A sequent calculus for preferential conditional logic based on neighbourhood semantics

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Abstract. The basic preferential conditional logic PCL, initially proposed by Burgess, finds an interest in the formalisation of both counterfactual and plausible reasoning, since it is at the same time more general than Lewis' systems for counterfactuals and it contains as a fragment the KLM preferential logic P for default reasoning. This logic is characterised by Kripke models equipped with a ternary relational semantics that represents a comparative similarity/normality assessment between worlds, relativised to each world. It is first shown that its semantics can be equivalently specified in terms of neighbourhood models. On the basis of this alternative semantics, a new labelled calculus is given that makes use of both world and neighbourhood labels. It is shown that the calculus enjoys syntactic cut elimination and that, by adding suitable termination conditions, it provides a decision procedure.

1 Introduction

Conditional logics have been studied since the 60's motivated by philosophical reasons, with seminal works due to Lewis, Nute, Stalnaker, Chellas, Pollock and Burgess, among others.[3] In all cases, the aim is to represent a kind of hypothetical implication $A > B$ different from classical material implication, but also from other non-classical implications, like the intuitionistic one. There are two kinds of interpretation of a conditional $A > B$: the first is hypothetical/counterfactual: "If A were the case then B would be the case". The second is prototypical: "Typically (normally) if A then B", or in other words "B holds in most normal/typical cases in which A holds". The applications of conditional logics to computer science, more specifically to artificial intelligence and knowledge representation, have followed these two interpretations: the hypothetical/counterfactual interpretation has lead to study the relation of conditional logics with the notion of *belief change* (with the crucial issue of the Ramsey Test), the prototypical interpretation has found an interest in the formalisation of default and non-monotonic reasoning (the well-known KLM systems) and has some relation with probabilistic reasoning. The range of conditional logics is however much more extensive and this brief account does not even touch the variety of conditional logics that have been studied in the literature in other context such as deontic and causal reasoning.

The semantics of conditional logics is defined in terms of various kinds of possible-world models, all of them comprising a notion of preference, comparative similarity or choice among worlds: intuitively, a conditional $A > B$ is true at a world x if B is true in all the worlds most normal/similar/close to x in which A is true. There are however different ways to formalise this notion of comparison/preference on worlds. Moreover, one may either assume that a most similar/close world to a given one always exists, or not: the first option is known as the controversial Limit Assumption, accepted for instance by Stalnaker but rejected by Lewis. For this reason, in contrast with the situation in standard modal logic, there is no unique semantics for conditional logics.

In this paper we consider the basic conditional logic **PCL** (Preferential Conditional Logic) defined by preferential models. In these models, every world x is associated with a set of accessible worlds W_x and a *preference* relation $y \leq_x z$ on this set; the intuition is that this relation assesses the relative normality/similarity of a pair of y, z with respect to x. A conditional $A > B$ is true at x if either there are no accessible A-worlds (i.e. worlds where A is true) or for each accessible A-world u there

[3] Cf. [11], [22], [23], [3], [20], [2].

is an accessible world y at least as normal as u and no worlds at least as normal as y satisfy $A \wedge \neg B$. This definition works no matter whether $\leq_x$-minimal worlds exist or not, making the aforementioned Limit Assumption superfluous. The logic **PCL** generalises Lewis' basic logic of counterfactuals, characterised by preferential models where the relation is connected (or equivalent sphere models). Moreover, its flat fragment corresponds to the preferential logic P of non-monotonic reasoning proposed by Kraus, Lehmann and Magidor [9]. Stronger logics, as those of the Lewis family, can be obtained by assuming further properties of the preference relation. An axiomatisation of **PCL** (and the respective completeness proof) has been originally presented by Burgess in [2], where the system is called S, and alternative completeness proofs are presented in [8] and in [6]. In particular, in the former a finite model property for **PCL** is proved, establishing also PSPACE complexity.

In sharp contrast with the simplicity of its Hilbert axiomatisation, the proof theory of **PCL** is largely unexplored and it is the object of this paper. Recent work on proof systems for other conditional logics includes [18], [19], [Lellman and Pattinson 2012], [1], but as far as we know only few systems are known for **PCL**: a labelled tableaux calculus has been given in [6] that makes use of pseudo-modalities indexed on worlds and of an explicit preference relation in the syntax, with termination obtained by relatively complex blocking conditions. Indexed modalities are used also in [16] where a labelled calculus for Lewis' logic VC (strictly stronger than **PCL**) is proposed: the calculus is based on the preference relations $\leq_x$ (considered as a ternary relations) and does not presuppose the limit assumption; it has good structural properties, first of all admissibility of cut, and termination is obtained by blocking conditions. An optimal unlabelled sequent calculus for **PCL** is presented in [21]: the calculus is obtained by closing one step rules by all possible cuts and by adding a specific rule for **PCL**; the resulting system is undoubtedly significant, but the rules have a highly combinatorial nature and are overly complicated.[4]

In this paper we take a different approach based on a reformulation of the semantics in terms of neighbourhood models. Neihghbuorhood semantics has been successfully employed to analyse non-normal modal logics, whose semantics cannot be defined in terms of ordinary relational Kripke models. In these models every world x is associated with a (possibly empty) set of neighbourhoods $I(x)$ and each $a \in I(x)$ is just an arbitrary (non-empty) set of worlds. The intuition is that each neighbourhood $a \in I(x)$ represents a state of information/knwowledge/affair to be taken into account to evaluate the truth of modal formulas in world x. Our starting point is a semantical characterisation of **PCL** in terms of Weak Neighbourhood Models (WNM). It can be shown on the one hand that each preferential model gives rise to a WNM and on the other hand that **PCL** is sound with respect to the WNM. Thus, since **PCL** is complete with respect to preferential models (as mentioned above), we obtain that it is also *sound* and *complete* with respect to WMN. Thus WNM can be considered as an 'official' semantics for this logic. This result is not unexpected: there is a known duality between partial orders and so-called Alexandrov topologies, so that the neighbourhood models can be built by associating to each world a topology of this kind, with the neighbourhoods being the *open* sets; for *conditional logics* this duality is studied in detail in [12]. However, the topological semantics of [12] imposes some closure conditions on the neighbourhoods (namely closure under arbitrary unions and non-empty intersections) that are not required by the logic and that we do not assume. That is why we call our neighbourhood models "weak". As remarked above, WNM suffices and provides a 'lightweight' semantics for **PCL**.

Building on WMN, we define a labelled sequent calculus for **PCL**. The calculus makes use of both world and neighbourhood labels to encode the relevant features of the semantics into the syntax. In particular, the calculus makes use of a new operator | for capturing the neighbourhood semantics that involves both world and neighbourhood labels and contains rules for handling neighbourhood inclusion. The obtained calculus is standard in the sense that each connective is handled exactly by a dual Left and Right rule, both justified through a clear meaning explanation that respects the general guidelines of inferentialism. In addition to simplicity and modularity, the calculus features good proof-theoretical properties such as height-preserving invertibility and admissibility of contraction and cut. We further show that the calculus can be made terminating by a simple (non-redundancy)

[4] In particular, a non-trivial calculation (although a polynomial algorithm) is needed to obtain *one backward instance* of the (S)-rule for a given sequent.

restriction on rule application and by a small change of the rules, thereby obtaining a decision procedure for **PCL**. No complex blocking conditions are needed. We also prove semantic completeness of the calculus: from a failed proof of a formula it is possible to extract a *finite* WNM countermodel, built directly from a suitable branch of the attempted proof. The last result provides a constructive proof of the finite model property of **PCL** with respect to the WNM semantics.

Full proofs can be found in `http://www.helsinki.fi/~negri/pclnstc.pdf`.

2 The logic PCL

The language of Preferential Conditional Logic **PCL** is generated from a set *Atm* of propositional atoms and boolean connectives plus the special connective > (conditional) by the following BNF:

$$A := P \in Atm \mid \bot \mid \neg B \mid B \wedge C \mid B \vee C \mid B \supset C \mid B > C$$

PCL is axiomatised by the following set of axioms and rules:

(Class) Any axiomatization of classical propositional logic

(R-And) $(A > B) \wedge (A > C) \supset (A > (B \wedge C))$ (ID) $A > A$

(CSO) $((A > B) \wedge (B > A)) \supset ((A > C) \supset (B > C))$ (CA) $((A > C) \wedge (B > C)) \supset ((A \vee B) > C)$

(ModPon) $\dfrac{A \quad A \supset B}{B}$ (RCEA) $\dfrac{A \supset\subset B}{(A > C) \supset\subset (B > C)}$

(RCK) $\dfrac{A \supset B}{(C > A) \supset (C > B)}$

Some quick comments on the axioms: (Class), (R-And), (ModPon), (RCEA), (RCK) form the axiomatisation of the minimal normal conditional logic CK. The remaining ones (ID), (CSO), (CA) are specific of **PCL**. (CSO) is equivalent to the pair of well-known axioms of *cumulative monotony* (CM) and *restricted transitivity* (RT):

$$\text{(CM) } ((A > B) \wedge (A > C)) \supset ((A \wedge B) > C) \quad \text{(RT) } ((A > B) \wedge ((A \wedge B) > C)) \supset (A > C)$$

that are usually assumed in conditional logics for non-monotonic reasoning (such as KLM systems). Axiom (CA) allows a kind a of reasoning by cases in conditional logics.

The standard semantics of **PCL** is defined in terms of preferential models that we define next.

Definition 1.
A preferential model M has the form $(W, \{W_x\}_{x \in W}, \{\leq_x\}_{x \in W}, [\ ])$, where W is a non-empty set whose elements are called worlds and

- *For every x in W, W_x is a subset of W;*
- *For every x in W, $\leq_x$ is a binary reflexive and transitive relation in W_x;*
- *For every (atomic) formula P in Atm, $[P]$ is a subset of W.*

Truth conditions of formulas are defined in the usual way in the boolean case:

$$[A \wedge B] = [A] \cap [B],\ [A \vee B] = [A] \cup [B],\ [\neg A] = W - [A],\ [A \supset B] = (W - [A]) \cup [B].$$

For conditional formulas we have:

(∗) *$x \in [A > B]$ iff $\forall u \in W_x$ if $u \in [A]$ then there is y such that $y \leq_x u$, $y \in [A]$, and for all z, if $z \leq_x y$ then $z \in [A \supset B]$.*

We say that a formula A is valid *in a model M if $[A] = W$.*

The truth definition of a conditional is more complicated than it could be: it takes into account the fact that minimal $\leq_x$ worlds in $[A]$ do not neccesarily exist, as the relation $\leq_x$ (or more precisely its strict version) is not assumed to be well-founded. If we make this assumption, called *Limit Assumption*, the truth condition of a conditional can be greatly simplified as follows:

($**$) $x \in [A > B]$ iff $Min_x(A) \subseteq [B]$

where $Min_x(A) = \{y \in W_x \cap [A] \mid \forall z \in W_x \cap [A](z \leq_x y \rightarrow y \leq_x z)\}$. The Limit Assumption just asserts that if $[A] \cap W_x \neq \emptyset$ then $Min_x(A) \neq \emptyset$. It is easy to show that for models satisfying the limit assumption truths conditions ($*$) and ($**$) for conditionals are equivalent. Moreover, on finite models the limit assumption is given for free. Finally, the preferential semantics enjoys the finite model property, thus the Limit Assumption is irrelevant for the validity of formulas. All in all to sum up the results known in the literature [2], [8], [6], we have:

Theorem 1. *A formula is a theorem of* **PCL** *iff it is valid in the class of preferential models (with or without Limit Assumption).*

The preferential semantics is not the only possible one. We introduce an alternative semantics, in the spirit of a neighbourhood or topological semantics. This semantics abstracts away from the comparison relation of the preferential semantics.

Definition 2. *A weak neighbourhood model (WNM) M has the form* $(W, I, [\ ])$, *where* $W \neq \emptyset$, $[\ ] : Atm \longrightarrow Pow(W)$ *is the propositional evaluation, and* $I : W \longrightarrow Pow(Pow(W))$. *We denote the elements of* $I(x)$ *by* α, β.... *We assume that for each* $\alpha \in I(x), \alpha \neq \emptyset$. *The truth definition for boolean connectives is the same as in preferential models, and for the conditional operator we have*

$x \in [A > B]$ *iff*
$\forall \alpha \in I(x)$ *if* $\alpha \cap [A] \neq \emptyset$ *then there is* $\beta \in I(x)$ *such that* $\beta \subseteq \alpha$, $\beta \cap [A] \neq \emptyset$ *and* $\beta \subseteq [A \supset B]$.

We say that a formula is valid in a WNM M if $[A] = W$.

No matter what is the kind of a model M, we use the notation $M, x \models A$ to indicate that in M it holds $x \in [A]$; when M it is clear from the context, we simply write $x \models A$. Moreover, given a WNM M and $\alpha \in I(x)$, we use the following notations:

$\alpha \models^{\forall} A$ if $\alpha \subseteq [A]$, i.e. $\forall y \in \alpha\ y \models A$
$\alpha \models^{\exists} A$ if $\alpha \cap [A] \neq \emptyset$, i.e. $\exists y \in \alpha$ such that $y \models A$

Observe that with this notation, the truth condition for $>$ becomes:

(1) $x \models A > B$ iff $\forall \alpha \in I(x)$ if $\alpha \models^{\exists} A$ then there is $\beta \in I(x)$ such that $\beta \subseteq \alpha$ and $\beta \models^{\forall} A \supset B$.

By the definition, weak neighbourhood models are faithful to Lewis's intuition of the conditional as a variably strict implication. Moreover, the above truth conditions of $>$ can be seen as a *crucial weakening*, needed for counterfactuals, of the most obvious definition of *strict* implication in neighbourhood models eg: $x \models A \Rightarrow B$ iff $\forall \alpha \in I(x), \alpha \models^{\forall} A \supset B$.

Our aim is to prove that they provide an adequate semantics for **PCL**, that is, **PCL** is sound and complete with respect to this semantics. For completeness we rely on the fact that preferential models give rise to WPN in a canonical way, by taking as neighbourhoods the downward closed sets with respect to the partial order.

Proposition 1. *For any preferential model* $M = (W, \{W_x\}_{x \in W}, \{\leq_x\}_{x \in W}, [\ ])$ *there is neighbourhood model* $M_{ne} = (W, I, [\ ])$ *such that for every* $x \in W$ *and every formula A we have:*

$$M, x \models A \text{ iff } M_{ne}, x \models A$$

Proof. Given M as in the statement, we define $M_{ne} = (W, I, [\ ])$ by letting
$I(x) \equiv \{S \subseteq W_x :$ S is downward closed wrt. $\leq_x$ and $S \neq \emptyset\}$.
The claim is proved by mutual induction on the complexity of formulas (defined in the standard way). The base of induction is by definition; the inductive case easily goes through the boolean cases, thus let us concentrate on the case of $>$. We use the notation $z \downarrow_{\leq_x} = \{u \in W_x \mid u \leq_x z\}$.

Suppose first that $M, x \models A > B$, let $\alpha \in I(x)$ such that $\alpha \models^{\exists} A$. Thus for some $y \in \alpha$, we have $M_{ne}, y \models A$, and by induction hypothesis, we have $M, y \models A$. But then by hypotehsis we have that there exists $z \leq_x y$ such that $M, z \models A$ and for every $u \leq_x z$, we have $M, u \models A \supset B$. Let $\beta = z \downarrow_{\leq_x}$, we have that $\beta \in I(x)$, $\beta \subseteq \alpha$ (since $z \leq_x y$ and $y \in \alpha$) and $\beta \models^{\forall} A \supset B$; thus $M_{ne}, x \models A > B$.

Conversely, suppose that $M_{ne}, x \models A > B$. Let $y \in W_x$ such that $M, y \models A$, by induction hypothesis, $M_{ne}, y \models A$, let $\alpha = y \downarrow_{\leq_x}$, we have that $\alpha \models^{\exists} A$. Thus by hypothesis there is $\beta \in I(x)$, with $\beta \subseteq \alpha$ such that $\beta \models^{\exists} A$ and $\beta \models^{\forall} A \supset B$. Thus for some $z \in \beta$, $M_n, z \models A$, whence $M, z \models A$ by induction hypothesis. Let $u \leq_x z$, we have $u \in \beta$ (as it is downward closed), thus we have $M_{ne}, u \models A \supset B$, so that $M, u \models A \supset B$ by induction hypothesis. This implies $M, x \models A > B$.

The converse proposition can also be proved by assuming that the neighbourhoods $I(x)$ are closed with respect to non-empty intersections. In this case we can define a preferential model M_{pref} from a WNM M by stipulating

$W_x \equiv \bigcup\{\alpha \in I(x)\}$ for any $x \in W$ and $y \leq_x z$ iff $\forall \gamma \in I(x)(z \in \gamma \rightarrow y \in \gamma)$.

and then we can prove that the set of valid formulas in the two models is the same.[5] However, for our purpose of showing the adequacy of the WNM semantics for **PCL** it is not necessary, and we have:

Theorem 2. *A formula is a theorem of* **PCL** *iff it is valid in the class of Weak Neighbourhood models.*

Proof. **(If)** direction: first we show that if a formula A is valid in the class of WNM, then it is valid in the class of preferential models and then we conclude by theorem 1. Let a formula A be valid in WNM and let M be a preferential model, as in proposition 1 we build a WNM, M_{ne}, then by hypothesis A is valid in M_{ne}, and by the same proposition it is also valid in M.
(Only if) direction: this is proved by checking that all **PCL** axioms and rules are valid in WNM models. As an example we show the case of (CSO) and (CA), the others are easy and left to the reader. For (CSO) let M be WNM, suppose that (i) $x \models A > B$, (ii) $x \models B > A$ and (iii) $x \models A > C$, suppose $\alpha \in I(x)$, by (i) we get that there is $\beta \in I(x)$, with $\beta \subseteq \alpha$ such that $\beta \models^{\exists} A$ and $\beta \models^{\forall} A \supset B$, thus also $\beta \models^{\exists} B$, whence by (ii) there is $\gamma \in I(x)$ with $\gamma \subseteq \beta$ such that $\gamma \models^{\exists} B$ and $\gamma \models^{\forall} B \supset A$, thus also $\gamma \models^{\exists} A$, whence by (iii), there is $\delta \in I(x)$, with $\delta \subseteq \gamma$ such that $\delta \models^{\exists} A$ and $\delta \models^{\forall} A \supset C$, but we also have $\delta \models^{\forall} B \supset A$, whence $\delta \models^{\forall} B \supset C$, since $\delta \subseteq \alpha$ we are done.
For (CA), let M be WNM, and suppose that (i) $x \models A > C$, (ii) $x \models B > C$, let $\alpha \in I(x)$ and suppose that $\alpha \models^{\exists} A \vee B$: suppose that $\alpha \models^{\exists} A$ then by (i) there is $\beta \in I(x)$, with $\beta \subseteq \alpha$ such that $\beta \models^{\exists} A$ and $\beta \models^{\forall} A \supset C$, thus also $\beta \models^{\exists} A \vee B$. If $\beta \models^{\forall} \neg B$ then $\beta \models^{\forall} B \supset C$, whence also $\beta \models^{\forall} (A \vee B) \supset C$ and we are done; if $\beta \models^{\exists} B$, then by (ii) there is $\gamma \in I(x)$ with $\gamma \subseteq \beta$ such that $\gamma \models^{\exists} B$ and $\gamma \models^{\forall} B \supset C$, thus also $\gamma \models^{\exists} A \vee B$, but since $\gamma \subseteq \beta$, we get $\gamma \models^{\forall} A \supset C$ as well, whence $\gamma \models^{\forall} (A \vee B) \supset C$ and we are done again. The other case when $\alpha \models^{\exists} B$ is symmetrical and left to the reader.

In the next section we give a labelled calculus for **PCL** based on **Weak Neighbourhood models**.

3 A labelled sequent calculus

The rules of labelled calculi are obtained by a translation of the semantic conditions, taking into account some further adjustments to obtain good structural properties. However, unlike in labelled systems defined in terms of a standard Kripke semantics, here the explanation of the conditional is given in terms of a neighbourhood semantics. The quantifier alternation implicit in the semantical explanation is rendered through the introduction of new primitives, each with its own rules in terms of the earlier one in the order of generalization. The idea is to unfold all the semantic clauses "outside in", starting from the outermost condition until the standard syntactic entities of Kripke semantics (forcing of a formula at a world) are reached. So we start with the clauses for the "global conditional", i.e.

$$x : A > B \equiv \forall a(a \in I(x) \;\&\; a \overset{\exists}{\Vdash} A \rightarrow x \Vdash_a A|B)$$

and proceed to those for the "local conditional"

[5] This correspondence is known as the duality between partial orders and Alexandrov topologies and for conditional logic is considered in [12].

$x \Vdash_a A|B \equiv \exists c(c \in I(x) \,\&\, c \subseteq a \,\&\, c \overset{\exists}{\Vdash} A \,\&\, c \overset{\forall}{\Vdash} A \supset B)$

and finally to the "local forcing conditions"

$a \overset{\forall}{\Vdash} A \equiv \forall x(x \in a \to x : A)$ and $a \overset{\exists}{\Vdash} A \equiv \exists x(x \in a \,\&\, x : A)$

The calculus, which we shall denote by **G3CL**, is obtained as an extension of the propositional part of the calculus **G3K** of [13], so we omit below the propositional rules (including $L\bot$); the contexts Γ, Δ are multisets:

Initial sequents

$$x : P, \Gamma \Rightarrow \Delta, x : P$$

Rules for local forcing

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \overset{\forall}{\Vdash} A}\ R\overset{\forall}{\Vdash}\ (x\ fresh) \qquad \frac{x \in a, x : A, a \overset{\forall}{\Vdash} A, \Gamma \Rightarrow \Delta}{x \in a, a \overset{\forall}{\Vdash} A, \Gamma \Rightarrow \Delta}\ L\overset{\forall}{\Vdash}$$

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \overset{\exists}{\Vdash} A}{x \in a, \Gamma \Rightarrow \Delta, a \overset{\exists}{\Vdash} A}\ R\overset{\exists}{\Vdash} \qquad \frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \overset{\exists}{\Vdash} A, \Gamma \Rightarrow \Delta}\ L\overset{\exists}{\Vdash}\ (x\ fresh)$$

Rules for the conditional

$$\frac{a \in I(x), a \overset{\exists}{\Vdash} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}{\Gamma \Rightarrow \Delta, x : A > B}\ R> (a\ fresh)$$

$$\frac{a \in I(x), x : A > B, \Gamma \Rightarrow \Delta, a \overset{\exists}{\Vdash} A \quad x \Vdash_a A|B, a \in I(x), x : A > B, \Gamma \Rightarrow \Delta}{a \in I(x), x : A > B, \Gamma \Rightarrow \Delta}\ L>$$

$$\frac{c \in I(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \overset{\exists}{\Vdash} A \quad c \in I(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \overset{\forall}{\Vdash} A \supset B}{c \in I(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}\ RC$$

$$\frac{c \in I(x), c \subseteq a, c \overset{\exists}{\Vdash} A, c \overset{\forall}{\Vdash} A \supset B, \Gamma \Rightarrow \Delta}{x \Vdash_a A|B, \Gamma \Rightarrow \Delta}\ LC(c\ fresh)$$

Rules for inclusion[6]:

$$\frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref \qquad \frac{c \subseteq a, c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}{c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}\ Trans$$

$$\frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ L\subseteq$$

Before launching into full generality, we give an example of a derivation in the calculus to get an idea of how the system works:

[6] Observe that the right rule for inclusion $\dfrac{x \in a, \Gamma \Rightarrow \Delta, x \in b}{\Gamma \Rightarrow \Delta, a \subseteq b} R\subseteq$ (x *fresh*) is not needed because in the logical rules inclusion atoms are never active in the right-hand side of sequents. In other words, root-first proof search of purely logical sequents does not introduce inclusion atoms in the succedent. This simplification of the calculus is made possible by the use of a rule such as RC. The rule has two premisses rather than four, as would result as a direct translation of the semantic truth condition for $x \Vdash_a A|B$, that would bring atomic formulas of the form $a \in I(x)$ and $c \subseteq a$ in the right-hand side; at the same time this move makes initial sequents for such atomic formulas superfluous. This simplification is analogous to the one for rule $L\Diamond$ of basic modal logic from one to two premisses (cf. [13]).

Example 1. We show a derivation (found by root-first application of the rules of the calculus) of the sequent $x : A > P, x : A > Q \Rightarrow x : A > P \wedge Q$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dots, d \overset{\exists}{\Vdash} A \Rightarrow d \overset{\exists}{\Vdash} A, \dots \qquad \dfrac{\dfrac{\dfrac{\dots, d \subseteq c, y \in c, y \in d, y : A \supset Q, y : A \supset P, d \overset{\forall}{\Vdash} A \supset Q, c \overset{\forall}{\Vdash} A \supset P \Rightarrow y : A \supset P \wedge Q}{\dots, d \subseteq c, y \in c, y \in d, d \overset{\forall}{\Vdash} A \supset Q, c \overset{\forall}{\Vdash} A \supset P \Rightarrow y : A \supset P \wedge Q}\, L\overset{\forall}{\Vdash}\ \text{(twice)}}{\dots, d \subseteq c, y \in d, d \overset{\forall}{\Vdash} A \supset Q, c \overset{\forall}{\Vdash} A \supset P \Rightarrow y : A \supset P \wedge Q}\, L\subseteq}{\dots, d \subseteq c, d \overset{\forall}{\Vdash} A \supset Q, c \overset{\forall}{\Vdash} A \supset P \Rightarrow d \overset{\forall}{\Vdash} A \supset P \wedge Q}\, R\overset{\forall}{\Vdash}}{d \subseteq a, d \in I(x), d \subseteq c, d \overset{\exists}{\Vdash} A, d \overset{\forall}{\Vdash} A \supset Q, c \in I(x), c \subseteq a, c \overset{\exists}{\Vdash} A, c \overset{\forall}{\Vdash} A \supset P, a \in I(x), a \overset{\exists}{\Vdash} A, x : A > P, x : A > Q \Rightarrow x \Vdash_a A|P \wedge Q}\, RC}{d \in I(x), d \subseteq c, d \overset{\exists}{\Vdash} A, d \overset{\forall}{\Vdash} A \supset Q, c \in I(x), c \subseteq a, c \overset{\exists}{\Vdash} A, c \overset{\forall}{\Vdash} A \supset P, a \in I(x), a \overset{\exists}{\Vdash} A, x : A > P, x : A > Q \Rightarrow x \Vdash_a A|P \wedge Q}\, Trans}{x \Vdash_a A|Q, c \in I(x), c \subseteq a, c \overset{\exists}{\Vdash} A, c \overset{\forall}{\Vdash} A \supset P, a \in I(x), a \overset{\exists}{\Vdash} A, x : A > P, x : A > Q \Rightarrow x \Vdash_a A|P \wedge Q}\, LC}{c \in I(x), c \subseteq a, c \overset{\exists}{\Vdash} A, c \overset{\forall}{\Vdash} A \supset P, a \in I(x), a \overset{\exists}{\Vdash} A, x : A > P, x : A > Q \Rightarrow x \Vdash_a A|P \wedge Q}\, L>}{x \Vdash_a A|P, a \in I(x), a \overset{\exists}{\Vdash} A, x : A > P, x : A > Q \Rightarrow x \Vdash_a A|P \wedge Q}\, LC}{a \in I(x), a \overset{\exists}{\Vdash} A, x : A > P, x : A > Q \Rightarrow x \Vdash_a A|P \wedge Q}\, L>}{x : A > P, x : A > Q \Rightarrow x : A > P \wedge Q}\, R>
$$

here the derivable left premisses of both applications of $L >$ have been omitted to save space and the topsequents are easily derivable (by the $\overset{\exists}{\Vdash}$ and the propositional rules respectively).

For the **soundness** of **G3LC** with respect to WNM we need the following:

Definition 3. *Given a set S of world labels x and a set of N of neighbourhood labels a, and a weak neighbourhood model $M = (W, I, [\ ])$, an SN-realisation (ρ, σ) is a pair of functions mapping each $x \in S$ into $\rho(x) \in W$ and mapping each $a \in N$ into $\sigma(a) \in I(w)$ for some $w \in W$. We introduce the notion "M satisfies a sequent formula F under an S realisation (ρ, σ)" and denote it by $M \models_{\rho,\sigma} F$, where we assume that the labels in F occurs in S, N. The definition is by cases on the form of F:*

- $M \models_{\rho,\sigma} a \in I(x)$ *if* $\sigma(a) \in I(\rho(x))$
- $M \models_{\rho,\sigma} a \subseteq b$ *if* $\sigma(a) \subseteq \sigma(b)$
- $M \models_{\rho,\sigma} x : A$ *if* $\rho(x) \models A$
- $M \models_{\rho,\sigma} a \overset{\exists}{\Vdash} A$ *if* $\sigma(a) \models^{\exists} A$
- $M \models_{\rho,\sigma} a \overset{\forall}{\Vdash} A$ *if* $\sigma(a) \models^{\forall} A$
- $M \models_{\rho,\sigma} x \Vdash_a A|B$ *if* $\sigma(a) \in \rho(x)$ *and for some* $\beta \subseteq \sigma(a)$ $\beta \models^{\exists} A$ *and* $\beta \models^{\forall} A \supset B$

*Given a sequent $\Gamma \Rightarrow \Delta$, let S, N be the sets of world and neighbourhood labels occurring in $\Gamma \cup \Delta$, and let (ρ, σ) be an SN-realisation, we define: $M \models_{\rho,\sigma} \Gamma \Rightarrow \Delta$ if either $M \not\models_{\rho,\sigma} F$ for some formula $F \in \Gamma$ or $M \models_{\rho,\sigma} G$ for some formula $G \in \Delta$. We further define M-*validity *by*

$M \models \Gamma \Rightarrow \Delta$ *iff* $M \models_{\rho,\sigma} \Gamma \Rightarrow \Delta$ *for every SN-realisation (ρ, σ)*

We finally say that a sequent $\Gamma \Rightarrow \Delta$ is valid *if $M \models \Gamma \Rightarrow \Delta$ for every neighbourhood model M.*

We assume that the forcing relation extends the one of classical logic. We have:

Theorem 3. *If $\Gamma \Rightarrow \Delta$ is derivable in* **G3CL** *then it is valid in the class of Weak Neighbourhood models.*

The proof of **admissibility of the structural rules** in **G3CL** follows the pattern presented in [15], section 11.4, but with some important non-trivial extra burden caused by the layering of rules for the conditional, as we shall see. Likewise, some preliminary results are needed, namely height-preserving admissibility of substitution (in short, hp-substitution) and height-preserving invertibility (in short, hp-invertibility) of the rules. We recall that the *height* of a derivation is its height as a tree, i.e. the length of its longest branch, and that $\vdash_n$ denotes derivability with derivation height bounded by n in a given system.

In many proofs we shall use an induction on formula weight, and finding the right definition of weight that takes into account all the constraints that we need for the induction to work is a subtle task. The following definition is found alongside the proofs of the structural properties, but for expository reasons it is here anticipated. Observe that the definition extends the usual definition of weight from (pure) formulas to labelled fomulas and local forcing relations, namely, to all formulas of the form $x : A$, $a \overset{\forall}{\Vdash} A$, $a \overset{\exists}{\Vdash} A$, $x \Vdash_a A|B$.

Definition 4. *The label of formulas of the form $x : A$ and $x \Vdash_a A|B$ is x. The label of formulas of the form $a \overset{\forall}{\Vdash} A$, $a \overset{\exists}{\Vdash} A$ is a. The label of a formula $\mathcal{F}$ will be denoted by $l(\mathcal{F})$. The pure part of a labelled formula $\mathcal{F}$ is the part without the label and without the forcing relation, either local ($\Vdash_a$) or worldwise ($:$) and will be denoted by $p(\mathcal{F})$.*

The weight of a labelled formula $\mathcal{F}$ *is given by the pair* $(\mathtt{w}(p(\mathcal{F})), \mathtt{w}(l(\mathcal{F})))$ *where*

- *For all worlds labels x and all neighbourhood labels a,* $\mathtt{w}(x) = 0$ *and* $\mathtt{w}(a) = 1$.
- $\mathtt{w}(P) = \mathtt{w}(\bot) = 1$,
  $\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1$ *for $\circ$ conjunction, disjunction, or implication,*
  $\mathtt{w}(A|B) = \mathtt{w}(A) + \mathtt{w}(B) + 2$,
  $\mathtt{w}(A > B) = \mathtt{w}(A) + \mathtt{w}(B) + 3$.

Weights of labelled formulas are ordered lexicographically.

From the definition of weight it is clear that the weight gets decreased if we move from a formula labelled by a neighbourhood label to the same formula labelled by a world label, or if we move (regardless the label) to a formula with a pure part of strictly smaller weight.

In our system, in addition to world labels, we have neighbourhood labels. The latter are subject to similar conditions, such as the conditions of being fresh in certain rules, as the world labels. Consequently, we shall need properties of hp-substitution in our analysis. Before stating and proving the property, we observe that the definition of substitution of labels given in [13] can be extended in an obvious way – that need not be pedantically detailed here – to all the formulas of our language and to neighbourhood labels. We'll have, for example, $x : A > B(y/x) \equiv y : A > B$ and $x \Vdash_a A|B(b/a) \equiv x \Vdash_b A|B$. Our calculus enjoys the property of hp-admissibility of substitution both of world and neighbourhood labels, that is:

Proposition 2. *1. If $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \Gamma(y/x) \Rightarrow \Delta(y/x)$;*
2. If $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \Gamma(b/a) \Rightarrow \Delta(b/a)$.

By a straightforward induction we can also prove:

Proposition 3. *The rules of left and right weakening are hp-admissible in* **G3CL**.

Hp-invertibility of the rules of a sequent calculus means that for every rule of the form $\frac{\Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta}$, if $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma' \Rightarrow \Delta'$, and for every rule of the form $\frac{\Gamma' \Rightarrow \Delta' \quad \Gamma'' \Rightarrow \Delta''}{\Gamma \Rightarrow \Delta}$ if $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma' \Rightarrow \Delta'$ and $\vdash_n \Gamma'' \Rightarrow \Delta''$. We have:

Proposition 4. *All the rules of* **G3CL** *are hp-invertible.*

The rules of contraction of **G3CL** have the following form, where ϕ is either a "relational" atom of the form $a \in I(x)$ or $x \in a$ or a labelled formula of the form $x : A$, $a \overset{\forall}{\Vdash} A$, $a \overset{\exists}{\Vdash} A$ or $x \Vdash_a A|B$:

$$\frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\, LC \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi}\, RC$$

Since relational atoms never appear on the right, the corresponding right contraction rules will no be needed. We do not need to give different names for these rules since we can prove that all of them are hp-admissible:

Theorem 4. *The rules of left and right contraction are hp-admissible in* **G3CL**.

Theorem 5. *Cut is admissible in* **G3CL**.

Proof. By double induction, with primary induction on the weight of the cut formula and subinduction on the sum of the heights of derivations of the premisses of cut. The cases in which the premisses of cut are either initial sequents or obtained through the rules for &, $\vee$, or $\supset$ follow the treatment Theorem 11.9 of [15]. For the cases in which the cut formula is a side formula in at least one rule used to derive the premisses of cut, the cut reduction is dealt with in the usual way by permutation of cut, with possibly an application of hp-substitution to avoid a clash with the fresh variable in rules with variable condition. In all such cases the cut height is reduced.

The only cases we shall treat in detail on those with cut formula principal in both premisses of cut and of the form $a \overset{\forall}{\Vdash} A$, $a \overset{\exists}{\Vdash} A$ or $x \Vdash_a A|B$, $x : A > B$. We thus have the following cases:

1. The cut formula is $a \overset{\forall}{\Vdash} A$, principal in both premisses of cut. We have a derivation of the form

$$\dfrac{\dfrac{\overset{\mathcal{D}}{y \in a, \Gamma \Rightarrow \Delta, y : A}}{\Gamma \Rightarrow \Delta, a \overset{\forall}{\Vdash} A}\; R\overset{\forall}{\Vdash} \qquad \dfrac{x : A, x \in a, a \overset{\forall}{\Vdash} A, \Gamma' \Rightarrow \Delta'}{x \in a, a \overset{\forall}{\Vdash} A, \Gamma' \Rightarrow \Delta'}\; L\overset{\forall}{\Vdash}}{x \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; Cut$$

This is converted into the following derivation:

$$\dfrac{\dfrac{\overset{\mathcal{D}(x/y)}{x \in a, \Gamma \Rightarrow \Delta, x : A} \qquad \dfrac{\Gamma \Rightarrow \Delta, a \overset{\forall}{\Vdash} A \quad x : A, x \in a, a \overset{\forall}{\Vdash} A, \Gamma' \Rightarrow \Delta'}{x : A, x \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; Cut_1}{x \in a, x \in a, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}\; Cut_2}{x \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; Ctr^*$$

Here $\mathcal{D}(x/y)$ denotes the result of application of hp-substitution to $\mathcal{D}$, using the fact that y is a fresh variable; compared to the original cut, Cut_1 is a cut of reduced height, Cut_2 is one of reduced size of cut formula, and Ctr^* denote repreated applications of (hp-)admissible contraction steps.

2. The cut formula is $a \overset{\exists}{\Vdash} A$, principal in both premisses of cut. The cut is reduced in a way similar to the one in the case above.

3. The cut formula is $x \Vdash_a A|B$, principal in both premisses of cut. We have the derivation

$$\dfrac{\dfrac{c \in I(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \overset{\exists}{\Vdash} A \quad c \in I(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \overset{\forall}{\Vdash} A \supset B}{c \in I(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B}\; RC \quad \dfrac{\overset{\mathcal{D}}{d \in I(x), d \subseteq a, d \overset{\exists}{\Vdash} A, d \overset{\forall}{\Vdash} A \supset B, \Gamma' \Rightarrow \Delta'}}{x \Vdash_a A|B, \Gamma' \Rightarrow \Delta'}\; LC}{c \in I(x), c \subseteq a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; Cut$$

The transformed derivation is obtained as follows: First we have the derivation $\mathcal{D}_2$

$$\dfrac{\dfrac{c \in I(x), c \subseteq a, \Gamma \Rightarrow \Delta, x \Vdash_a A|B, c \overset{\exists}{\Vdash} A \quad x \Vdash_a A|B, \Gamma' \Rightarrow \Delta'}{c \in I(x), c \subseteq a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c \overset{\exists}{\Vdash} A}\; Cut_1 \qquad \overset{\mathcal{D}(c/d)}{c \in I(x), c \subseteq a, c \overset{\exists}{\Vdash} A, c \overset{\forall}{\Vdash} A \supset B, \Gamma' \Rightarrow \Delta'}}{c \in I(x)^2, c \subseteq a^2, c \overset{\forall}{\Vdash} A \supset B, \Gamma, \Gamma'^2 \Rightarrow \Delta, \Delta'^2}\; Cut_2$$

where the upper cut Cut_1 is of reduced height and Cut_2 of reduced weight. Second, we have the following derivation $\mathcal{D}_3$ which uses a cut or reduced height:

$$\dfrac{c \in I(x), c \subseteq a, \Gamma \Rightarrow \Delta, c \overset{\forall}{\Vdash} A \supset B, x \Vdash_a A|B \quad x \Vdash_a A|B, \Gamma' \Rightarrow \Delta'}{c \in I(x), c \subseteq a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c \overset{\forall}{\Vdash} A \supset B}\; Cut_2$$

A cut (of reduced weight) of the conclusion of $\mathcal{D}_2$ with that of $\mathcal{D}_3$ gives the sequent

$$c \in I(x)^3, c \subseteq a^3, \Gamma^2, \Gamma'^3 \Rightarrow \Delta^2, \Delta'^3$$

from which the conclusion of the original derivation is obtained though (hp-)admissible steps of contraction.

4. The cut formula is $x : A > B$, principal in both premisses of cut.

$$\dfrac{\dfrac{\overset{\mathcal{D}}{b \in I(x), b \overset{\exists}{\Vdash} A, \Gamma \Rightarrow \Delta, x \Vdash_b A|B}}{\Gamma \Rightarrow \Delta, x : A > B}\; R> \quad \dfrac{a \in I(x), x : A > B, \Gamma' \Rightarrow \Delta', a \overset{\exists}{\Vdash} A \quad x \Vdash_a A|B, a \in I(x), x : A > B, \Gamma' \Rightarrow \Delta'}{a \in I(x), x : A > B, \Gamma' \Rightarrow \Delta'}\; L>}{a \in I(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; Cut$$

The cut is converted into four cuts of reduced height or weight of cut formula as follows: First we have the derivation (call it $\mathcal{D}_2$)

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : A > B \quad a \in I(x), x : A > B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', a \overset{\exists}{\Vdash} A}{a \in I(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', a \overset{\exists}{\Vdash} A}\, Cut_1 \qquad \begin{array}{c}\mathcal{D}(a/b)\\ a \in I(x), a \overset{\exists}{\Vdash} A, \Gamma \Rightarrow \Delta, x \Vdash_a A|B\end{array}}{a \in I(x)^2, \Gamma^2, \Gamma' \Rightarrow \Delta^2, \Delta', x \Vdash_a A|B}\, Cut_2$$

where Cut_1 is of reduced cut height and Cut_2 of reduced weight of cut formula. Second we have the derivation (call it $\mathcal{D}_3$) obtained from the given one with reduced weight of cut formula:

$$\dfrac{\Gamma \Rightarrow \Delta, x : A > B \quad x \Vdash_a A|B, a \in I(x), x : A > B, \Gamma' \Rightarrow \Delta'}{a \in I(x), x \Vdash_a A|B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut_3$$

Finally the two conclusions of $\mathcal{D}_2$ and $\mathcal{D}_3$ are used as premisses of a fourth cut (of reduced weight) to obtain the sequent

$$a \in I(x)^3, \Gamma^3, \Gamma'^2 \Rightarrow \Delta^3, \Delta'^2$$

and the original conclusion is obtained though applications of (hp-)admissible contraction steps.

To ensure the consequences of cut elimination we observe another crucial property of our system. We say that a labelled system has the *subterm property* if every world or neighbourhood variable occurring in any derivation is either an eigenvariable or occurs in the conclusion.[7] By inspection of the rules of **G3CL**, we have:

Proposition 5. *Every derivation in* **G3CL** *satisfies the subterm property.*

4 Completeness and termination

The calculus **G3CL** is not terminating as unrestricted root-first proof search may give rise to indefinetely growing branches. Consider rules $L \overset{\forall}{\Vdash}$ and $R \overset{\exists}{\Vdash}$. Root-first repeated applications of those rules on the same pair of principal formulas is *a priori* possible and it would be desirable, to restrict the search space, to show that they need to be applied only once on a given pair of matching principal formulas.[8] In fact, we have:

Lemma 1. *In* **G3CL** *rules* $L \overset{\exists}{\Vdash}$ *and* $R \overset{\exists}{\Vdash}$ *need to be applied only once on the same pair of principal formulas.*

The avoidance of indefinitely applicable rules covered by the above lemma is not the only case of restrictions that can be imposed to the calculus. Consider the following example:

$$\dfrac{\dfrac{a \in I(x), a \overset{\exists}{\Vdash} P, x : P > Q \Rightarrow a \overset{\exists}{\Vdash} P, x \Vdash_a P|R \qquad \dfrac{\dfrac{\dots \Rightarrow c \overset{\exists}{\Vdash} P, x \Vdash_a P|R \qquad \dfrac{\begin{array}{c}\vdots\\ d \in I(x), d \subseteq c, d \overset{\exists}{\Vdash} P, d \overset{\forall}{\Vdash} P \supset Q, c \in I(x), c \subseteq a, c \overset{\exists}{\Vdash} P, c \overset{\forall}{\Vdash} P \supset Q, a \in I(x), a \overset{\exists}{\Vdash} P, x : P > Q \Rightarrow x \Vdash_a P|R\end{array}}{x \Vdash_c P|Q, c \in I(x), c \subseteq a, c \overset{\exists}{\Vdash} P, c \overset{\forall}{\Vdash} P \supset Q, a \in I(x), a \overset{\exists}{\Vdash} P, x : P > Q \Rightarrow x \Vdash_a P|R}\, LC}{c \in I(x), c \subseteq a, c \overset{\exists}{\Vdash} P, c \overset{\forall}{\Vdash} P \supset Q, a \in I(x), a \overset{\exists}{\Vdash} P, x : P > Q \Rightarrow x \Vdash_a P|R}\, L>}{x \Vdash_a P|Q, a \in I(x), a \overset{\exists}{\Vdash} P, x : P > Q \Rightarrow x \Vdash_a P|R}\, LC}{a \in I(x), a \overset{\exists}{\Vdash} P, x : P > Q \Rightarrow x \Vdash_a P|R}\, L>}{x : P > Q \Rightarrow x : P > R}\, R>$$

We can see in this special case how the proof search can be truncated, and then generalize the argument through a suitable definition of *saturated branch*; this will be then strengthened to a proof that in proof search saturated branches can always be obtained in a finite number of steps.

[7] This property, restricted to world variables, is called *analyticity* in [4].

[8] This desirable property is analogous to the property for basic modal systems established for rules $L\Box$ and $R\Diamond$ in Lemma 6.3 and 6.4 [13].

Without loss of generality we can assume that a derivation of a given sequent is of minimal height. Let $\mathcal{D}$ be the derivation of the upper rightmost sequent, and assume it has height n. Then by hp-substitution we get a derivation $\mathcal{D}(c/d)$ of the same height of the sequent

$$c \in I(x), c \subseteq c, c \overset{\exists}{\Vdash} P, c \overset{\forall}{\Vdash} P \supset Q, c \in I(x), c \subseteq a, c \overset{\exists}{\Vdash} P, c \overset{\forall}{\Vdash} P \supset Q, a \in I(x), a \overset{\exists}{\Vdash} P, x : P > Q \Rightarrow x \Vdash_a P|R$$

and by hp-contraction we obtain a derivation of height n of the sequent

$$c \in I(x), c \subseteq c, c \overset{\exists}{\Vdash} P, c \overset{\forall}{\Vdash} P \supset Q, c \in I(x), c \subseteq a, a \in I(x), a \overset{\exists}{\Vdash} P, x : P > Q \Rightarrow x \Vdash_a P|R$$

and therefore, by a step of reflexivity, of height $n + 1$ of

$$c \in I(x), c \overset{\exists}{\Vdash} P, c \overset{\forall}{\Vdash} P \supset Q, c \in I(x), c \subseteq a, a \in I(x), a \overset{\exists}{\Vdash} P, x : P > Q \Rightarrow x \Vdash_a P|R$$

Observe however that this is the same as the sequent that was obtained in the attempted derivation in $n + 2$ steps, thus contradicting the assumption of minimality.

A saturated sequent is obtained by applying all the available rules with the exception of rules application that would produce a redundancy such as a loop or a duplication of already existing formulas modulo a suitable substitution of labels. There are two ways to treat uniformly the case of redundancies arising from loops ad those ones arising from duplications: one is to write all the rules in a *cumulative style*, i.e. by always copying the principal formulas of each rules in the premisses, a choice pursued in [5]; another is to consider branches rather than sequents, as in [14]. Here we follow the latter choice, and indicate $\downarrow\Gamma$ ($\downarrow\Delta$) the union of the antecedents (succedents) in the branch from the end-sequent up to $\Gamma \Rightarrow \Delta$.

Definition 5. *We say that a branch in a proof search from the endsequent up to a sequent $\Gamma \Rightarrow \Delta$ is* saturated *if the following conditions hold:*

- *(Init) There is no $x : P$ in $\Gamma \bigcap \Delta$.*
- *(L$\bot$) There is no $x \in \bot$ in Γ.*
- *(Ref) If a is in Γ, Δ, then $a \subseteq a$ is in Γ.*
- *(Trans) If $a \subseteq b$ and $b \subseteq c$ are in Γ, then $a \subseteq c$ is in Γ.*
- *(L$\wedge$) If $x : A \wedge B$ is in $\downarrow\Gamma$, then $x : A$ and $x : B$ are in $\downarrow\Gamma$.*
- *(R$\wedge$) If $x : A \wedge B$ is in $\downarrow\Delta$, then either $x : A$ or $x : B$ is in $\downarrow\Delta$.*
- *(L$\vee$) If $x : A \vee B$ is in $\downarrow\Gamma$, then either $x : A$ or $x : B$ is in $\downarrow\Delta$.*
- *(R$\vee$) If $x : A \vee B$ is in $\downarrow\Delta$, then $x : A$ and $x : B$ are in $\downarrow\Gamma$.*
- *(L$\supset$) If $x : A \supset B$ is in $\downarrow\Gamma$, then either $x : A$ is in $\downarrow\Delta$ or $x : B$ is in $\downarrow\Gamma$.*
- *(R$\supset$) If $x : A \supset B$ is in $\downarrow\Delta$, then $x : A$ is in $\downarrow\Gamma$ and $x : B$ is in $\downarrow\Delta$.*
- *(R$\overset{\forall}{\Vdash}$) If $a \overset{\forall}{\Vdash} A$ is in $\downarrow\Delta$, then for some x there is $x \in a$ in Γ and $x : A$ in $\downarrow\Delta$.*
- *(L$\overset{\forall}{\Vdash}$) If $x \in a$ and $a \overset{\forall}{\Vdash} A$ and are in Γ, then $x : A$ is in $\downarrow\Gamma$.*
- *(R$\overset{\exists}{\Vdash}$) If $x \in a$ is in Γ and $a \overset{\exists}{\Vdash} A$ is in Δ, then $x : A$ is in $\downarrow\Delta$.*
- *(L$\overset{\exists}{\Vdash}$) If $a \overset{\exists}{\Vdash} A$ is in $\downarrow\Gamma$, then for some x there is $x \in a$ in Γ and $x : A$ is in $\downarrow\Gamma$.*
- *(R$>$) If $x : A > B$ is in $\downarrow\Delta$, then there is a such that $a \in I(x)$ is in Γ, $a \overset{\exists}{\Vdash} A$ is in $\downarrow\Gamma$, and $x \Vdash_a A|B$ is in $\downarrow\Delta$.*
- *(L$>$) If $a \in I(x)$ and $x : A > B$ are in Γ, then either $a \overset{\exists}{\Vdash} A$ is in $\downarrow\Delta$ or $x \Vdash_a A|B$ is in $\downarrow\Gamma$.*
- *(RC) If $c \in I(x)$ and $c \subseteq a$ are in Γ and $x \Vdash_a A|B$ is in $\downarrow\Delta$, then either $c \overset{\exists}{\Vdash} A$ or $c \overset{\forall}{\Vdash} A \supset B$ is in $\downarrow\Delta$.*
- *(LC) If $x \Vdash_a A|B$ is in $\downarrow\Gamma$, then for some c in $I(x)$, we have $c \subseteq a$ in Γ and $c \overset{\exists}{\Vdash} A$, $c \overset{\forall}{\Vdash} A \supset B$ in $\downarrow\Gamma$.*
- *(L$\subseteq$) If $x \in a$ and $a \subseteq b$ are in Γ, then $x \in b$ is in Γ.*

Given a root sequent $\Rightarrow x : A$ we build backwards a branch by application of the rules; the branch is a sequence of sequents $\Gamma_i \Rightarrow \Delta_i$ where $\Gamma_0 \Rightarrow \Delta_0 \equiv \Rightarrow x : A$ and each $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is obtained by application of a rule R to $\Gamma_i \Rightarrow \Delta_i$.

To obtain a terminating proof search we modify (slightly) the calculus as follows:

– We replace the rule L> by the following rule:

$$\frac{a \in I(x), x : A > B, \Gamma \Rightarrow \Delta, a \overset{\exists}{\Vdash} A \quad \underline{a \overset{\exists}{\Vdash} A}, x \Vdash_a A|B, a \in I(x), x : A > B, \Gamma \Rightarrow \Delta}{a \in I(x), x : A > B, \Gamma \Rightarrow \Delta} \; L' >$$

– We add the rule Mon∀

$$\frac{b \subseteq a, b \overset{\forall}{\Vdash} A, a \overset{\forall}{\Vdash} A, \Gamma \Rightarrow \Delta}{b \subseteq a, a \overset{\forall}{\Vdash} A, \Gamma \Rightarrow \Delta} \; Mon\forall$$

and we consider the respective saturation conditions:

(L>′) If $a \in I(x)$ and $x : A > B$ are in Γ, then either $a \overset{\exists}{\Vdash} A$ is in $\downarrow\Delta$ or $a \overset{\exists}{\Vdash} A$ and $x \Vdash_a A|B$ are in $\downarrow\Gamma$.
(*Mon*∀) If $b \subseteq a$, $a \overset{\forall}{\Vdash} A$ are in Γ, then $b \overset{\forall}{\Vdash} A$ is in Γ.

We also distinguish between *dynamic* rules, i.e. rules that, root-first, introduce new world or neighbourhood labels, and *static* rules, those that operate only on the given labels. Moreover we consider the following **strategy** of application of the rules:

1. No rule can be applied to an initial sequent,
2. Static rules are applied before dynamic rules,
3. R> is applied before LC,
4. A rule R cannot be applied to $\Gamma_i \Rightarrow \Delta_i$ if $\downarrow\Gamma_i$ and/or $\downarrow\Delta_i$ satisfy the saturation condition associated to R.

Proposition 6. *Any branch $\Gamma_0 \Rightarrow \Delta_0, \ldots, \Gamma_i \Rightarrow \Delta_i, \Gamma_{i+1} \Rightarrow \Delta_{i+1}, \ldots$ of a derivation built in accordance with the strategy, with $\Gamma_0 \Rightarrow \Delta_0 \equiv \Rightarrow x_0 : A$, is finite.*

Proof. Consider any branch of any derivation of $\Rightarrow x_0 : A$. If the branch contains an initial sequent, this sequent is the last one and the branch is finite. If the branch does not contain an initial sequent, we observe the following facts: any label (world or neighbourhood) appears in the R part of a sequent of the derivation only if it appears also in the L part (with the possible exception of x_0 if the branch contains *only* the root sequent $\Rightarrow x_0 : A$). Observe also that given any sequent $\Gamma_i \Rightarrow \Delta_i$ occurring in a derivation branch, if $a \in I(x), y \in a, b \in I(y), u \in b$ all belong to $\downarrow\Gamma_i$, then we can assume, in virtue of the variable conditions in dynamic rules, that none of $b \in I(x)$, $u \in a$, $a \in I(y)$ is in $\downarrow\Gamma_i$ and moreover if $g \in I(x)$, $h \in I(x)$ are in $\downarrow\Gamma_i$, and neither $g \subseteq h$, nor $h \subseteq g$ are in $\downarrow\Gamma_i$, then there is no u such that $u \in g$ and $u \in h$ are both in $\downarrow\Gamma_i$. These remarks are aimed at preparing the following: given a branch $\Gamma_0 \Rightarrow \Delta_0, \ldots, \Gamma_i \Rightarrow \Delta_i, \Gamma_{i+1} \Rightarrow \Delta_{i+1}, \ldots$, let $\downarrow\Gamma$ and $\downarrow\Delta$ be the unions of all the Γ_i and Δ_i respectively; let us define the relation:

$a \prec x$ if $a \in I(x)$ is in $\downarrow\Gamma$ and $y \prec b$ if $y \in b$ is in $\downarrow\Gamma$

Fact: Then the relation $\prec$ does not contain cycles, has a tree-like structure with root x_0, and the length of any $\prec$-chain is bounded by the $2d(A)$ where $d(A)$ is *degree* of the formula A in the root sequent $\Rightarrow x_0 : A$, that is the maximum level of nesting of $>$ in A, defined as usual: $d(P) = 0$ if $P \in Atm$, $d(\neg C) = d(C)$, $d(C\#D) = max\{d(C), d(D)\}$ with $\# \in \{\wedge, \vee, \supset\}$ and $d(C > D) = max\{d(C), d(D)\} + 1$.

The last claim of **Fact** can be proved formally as follows: for any u occurring in $\downarrow\Gamma$ we define

$d(u) = max\{d(C) \mid u : C \in \downarrow\Gamma \cup \downarrow\Delta\}$.

By induction on $d(u)$ we show that the length of any chain beginning with u (downwards) has length $\leq 2d(u)$. If $d(u) = 0$, then the claim is obvious, since there are no chains beginning with u of length > 0. If $d(u) > 0$ consider any chain beginning with u of length > 0, the chain will contain a neighbour $a \in I(u)$ as immediate successor of u; observe that for all formulas $a \overset{\forall}{\Vdash} G$ or $a \overset{\exists}{\Vdash} G$ in $\downarrow\Gamma \cup \downarrow\Delta$ it holds $d(G) < d(u)$ as $G = E$ or $G = E \supset F$, for some $E > F$ such that $u : E > F \in \downarrow\Gamma \cup \downarrow\Delta$ with $d(E > F) \leq d(u)$. If the chain goes on further with a successor of a, it will be one $y \in a$, but all formulas $y : D \in \downarrow\Gamma \cup \downarrow\Delta$ may only be subformulas of a formula G, such that $a \overset{\forall}{\Vdash} G$ or $a \overset{\exists}{\Vdash} G$ are in $\downarrow\Gamma \cup \downarrow\Delta$. Thus $d(y) < d(u)$, and by inductive hypothesis all chains beginning with y have length $\leq 2d(y)$. Thus the chain beginning with u will have length $\leq 2d(y) + 2 \leq 2(d(u) - 1) + 2 = 2d(u)$.

Our purpose is to show that $\downarrow\Gamma$, $\downarrow\Delta$ are indeed finite. Since the labels can only be attached to subformulas of the initial A in $\Rightarrow x_0 : A$ (that are finitely many), we are left to show that the $\prec$ relation forms a *finite* tree. But we have just proved that every $\prec$-chain is finite, thus it is sufficient to show that every node in this tree has a finite number of immediate successors, and then we obtain the desired conclusion. In other words we must show that:

1. for each a occurring $\downarrow\Gamma$, the set $\{u \mid u \in a \in\downarrow\Gamma\}$ is finite.
2. for each x occurring in $\downarrow\Gamma \cup \{x_0\}$, the set $\{a \mid a \in I(x) \in\downarrow\Gamma\}$ is finite.

Let us consider 1: take a label a occurring in some Γ_i; worlds u can be added to a (i.e. $u \in a$ will appear in some Γ_k with $k > i$) only because of the application of a the rule (L$\overset{\exists}{\Vdash}$) to some $a \overset{\exists}{\Vdash} C \in \Gamma_j$ or (R$\overset{\forall}{\Vdash}$) to $a \overset{\forall}{\Vdash} D \in \Delta_j$, $j \geq i$. But there is only a finite number of such formulas, and they are treated only once, so the result follows.
Let us consider 2: take a label x occurring in some Γ_i. A neighbour a can be added to $I(x)$ (meaning that $a \in I(x)$ will appear in some Γ_k with $k > i$) only because of rule R> applied to some $x : C > D \in \Delta_i$ or because of rule LC applied to some $x \Vdash_b E|F$ with $b \in I(x)$ also in Γ_i. In the former case we note that the the number of formulas $x : C > D \in \Delta_i$ if finite and each is treated only once, by the saturation restriction. Thus only finitely many neighbours b will be added to $I(x)$.

The latter case is slightly more complicated: each $x \Vdash_b E|F$ is generated by a formula $x : E > F$, with $b \in I(x)$ also in Γ_i by an application of rule L$'$ >, taking the right premisse of this rule. The formulas $x : E > F$ are finitely many, say $x : E_1 > F_1, \dots, x : E_k > F_k$. Thus in the worst case, for a given $b \in I(x) \in \Gamma_i$, all k formulas $x \Vdash_b E_1|F_1, \dots, x \Vdash_b E_k|F_k$ will appear in some Γ_j for some $j > i$. Suppose next that LC is applied first for some l to $x \Vdash_b E_l|F_l$, to keep the indexing easy we let $l = 1$, this will generate a new neighbour d, introducing $d \in I(x), d \subseteq b$, $d \overset{\exists}{\Vdash} E_1$ and $d \overset{\forall}{\Vdash} E_1 \supset F_1$. The static rule L$'$ > can be applied again to d, generating in the worst case (it corresponds to taking always the right premiss) $x \Vdash_d E_1|F_1, \dots, x \Vdash_d E_k|F_k$. Let us denote by $\Gamma_p \Rightarrow \Delta_p$ the sequent further up in the branch containing $x \Vdash_d E_1|F_1, \dots, x \Vdash_d E_k|F_k$; by saturation we have that $d \in I(x), d \subseteq d, d \overset{\exists}{\Vdash} E_1, d \overset{\forall}{\Vdash} E_1 \supset F_1 \in\downarrow\Gamma_p$, thus LC cannot be applied to $x \Vdash_d E_1|F_1$ and only $k - 1$ applications of LC are possible, namely to $x \Vdash_d E_2|F_2, \dots, x \Vdash_d E_k|F_k$.

Suppose next, to keep the indexing simple, that LC is applied then to $x \Vdash_d E_2|F_2$, then it will add a new e with $e \in I(x), e \subseteq d$, $e \overset{\exists}{\Vdash} E_2$ and $e \overset{\forall}{\Vdash} E_2 \supset F_2$. Again, the rule L'> can be applied, and in the worst case it will add $x \Vdash_e E_1|F_1, \dots, x \Vdash_e E_k|F_k$. But here the new version L'>, becomes significant: also $e \overset{\exists}{\Vdash} E_1, \dots, e \overset{\exists}{\Vdash} E_k$ will be added to the (antecedent) of the sequent containing $x \Vdash_e E_1|F_1, \dots, x \Vdash_e E_k|F_k$. Moreover by saturation with respect to Mon$\forall$, the antecedent $e \overset{\forall}{\Vdash} E_1 \supset F_1$ will be added, as well as $e \subseteq e$. Thus at this point, by saturation restriction, LC cannot be applied neither to $x \Vdash_e E_2|F_2$, nor to $x \Vdash_e E_1|F_1$, and only (k-2) applications are possible.

A simple generalisation of the previous argument shows that after any application of LC which generates new subneighbours d of a given neighbour b, the number of applications of LC to each d strictly decreases, whence the number of further neighbours which can be subsequently generated: if there are $x \Vdash_b E_1|F_1, \dots, x \Vdash_b E_k|F_k$ they will produce at most k $d_1, \dots, d_k \subseteq b$, but each d_l can produce at most $k - 1$ $e_1, \dots, e_{k-1} \subseteq d_l$, and each e_m can produce at most $k - 2$ $g_1, \dots, g_{k-2} \subseteq e_m$, and so on. Thus the process must terminate and there will be a sequent $\Gamma_q \Rightarrow \Delta_q$ such that $\downarrow\Gamma_q$ is saturated with respect to all $x \Vdash_a E_j|F_j$, for all a such that $a \in I(x) \in\downarrow\Gamma_q$, and this shows that $\{a \mid a \in I(x) \in\downarrow\Gamma\}$ is finite.

The following is an easy consequence.

Theorem 6. *Any proof search for $\Rightarrow x : A$ is finite. Moreover every branch either contains an initial sequent or is saturated.*

Proof. By the previous proposition every branch is finite; let us consider any branch $\Gamma_0 \Rightarrow \Delta_0, \dots, \Gamma_m \Rightarrow \Delta_m$. The branch ends with $\Gamma_m \Rightarrow \Delta_m$, no rule is applicable to it, thus, trivially, either $\Gamma_m \Rightarrow \Delta_m$ is an initial sequent or the branch is saturated, otherwise some rule would be applicable to $\Gamma_m \Rightarrow \Delta_m$.

Observe that the number of labelled formulas in a saturated branch may be exponential in the size of the root sequent. For this reason, our calculus is not optimal, since the complexity of **PCL** is PSPACE [8].

As mentioned in the introduction, [21] give a (very complicated) optimal calculus. Beyond the technicality of the calculus, the essential ingredient, which goes back to Lehmann, is to restrict the semantics to *linearly ordered* preferential models. This restriction preserves soundness for *flat* sequent with at most one positive conditional on the right (after propositional unravelling), whereas for (flat) sequents with several positive conditionals on the right one has to consider "multi-linear" models as defined in [7]. Then one can study a calculus matching this strengthened semantics. This idea is developed also in [7] where an optimal calculus for KLM logic P, the flat version of **PCL** is given. We conjecture that a similar idea can be adopted for **PCL** based on WNM semantics: we should first restrict the semantics to a special type of neighbourhood models, show that the restriction preserves soundness and then develop a calculus with respect the sharpened semantics, with the hope of obtaining an optimal one. All of this will be object of future research.

The following lemma shows how to define finite countermodels from saturated branches:

Lemma 2. *For any saturated branch leading to a sequent $\Gamma \Rightarrow \Delta$ there exists a (finite) countermodel $\mathcal{M}$ to $\Gamma \Rightarrow \Delta$, which makes all the formulas in $\downarrow\Gamma$ true and all the formulas in $\downarrow\Delta$ false.*

Proof. Define the countermodel $\mathcal{M} \equiv (W, N, I, \Vdash)$ as follows:

1. The set W of worlds consists of all the world labels in Γ;
2. The set N of neighbourhood consists of all the neighbourhood labels in Γ;
3. For each x in W, the set of neighbourood $I(x)$ consists of all the a in N such that $a \in I(x)$ is Γ;
4. For each a in N, a consists of all the y in W such that $y \in a$ in is Γ;
5. The valuation is defined on atomic formulas by $x \Vdash P$ if $x : P$ in Γ and is extended to arbitrary labelled formulas following the clauses of neighbourhood semantics for conditional logic (cf. beginning of Section 3).

Next we can prove the following (cf. Definition 3: here ρ and σ are the identity maps, and we leave them unwritten):

1. If A is in $\downarrow\Gamma$, then $\mathcal{M} \models A$.
2. If A is in $\downarrow\Delta$, then $\mathcal{M} \not\models A$.

The two claims are proved simultaneously by cases/induction on the weight of A (cf. Def. 4):

(a) If A is a formula of the form $a \in I(x)$, $x \in a$, $a \subseteq b$, claim 1. holds by definition of $\mathcal{M}$ and claim 2. is empty.

(b) If A is a labelled atomic formula $x : P$, the claims hold by definition of $\Vdash$ and by the saturation clause *Init* no inconsistency arises. If A is $\bot$, it holds by definition of the forcing relation that it is never forced, and therefore 2. holds, whereas 1. holds by the saturation clause for $L\bot$. If A is a conjunction, or a disjunction, or an implication, the claim holds by the corresponding saturation clauses and inductive hypothesis on smaller formulas.

(c) If $a \overset{\exists}{\Vdash} A$ is in $\downarrow\Gamma$, by the saturation clause ($L\overset{\exists}{\Vdash}$), for some x there is $x \in a$ in Γ and $x : A$ is in $\downarrow\Gamma$. Then $\mathcal{M} \models x \in a$ by (a) and by IH $\mathcal{M} \models x : A$, therefore $\mathcal{M} \models a \overset{\exists}{\Vdash} A$. If $a \overset{\exists}{\Vdash} A$ is in $\downarrow\Delta$, then it is in Δ because such formulas are always copied to the premisses in the right-hand side of sequents. Consider an arbitrary world x in a. Then by definition of $\mathcal{M}$ we have $x \in a$ in Γ and thus by the saturation clause ($R\overset{\exists}{\Vdash}$) we also have $x : A$ is in $\downarrow\Delta$. By IH we have $\mathcal{M} \not\models x : A$ and therefore $\mathcal{M} \not\models a \overset{\exists}{\Vdash} A$. The proof for formulas of the form $a \overset{\forall}{\Vdash} A$ is similar.

(d) If $x \Vdash_a A|B$ is in $\downarrow\Gamma$, then by saturation for some c in $I(x)$, we have $c \subseteq a$ in Γ and $c \overset{\exists}{\Vdash} A$, $c \overset{\forall}{\Vdash} A \supset B$ in $\downarrow\Gamma$. By IH this gives $\mathcal{M} \models c \overset{\exists}{\Vdash} A, c \overset{\forall}{\Vdash} A \supset B$ and by definition of $\mathcal{M}$ we obtain $\mathcal{M} \models x \Vdash_a A|B$.

If $x \Vdash_a A|B$ is in $\downarrow\Delta$, consider an arbitrary c in $I(x)$ with $c \subseteq a$ in the model. By definition of $\mathcal{M}$ we have that $c \in I(x)$ and $c \subseteq a$ are in Γ, and therefore by saturation clause (RC) we obtain then either $c \overset{\exists}{\Vdash} A$ or $c \overset{\forall}{\Vdash} A \supset B$ is in $\downarrow\Delta$. By IH we have that either $\mathcal{M} \not\models c \overset{\exists}{\Vdash} A$ or $\mathcal{M} \not\models c \overset{\forall}{\Vdash} A \supset B$. Overall, this means that $\mathcal{M} \not\models x \Vdash_a A|B$.

(e) If $x : A > B$ is in $\downarrow\Gamma$, then because of the form of the rules of the calculus it actually is in Γ; let a be a in $I(x)$ in the model. Then $a \in I(x)$ and $x : A > B$ are in Γ and the saturation clause (L>) applies, giving that either $a \overset{\exists}{\Vdash} A$ is in $\downarrow\Delta$ or $x \Vdash_a A|B$ is in $\downarrow\Gamma$. By IH we that have that either $\mathcal{M} \not\models a \overset{\exists}{\Vdash} A$ or $\mathcal{M} \models x \Vdash_a A|B$. It follows that $\mathcal{M} \models x : A > B$.

If $x : A > B$ is in $\downarrow\Delta$, then by (R>) there is a such that $a \in I(x)$ is in Γ, $a \overset{\exists}{\Vdash}$ is in $\downarrow\Gamma$, and $x \Vdash_a A|B$ is in $\downarrow\Delta$. By IH we obtain $\mathcal{M} \models a \overset{\exists}{\Vdash}$ and $\mathcal{M} \not\models x \Vdash_a A|B$, and therefore $\mathcal{M} \not\models x : A > B$.

We are ready to prove the completeness of the calculus.

Theorem 7. *If A is valid then there is a derivation of $\Rightarrow x : A$, for any label x.*

Proof. By Theorem 6 for every A there is (a finite procedure that leads to) either a derivation for $\Rightarrow x : A$ or to a saturated branch. By the above lemma a saturated branch gives a countermodel of A. It follows that if A is valid it has to be derivable.

The proof of the above theorem shows not only the completeness of the calculus, but more specifically that for any unprovable formula the calculus provides a finite countermodel. Given the soundness of the calculus, as a by product we obtain a constructive proof of the finite model property for this logic.

5 Conclusions

In this paper we have given a labelled sequent calculus for the basic preferential conditional logic **PCL**. The calculus stems from a new semantics for this logic in terms of Weak Neighborhood Systems, a semantics of independent interest. The calculus has good proof-theoretical properties, such as the admissibility of cut and contraction. Completeness follows from the cut-elimination theorem and derivations of the axioms and rules of PCL and is also shown by a direct proof search/countermodel construction. The calculus can be made terminating by adopting a suitable search strategy and by slightly changing the rules. In comparison with other proposals such as [6] and [16], no complex blocking conditions are necessary to ensure termination. The calculus however is not optimal as the size of a derivation branch may grow exponentially. We shall study how to refine it in order to obtain an optimal calculus; as briefly discussed in the previous section, a sharper semantical analysis of **PCL** might be needed to this purpose.

In future research, we also intend to extend the Weak Neighbourhood Semantics and find corresponding calculi for the main extensions of **PCL**.

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