ours for the second fragment were obtained in [1] for an unlabelled Tait-style calculus for the logic of common knowledge. However, our calculus allows for a syntactical cut elimination, whereas [1] shows through a semantical argument that the rule of cut is not needed. The problem of finding a proof-theoretically justified finitary calculus for the complete system of LTL is still open. All the different approaches proposed so far in the literature have provided only partial solutions and it seems that the difficulties stand in the matter rather than in the method applied.

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