The Logic of Conditional Beliefs: Neighbourhood Semantics and Sequent Calculus

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Abstract

The logic of Conditional Beliefs has been introduced by Board, Baltag and Smets to reason about knowledge and revisable beliefs in a multi-agent setting. It is shown how the semantics of this logic, defined in terms of plausibility models, can be equivalently formulated in terms of neighbourhood models, a multi-agent generalisation of Lewis' spheres models. On the basis of this new semantics, a labelled sequent calculus for the logic of Conditional Beliefs is developed. The calculus has strong proof-theoretic properties, in particular admissibility of contraction and cut, and it provides a direct decision procedure for the logic. Furthermore, its semantic completeness is used to obtain a constructive proof of the finite model property of the logic.

Keywords: Epistemic logic, conditional logic, neighbourhood semantics, sequent calculus, decision procedure.

1 Introduction

Modal epistemic logic has been studied for a long time in formal epistemology, computer science, and notably in artificial intelligence. In this logic, to each agent i is associated a knowledge modality K_i, so that the formula $K_i A$ expresses that "agent i knows A." Through agent-indexed modal operators, epistemic logic can be employed to reason about the mutual knowledge of a set of agents. The logic has been further extended by other modalities to encode various types of combined knowledge of agents (e.g., common knowledge). However, knowledge is not the only propositional attitude, and belief is equally significant to reason about epistemic interaction among agents. Board [5], and then Baltag and Smets [2], [3], [4] have proposed a logic called *CDL* (Conditional Doxastic Logic) for modelling both belief and knowledge in a multi-agent

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setting. The essential feature of beliefs is that they are *revisable* whenever the agent learns new information. To capture this revisable nature of beliefs, *CDL* contains the conditional belief operator $Bel_i(C|B)$, the meaning of which is that agent i believes C if she learnt B. Thus the conditional belief has an hypothetical meaning: if agent i learnt B, she would believe that C was true in the state of the world *before* the act of learning B. The logic captures the agent's changing beliefs in an unchanging world. For this reason Baltag and Smets [3] qualify this logic as "static" in contrast to "dynamic" epistemic logic, where the very act of learning (by some form of announcement) may change the agent's beliefs. The logic *CDL* in itself is used as the basic formalism to study further dynamic extensions of epistemic logics, determined by several kinds of epistemic/doxastic actions. Notice that both unconditional beliefs and knowledge can be defined in *CDL*: Bel_iB (agent i believe B) as $Bel_i(B|\top)$, and K_iB (agent i knows B) as $Bel(\bot|\neg B)$, the latter meaning that i considers impossible (inconsistent) to learn $\neg B$.

To exemplify the language, consider a variant of the three-wise-men puzzle, where agent a may initially believe that she has a white hat: Bel_aW_a. However, if a learns that agent b knows the colour of the hat b herself wears, she might change her beliefs and be convinced that she is wearing a black hat instead: $Bel_a(B_a|K_bW_b \vee K_bB_b)$. The example shows that the conditional operator is non-monotonic in the sense that $Bel_i(C|A)$ does not entail $Bel_i(C|A \wedge B)$ (here $A = \top$).

The axiomatization of the operator Bel_i in *CDL* internalises the well-known AGM postulates of belief revision [3].

The semantic interpretation of *CDL* is defined in terms of the so-called epistemic plausibility models. In these models, to each agent i is associated an equivalence relation $\sim_i$, used to interpret knowledge, and a well-founded preorder $\preceq_i$ on worlds. The relation $\preceq_i$ assesses the relative plausibility of worlds according to an agent i and is used to interpret conditional beliefs: i believes B conditionally on A in a world x if B holds in *the most plausible worlds* accessible from x in which A holds, where the "most plausible worlds" for an agent i are the $\preceq_i$-minimal ones. This semantic approach has been dominant in the studies of *CDL*; in addition to [5] and [3] we mention works by Pacuit [19], Van Ditmarsch et al. [20] and Demay [6].

In this paper we provide an alternative semantics, based on neighbourhood models, for *CDL*. Neighbourhood models are often used in the interpretation of non-normal modal logics. In the present setting they can be seen as a multi-agent generalization of Lewis' spheres models for counterfactual logics. Notice that finite sphere models have been used to define semantically (mono-agent) belief revision since Grove' seminal work [8]. In neighbourhood models to each world x and agent i is associated a set $I_i(x)$ of nested sets of worlds; each set $\alpha \in I_i(x)$ represents, so to say, a relevant piece of information that can

[3] We cannot mention here the vast literature on the relation between belief revision, conditional logics, the Ramsey Test, and Gärdenfors Triviality Result.

be used to establish the truth of an epistemic/doxastic statement. The interpretation of the conditional belief operator Bel_i then coincides with Lewis' semantics of the counterfactual operator. The equivalence between plausibility models and neighbourhood models does not come as totally unexpected: for the mono-agent case, it was suggested or stated without proof by Board [5], Pacuit [19], Marti et al [11], and it is based on an old result about the correspondence between partial orders and Alexandroff topologies [1]. We will detail the correspondence for the multi-agent case.

We believe that neighbourhood models provide by themselves a terse interpretation of the epistemic and doxastic modalities, abstracting away from the relational information specified in plausibility models. Moreover, it is worth noticing that in these models the interpretation of unconditional beliefs and knowledge results in the standard universal/existential neighbourhood modalities.

Up to this moment, the logic *CDL* has been studied only from a semantic viewpoint, and no proof-system or calculus is known for it. Our main goal is to provide one. On the basis of neighbourhood semantics we develop a labelled sequent calculus called **G3CDL**. We follow the general methodology of [13] to develop labelled calculi for modal logics. Similarly to [14], the calculus **G3CDL** makes use of world and neighbourhood labels, thereby importing the semantics, limited to the essential, into the syntax. In **G3CDL**, each connective is handled by symmetric left/right rules, whereas the properties of neighbourhood models are handled by additional rules independent of the language of *CDL*. The resulting calculus is analytical and enjoys strong proof-theoretical properties, the most important being admissibility of cut and contraction, for which we provide a syntactical proof. Through the adoption of a standard strategy, we show that the calculus **G3CDL** provides a decision procedure for *CDL*. We shall also prove the semantic completeness of the calculus: it is possible to extract from a failed derivation a finite countermodel of the initial formula. This result combined with the soundness of the calculus yields a constructive proof of the finite model property of *CDL*.

2 The logic of conditional beliefs: Axiomatization and semantics

The language of *CDL* is defined from a denumerable set of atoms Atm by means of propositional connectives and the conditional operator Bel_i, where i ranges over a set of agents $\mathcal{A}$. In the following, P denotes an atom and i an agent. The formulas of the language are generated according to the following definition:

$$A \ := \ P \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid A \supset A \mid Bel_i(A|A)$$

The conditional belief operator $Bel_i(C|B)$ is read "agent i believes C, given B." As mentioned in the introduction, we may define the unconditional belief and knowledge operator in terms of conditional belief:

$Bel_i A =_{def} Bel_i(A|\top)$ (belief)
$K_i A =_{def} Bel_i(\bot|\neg A)$ (knowledge)

An axiomatization of *CDL* has been discussed in [5], [19], [3]. We present below Board's axiomatization, which is formulated using only the conditional belief operator. Equivalent axiomatizations that make use of both the belief operator and the knowledge operator have been given by Baltag and Smets [2], [4], [3], and Pacuit [19]. The axiomatization of *CDL* extends the classical propositional calculus by the following axioms and rules:

(1) If $\vdash B$, then $\vdash Bel_i(B|A)$
(2) If $\vdash A \supset\subset B$, then $\vdash Bel_i(C|A) \supset\subset Bel_i(C|B)$
(3) $(Bel_i(B|A) \wedge Bel_i(B \supset C|A)) \supset Bel_i(C|A)$
(4) $Bel_i(A|A)$
(5) $Bel_i(B|A) \supset (Bel_i(C|A \wedge B) \supset\subset Bel_i(C|A))$
(6) $\neg Bel_i(\neg B|A) \supset (Bel_i(C|A \wedge B) \supset\subset Bel_i(B \supset C|A))$
(7) $Bel_i(B|A) \supset Bel_i(Bel_i(B|A)|C)$
(8) $\neg Bel_i(B|A) \supset Bel_i(\neg Bel_i(B|A)|C)$
(9)$A \supset \neg Bel_i(\bot|A)$

In terms of Belief Revision, the above axioms may be understood as an epistemic and internalized version of the AGM postulates. Some quick remarks (cf. [5] for a deeper discussion): The distribution axiom (3) and the epistemization rule (2) express deductive closure of beliefs. The success axiom (4) ensures that the learned information is included in the set of beliefs. Axioms (5) and (6) encode the *minimal change principle*, a basic assumption of belief revision (see the correspondence with AGM postulates K*7 and K*8). Axiom (9) ensures that learning a true information cannot lead to inconsistent beliefs (it roughly corresponds to AGM K*5). Axioms (7) and (8) express positive and negative introspection for belief. Observe that from the above axioms it is possible to derive the standard S5 characterization of knowledge:

$$K_iA \supset A \qquad K_iA \supset K_iK_iA \qquad \neg K_iA \supset K_i\neg K_iA$$

The semantics of *CDL* is defined in terms of *epistemic plausibility models* (*P*-models for short; they were originally called Belief Revision Structures by Board). These are Kripke structures that comprise for each agent two relations over worlds, namely an equivalence relation, which defines knowledge (as in standard epistemic models) and a plausibility relation, which is used to define beliefs. The intuition is that the beliefs of an agent are the propositions which hold in the worlds considered as the most plausible by the agent.

A *pre-order* $\preceq$ over a set W is a reflexive and transitive relation over W. Given $S \subseteq W$, $\preceq$ is *connected* over S if for all $x, y \in S$ either $x \preceq y$ or $y \preceq x$. An *infinite descending* $\preceq$*-chain* over W is a sequence of elements of W $\{x_n\}_{n\geq 0}$ such that for all n, $x_{n+1} \preceq x_n$ but $x_n \not\preceq x_{n+1}$. We say that $\preceq$ is *well-founded* over W if there are no infinite descending $\preceq$-chains over W. Given $S \subseteq W$, let $Min_{\preceq}(S) \equiv \{u \in S \,|\, \forall z \in S(z \preceq u \text{ implies } u \preceq z)\}$. Observe that whenever $\preceq$ is connected over S the definition $Min_{\preceq}(S)$ can be simplified to $Min_{\preceq}(S) = \{u \in S \,|\, \forall z \in S\ (u \preceq z)\}$. Finally, the well-foundedness property can be equivalently stated as: for each $S \subseteq W$ if $S \neq \emptyset$ then $Min_{\preceq}(S) \neq \emptyset$.

Definition 2.1 Let $\mathcal{A}$ be a set of agents; an *epistemic plausibility model*

$\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ consists of the following: a non-empty set W of elements called "worlds" or "states"; for each $i \in \mathcal{A}$, an equivalence relation $\sim_i$ over W (with $[x]_{\sim_i} \equiv \{w \mid w \sim_i x\}$); for each $i \in \mathcal{A}$, a well-founded pre-order $\preceq_i$ over W; a valuation function $[\![\ ]\!] : Atm \to \mathcal{P}(W)$. We assume $\preceq_i$ to satisfy the following properties:

- *Plausibility implies possibility*: If $w \preceq_i v$ then $w \sim_i v$;
- *Local connectedness*: If $w \sim_i v$ then $w \preceq_i v$ or $v \preceq_i w$ (in other words, $\preceq_i$ is connected over every equivalence class of $\sim_i$).

The truth conditions for formulas of the language are given by inductively extending the evaluation function $[\![\ ]\!]$ as follows:

- For the Boolean case we have the standard clauses, $[\![A \wedge B]\!] \equiv [\![A]\!] \cap [\![B]\!]$, $[\![\neg A]\!] \equiv W - [\![A]\!]$, etc.
- $[\![Bel_i(B|A)]\!] \equiv \{x \in W \mid Min_{\preceq_i}([x]_{\sim_i} \cap [\![A]\!]) \subseteq [\![B]\!]\}$.

We say that a formula A is *valid* in a model $\mathcal{M}$ if $[\![A]\!] = W$ and that A is *valid in the class of epistemic plausibility models* if A is valid in every P-model.
Notational convention: We often write $\mathcal{M}, x \Vdash A$ meaning $x \in [\![A]\!]$. The notation is further shortened to $x \Vdash A$ whenever $\mathcal{M}$ is unambiguous.

The axiomatization of *CDL* is sound and complete w.r.t. epistemic plausibility models [5].

Theorem 2.2 (Completeness of the axiomatization) *A formula A is a theorem of CDL if and only if it is valid in the class of P-models.*

The following proposition, proved by unfolding the definitions, gives an equivalent formulation of the truth condition of the conditional operator Bel_i provided in Definition 2.1. From now on, we shall use this formulation.

Proposition 2.3 *Given any P-model $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$, with $x \in W$, we have that $\mathcal{M}, x \Vdash Bel_i(B|A)$ iff: either for all y, $y \sim_i x$ implies $y \Vdash \neg A$ or there is y with $y \sim_i x$ such that $y \Vdash A$ and $\forall z, z \preceq_i y$ implies $z \Vdash A \supset B$.*

We introduce an alternative semantics for *CDL* based on neighbourhood models (N-models for short). As explained in the introduction, these are a multi-agent version of the spheres models introduced by Lewis for counterfactual logic.

Definition 2.4 Let $\mathcal{A}$ be a set of agents; a *multi-agent neighbourhood model* has the form $\mathcal{M} = \langle W, \{I\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ where W is a non empty set of elements; for each $i \in \mathcal{A}$, I_i is a function $I_i : W \to \mathcal{P}(\mathcal{P}(W))$, and $[\![\ ]\!] : Atm \to \mathcal{P}(W)$ is the propositional evaluation.
For $i \in \mathcal{A}$, $x \in W$, I_i satisfies the following properties:

- *Non-emptiness*: $\forall \alpha \in I_i(x), \alpha \neq \emptyset$
- *Nesting*: $\forall \alpha, \beta \in I_i(x), \alpha \subseteq \beta$ or $\beta \subseteq \alpha$
- *Total reflexivity*:[4] $\exists \alpha \in I_i(x)$ such that $x \in \alpha$

[4] Total reflexivity entails $\forall x \in W, I_i(x) \neq \emptyset$.

- *Local absoluteness*: If $\alpha \in I_i(x)$ and $y \in \alpha$ then $I_i(x) = I_i(y)$
- *Strong closure under intersection*: If $S \subseteq I_i(x)$ and $S \neq \emptyset$ then $\bigcap S \in S$.

The truth conditions for Boolean combinations of formulas are the standard ones, as in P-models; for conditional belief we have:

$x \in [\![Bel_i(B|A)]\!]$ iff $\forall \alpha \in I_i(x)$ it holds that $\alpha \cap [\![A]\!] = \emptyset$ or $\exists \beta \in I_i(x)$ such that $\beta \cap [\![A[\![\neq \emptyset$ and $\beta \subseteq [\![A \supset B]\!]$

A formula A is *valid* in $\mathcal{M}$ if $[\![A]\!] = W$. We say that A *is valid in the class of neighbourhood models* if A is valid in every N-model.

Observe that strong closure under intersection always holds in finite models, because of non-emptiness and nesting. To simplify the notation, we use the local forcing relations introduced in [12]:

$\alpha \Vdash^{\forall} A$ iff $\forall y \in \alpha, y \Vdash A$
$\alpha \Vdash^{\exists} A$ iff $\exists y \in \alpha, y \Vdash A$

With this notation, the truth condition for the conditional belief operator belief Bel_i becomes:

$x \Vdash Bel_i(B|A)$ iff $(\forall \alpha \in I_i(x),\ \alpha \Vdash^{\forall} \neg A)$ or $(\exists \beta \in I_i(x),\ \beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \supset B)$

With the notation just introduced the semantic definition of unconditional belief and knowledge operators can be stated as follows:

$x \Vdash Bel_i B$ iff $\exists \beta \in I_i(x), \beta \Vdash^{\forall} B$
$x \Vdash K_i B$ iff $\forall \beta \in I_i(x), \beta \Vdash^{\forall} B$

Notice that these operators correspond to the standard modalities in neighbourhood models.

We now show the equivalence between neighbourhood models and epistemic plausibility models. The proofs make use of the basic correspondence between partial orders and topologies recalled in Marti and Pinosio [11] and Pacuit [18], and that dates back to Alexandroff [1]. However, the result must be adapted to the present setting of multi-agent epistemic neighbourhood models.

Theorem 2.5 *A formula A is valid in the class P-models if and only if it is valid in the class of multi-agent N-models.*

Proof. We first define the measure of *weight* of a *CDL* formula as follows: $\mathtt{w}(P) = \mathtt{w}(\bot) = 1$; $\mathtt{w}(\neg A) = \mathtt{w}(A) + 2$; $\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1$ for $\circ = \{\wedge, \vee, \supset\}$; $\mathtt{w}(Bel_i(B|A)) = \mathtt{w}(A) + \mathtt{w}(B) + 3$ (cf. Definition 3.2).
[**only if**] Given a N-model $\mathcal{M}_N$ we build a P-model $\mathcal{M}_P$ and we show that for any formula A, if A is valid in $\mathcal{M}_P$ then A is valid in $\mathcal{M}_N$.
Let $\mathcal{M}_N = \langle W, \{I\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$ be a multi-agent N-model. We construct a P-model $\mathcal{M}_P = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$, by stipulating:

- $x \sim_i y$ iff $\exists \alpha \in I_i(x), y \in \alpha$;
- $x \preceq_i y$ iff $\forall \alpha \in I_i(y)$, if $y \in \alpha$ then $x \in \alpha$.

We can easily show that $\sim_i$ is an equivalence relation and that $\preceq_i$ satisfies the properties of reflexivity, transitivity, and plausibility implies possibility. Properties of local connectedness and well-foundedness for $\preceq_i$ require some additional work.

Local connectedness: suppose that $x \sim_i y$ holds, but neither $x \preceq_i y$ nor $y \preceq_i x$ hold. By definition of $\preceq_i$ we have for some $\beta \in I_i(y)$, $y \in \beta$ and $x \notin \beta$ and for some $\gamma \in I_i(x)$, $x \in \gamma$ and $y \notin \gamma$. Since $x \sim_i y$, by reflexivity $\exists \alpha \in I_i(x), y \in \alpha$, whence by local absoluteness $I_i(y) = I_i(x)$. Thus both $\beta, \gamma \in I_i(x)$, and by nesting either $\beta \subseteq \gamma$ or $\gamma \subseteq \beta$ holds. In case the former holds we get $y \in \gamma$, and in case the latter holds we have $x \in \beta$. In both cases we reach a contradiction.

Well-foundedness: If $\mathcal{M}_N$ is finite there is nothing to prove. Suppose then that $\mathcal{M}_N$ is *infinite*. Suppose that there is an infinite descending chain $\{z_k\}_{k \geq 0}$ w.r.t. $\preceq_i$, with all $z_k \in W$, so that for all k it holds that $z_{k+1} \preceq_i z_k$ and $z_k \not\preceq_i z_{k+1}$. Observe that by definition of $\preceq_i$, plausibility implies possibility and local absoluteness we obtain that for all $k, h \geq 0$, it holds $I_i(z_k) = I_i(z_h) = \ldots = I_i(z_0)$. Thus by definition of $\preceq_i$, for all $k \geq 0$ since $z_k \not\preceq_i z_{k+1}$, we get that for all $z_k \in \{z_k\}_{k \geq 0}$ there exists $\beta_{z_{k+1}} \in I_i(z_0)$ such that: $(*)$ $z_{k+1} \in \beta_{z_{k+1}}$ and $z_k \notin \beta_{z_{k+1}}$. Consider the set $T = \{\beta_{z_{k+1}} | z_k \in \{z_k\}_{k \geq 0}\}$. T is non-empty; thus by the strong closure under intersection it follows that $\bigcap T \in T$, and also $\bigcap T \neq \emptyset$. Obviously, we have that $(**)$ for all $\beta \in T$, $\bigcap T \subseteq \beta$. Since $\bigcap T \in T$ it must be $\bigcap T = \beta_{z_{t+1}}$ for some $z_t \in \{z_k\}_{k \geq 0}$. But by using $(*)$ *twice* (namely for z_{t+1} and for z_{t+2}) we have $z_{t+1} \in \beta_{z_{t+1}}$ and $z_{t+1} \notin \beta_{z_{t+2}}$, thus $\bigcap T = \beta_{z_{t+1}} \not\subseteq \beta_{z_{t+2}}$ against $(**)$.

We now prove that for any $x \in W$ and formula A it holds that

(a) $\mathcal{M}_N, x \Vdash A$ iff $\mathcal{M}_P, x \Vdash A$

We proceed by induction on the weight of A. The base case (A atomic) holds by definition; for the inductive cases, we consider only $A = Bel_i(C|B)$. To simplify notation we write $u \Vdash_P B$ instead of $\mathcal{M}_P, u \Vdash B$ and $u \Vdash_N B$ instead of $\mathcal{M}_N, u \Vdash B$. Direction $[\Rightarrow]$ of statement (a) easily follows from the definitions. As for the opposite direction, suppose that $x \Vdash_P Bel_i(C|B)$ holds. This means that either $\forall y\ y \sim_i x$ implies $y \Vdash_P \neg B$ or there exists w such that $w \sim_i x$ and $w \Vdash_P B$ and $\forall z, z \preceq_i w$ implies $z \Vdash_P B \supset C$. There are two cases to consider. If the first disjunct holds, by definition and by inductive hypothesis statement (a) is met. We explicitly prove the case in which the second disjunct holds. Suppose that there exists w such that $w \sim_i x$ and $w \Vdash_P B$ and $\forall z, z \preceq_i w$ implies $z \Vdash_P B \supset C$. From $w \sim_i x$ (hypothesis) it follows by definition that $\exists \alpha \in I(x), w \in \alpha$. By local absoluteness, $I(x) = I(w)$. Now consider the set $S = \{\beta \in I(x) | w \in \beta\}$. It holds that $\alpha \in S$, and that $S \neq \emptyset$. Let $\gamma = \cap S$. By strong closure under intersection, $\gamma \in S \subseteq I_i(x)$; thus $\gamma \in I_i(x)$. But $w \in \gamma$ and since we have $w \Vdash_P B$, by inductive hypothesis we also have $w \Vdash_N B$. We have obtained that $\gamma \Vdash^{\exists} B$. We still have to prove that $\gamma \Vdash^{\forall} B \supset C$. Given $u \in \gamma$, we want to prove that $u \Vdash_N B \supset C$. We first show that $u \preceq_i w$. To

this purpose (by definition of $\preceq_i$), let $\delta \in I(w)$ with $w \in \delta$ we have to show that $u \in \delta$: since $I(x) = I(w)$, also $\delta \in I(x)$, whence, $\delta \in S$, so that $\gamma \subseteq \delta$, and therefore $u \in \delta$. Since $u \preceq_i w$ by the hypothesis we have $u \Vdash_P B \supset C$ and finally by induction hypothesis $u \Vdash_N B \supset C$.
Next, we show that if A is valid in $\mathcal{M}_P$ then A is also valid in $\mathcal{M}_N$. Suppose that A is valid in $\mathcal{M}_P$. This means that for all $w \in W$, we have $w \Vdash_P A$, thus by (a) we have also $w \Vdash_N A$ for all $w \in W$, which means that A is valid in $\mathcal{M}_N$. Finally, let A be valid in the class of P-models. We want to show that A is also valid in the class of N-models. Given a N-model $\mathcal{M}_N$, we build an P-model $\mathcal{M}_P$ as above. By hypothesis A is valid in $\mathcal{M}_P$ and for what we have just shown A is valid in $\mathcal{M}_N$.

[**If**] Given a P-model $\mathcal{M}_P$ we build an N-model $\mathcal{M}_N$ and we show that for any A, if A is valid in $\mathcal{M}_N$ then A is valid in $\mathcal{M}_P$. Let $\mathcal{M}_P = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{\preceq_i\}_{i \in \mathcal{A}}, [\![\ ]\!] \rangle$ be an P-model. We build an N-model $\mathcal{M}_N$ as follows. Let $u \in W$, and define its downward closed set $\downarrow^{\preceq_i} u$ w.r.t. $\preceq_i$ as $\downarrow^{\preceq_i} u = \{v \in W | v \preceq_i u\}$. We now define the model $\mathcal{M}_N = \langle W, \{I\}_{i \in \mathcal{A}}, [\![\ ]\!] \rangle$, where the neighbourhood for any $x \in W$ is $I_i(x) = \{\downarrow^{\preceq_i} u | u \sim_i x\}$.

It can be easily proved that $\mathcal{M}_N$ satisfies all the properties of an N-model; we show only the case of the *strong closure under intersection*. In the finite case, this property immediately follows from properties of non-emptiness and nesting. Let us consider the infinite case. Let $S \subseteq I_i(x)$, $S \neq \emptyset$, with S countable so that $S = \{\alpha_h | h \geqslant 0\}$ where $\alpha_h = \downarrow^{\preceq_i} x_h$ for $x_h \sim_i x$. We prove that $(*)$ $\exists \alpha_h \in S$ such that $\forall \alpha_k \in S, \alpha_h \subseteq \alpha_k$. If $(*)$ holds then $\alpha_h = \bigcap S$ and $\alpha_h \in S$ and the proof is over. Suppose by contradiction that $(*)$ does not hold. This means that 1) $\forall \alpha_h \in S\ \exists \alpha_k \in S, \alpha_h \nsubseteq \alpha_k$. Thus, by the property of spheres nesting 2) $\forall \alpha_h \in S\ \exists \alpha_k \in S, \alpha_k \subset \alpha_h$. From 2), by denumerable dependent choice we build an infinite (strictly decreasing) chain of neighbourhoods $\alpha_1 \supset \alpha_2 \supset \alpha_3 \supset \dots$. For every $n \geq 1$ we have by definition that $\alpha_n = \downarrow^{\preceq_i} u_n$. Let $v_n \in \alpha_n - \alpha_{n+1}$, $v_{n+1} \in \alpha_{n+1} - \alpha_{n+2}$, etc. We have $v_{n+1} \preceq_i u_{n+1}$ by construction and it is enough to prove that $u_{n+1} \preceq_i v_n$ to conclude by transitivity that $v_{n+1} \preceq_i v_n$. By construction, we have $v_n \npreceq_i u_{n+1}$ and therefore by local connectedness, $u_{n+1} \preceq_i v_n$. Moreover by $v_n \npreceq_i u_{n+1}$ it also follows that $v_n \npreceq_i v_{n+1}$. We have thus an infinitely descending $\preceq_i$-chain of worlds $\{v_n\}_{n \geq 1}$, against the assumption of well-foundedness of W. We reached a contradiction from the negation of $(*)$; therefore, $(*)$ holds.

We now have to prove that for any $x \in W$ and formula A, it holds that (b) $\mathcal{M}_P, x \Vdash A$ iff $\mathcal{M}_N, x \Vdash A$. The proof strategy is the same employed in the previous case. Next, as above, we show that if A is valid in $\mathcal{M}_N$ then A is also valid in $\mathcal{M}_P$. Finally, let A be valid in the class of N-models. We want to show that A is also valid in the class of P-models. Given an P-model $\mathcal{M}_P$, we build an N-model $\mathcal{M}_N$ as described. By hypothesis A is valid in $\mathcal{M}_N$ and by what we have just shown A is valid in $\mathcal{M}_P$. □

Corollary 2.6 *A formula A is a theorem of CDL if and only if it is valid in the class of neighbourhood models.*

Observe that the correspondence between plausibility and neighbourhood models holds for infinite models as well. For this reason, the correspondence can probably be used to establish *strong completeness* of *CDL*, which at present is an open issue, with respect to any of the two semantics.

Initial sequents

$$x : P, \Gamma \Rightarrow \Delta, x : P \qquad x : \bot, \Gamma \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, x : \top$$

Rules for local forcing

$$\frac{x : A, x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\Vdash^{\forall}} \qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\ {\scriptstyle R\Vdash^{\forall}\ (x\ \text{fresh})}$$

$$\frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\Vdash^{\exists}\ (x\ \text{fresh})} \qquad \frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ {\scriptstyle R\Vdash^{\exists}}$$

Propositional rules

$$\frac{\Gamma \Rightarrow \Delta, x : A}{x : \neg A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\neg} \qquad \frac{x : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : \neg A}\ {\scriptstyle R\neg}$$

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\wedge} \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B}\ {\scriptstyle R\wedge}$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\vee} \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\ {\scriptstyle R\vee}$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\supset} \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\ {\scriptstyle R\supset}$$

Rules for conditional belief

$$\frac{a \in I_i(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A}{\Gamma \Rightarrow \Delta, x : Bel_i(B|A)}\ {\scriptstyle RB\ (a\ \text{fresh})}$$

$$\frac{a \in I_i(x), x : Bel_i(B|A), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad x \Vdash_i B|A, a \in I_i(x), x : Bel_i(B|A), \Gamma \Rightarrow \Delta}{a \in I_i(x), x : Bel_i(B|A), \Gamma \Rightarrow \Delta}\ {\scriptstyle LB}$$

$$\frac{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\exists} A \quad a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\forall} A \supset B}{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A}\ {\scriptstyle RC}$$

$$\frac{a \in I_i(x), a \Vdash^{\exists} A, a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta}{x \Vdash_i B|A, \Gamma \Rightarrow \Delta}\ {\scriptstyle LC (a\ \text{fresh})}$$

Rules for inclusion

$$\frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle Ref} \qquad \frac{c \subseteq a, c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}{c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}\ {\scriptstyle Tr}$$

$$\frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\subseteq}$$

Rules for semantic conditions

$$\frac{a \subseteq b, a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta \quad b \subseteq a, a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta}{a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta}\ {\scriptstyle S}$$

$$\frac{x \in a, a \in I_i(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle T\ (a\ \text{fresh})}$$

$$\frac{a \in I_i(x), y \in a, b \in I_i(x), b \in I_i(y), \Gamma \Rightarrow \Delta}{a \in I_i(x), y \in a, b \in I_i(x), \Gamma \Rightarrow \Delta}\ {\scriptstyle A_1} \qquad \frac{a \in I_i(x), y \in a, a \in I_i(y), \Gamma \Rightarrow \Delta}{a \in I_i(x), y \in a, \Gamma \Rightarrow \Delta}\ {\scriptstyle A_2}$$

Table 1. Sequent calculus **G3CDL**

3 Sequent calculus

In this section we present a labelled sequent calculus **G3CDL** for *CDL* based on neighbourhood semantics. As shown in Table 1, the calculus **G3CDL** has two kinds of labels: labels for worlds $x, y \ldots$ and labels for neighbourhoods $a, b \ldots$, as in the ground calculus for neighbourhood semantics introduced in [12].

The meaning of the expressions employed in the calculus is defined as follows:

$$a \Vdash^{\exists} A \;\equiv\; \exists x(x \in a \;\&\; x \Vdash A); \quad a \Vdash^{\forall} A \;\equiv\; \forall x(x \in a \longrightarrow x \Vdash A)$$
$$x \Vdash_i B|A \;\equiv\; \exists c(c \in I_i(x) \;\&\; c \Vdash^{\exists} A \;\&\; c \Vdash^{\forall} A \supset B)$$
$$x : Bel_i(B|A) \equiv \forall a \in I_i(x)(a \Vdash^{\forall} \neg A) \text{ or } \exists b \in I_i(x)(b \Vdash^{\exists} A \;\&\; b \Vdash^{\forall} A \supset B)$$

Here $\Vdash$ denotes the forcing condition of relational semantics; to distinguish the semantic notion and its syntactic counterpart, and for the sake of a more compact notation, we employ a colon in the labelled calculus. The propositional rules of **G3CDL**, the basic labelled modal system, are given as in [13], while the rules for the local forcing relation are defined as in [12].

Furthermore, each semantic condition on neighbourhood models (Definition 2.4) is in correspondence with a rule in the calculus. Rule (S) corresponds to the property of nesting in Definition 2.4; (T) corresponds to total reflexivity, and (A_1) and (A_2) to local absoluteness. As for non-emptiness, the property is expressed by the rules for local forcing. The property of strong closure under intersection needs not be expressed, since the property holds in finite models and we shall prove that the logic has the *finite model property*.

Observe that some rules maintain their principal formula in the premisses: this is needed to ensure invertibility of the rules and admissibility of contraction.

Example 3.1 We show a derivation of the left-to-right direction of axiom (6). We omit the derivable left premisses of rule (RC) in $\mathcal{D}$ and of rule (LB) in the final derivation.

$\mathcal{D}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : A \cdots \Rightarrow \ldots y : A \qquad y : B \cdots \Rightarrow \ldots y : B}{y : A, y : B, y \in b, c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots y : A \wedge B}\,R\wedge}{y : A, y : B, y \in b, c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots b \Vdash^{\exists} A \wedge B}\,R\Vdash^{\exists}}{y \in b, c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots b \Vdash^{\exists} A \wedge B, y : A \supset \neg B}\,R\supset, R\neg}{c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots b \Vdash^{\exists} A \wedge B, b \Vdash^{\forall} A \supset \neg B}\,R\Vdash^{\forall}}{c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots b \Vdash^{\exists} A \wedge B, x \Vdash_i \neg B|A}\,RC}{b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots x : Bel_i(\neg B|A), b \Vdash^{\exists} A \wedge B}\,RB$$

$\mathcal{E}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{z : A \cdots \Rightarrow \ldots z : A \qquad z : c \cdots \Rightarrow \ldots z : C}{z : A \supset C, z : A, z : B, z \in b, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B, \cdots \Rightarrow \ldots z : C}\,L\supset}{z : A, z : B, z \in b, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots z : C}\,L\Vdash^{\forall}}{z \in b, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots z : (A \wedge B) \supset C}\,R\supset, L\wedge}{b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots b \Vdash^{\forall} (A \wedge B) \supset C}\,R\Vdash^{\forall}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ \mathcal{D} & \mathcal{E}\end{array}}{b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \in I_i(x), a \Vdash^{\exists} A \wedge B, x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x \Vdash_i C|A \wedge B}\, RC}{x \Vdash_i C|A, a \in I_i(x), a \Vdash^{\exists} A \wedge B, x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x \Vdash_i C|A \wedge B}\, LC}{a \in I_i(x), a \Vdash^{\exists} A \wedge B, x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x \Vdash_i C|A \wedge B}\, LB}{x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x : Bel_i(C|A \wedge B)}\, RB}{x : \neg(Bel_i(\neg B|A)), x : Bel_i(C|A) \Rightarrow x : Bel_i(C|A \wedge B)}\, L\neg$$

Rules for unconditional belief and knowledge

The modal operators of belief and knowledge can be defined semantically in terms of the conditional belief operator: $Bel_i A = Bel_i(A|\top)$ and $K_i A = Bel_i(\bot|\neg A)$. By adopting these definitions, we can extend **G3CDL** with the rules displayed below, which correspond to the interpretation of the two operations in the neighbourhood semantics.

$$\dfrac{a \in I_i(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}{\Gamma \Rightarrow \Delta, x : K_i A}\, LK\ (a \text{ fresh}) \qquad \dfrac{a \in I_i(x), x : K_i A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{a \in I_i(x), x : K_i A, \Gamma \Rightarrow \Delta}\, RK$$

$$\dfrac{a \in I_i(x), \Gamma \Rightarrow \Delta, x : Bel_i A, a \Vdash^{\forall} A}{a \in I_i(x), \Gamma \Rightarrow \Delta, x : Bel_i A}\, LUB \qquad \dfrac{a \in I_i(x), a \Vdash^{\forall} A \Rightarrow \Delta}{x : Bel_i A, \Gamma \Rightarrow \Delta}\, RUB\ (a \text{ fresh})$$

These rules are *admissible* in **G3CDL**, i.e., whenever the premiss is derivable, also the conclusion is. This can be proved employing the rules of **G3CDL** and the rules of weakening and contraction, shown admissible in next section. By means of example, we show admissibility of (LK) (the other rules can be obtained in a similar way).

$$\dfrac{\dfrac{...a \Vdash^{\exists} \neg A \Rightarrow a \Vdash^{\exists} \neg A... \qquad \dfrac{a \in I_i(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}{a \in I_i(x), a \Vdash^{\exists} \neg A, \Gamma \Rightarrow \Delta, x \Vdash_i \bot|\neg A, a \Vdash^{\forall} A}\, Wk}{a \in I_i(x), a \Vdash^{\exists} \neg A, \Gamma \Rightarrow \Delta, x \Vdash_i \bot|\neg A}\, RC}{\Gamma \Rightarrow \Delta, x : Bel_i(\bot|\neg A)}\, RB$$

The left premiss of (RC), which we have not detailed, is derivable.

Structural properties

Definition 3.2 The label of formulas of the form $x : A$ is x. The label of formulas of the form $a \Vdash^{\forall} A$ and $a \Vdash^{\exists} A$ is a. The label of a formula $\mathcal{F}$ will be denoted by $l(\mathcal{F})$. The pure part of a labelled formula $\mathcal{F}$ is the part without the label and without the forcing relation, either local ($\Vdash^{\exists}$, $\Vdash^{\forall}$) or worldwide (:) and will be denoted by $p(\mathcal{F})$.
The *weight of a labelled formula* $\mathcal{F}$ is the pair $(\mathtt{w}(p(\mathcal{F})), \mathtt{w}(l(\mathcal{F})))$ where:

(i) for all world labels x and all neighbourhood labels a, $\mathtt{w}(x) = 0$, $\mathtt{w}(a) = 1$;
(ii) $\mathtt{w}(P) = \mathtt{w}(\bot) = 1$; $\mathtt{w}(\neg A) = \mathtt{w}(A) + 2$; $\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1$ for $\circ$ conjunction, disjunction, or implication; $\mathtt{w}(B|A) = \mathtt{w}(A) + \mathtt{w}(B) + 2$; $\mathtt{w}(Bel_i(B|A)) = \mathtt{w}(B|A) + 1$.

Weights of labelled formulas are ordered lexicographically.

From the definition of weight it is clear that the weight gets decreased if we move from a formula labelled by a neighbourhood label to the same formula labelled by a world label, or if we move (regardless the label) to a formula with a pure part of strictly smaller weight. The following lemma is proved by induction on formula weights:

Lemma 3.3 *Sequents of the following form are derivable in* **G3CDL** *for arbitrary neighbourhoods labels a, b and formulas A and B:*
(i) $a \subseteq b, \Gamma \Rightarrow \Delta, a \subseteq b$ *(ii)* $a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A$ *(iii)* $a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$
(iv) $x \Vdash_i B|A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A$ *(v)* $x : A, \Gamma \Rightarrow \Delta, x : A$

The definition of substitution of labels given in [13] can be extended in an obvious way – that need not be pedantically detailed here – to all the formulas of our language and to neighbourhood labels. With this definition we have, for example, $(a \Vdash^{\exists} A)(b/a) \equiv b \Vdash^{\exists} A$, and $(x \Vdash_i B|A)(y/x) \equiv y \Vdash_i B|A$.

We denote by $\vdash_n \Gamma \Rightarrow \Delta$ a derivation whose endsequent is $\Gamma \Rightarrow \Delta$ and which has height n, where the height of a derivation is the number of nodes occurring in the longest derivation branch. The calculus is routinely shown to enjoy the property of height preserving (hp for short) substitution both of world and neighbourhood labels:

Proposition 3.4

(i) *If* $\vdash_n \Gamma \Rightarrow \Delta$, *then* $\vdash_n \Gamma(y/x) \Rightarrow \Delta(y/x)$;
(ii) *If* $\vdash_n \Gamma \Rightarrow \Delta$, *then* $\vdash_n \Gamma(b/a) \Rightarrow \Delta(b/a)$.

Hp-admissibility of weakening and contraction are then obtained by an easy induction on derivation height:

Proposition 3.5 *The rules of left and right weakening are hp-admissible in* **G3CDL**.

Theorem 3.6 *All the rules of* **G3CDL** *are* hp-invertible*, i.e. for every rule of the form* $\frac{\Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta}$, *if* $\vdash_n \Gamma \Rightarrow \Delta$ *then* $\vdash_n \Gamma' \Rightarrow \Delta'$, *and for every rule of the form* $\frac{\Gamma' \Rightarrow \Delta' \quad \Gamma'' \Rightarrow \Delta''}{\Gamma \Rightarrow \Delta}$ *if* $\vdash_n \Gamma \Rightarrow \Delta$ *then* $\vdash_n \Gamma' \Rightarrow \Delta'$ *and* $\vdash_n \Gamma'' \Rightarrow \Delta''$.

The rules of contraction of **G3CDL** have the following form, where $\mathcal{F}$ is either a "relational" atom of the form $a \in I(x)$ or $x \in a$ or a labelled formula of the form $x : A$, $a \Vdash^{\forall} A$, $a \Vdash^{\exists} A$ or a formula of the form $x \Vdash_i B|A$ or $x : Bel_i(B|A)$:

$$\frac{\mathcal{F}, \mathcal{F}, \Gamma \Rightarrow \Delta}{\mathcal{F}, \Gamma \Rightarrow \Delta}\,{\scriptstyle LCtr} \qquad \frac{\Gamma \Rightarrow \Delta, \mathcal{F}, \mathcal{F}}{\Gamma \Rightarrow \Delta, \mathcal{F}}\,{\scriptstyle RCtr}$$

Theorem 3.7 *The rules of left and right contraction are hp-admissible in* **G3CDL**.

Theorem 3.8 *Cut is admissible in* **G3CDL**.

Proof. By double induction, with primary induction on the weight of the cut formula and subinduction on the sum of the heights of derivations of the premisses of cut. The cases in which the premisses of cut are either initial sequents or obtained through the rules for $\wedge$, $\vee$, or $\supset$ follow the treatment of Theorem 11.9 of [16]. For the cases in which the cut formula is a side formula in at least one rule used to derive the premisses of cut, the cut reduction is dealt with in the usual way by permutation of cut, with possibly an application of hp-substitution to avoid a clash with the fresh variable in rules with variable condition. In all such cases the cut height is reduced.

For space limitations, we treat only the cases in wich the cut formula is principal in both premisses and has the form $x \Vdash_i B|A$ or $x : Bel_i(B|A)$.

(1) The cut formula is $x \Vdash_i B|A$, principal in both premisses of cut:

$$\dfrac{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\exists} A \quad a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\forall} A \supset B}{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A}\ RC$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset B, \Gamma' \Rightarrow \Delta'\end{matrix}}{x \Vdash_i B|A, \Gamma' \Rightarrow \Delta'}\ LC$$

The conclusion of the cut is the sequent $a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. The derivation is converted into the following:

$$\dfrac{\dfrac{\dfrac{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\exists} A \quad x \Vdash_i B|A, \Gamma' \Rightarrow \Delta'}{a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', a \Vdash^{\exists} A}\ Cut_1 \quad (1)}{a \in I_i(x)^3, \Gamma^2, \Gamma'^3 \Rightarrow \Delta^2, \Delta'^3}\ Cut_4}{a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Ctr^*$$

where (1) is the derivation:

$$\dfrac{a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', a \Vdash^{\forall} A \supset B \quad \begin{matrix}\mathcal{D}(a/b)\\ a \in I_i(x), a \Vdash^{\exists} A, a \Vdash^{\forall} A \supset B, \Gamma' \Rightarrow \Delta'\end{matrix}}{a \in I_i(x)^2, a \Vdash^{\exists} A, \Gamma, \Gamma'^2 \Rightarrow \Delta, \Delta'^2}\ Cut_3$$

where the left premiss is obtained by Cut_2 from the sequent $a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\forall} A \supset B$ and $x \Vdash_i B|A, \Gamma' \Rightarrow \Delta'$. Observe that all four cuts are of reduced height (Cut_1 and Cut_2) or reduced weight (Cut_3 and Cut_4) because $\mathtt{w}(a \Vdash^{\exists} A) < \mathtt{w}(a \Vdash^{\forall} A \supset B) < \mathtt{w}(x \Vdash_i B|A)$.

(2) The cut formula is $x : Bel_i(B|A)$, principal in both premisses of cut:

$$\dfrac{\begin{matrix}\mathcal{D}\\ b \in I_i(x), b \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A\end{matrix}}{\Gamma \Rightarrow \Delta, x : Bel_i(B|A)}\ RB$$

$$\dfrac{a \in I_i(x), x : Bel_i(B|A), \Gamma' \Rightarrow \Delta', a \Vdash^{\exists} A \quad a \in I_i(x), x \Vdash_i B|A, x : Bel_i(B|A), \Gamma' \Rightarrow \Delta'}{a \in I_i(x), x : Bel_i(B|A), \Gamma' \Rightarrow \Delta'}\ LB$$

The conclusion is the sequent $a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. The cut is converted to four smaller cuts as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : Bel_i(B|A) \quad a \in I_i(x), x : Bel_i(B|A), \Gamma' \Rightarrow \Delta', a \Vdash^{\exists} A}{a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', a \Vdash^{\exists} A}\ Cut_2 \quad (2)}{a \in I_i(x)^3, \Gamma^3, \Gamma'^2 \Rightarrow \Delta^3, \Delta'^2}\ Cut_4}{a \in I_i(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Ctr^*$$

where (2) is the derivation:

$$\dfrac{\begin{matrix}\mathcal{D}(a/b)\\ a \in I_i(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A\end{matrix} \quad a \in I_i(x), x \Vdash_i B|A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{a \in I_i(x)^2, a \Vdash^{\exists} A, \Gamma^2, \Gamma' \Rightarrow \Delta^2, \Delta'}\ Cut_3$$

where the right premiss is derived by Cut_1 from $\Gamma \Rightarrow \Delta, x : Bel_i(B|A)$ and $a \in I_i(x), x \Vdash_i B|A, x : Bel_i(B|A), \Gamma' \Rightarrow \Delta'$. Cut_1 and Cut_2 have reduced height and the other cuts are performed on formulas of reduced weight, because $\mathtt{w}(a \Vdash^{\exists} A) < \mathtt{w}(x \Vdash_i B|A) < \mathtt{w}(x : Bel_i(B|A))$. □

4 Soundness, termination, and completeness

We first show soundness of the calculus. We need to interpret labelled sequents in neighbourhood models, and to this purpose we define the notion of realization.

Definition 4.1 Let $\mathcal{M} = \langle W, \{I\}_{i \in \mathcal{A}}, [\![\ ]\!] \rangle$ be a neighbourhood model, S a set of world labels, and N a set of neighbourhood labels. An SN-realization over $\mathcal{M}$ consists of a pair of functions (ρ, σ) such that

- $\rho : S \to W$ is a function which assigns to each $x \in S$ an element $\rho(x) = w \in W$;
- $\sigma : N \to \mathcal{P}(W)$, i.e. a function which assigns to each $a \in N$ an element $\sigma(a) \in I(w)$, for some $w \in W$.

Given a sequent $\Gamma \Rightarrow \Delta$, with S, N as above, and (ρ, σ) an SN-realization, we say that $\Gamma \Rightarrow \Delta$ is satisfied in $\mathcal{M}$ under the SN-realization (ρ, σ) if the following conditions hold:

- $\mathcal{M} \vDash_{\rho,\sigma} a \in I_i(x)$ if $\sigma(a) \in I_i(\rho(x))$ and $\mathcal{M} \vDash_{\rho,\sigma} a \subseteq b$ if $\sigma(a) \subseteq \sigma(b)$;
- $\mathcal{M} \vDash_{\rho,\sigma} x : A$ if $\rho(x) \Vdash A$;
- $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\exists} A$ if $\sigma(a) \Vdash^{\exists} A$ and $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\forall} A$ if $\sigma(a) \Vdash^{\forall} A$;
- $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash_i B|A$ if for some c $\in I_i(\rho(x))$, $c \Vdash^{\exists} A$ and $c \Vdash^{\forall} A \supset B$;
- $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash_i Bel_i(B|A)$ if for all $a \in I_i(\rho(x))$, $a \Vdash^{\forall} A$ or $\mathcal{M} \vDash_{\rho,\sigma} x \Vdash_i B|A$;
- $\mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta$ if either $\mathcal{M} \nvDash_{\rho,\sigma} F$ for some formula $F \in \Gamma$ or $\mathcal{M} \vDash_{\rho,\sigma} G$ for some formula $G \in \Delta$.

Then, define $\mathcal{M} \vDash \Gamma \Rightarrow \Delta$ iff $\mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta$ for every SN- realization (ρ, σ). A sequent $\Gamma \Rightarrow \Delta$ is said to be *valid* if $\mathcal{M} \vDash \Gamma \Rightarrow \Delta$ holds for every neighbourhood model $\mathcal{M}$, i.e. if $\Gamma \Rightarrow \Delta$ is satisfied for every model $\mathcal{M}$ and for every SN-realization (ρ, σ).

Theorem 4.2 (Soundness) *If a sequent $\Gamma \Rightarrow \Delta$ is derivable in the calculus, then it is valid in the class of multi-agent neighbourhood models.*

We now show that, by adopting a suitable proof search strategy, the calculus yields a decision procedure for *CDL*. We also prove the completeness of the calculus under the same strategy. The adoption of the strategy is not strictly necessary for completeness; however, it ensures that we can extract a finite countermodel from an open or failed derivation branch. Although the termination proof has some similarity with the one in [14], for **G3CDL** it is more difficult due to the specific semantic rules, in particular local absoluteness.

As often happens with labelled calculi, the calculus **G3CDL** in itself is non-terminating in the sense that a root-first (i.e. upwards) construction of a derivation may generate infinite branches. Here below is an example (we omit writing the derivable left premisses of LB):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{c \in I_i(x), c \Vdash^{\exists} A, c \Vdash^{\forall} A \supset B ... x : Bel_i(B|A) \Rightarrow x \Vdash_i C|A}}{x \Vdash_i B|A, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset B, a \in I_i(x), a \Vdash^{\exists} A, x : Bel_i(B|A) \Rightarrow x \Vdash_i C|A}\, {\scriptstyle LC}}{b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset B, a \in I_i(x), a \Vdash^{\exists} A, x : Bel_i(B|A) \Rightarrow x \Vdash_i C|A}\, {\scriptstyle LB}}{x \Vdash_i B|A, a \in I_i(x), a \Vdash^{\exists} A, x : Bel_i(B|A) \Rightarrow x \Vdash_i C|A}\, {\scriptstyle LC}}{\dfrac{a \in I_i(x), a \Vdash^{\exists} A, x : Bel_i(B|A) \Rightarrow x \Vdash_i C|A}{x : Bel_i(B|A) \Rightarrow x : Bel_i(C|A)}\, {\scriptstyle RB}}\, {\scriptstyle LB}$$

The loop is generated by the application of rules (LB) and (LC). Our aim is to specify a strategy which ensures termination by preventing any kind of loop. The main point is to avoid redundant (backwards) applications of rules. To define precisely this notion we associate to each rule a saturation condition.

Definition 4.3 Given a derivation branch $\mathcal{B}$ of the form $\Gamma_0 \Rightarrow \Delta_0, ..., \Gamma_k \Rightarrow \Delta_k, \Gamma_{k+1} \Rightarrow \Delta_{k+1}, ...$ where $\Gamma_0 \Rightarrow \Delta_0$ is the sequent $\Rightarrow x_0 : A_0$, let $\downarrow \Gamma_k$ (respectively $\downarrow \Delta_k$) denote the union of the antecedents (respectively the succedents) occurring in the branch from the root $\Gamma_0 \Rightarrow \Delta_0$ up to $\Gamma_k \Rightarrow \Delta_k$.

For each rule (R), we say that a sequent $\Gamma \Rightarrow \Delta$ *satisfies the saturation condition associated to* (R) if the following hold: for rule $(L\wedge)$, if $x : A \wedge B \in \Gamma$, then $x : A \in\downarrow \Gamma$ and $x : B \in\downarrow \Gamma$. The other propositional conditions are similar, and can be found in [14]. Conditions for the other rules are the following: (Rf) If a is in Γ, Δ then $a \subseteq a$ is in Γ; (Tr) If $a \subseteq b$ and $b \subseteq c$ are in Γ, then $a \subseteq c$ is in Γ; $(L \subseteq)$ If $x \in a$ and $a \subseteq b$ are in Γ, then $x \in b$ is in Γ; $(R \Vdash^{\forall})$ If $a \Vdash^{\forall} A$ is in $\downarrow \Delta$, then for some x there is $x \in a$ in Γ and $x : A$ in $\downarrow \Delta$; $(L \Vdash^{\forall})$ If $x \in a$ and $a \Vdash^{\forall} A$ are in Γ, then $x : A$ is in Γ; $(R \Vdash^{\exists})$ If $x \in a$ is in Γ and $a \Vdash^{\exists} A$ is in Δ, then $x : A$ is in $\downarrow \Delta$; $(L \Vdash^{\exists})$ If $a \Vdash^{\exists} A$ is in $\downarrow \Gamma$, then for some x there is $x \in a$ in Γ and $x : A$ is in $\downarrow \Gamma$; (RB) If $x : Bel_i(B|A)$ is in $\downarrow \Delta$, then for some $i \in \mathcal{A}$ and for some a, $a \in I_i(x)$ is in Γ, $a \Vdash^{\exists} A$ is in $\downarrow \Gamma$ and $x \Vdash_i B|A$ is in $\downarrow \Delta$; (LB) If $a \in I_i(x)$ and $x : Bel_i(B|A)$ are in Γ, then either $a \Vdash^{\exists} A$ is in $\downarrow \Delta$ or $x \Vdash_i B|A$ is in $\downarrow \Gamma$; (RC) If $a \in I_i(x)$ is in Γ and $x \Vdash_i B|A$ is in Δ, then either $a \Vdash^{\exists} A$ or $a \Vdash^{\forall} A \supset B$ are in $\downarrow \Delta$; (LC) If $x \Vdash_i B|A$ is in $\downarrow \Gamma$, then for some $i \in \mathcal{A}$ and for some a, $a \in I_i(x)$ is in Γ, $a \Vdash^{\exists} A$ and $a \Vdash^{\forall} A \supset B$ are in $\downarrow \Gamma$; (T) For all x occurring in $\downarrow \Gamma \cup \downarrow \Delta$, for all $i \in \mathcal{A}$ there is an a such that $a \in I_i(x)$ and $x \in a$ are in Γ; (S) If $a \in I_i(x)$ and $b \in I_i(x)$ are in Γ, then $a \subseteq b$ or $b \subseteq a$ are in Γ; $(A1)$ If $a \in I_i(x)$ and $y \in a$ are in Γ, then if $b \in I_i(x)$ is in Γ also $b \in I_i(y)$ is in Γ; If $b \in I_i(y)$ is in Γ also $b \in I_i(x)$ is in Γ (for $(A2)$ is similar).
Furthermore, a sequent $\Gamma \Rightarrow \Delta$ is *saturated* if

(Init) There is no $x : P$ in $\Gamma \cap \Delta$;
$(L\bot)$ There is no $x : \bot$ in Γ;
$\Gamma \Rightarrow \Delta$ satisfies *all* saturation conditions listed above.

To analyse the interdependencies between labels in a sequent we introduce the following:

Definition 4.4 Given a branch $\mathcal{B}$ as in Definition 4.3, a neighbourhood label a and world labels x, y, all occurring in $\downarrow \Gamma_k$, we define:

- $k(x) = min\{t \mid x$ occurs in $\Gamma_t\}$; we similarly define $k(a)$.
- $x \rightarrow_g a$ (read "x generates a") if for some $t \leq k$ and $i \in \mathcal{A}$, $k(a) = t$ and $a \in I_i(x)$ occurs in Γ_t.
- $a \rightarrow_g x$ (read "a generates x") if for some $t \leq k$ and $i \in \mathcal{A}$, $k(x) = t$ and $x \in a$ occurs in Γ_t.
- $x \overset{w}{\rightarrow} y$ (read "x generates y") if for some a it holds that $x \rightarrow_g a$ and $a \rightarrow_g y$.

Lemma 4.5 *Given a branch $\mathcal{B}$ as in Definition 4.3, we have that (a) the relation $\overset{w}{\rightarrow}$ is acyclic and forms a tree with root x_0 and (b) all world labels occurring in $\mathcal{B}$ are nodes of the tree, that is letting $\overset{w}{\rightarrow}^*$ be the transitive closure of $\overset{w}{\rightarrow}$, if u occurs in $\downarrow \Gamma_k$, then $x_0 \overset{w}{\rightarrow}^* u$.*

Proof. (a) immediately follows from the definition of relation $\rightarrow_g$ and from the sequent calculus rules, (b) easily proven by induction on $k(u) \leqslant k$. □

We can now detail the proof-search strategy. A rule (R) is said to be *applicable* to a world label x if R is applicable to a labelled formula with label x occurring in the denominator of a rule. In case of rules $(A1), (A2)$ of local absoluteness, we say the rule is applied to x (rather than to y).

Definition 4.6 When constructing root-first a derivation tree for a sequent $\Rightarrow x_0 : A$, apply the following strategy:

(i) No rule can be applied to an initial sequent;
(ii) If $k(x) < k(y)$ all rules applicable to x are applied before any rule applicable to y.
(iii) Rule (T) is applied as the first one to each world label x.
(iv) Rules which do not introduce a new label (static rules) are applied *before* the rules which do introduce new labels (dynamic rules), with the exception of (T), as in (iii);
(v) Rule (RB) is applied *before* rule (LC);
(vi) A rule (R) cannot be applied to a sequent $\Gamma_i \Rightarrow \Delta_i$ if $\downarrow \Gamma_i$ and / or $\downarrow \Delta_i$ satisfy the saturation condition associated to (R).

It follows from the strategy that if $x \overset{w}{\rightarrow} y$, every rule applicable to x is applied before any every rule applicable to y. In the previous example, the loop would have been stopped at the second application (root-first) of (LB), because the application of (LB) would violate condition (vi): the branch already satisfies the saturation condition for (LB), because $x \Vdash_i B|A$ is already in $\downarrow \Gamma$.
As an easy consequence of conditions (ii) and (iv) of the strategy, we have:

Lemma 4.7 *Let us consider a branch $\mathcal{B}$ as in Definition 4.3 and two labels x, y such that $x \overset{w}{\rightarrow}^* y$. Then for all b, if $b \in I_i(x) \in \Gamma_k$ then also $b \in I_i(y) \in \Gamma_k$.*

As usual, the size of a formula A, denoted by $|A|$, is the number of symbols occurring in A. The size of a sequent $\Gamma \Rightarrow \Delta$ is the sum of all the sizes of the formulas occurring in it. The following Lemma and Proposition are needed to prove termination.

Lemma 4.8 *Given a branch $\mathcal{B}$ as in Definition 4.3 and a world label x, we define $N(x) = \{a \mid x \to_g a\}$ as the set of neighbourhood labels generated by x, and $W(x) = \{y \mid x \overset{w}{\to} y\}$ as the set of world labels generated by x. The size of $N(x)$ and $W(x)$ is finite, more precisely: $Card(N(x)) = O(|A_0|)$ and $Card(W(x)) = O(|A_0|^2)$.*

Proposition 4.9 *Any derivation branch $\mathcal{B} = \Gamma_0 \Rightarrow \Delta_0, ..., \Gamma_k \Rightarrow \Delta_k, \Gamma_{k+1} \Rightarrow \Delta_{k+1}, ...$ of a derivation which starts from $\Gamma_0 \Rightarrow \Delta_0$, where $\Gamma_0 \Rightarrow \Delta_0$ has the form $\Rightarrow x_0 : A_0$, and which is built in accordance with the Strategy, is finite.*

Proof. Let us consider a branch $\mathcal{B}$. Suppose by contradiction that $\mathcal{B}$ is not finite, let $\Gamma^* = \bigcup_k \Gamma_k$ and $\Delta^* = \bigcup_k \Delta_k$. Then Γ^* is infinite. All formulas occurring with a label in Γ^* are subformulas of A_0, but the subformulas of A_0 are finitely many (namely they are $O(|A_0|)$. Thus Γ^* must contain infinitely many labels. In the light of Lemma 4.8, we have that Γ^* must contain infinitely many *world* labels, since each world label x generates only $O(|A_0|)$ neighbourhood labels. Let us consider now the tree determined by the relation $\overset{w}{\to}^*$ with root x_0. By Lemma 4.5, each label in any Γ_k occurs in that tree, therefore the tree determined by $\overset{w}{\to}^*$ is infinite. By previous lemma, every label in the tree has $O(|A_0|^2)$ successors, thus a finite number. By König's lemma, the tree must contain an *infinite path*: $x_0 \overset{w}{\to} x_1 \overset{w}{\to} \dots \overset{w}{\to} x_t \overset{w}{\to} x_{t+1} \dots$, with all x_t being different. We observe that (a) infinitely many x_t must be generated by dynamic rules using some subformulas of A_0, but (b) these formulas are finitely many, thus there must be a subformula of A_0 which is used infinitely many times to "generate" world labels (or better to generate a neighbourhood label from which a further world label is generated). There are two cases: this subformula is of type $Bel_i(D|C)$ occurring in Δ^* or it is of type $\Vdash_i B|A$ occurring in Γ^* (in this latter case it is not properly a subformulas of A_0 but it comes from one of them). In the first case it must occur that for some x_t we have that $x_t : Bel_i(D|C)$ occurs in some $\Delta_{s(x_t)}$ and for some a, such that $k(a) = s(x_t)+1$, we have that $a \in I_i(x_t), a \Vdash^{\exists} C \in \Gamma_{s(x_t)+1}$ and $x_t \Vdash_i D|C \in \Delta_{s(x_t)+1}$. Moreover, we have $a \to_g x_{t+1}$. But at the same time there must be in the sequence an x_r with $r > t$, such that $x_r : Bel_i(D|C)$ occurs in some $\Delta_{s(x_r)}$ and for a new b, that is with $k(b) = s(x_r) + 1$, we have that $(*)$ $b \in I_i(x_r), b \Vdash^{\exists} C \in \Gamma_{s(x_r)+1}$ and $x_r \Vdash_i D|C \in \Delta_{s(x_r)+1}$ and $b \to_g x_{t+1}$. By Lemma 4.7, we have that $a \in I_i(x_r)$; thus a itself fulfils the saturation condition for (RB) applied to $x_r : Bel_i(D|C) \in \Delta_{s(x_r)}$, and step $(*)$ violates the strategy. We have thus reached a contradiction.

In the second case the situation is similar: for some t, $x_t \Vdash_i D|C$ occurs in some $\Gamma_{s(x_t)}$ and for a new a, with $k(a) = s(x_t)+1$, we have that $a \in I_i(x_t), a \Vdash^{\exists} C \in \Gamma_{s(x_t)+1}$ and $a \Vdash^{\forall} C \supset D \in \Gamma_{s(x_t)+1}$. Moreover, we have that $a \to_g x_{t+1}$. Similarly there must be an x_r in the sequence with $r > t$, such that $x_r \Vdash_i D|C$ occurs in some $\Gamma_{s(x_r)}$ and for a new b, with $k(b) = s(x_r) + 1$, we have that we have that (**) $b \in I_i(x_r), b \Vdash^{\exists} C \in \Gamma_{s(x_r)+1}$ and $b \Vdash^{\forall} C \supset D \in \Gamma_{s(x_r)+1}$. By Lemma 4.7, we have that $a \in I_i(x_r)$; thus a itself fulfils the saturation condition for (LC) applied to $x_r \Vdash_i D|C \in \Gamma_{s(x_r)}$, so that step (**) violates the strategy. Again, we have reached a contradiction. □

Termination of proof search under the strategy is now an obvious consequence:

Theorem 4.10 *Proof search for any sequent of the form* $\Rightarrow x_0 : A_0$ *always comes to an end after a finite number of steps. Furthermore, each sequent that occurs as a leaf of the derivation tree is either an initial sequent or a saturated sequent.*

The above theorem provides a decision procedure for *CDL*. Even without a precise analysis of its complexity, it is easy to see that each proof branch may have an exponential size with respect to the size of the formula A_0 at the root of the derivation. The exact complexity of logic *CDL* has not been determined. In [9] it is shown that the single-agent version of *CDL* is CoNP. However, since $S5_n$, the multi-agent version of $S5$, is embeddable in *CDL* via the definition of the knowledge operator K_i, by the results in [10] we get that PSPACE is a lower bound for the complexity of *CDL*. We strongly conjecture that this is also its upper bound; this will be the object of future research, together with a strategy to obtain from **G3CDL** an optimal decision procedure for *CDL*.
The calculus is complete under the terminating strategy.

Theorem 4.11 *Let* $\Gamma \Rightarrow \Delta$ *be the upper sequent of a saturated branch* $\mathcal{B}$ *in a derivation tree. Then there exists a finite countermodel* $\mathcal{M}$ *to* $\Gamma \Rightarrow \Delta$ *that satisfies all formulas in* $\downarrow \Gamma$ *and falsifies all formulas in* $\downarrow \Delta$.

Proof. Let $\Gamma \Rightarrow \Delta$ be the upper sequent of a saturated branch $\mathcal{B}$. By theorem 4.10 , $\mathcal{B}$ is finite. We construct a model $\mathcal{M}_{\mathcal{B}}$ and an $SN_{\mathcal{B}}$-realization (ρ, σ), and show that it satisfies all formulas in $\downarrow \Gamma$ and falsifies all formulas in $\downarrow \Delta$.
Let $S_{\mathcal{B}} = \{x \mid x \ \in (\downarrow \Gamma \cup \downarrow \Delta)\}$ and $N_{\mathcal{B}} = \{a \mid a \in (\downarrow \Gamma \cup \downarrow \Delta)\}$. Then, associate to each $a \in N_{\mathcal{B}}$ a neighbourhood α_a, such that $\alpha_a = \{y \in S_{\mathcal{B}} | y \in a$ belongs to $\Gamma\}$, thus $\alpha_a \subseteq S_{\mathcal{B}}$. We define a neighbourhood model $\mathcal{M}_{\mathcal{B}} = \langle W, I_i, [\![\ ]\!] \rangle$ as

- $W = S_{\mathcal{B}}$, i.e. the set W consists of all the labels occurring in the saturated branch $\mathcal{B}$;
- For each $x \in W$, $I_i(x) = \{\alpha_a | a \in I_i(x)$ belongs to $\downarrow \Gamma\}$;
- For P atomic, $[\![P]\!] = \{x \in W | x : P$ belongs to $\downarrow \Gamma\}$.

Employing the saturation conditions we can easily prove that if $a \subseteq b$ belongs to Γ, then $\alpha_a \subseteq \beta_b$ and that $\mathcal{M}_{\mathcal{B}}$ satisfies all properties of a multi-agent neighbourhood model, namely non-emptiness, total reflexivity, nesting, and local absoluteness (strong closure under intersection follows from finiteness). We define a realization (ρ, σ) such that $\rho(x) = x$ and $\sigma(a) = \alpha_a$. We then prove that

[**Claim 1**] if $\mathcal{F}$ is in $\downarrow \Gamma$, then $\mathcal{M}_{\mathcal{B}} \vDash \mathcal{F}$
[**Claim 2**] if $\mathcal{F}$ is in $\downarrow \Delta$, then $\mathcal{M}_{\mathcal{B}} \nvDash \mathcal{F}$

where $\mathcal{F}$ denotes any formula of the language, i.e. $\mathcal{F} = a \in I_i(x), x \in A, a \subseteq b, x \Vdash^{\forall} A, x \Vdash^{\exists} A, x \Vdash_i B|A, x : A$. The two claim are routinely proved by induction on the weight of the formula $\mathcal{F}$ using the fact that $\Gamma \Rightarrow \Delta$ is saturated and employing, whenever needed, the induction hypothesis.

□

The completeness of the calculus is an obvious consequence:

Theorem 4.12 *If A is valid then it is provable in* **G3CDL**.

Theorem 4.11 together with soundness of **G3CDL** provide a constructive proof of *the finite model property* of the logic *CDL*: if A is satisfiable in a model, then by the soundness of **G3CDL** we have that $\neg A$ is not provable. Thus by Theorem 4.11 we can build a finite countermodel that falsifies $\neg A$, i.e. that satisfies A.

5 Conclusions, related works, and further research

We have proposed an alternative semantics, based on neighbourhood models, for the logic *CDL* of conditional beliefs. On the basis of this semantics, which is is a multi-agent version of Lewis' spheres models, we have developed the labelled sequent calculus **G3CDL**, following the methodology of [13], [12], [14]. The calculus **G3CDL** is analytical and enjoys cut elimination, admissibility of the other structural rules, and invertibility of all the rules. Moreover, on the basis of this calculus, we obtain a decision procedure for the logic *CDL* under a natural proof search strategy. The completeness of the calculus is established by means of a finite procedure which constructs a countermodel from a failed (or open) derivation branch. The finite countermodel construction provides in itself a constructive proof of the finite model property of the logic.

Although no proof-system for *CDL* was known before the calculus **G3CDL**, a few labelled calculi for conditional logics have been studied in the literature, in [15], [7], [17]. Observe however that all these calculi are based either on the relational semantics or on the selection function semantics; thus there is no direct relation between the calculus presented in this paper and these works.

A number of issues are open to further investigation. On the semantical side, other doxastic operators have been considered in the literature, such as *safe* belief and *strong* belief [3]. We conjecture that also these operators can be naturally interpreted in neighbourhood models and consequently captured by extensions of the calculus **G3CDL**. Furthermore, *CDL* is the "static" logic that underlies dynamic extensions by *doxastic actions* [3]. It should be worth studying if our calculus can be extended to deal with the dynamic systems as well.

Finally, from a computational side, to the best of our knowledge the exact complexity of *CDL* is not known. We conjecture its upper bound to be PSPACE; however, further investigations are needed to confirm this result.

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