A Gentzen-style analysis of Public Announcement Logic

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1 Introduction

In recent years a distinction in the field of studies on epistemic logic has been made between the static and the dynamic approach to the problems of knowledge and belief. The static aspects of knowledge have been investigated for a long time, at least since Hintikka's monograph of the early 1960's (cf. Hintikka 1962). The attention has been focused on what the agents know statically, that is, on the state of information available to them in a fixed epistemic model. The most evident limitation of the static approach is that knowledge is supposed to be unchanging and exempt from modifications, whereas it is natural to think that it may change for several reasons: events from the outside, actions and observation by the agents, and, especially, by communication among them. This is why in the last few years several authors (cf. van Ditmarsch et al. 2007 for a detailed bibliography) have introduced the notion of *information update* as a dynamic component in epistemic logic. Intuitively, an update is the result of an action performed by the agents and corresponds to an operation that yields a modification on the model. What is known as dynamic epistemic logic is a large family of logics that extend standard epistemic logic with dynamic operators for information updates.

In this paper we shall deal with a type of dynamic epistemic logic, the logic of public announcements (PAL, for short). An announcement is the simplest epistemic action by means of which the state of an agent's information can be updated: After an announcement that A, the state of an agent's knowledge is not any longer the same, but it is modified in such a way that every not-A world is simply deleted from the model. The most popular example involving public announcements is the well-known Muddy Children Puzzle (cf. van Ditmarsch et al. 2007). In the logic of public announcement it is assumed that not any fact that agents tell to each other can be considered as an announcement: Announcements are required to be public and truthful. Thus, the aim of PAL is to model acquisition of information within a group of agents by dealing with situations in which agents get new information by the epistemic actions of announcing publicly to each other certain true facts.

The first attempt to formalize reasoning about public announcements was presented in Palza (1989). The language of multi-agent epistemic logic is enriched with a binary connective $+$ with the following intended meaning: $A+B$ is true if and only if B is true after the truthful announcement that A. In the current literature (cf. van Ditmarsch et al. 2007) the notation for $A+B$ is $\langle A\rangle B$. The announced formula A yields a restriction of the model to those worlds and accessibility relations in which A holds. Then $\langle A\rangle B$ is true at a world w of the model M if and only if B is true at w of M' where M' contains only those worlds in which A is true and the accessibility relations are restricted to those worlds in which A holds.

The standard axiom system for Public Announcement Logic (**PAL**) extends the multi-agent epistemic **S5** system. Traditionally, **PAL** axioms are reduction axioms: Every formula that contains announcements can be rewritten into a formula without announcements. A completeness result for **PAL** is given in Plaza (1989) and in van Ditmarsch et al. (2007).

In the literature on PAL, proof theory does not play a central role. The studies have been focused mostly on model-theoretic aspects and the proof-theoretical part has been limited to providing Hilbert-style axiom systems, an exception being the tableaux calculus of Balbiani et al. (2010). In this paper, we focus on the proof-theoretical aspects of PAL. Our aim here is to provide a proof system for PAL directly justified by the semantics of epistemic model updates and without use of reduction axioms. Following the labelled approach of Negri (2005), here briefly recalled in Section 1, we present in Section 2 a Gentzen-style proof system for PAL (**G3PAL**) and prove that it enjoys all the structural properties usually required of sequent systems, in particular admissibility of the rule of Cut. (Section 3). It is shown that in **G3PAL** all the axioms of **PAL** are derivable. These derivations are obtained by deterministic root-first proof search and give an illustration of the algorithmic properties of calculus. Restriction of the search space of proofs is the most important consequence of cut elimination and it makes the calculus amenable to automatization. Moreover, the derivability of the **PAL** axioms, together with the soundness of the rules, gives indirectly a completeness result for **G3PAL**. Completeness with respect to the Kripke semantics of PAL is proved also directly, in Section 4.

2 Preliminaries on Labelled Sequent Calculi

In order to give a self-contained presentation, we briefly recall in this section the background of our method, presented in Negri and von Plato (2001) and Negri (2005), for the development of cut-free labelled systems for multi-modal logics. For extensions of classical predicate logic, the starting point is the contraction- and cut-free sequent calculus **G3c** (cf. Negri and von Plato 2001 for the rules and the basic properties). We recall that all the rules of **G3c** are invertible and all the structural rules are admissible. Weakening and contraction are in addition *height-preserving-* (*hp-*) admissible, that is, whenever their premisses are derivable, so also is their conclusion, with at most the same derivation height (the *height* of a derivation is its height as a tree, that is, the length of its longest branch). Moreover, the calculus enjoys hp-admissibility of substitution of individual variables. Invertibility of the rules of **G3c** is also height-preserving (*hp-invertible*). Detailed proofs can be found in chapters 3 and 4 of Negri and von Plato (2001). It is possible to prove that the extension of **G3c** with suitably formulated rules that correspond to axioms for specific theories maintains all the structural properties of the basic **G3c** system. Universal axioms are first transformed, through the rules of **G3c**, into a normal form that consists of conjunctions of formulas of the form

$$P_1 \,\&\, \ldots \,\&\, P_m \supset Q_1 \vee \ldots \vee Q_n$$

where all P_i, Q_j are atomic; if $m = 0$ then the implication reduces to the succedent which is $\perp$ if $n = 0$. The universal closure of any such formula is called a *regular formula*. We abbreviate the multiset $P_1, \ldots, P_m$ as $\overline{P}$. Each conjunct is then converted into a schematic rule, called the *regular rule scheme*, of the form

$$\frac{Q_1, \overline{P}, \Gamma \Rightarrow \Delta \quad \cdots \quad Q_n, \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}\ {\scriptstyle Reg}$$

By this method, all universal theories can be formulated as contraction- and cut-free systems of sequent calculi.

In Negri (2005), the method is extended to cover also geometric theories, that is, theories axiomatized by geometric implications. We recall that a *geometric formula* is a formula that does not contain $\supset$, $\neg$, or $\forall$, and a *geometric implication* is a sentence of the form $\forall \overline{z}(A \supset B)$ where A and B are geometric formulas. Geometric implications can be reduced to a normal form that consists of conjunctions of formulas, called *geometric axioms*, of the form

$$\forall \overline{z}(P_1 \,\&\, \ldots \,\&\, P_m \supset \exists \overline{x}_1 M_1 \vee \ldots \vee \exists \overline{x}_n M_n)$$

in which $\overline{z}$ is a vector of variables $z_1, \ldots, z_r$, $\forall \overline{z}$ abbreviates $\forall z_1 \ldots \forall z_r$, and similarly for the $\overline{x}_j$ and the existential quantifiers, each P_i is an atomic formula, each M_j is a conjunction of a list of atomic formulas $\overline{Q}_j$, and none of the variables in the vectors $\overline{x}_j$ are free in any P_i. Note that regular formulas are degenerate cases of geometric implications, with neither conjunctions nor existential quantifications to the right of the implication. The geometric rule scheme for geometric axioms takes the form

$$\frac{\overline{Q_1}(\overline{y}_1/\overline{x}_1), \overline{P}, \Gamma \Rightarrow \Delta \quad \cdots \quad \overline{Q_n}(\overline{y}_n/\overline{x}_n), \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}\ {\scriptstyle GRS}$$

where $\overline{Q_j}$ and $\overline{P}$ indicate the multisets of atomic formulas $Q_{j_1}, \ldots, Q_{j_{k_j}}$ and $P_1, \ldots, P_m$, respectively, and the vectors of eigenvariables $\overline{y}_1, \ldots, \overline{y}_n$ of the premisses are not free in the conclusion. We use the notation $A(\overline{y}/\overline{x})$ to indicate A after the substitution of the vector of variables $\overline{y}$ for the vector of variables $\overline{x}$.

In order to maintain admissibility of contraction in the extensions with regular and geometric rules, the formulas $P_1, \ldots, P_m$ in the antecedent of the conclusion of the scheme have, as indicated, to be repeated in the antecedent of each of the premisses. In addition, whenever an instantiation of free parameters in atoms produces a duplication (two identical atoms) in the conclusion of a rule instance, say $P_1, \ldots, P, P, \ldots, P_m, \Gamma \Rightarrow \Delta$ there is of course a corresponding duplication in each premiss. The *closure condition* imposes the requirement that the rule with the duplication P, P contracted into a single P, both in the premisses and in the conclusion, be added to the system of rules. For each axiom system, there is only a bounded number of possible cases of contracted rules to be added, very often none at all, so the condition is unproblematic. The main result for such extensions is the following:

Theorem 2.1. *The structural rules of* weakening, contraction *and* cut *are admissible in all extensions of* **G3c** *with the geometric rule-scheme and satisfying the closure condition.* Weakening *and* contraction *are hp-admissible.*

The method of extension of sequent calculi can be applied not only to the proof theory of specific theories such as lattice theory, arithmetic, and geometry (cf. Negri and von Plato 2001), but also to the proof theory of modal logics. The first step is to formulate an adequate sequent calculus for the basic modal logic **K** and successively extend the set of rules in order to cover the cases of modal logics as **T**, **K4**, **KB**, **S4**, **B**, **S5**, and so on. The basic modal logic **K** is formulated as a labelled sequent calculus through an internalization of the possible worlds semantics within the syntax: The language is enriched so that sequents are expressions of the form $\Gamma \Rightarrow \Delta$ where the multisets Γ and Δ consist of relational atoms wRv and labelled formulas $w : A$. Relational atoms and labelled formulas are the syntactic counterparts of the accessibility relations between worlds and the forcing $w \Vdash A$ of Kripke models, respectively. The rules for the modalities $\Box$ and $\Diamond$ are obtained through a meaning explanation in terms of Kripke semantics and an inversion principle. The other logics are formulated by adding to the basic calculus rules expressing properties of binary relations in such a way that complete systems for all the modal logics characterized by geometric frame conditions are obtained.

Basic Epistemic Logic

We start from the cut-free calculus **G3K** given in Negri (2005) and replace the alethic modality $\Box$ with the knowledge operator $\mathcal{K}_a$ for each agent a. Observe that a multi-modal generalization of **G3K** is obtained by allowing an accessibility relation R_a and a corresponding epistemic attitude $\mathcal{K}_a$ for each agent a, as in Hakli and Negri (2008). The rules for each connective and modality are obtained from their meaning explanation in terms of the Kripke model.

Definition 2.2. *Let $\mathcal{P}$ be a set of atomic formulas and $\mathcal{A}$ a set of agents, a **Kripke model** is a structure $M = \langle W, R_a, V\rangle$ where W is a non-empty set; for every $a \in \mathcal{A}$, R_a is a binary relation on W; V is an evaluation function that assigns to every atom $P \in \mathcal{P}$ the set of worlds in which P holds. The standard notation for $w \in V(P)$ is $w \Vdash P$.*

The evaluation function on atoms is extended in a unique way to arbitrary formulas by means of inductive clauses. The clauses for the propositional connectives are the standard ones. The inductive step for the knowledge operator is as follows:

$$w \Vdash \mathcal{K}_a A \text{ if and only if for all } v,\ wR_av \text{ implies } v \Vdash A$$

The left-to-right direction in the explanation above justifies the left rules, the right-to-left direction the right rules. The role of the quantifier is reflected in the variable condition for rule $R\mathcal{K}_a$ that v is the eigenvariable and it does not appear in the conclusion. The definition thus gives the following rules:

$$\frac{v : A, w : \mathcal{K}_a A, wR_av, \Gamma \Rightarrow \Delta}{w : \mathcal{K}_a A, wR_av, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathcal{K}_a} \qquad \frac{wR_av, \Gamma \Rightarrow \Delta, v : A}{\Gamma \Rightarrow \Delta, w : \mathcal{K}_a A}\ {\scriptstyle R\mathcal{K}_a}$$

Mathematical Rules

Although there is no primitive rule for deriving the standard properties of knowledge (veridical knowledge, positive and negative introspection, etc.) such extensions are always possible by Theorem 2.1. For instance, the veridical knowledge property (what is known is true) corresponds to axiom T, $\mathcal{K}_a A \supset A$, and imposes that all the accessibility relations are reflexive: wR_aw, for every world w. The reflexivity condition is an universal axiom and it is possible to convert it into a sequent rule by the method just presented, as follows:

$$\frac{wR_aw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle Ref}$$

System **G3T** is **G3K** plus Ref. It is easy to see that in the presence of Ref axiom T is derivable. The positive and negative introspection properties expressed by the axioms $\mathcal{K}_a A \supset \mathcal{K}_a\mathcal{K}_a A$ and $\neg\mathcal{K}_a A \supset \mathcal{K}_a\neg\mathcal{K}_a A$, respectively, correspond to transitivity and euclideannes of the accessibility relations and give system **S5**. A sequent calculus for epistemic **S5** is obtained by adding to **G3T** the following rules:

$$\frac{wR_az, wR_av, vR_az, \Gamma \Rightarrow \Delta}{wR_av, vR_az, \Gamma \Rightarrow \Delta}\ \mathit{Trans} \qquad \frac{vR_az, wR_av, wR_az, \Gamma \Rightarrow \Delta}{wR_av, wR_az, \Gamma \Rightarrow \Delta}\ \mathit{Eucl}$$

3 A Gentzen System for PAL

The next step is to introduce rules for announcements. The set of formulas is built in usual way, adding a clause for announcements: If A and B are formulas, also $[A]B$ is a formula. The intended meaning is: B is true, after every public announcement that A. Recall from van Ditmarsch et al. (2007) the definition of Kripke model and of restricted Kripke model. The restriction is the dynamic component of the semantics and it corresponds to the operation of information update in the model. Here we shall indicate the restriction on a Kripke model M by M^A, whereas standard literature (cf. van Ditmarsch et al. 2007) uses $M|A$.

Definition 3.1. *Let M be a Kripke model and A a formula, a* ***restricted Kripke model*** *is $M^A = \langle W^A, R_a^A, V^A \rangle$ where $W^A = \{w \in W : w \Vdash A\}$; $R_a^A = R_a \cap (W^A \times W^A)$; and $V^A = V \cap W^A$. The standard notation for $w \in V^A(P)$ is $w \Vdash^A P$.*

Thus, M^A is the Kripke model M restricted to those worlds in which A holds and M^A is called the model M *updated* with A and the inductive clause for announcements can be formulated as follows:

$$w \Vdash [A]B \text{ if and only if } w \Vdash A \text{ implies } w \Vdash^A B$$

Note that by Definition 3.1 we have two forcing relation, $w \Vdash B$ and $w \Vdash^A B$, for forcing and restricted forcing, respectively. Restricted forcing is given on atomic formulas by the restricted evaluation function and it is extended to arbitrary formulas by induction on the announced formula. For reason related to the admissibility of the structural rules (cf. Lemma 4.4), we have to take into account the general case of a (possibly empty) list of successive announcements and then of a Kripke model restricted to a (possibly empty) list of formulas. Let φ be a list of formulas we indicate with M^φ the Kripke model restricted to φ. If φ is the empty list ϵ then $M^\epsilon = M$ and the restricted forcing coincides with the unrestricted one. The notion of $w \Vdash^\varphi B$ is defined by induction on the announced formula B as follows. Observe that in the case of B atomic, φ must be not empty, because the empty case is already given by the model. Let φ, A be a list of formulas the last element of which is A we indicate with $M^{\varphi,A}$ a Kripke model restricted to not empty list of formulas. Note that $M^{\varphi,A}$ should be written M^{φ^A}, but we prefer a linear notation where the comma has the same role of the concatenation operator $\bullet$ in Balbiani et al. (2010).

$w \Vdash^{\varphi,A} P$	if and only if	$w \Vdash^{\varphi} A$ and $w \Vdash^{\varphi} P$
$w \Vdash^{\varphi} B \,\&\, C$	if and only if	$w \Vdash^{\varphi} B$ and $w \Vdash^{\varphi} C$
$w \Vdash^{\varphi} B \vee C$	if and only if	$w \Vdash^{\varphi} B$ or $w \Vdash^{\varphi} C$
$w \Vdash^{\varphi} B \supset C$	if and only if	$w \Vdash^{\varphi} B$ implies $w \Vdash^{\varphi} C$
$w \Vdash^{\varphi} \mathcal{K}_a B$	if and only if	for all v, $wR_a^{\varphi}v$ implies $v \Vdash^{\varphi} B$
$w \Vdash^{\varphi} [B]C$	if and only if	$w \Vdash^{\varphi} B$ implies $w \Vdash^{\varphi,B} C$

By $wR_a^{\varphi}v$ we indicate the restriction on the accessibility relations, where $wR_a^{\epsilon}v$ is wR_av and $wR_a^{\varphi,A}v$ is $R_a^{\varphi} \cap (W^A \times W^A)$. Both the semantics of Kripke models and of restricted Kripke models are exploited in order to obtain a sequent system (**G3PAL**) for PAL, following the method of Negri (2005). From the definition of forcing restricted to a list of announcements we obtain the following set of rules. Recall that sequents with atoms P as active formulas are initial and v does not appear in the conclusion of $R\mathcal{K}_a :^{\varphi}$.

Initial sequent

$$w : P, \Gamma \Rightarrow \Delta, w : P$$

Rules

$$\frac{}{w :^{\varphi} \bot, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\bot:^{\varphi}}$$

$$\frac{w :^{\varphi} A, w :^{\varphi} P, \Gamma \Rightarrow \Delta}{w :^{\varphi,A} P, \Gamma \Rightarrow \Delta}\ {\scriptstyle LAt:^{\varphi,A}} \qquad \frac{\Gamma \Rightarrow \Delta, w :^{\varphi} A \quad \Gamma \Rightarrow \Delta, w :^{\varphi} P}{\Gamma \Rightarrow \Delta, w :^{\varphi,A} P}\ {\scriptstyle RAt:^{\varphi,A}}$$

$$\frac{w :^{\varphi} B, w :^{\varphi} C, \Gamma \Rightarrow \Delta}{w :^{\varphi} B \,\&\, C, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\&:^{\varphi}} \qquad \frac{\Gamma \Rightarrow \Delta, w :^{\varphi} B \quad \Gamma \Rightarrow \Delta, w :^{\varphi} C}{\Gamma \Rightarrow \Delta, w :^{\varphi} B \,\&\, C}\ {\scriptstyle R\&:^{\varphi}}$$

$$\frac{w :^{\varphi} B, \Gamma \Rightarrow \Delta \quad w :^{\varphi} C, \Gamma \Rightarrow \Delta}{w :^{\varphi} B \vee C, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\vee:^{\varphi}} \qquad \frac{\Gamma \Rightarrow \Delta, w :^{\varphi} B, w :^{\varphi} C}{\Gamma \Rightarrow \Delta, w :^{\varphi} B \vee C}\ {\scriptstyle R\vee:^{\varphi}}$$

$$\frac{\Gamma \Rightarrow \Delta, w :^{\varphi} B \quad w :^{\varphi} C, \Gamma \Rightarrow \Delta}{w :^{\varphi} B \supset C, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\supset:^{\varphi}} \qquad \frac{w :^{\varphi} B, \Gamma \Rightarrow \Delta, w :^{\varphi} C}{\Gamma \Rightarrow \Delta, w :^{\varphi} B \supset C}\ {\scriptstyle R\supset:^{\varphi}}$$

$$\frac{v :^{\varphi} B, w :^{\varphi} \mathcal{K}_a B, wR_a^{\varphi}v, \Gamma \Rightarrow \Delta}{w :^{\varphi} \mathcal{K}_a B, wR_a^{\varphi}v, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathcal{K}_a:^{\varphi}} \qquad \frac{wR_a^{\varphi}v, \Gamma \Rightarrow \Delta, v :^{\varphi} B}{\Gamma \Rightarrow \Delta, w :^{\varphi} \mathcal{K}_a B}\ {\scriptstyle R\mathcal{K}_a:^{\varphi}}$$

$$\frac{w :^{\varphi,B} C, w :^{\varphi} [B]C, w :^{\varphi} B, \Gamma \Rightarrow \Delta}{w :^{\varphi} [B]C, w :^{\varphi} B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L[\,]:^{\varphi}} \qquad \frac{w :^{\varphi} B, \Gamma \Rightarrow \Delta, w :^{\varphi,B} C}{\Gamma \Rightarrow \Delta, w :^{\varphi} [B]C}\ {\scriptstyle R[\,]:^{\varphi}}$$

Table 1

These rules do not give a complete system for PAL. At some point of the derivation of the so-called *announcements composition axiom*

$$[A][B]C \supset\subset [A \,\&\, [A]B]C$$

we need two more rules. Intuitively, the axiom states that the sequence of two announcements $[A][B]$ can be reduced to the single announcement that A holds and after that A is announced, B holds. Observe that no rule of the above set can derive formulas of type $w \Vdash^{A\,\&\,[A]B} C$. To find the appropriate rules to be added we first establish the equivalence between $w \Vdash^{A\,\&\,[A]B} C$ and $w \Vdash^{A,B} C$. The same result can be found in van Ditmarsch et al. (2007).

Lemma 3.2. *For all Kripke models M, $W^{A\,\&\,[A]B} = W^{A,B}$*

Proof. From Definition 3.1 we have $W^{A\,\&\,[A]B} = \{w \in W \mid w \Vdash A\,\&\,[A]B\}$ and $W^{A,B} = \{w \in W^A \mid w \Vdash^A B\}$. We prove that for an arbitrary world w, $w \in W^{A\,\&\,[A]B}$ if and only if $w \in W^{A,B}$. By 3.1, we have that $w \in W^{A\,\&\,[A]B}$ if and only if $w \Vdash A\,\&\,[A]B$; by the semantics of $\&$ and $[\,]$, this is equivalent to $w \Vdash A$ and $w \Vdash A$ *implies* $w \Vdash^A B$; in turn, this is classically equivalent to $w \Vdash A$ and $w \Vdash^A B$. The first conjunct gives $w \in W^A$ and by 3.1 $w \in W^A$ and $w \Vdash^A B$ if and only if $w \in W^{A,B}$.

$\square$

Lemma 3.2 justifies the following rules and the definition below:

$$\frac{w :^{\varphi,A,B} C, \Gamma \Rightarrow \Delta}{w :^{\varphi,A\,\&\,[A]B} C, \Gamma \Rightarrow \Delta}\ L_{cmp} \qquad \frac{\Gamma \Rightarrow \Delta, w :^{\varphi,A,B} C}{\Gamma \Rightarrow \Delta, w :^{\varphi,A\,\&\,[A]B} C}\ R_{cmp}$$

Definition 3.3. G3PAL *is the sequent system given in* Table 1 *augmented with the two rules of composition L_{cmp} and R_{cmp}.*

4 Admissibility of the structural rules

In this section we prove that the structural rules of weakening and contraction are hp-admissible and all the logical rules of **G3PAL** hp-invertible. Moreover, the rule of cut is proved admissible. For this purpose, we need a definition and a preliminary result about the substitution of labels.

Definition 4.1. *The substitution of labels in relational atoms and labelled formulas is defined as follows:*

$$\begin{array}{lcl}
wR^{\varphi}v[u/z] & \equiv & wR^{\varphi}v \text{ if } z \neq w \text{ and } z \neq v \\
wR^{\varphi}v[u/w] & \equiv & uR^{\varphi}v \text{ if } w \neq v \\
wR^{\varphi}v[u/v] & \equiv & wR^{\varphi}u \text{ if } w \neq v \\
wR^{\varphi}w[u/w] & \equiv & uR^{\varphi}u \\
w :^{\varphi} A[u/v] & \equiv & w :^{\varphi} A \text{ if } v \neq w \\
w :^{\varphi} A[u/w] & \equiv & u :^{\varphi} A
\end{array}$$

The definition is extended to multisets component-wise. We have

Lemma 4.2. *The rule of substitution of labels*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma[v/w] \Rightarrow \Delta[v/w]}\ {\scriptstyle Subst}$$

is hp-admissible in **G3PAL**.

Proof. By induction on the height h of the derivation of the premiss as in Negri (2005). If $h = 0$ then the premiss is initial or an istance of $L\bot :^{\varphi}$ and also the conclusion is initial or conclusion of $L\bot :^{\varphi}$. If $h = n+1$ suppose the claim holds for derivations of height n and consider the last rule applied in the derivation. If the last rule is a propositional rule or a modal rule without variable conditions, apply the inductive hypothesis to the premisses and then the rule. If the last rule is a rule with a variable condition as $R\mathcal{K}_a$, we must be careful with the the cases in which either w or v is the eigenvariable of the rule, because a straightforward substitution may result in a violation of the restriction. In those cases we must apply the inductive hypothesis to the premiss and replace the eigenvariable with a fresh variable that does not appear in the derivation. The details are omitted here but similar cases are considered in Lemma 4.3 of Negri (2005). □

We have limited initial sequents to atoms, but obviously those with arbitrary formulas should be derivable. The following lemma is necessary to prove the correspondence between **G3PAL** and its equivalent Hilbert-style system and will be used in the proof of the completeness theorem. The straightforward definition of formula length has to be extended as follows:

Definition 4.3. *The length ℓ of a formula B is defined by induction of the structure of B as follows: $\ell(\bot) \equiv 0$; $\ell(P) \equiv 1$; $\ell(B \circ C) \equiv \ell(B) + \ell(C) + 1$, for $\circ$ a propositional connectives; $\ell(\mathcal{K}_a B) \equiv \ell(B) + 1$; $\ell([B]C) \equiv \ell(B) + \ell(C) + 1$. For labelled formulas $\ell(w : B) \equiv \ell(B)$ and if φ is a list of formulas $A_1, \ldots, A_n$ then $\ell(w :^{\varphi,A} B) \equiv \ell(A_1) + \cdots + \ell(A_n) + \ell(A) + \ell(B)$.*

Now we can establish the following

Lemma 4.4. *All sequents of the form $w :^{\varphi} B, \Gamma \Rightarrow \Delta, w :^{\varphi} B$ are derivable in* **G3PAL**.

Proof. By induction on the length of φ with subinduction on B. If B is atomic and φ has length zero, we have an initial sequent. If B is an atomic formula P and φ is not empty, rules $LAt :^{\varphi,A}$ and $RAt :^{\varphi,A}$ are used to reduce the length of the list of announcements and the inductive hypothesis is applied. If B is $\bot$ then $w :^{\varphi} \bot, \Gamma \Rightarrow \Delta, w :^{\varphi} \bot$ is an istance of $L\bot :^{\varphi}$. If B is a propositional, epistemic or announcement formula apply root-first the appropriate rules and observe that similar sequents, of reduced length, appear in the premisses; then the claim holds by the inductive hypothesis. For istance, suppose that B is an atom P consider the following derivation

$$\cfrac{\cfrac{w :^{\varphi} A, w :^{\varphi} P, \Gamma \Rightarrow \Delta, w :^{\varphi} A \quad w :^{\varphi} A, w :^{\varphi} P, \Gamma \Rightarrow \Delta, w :^{\varphi} P}{w :^{\varphi} A, w :^{\varphi} P, \Gamma \Rightarrow \Delta, w :^{\varphi,A} P}\, RAt :^{\varphi,A}}{w :^{\varphi,A} P, \Gamma \Rightarrow \Delta, w :^{\varphi,A} P}\, LAt :^{\varphi,A}$$

The right topmost is initial and the left one is derivable by the inductive hypothesis because $\ell(w :^{\varphi,A} P) > \ell(w :^{\varphi} A)$.

□

Theorem 4.5. *The rules of weakening*

$$\frac{\Gamma \Rightarrow \Delta}{w :^{\varphi} B, \Gamma \Rightarrow \Delta}\, Wk \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w :^{\varphi} B}\, Wk$$

are hp-admissible in **G3PAL**.

Proof. The proof is by induction on the height of the derivation of the premiss. The cases with propositional rules and announcement rules and the modal rules without variable conditions are straightforward. As in Negri (2005), if the last step is a rule with a variable condition, we need to apply Lemma 4.2 to the premisses of the rule in order to avoid a clash with the variables in $w :^{\varphi} B$. The conclusion is then obtained by applying the inductive hypothesis and the rule in question.

□

One of the most useful properties of **G3**-systems is the hp-invertibility of all the logical rules. The result holds also in **G3PAL** and it is necessary for admissibility of contraction.

Lemma 4.6. *All the rules of* **G3PAL** *are hp-invertible.*

Proof. By induction on the height h of the derivation. The proof of the cases corresponding to the rule for $\&$, $\vee$ and $\supset$ is similar to the proof of Theorem 3.11 of Negri and von Plato (2001). Rules $L[\,] :^{\varphi}$ and $L\mathcal{K}_a :^{\varphi}$ are hp-invertible, since their premisses are obtained by Theorem 4.5 from the conclusion. The proof of the lemma for $R\mathcal{K}_a :^{\varphi}$ needs some care for the variable condition. If $\Gamma \Rightarrow \Delta, w :^{\varphi} \mathcal{K}_a B$ is an initial sequent or an istance of $L\bot :^{\varphi}$ then $w :^{\varphi} \mathcal{K}_a B$ is not principal and also $wR^{\varphi}v, \Gamma \Rightarrow \Delta, v :^{\varphi} B$ is initial or an istance of $L\bot :^{\varphi}$. If $\Gamma \Rightarrow \Delta, w :^{\varphi} \mathcal{K}_a B$ is concluded by a derivation of height $h > 0$ we have to consider the rule that introduced it. If $w :^{\varphi} \mathcal{K}_a B$ is principal then the rule is $R\mathcal{K}_a :^{\varphi}$ and the premiss $wR^{\varphi}_a v, \Gamma \Rightarrow \Delta, v :^{\varphi} B$ has a derivation of a lower height. If $w :^{\varphi} \mathcal{K}_a B$ is not principal and it has been introduced by a rule without variable condition, apply the inductive hypothesis and then the rule.If $\Gamma \Rightarrow \Delta, w :^{\varphi} \mathcal{K}_a B$ is a conclusion of $R\mathcal{K}_a :^{\varphi}$ we apply first Lemma 4.2 in order to avoid clash of variables and then the inductive hypothesis and $R\mathcal{K}_a :^{\varphi}$ again. The last step is

$$\frac{wR_a^{\varphi}v, \Gamma \Rightarrow \Delta, w :^{\varphi} \mathcal{K}_a B, v :^{\varphi} C}{\Gamma \Rightarrow \Delta, w :^{\varphi} \mathcal{K}_a B, w :^{\varphi} \mathcal{K}_a C} \; {\scriptstyle R\mathcal{K}_a :^{\varphi}}$$

By Lemma 4.2 applied to the premiss we have $wR_a^{\varphi}z, \Gamma \Rightarrow \Delta, w :^{\varphi} \mathcal{K}_a B, z :^{\varphi} C$. By the inductive hypothesis and an application of $R\mathcal{K}_a :^{\varphi}$ we can conclude the sequent $wR_a^{\varphi}v, \Gamma \Rightarrow \Delta, v :^{\varphi} B, w :^{\varphi,A} \mathcal{K}_a C$. The hp-inversion of $R[] :^{\varphi}$, $LAt :^{\varphi,A}$, $RAt :^{\varphi,A}$, L_{cmp} and R_{cmp} is done exactly as for the propositional cases. Apply the inductive hypothesis on the premiss(es) of the last rule applied and then the rule. □

Now we can prove admissibility of contraction. It plays a central role in the proof the the main result of cut elimination and it is useful also for the practical reason that it guarantees that in the proof search possible steps which produce a duplication formulas can be ignored: they have the same effect of a contraction, and by height-preserving admissibility of contraction the conclusion of the rule can be obtained in one step less from its premiss. The possible applicable rule can thus be discarded if we reasonably assume that the derivation we are looking for is a minimal one, i.e. one that does not admit any local shortening through the elimination of contraction steps.

Theorem 4.7. *The rules of contraction*

$$\frac{w :^{\varphi} B, w :^{\varphi} B, \Gamma \Rightarrow \Delta}{w :^{\varphi} B, \Gamma \Rightarrow \Delta} \; {\scriptstyle Ctr} \qquad \frac{\Gamma \Rightarrow \Delta, w :^{\varphi} B, w :^{\varphi} B}{\Gamma \Rightarrow \Delta, w :^{\varphi} B} \; {\scriptstyle Ctr}$$

are hp-admissible in **G3PAL***.*

Proof. By simultaneous induction on the height h of the derivation for left and right contraction. If the height is 0 $w :^{\varphi} B, w :^{\varphi} B, \Gamma \Rightarrow \Delta$ (respectively, $\Gamma \Rightarrow \Delta, w :^{\varphi} B, w :^{\varphi} B$) is initial or istance of $L\bot :^{\varphi}$ and also the conclusion $w :^{\varphi} B, \Gamma \Rightarrow \Delta$ (respectively, $\Gamma \Rightarrow \Delta, w :^{\varphi} B$) is initial or istance of $L\bot :^{\varphi}$. For the inductive step, two cases are distinguished: The case with none of the contraction formulas principal in the last rule, and the case with one principal. In the former, apply the inductive hypothesis to the premiss of the rule, then the rule. In the latter, apply the matching height-preserving inversion to the premiss(es) of the rule, the inductive hypothesis, and the rule. For istance, if the last step is an application of $R[\,] :^{\varphi}$ we have

$$\frac{w :^{\varphi} B, \Gamma \Rightarrow \Delta, w :^{\varphi} [B]C, w :^{\varphi,B} C}{\Gamma \Rightarrow \Delta, w :^{\varphi} [B]C, w :^{\varphi} [B]C} \; {\scriptstyle R[\,]:^{\varphi}}$$

By Lemma 4.6 we derive $w :^{\varphi} B, w :^{\varphi} B, \Gamma \Rightarrow \Delta, w :^{\varphi,B} C, w :^{\varphi,B} C$ and by the inductive hypothesis on left and right contraction simultaneously we have also $w :^{\varphi} B, \Gamma \Rightarrow \Delta, w :^{\varphi,B} C$. Now with an application of $R[\,] :^{\varphi}$ we conclude $\Gamma \Rightarrow \Delta, w :^{\varphi} [B]C$. □

We are now in a position to prove the most important result concerning proof analysis for **G3PAL**, namely the admissibility of cut. Admissibility of cut is crucial for delimiting the space of proof search, because it guarantees that no arbitrary new formulas need to be constructed during the search.

Theorem 4.8. *The rule of cut*

$$\dfrac{\Gamma \Rightarrow \Delta w :^{\varphi} B \quad w :^{\varphi} B, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\ {\scriptstyle Cut}$$

is admissible in **G3PAL**.

Proof. The proof has the same structure as the proof of admissibility of cut for the modal systems **G3K** of Negri (2005). We recall that the proof is by induction on the structure of the cut formula with sub-induction on the sum of the heights of the derivations of the premisses of cut. The proof is to a large extent similar to the cut-elimination proofs in Negri and von Plato (2001) (e.g. Theorem 3.2.3) so we shall consider in detail only the case in which the cut formula is either $w :^{\varphi,A} P$ or $w :^{\varphi} [B]C$ and it is principal in both premisses. With $Ctr^{\star}$ we denote multiple applications of contraction.

If the cut formula is an atom P, the derivation is of the form

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, w :^{\varphi} A \quad \Gamma \Rightarrow \Delta, w :^{\varphi} P}{\Gamma \Rightarrow \Delta, w :^{\varphi,A} P}\ {\scriptstyle RAt:^{\varphi,A}} \quad \dfrac{w :^{\varphi} A, w :^{\varphi} P, \Gamma' \Rightarrow \Delta'}{w :^{\varphi,A} P, \Gamma' \Rightarrow \Delta'}\ {\scriptstyle LAt:^{\varphi,A}}}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\ {\scriptstyle Cut}$$

and it can be converted into a derivation with two cuts on smaller formulas

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, w :^{\varphi} P \quad \dfrac{\Gamma \Rightarrow \Delta, w :^{\varphi} A \quad w :^{\varphi} A, w :^{\varphi} P, \Gamma' \Rightarrow \Delta'}{w :^{\varphi} P, \Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ {\scriptstyle Cut}}{\Gamma, \Gamma, \Gamma' \Rightarrow \Delta', \Delta, \Delta}\ {\scriptstyle Cut}}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\ {\scriptstyle Ctr^{\star}}$$

If the cut formula is an announcement, then $\Gamma' \equiv w :^{\varphi} B, \Gamma''$ and the derivation is

$$\dfrac{\dfrac{w :^{\varphi} B, \Gamma \Rightarrow \Delta, w :^{\varphi,B} C}{\Gamma \Rightarrow \Delta, w :^{\varphi} [B]C}\ {\scriptstyle R[\,]:^{\varphi}} \quad \dfrac{w :^{\varphi,B} C, w :^{\varphi} [B]C, w :^{\varphi} B, \Gamma'' \Rightarrow \Delta'}{w :^{\varphi} [B]C, w :^{\varphi} B, \Gamma'' \Rightarrow \Delta'}\ {\scriptstyle L[\,]:^{\varphi}}}{w :^{\varphi} B, \Gamma'', \Gamma \Rightarrow \Delta, \Delta'}\ {\scriptstyle Cut}$$

We first apply cut of reduced height to the conclusion of $R[\,] :^{\varphi}$ and the premiss of $L[\,] :^{\varphi}$ as follows

$$\dfrac{\dfrac{w :^{\varphi} B, \Gamma \Rightarrow \Delta, w :^{\varphi,B} C}{\Gamma \Rightarrow \Delta, w :^{\varphi} [B]C}\ R[\,]{:}^{\varphi} \quad w :^{\varphi,B} C, w :^{\varphi} [B]C, w :^{\varphi} B, \Gamma'' \Rightarrow \Delta'}{w :^{\varphi,B} C, w :^{\varphi} B, \Gamma'', \Gamma \Rightarrow \Delta, \Delta'}\ \mathit{Cut}$$

Successively, we apply a cut with smaller cut formula to the conclusion of the first cut and the premiss of $R[\,] :^{\varphi}$

$$\dfrac{\dfrac{w :^{\varphi} B, \Gamma \Rightarrow \Delta, w :^{\varphi,B} C \quad w :^{\varphi,B} C, w :^{\varphi} B, \Gamma'', \Gamma \Rightarrow \Delta, \Delta'}{w :^{\varphi} B, \Gamma'', \Gamma, \Gamma \Rightarrow \Delta, \Delta, \Delta'}\ \mathit{Cut}}{w :^{\varphi} B, \Gamma'', \Gamma \Rightarrow \Delta, \Delta'}\ \mathit{Ctr}^{\star}$$

□

All the rules **G3PAL** derive directly from the meaning explanation of the connectives and the modal operators in terms of Kripke models. The rules of composition of announcements (L_{cmp} and R_{cmp}) have been obtained from a result concerning restricted Kripke models (Lemma 3.1). We prove here two results about the property of composition for announcements which will be useful in what follows. By Lemma 4.4 we have:

Lemma 4.9. *All sequents of the form*

$$w : A, w :^{A} B, \Gamma \Rightarrow \Delta, w : A \,\&\, [A]B$$

are derivable.

Corollary 4.10. *The rule*

$$\dfrac{w : A \,\&\, [A]B, \Gamma \Rightarrow \Delta}{w : A, w :^{A} B, \Gamma \Rightarrow \Delta}\ \star$$

is admissible.

5 Completeness

In the table below, we recall from van Ditmarsch et al. (2007) the standard Hilbert-style system for Public Announcement Logic (**PAL**). We shall prove that all **PAL** axioms are derivable and all **PAL** rules are admissible in **G3PAL**. Given that **PAL** is complete (cf. Plaza 1989 and van Ditmarsch et al. 2007) and that the rules of **G3PAL** are sound with respect to the Kripke semantics, the derivability and admissibility in **G3PAL** of **PAL** axioms and rules can be considered an indirect proof of the completeness theorem for **G3PAL**. The results of this section also exemplify how **G3PAL** is used for making proofs in PAL. The admissibility results of the previous section allow a proof-search procedure for **G3PAL** derivations, that is, permit to construct a derivation starting from the conclusion: the end-sequent is analysed in order to determine a last possible rule of inference and thus its premiss(es). The procedure is iterated until a node at which no rule can be applied is reached: If every leaf is an initial sequent or a conclusion of $L\bot :^{\varphi}$, we obtain a derivation. Otherwise, the procedure fails if at least one of the leaves is not an initial sequent or a conclusion $L\bot :^{\varphi}$, or if the proof search does not stop.

A1	All theorems of classical propositional logic	
A2	$\mathcal{K}_a(A \supset B) \supset (\mathcal{K}_a A \supset \mathcal{K}_a B)$	Distribution
A3	$\mathcal{K}_a A \supset A$	Veridical Knowledge
A4	$\mathcal{K}_a A \supset \mathcal{K}_a \mathcal{K}_a A$	Positive Introspection
A5	$\neg\mathcal{K}_a A \supset \mathcal{K}_a \neg\mathcal{K}_a A$	Negative Introspection
A6	$[A]P \supset\subset (A \supset P)$	Atomic Permanence
A7	$[A]\neg B \supset\subset (A \supset \neg[A]B)$	Announcements and Negation
A8	$[A](B \,\&\, C) \supset\subset ([A]B \,\&\, [A]C)$	Announcements and Conjunction
A9	$[A]\mathcal{K}_a B \supset\subset (A \supset \mathcal{K}_a[A]B)$	Announcements and Knowledge
A10	$[A][B]C \supset\subset [A \,\&\, [A]B]C$	Announcements Composition
R1	From $\Gamma \vdash A \supset B$ and $\Delta \vdash A$ infer $\Gamma, \Delta \vdash B$	Modus Ponens
R2	From $\vdash A$ infer $\vdash \mathcal{K}_a A$	Necessitation

Axiom A7 corresponds to the property that announcements are partial functions and A8 is known also as Ramsey's axiom. Note the formulation of the necessitation rule, with an empty set of assumptions: This restriction is required to obtain the validity of the deduction theorem in an Hilbert system. It is crucial to have the corresponding rule admissible rather than primitive in the sequent system: The addition of a context-dependent rule such as R2 to **G3PAL** would impair modularity the system and thus complicate the proof of its structural properties. For a detailed on the deduction theorem in modal logic see Hakli and Negri (2009).

All the axioms of **PAL** find a counterpart in **G3PAL**. Axioms A1-A5 are derivable in **G3K** of Negri (2005) with mathematical rules for reflexivity, transitivity and symmetry. For the derivability of the other **PAL** axioms consider the rules for restricted forcing where $\varphi = \epsilon$. All the axioms are derivable by a proof-search from the conclusion. The derivations of axioms A6, A7 and A8 are similar to those presented in the following lemmas:

Lemma 5.1. *The sequent* $\Rightarrow w : [A]\mathcal{K}_a B \supset\subset (A \supset \mathcal{K}_a[A]B)$ *is derivable in* **G3PAL**.

Proof. We have two derivations, one for each direction of the equivalence.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v :^A B, w :^A \mathcal{K}_a B, v : A, wR_a v, w : [A]\mathcal{K}_a B, w : A \Rightarrow v :^A B}{w :^A \mathcal{K}_a B, v : A, wR_a v, w : [A]\mathcal{K}_a B, w : A \Rightarrow v :^A B}\;{\scriptstyle L\mathcal{K}_a :^A}}{v : A, wR_a v, w : [A]\mathcal{K}_a B, w : A \Rightarrow v :^A B}\;{\scriptstyle L[\,]}}{wR_a v, w : [A]\mathcal{K}_a B, w : A \Rightarrow v : [A]B}\;{\scriptstyle R[\,]}}{w : [A]\mathcal{K}_a B, w : A \Rightarrow w : \mathcal{K}_a[A]B}\;{\scriptstyle R\mathcal{K}_a}}{w : [A]\mathcal{K}_a B \Rightarrow w : A \supset \mathcal{K}_a[A]B}\;{\scriptstyle R\supset}}{\Rightarrow w : [A]\mathcal{K}_a B \supset (A \supset \mathcal{K}_a[A]B)}\;{\scriptstyle R\supset}
$$

$$
\dfrac{\dfrac{\dfrac{w : A \Rightarrow w :^A \mathcal{K}_a B, w : A \qquad \dfrac{\dfrac{\dfrac{v :^A B, v : [A]B, wR_a v, v : A, w : A, w : \mathcal{K}_a[A]B \Rightarrow v :^A B}{v : [A]B, wR_a v, v : A, w : A, w : \mathcal{K}_a[A]B \Rightarrow v :^A B}\;{\scriptstyle L[\,]}}{wR_a v, v : A, w : A, w : \mathcal{K}_a[A]B \Rightarrow v :^A B}\;{\scriptstyle L\mathcal{K}_a}}{w : A, w : \mathcal{K}_a[A]B \Rightarrow w :^A \mathcal{K}_a B}\;{\scriptstyle R\mathcal{K}_a :^A}}{w : A, w : A \supset \mathcal{K}_a[A]B \Rightarrow w :^A \mathcal{K}_a B}\;{\scriptstyle L\supset}}{w : A \supset \mathcal{K}_a[A]B \Rightarrow w : [A]\mathcal{K}_a B}\;{\scriptstyle R[\,]}}{\Rightarrow w : (A \supset \mathcal{K}_a[A]B) \supset [A]\mathcal{K}_a B}\;{\scriptstyle R\supset}
$$

where the uppermost sequents are derivable by Lemma 4.4. □

Lemma 5.2. *The sequent* $\Rightarrow w : [A][B]C \supset\subset [A \,\&\, [A]B]C$ *is derivable in* **G3PAL**.

Proof. The derivations are as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{w :^{A,B} C, w :^A [B]C, w :^A B, w : A, w : [A]B, w : [A][B]C \Rightarrow w :^{A,B} C}{w :^A [B]C, w :^A B, w : A, w : [A]B, w : [A][B]C \Rightarrow w :^{A,B} C}\;{\scriptstyle L[\,]:^A}}{w :^A B, w : A, w : [A]B, w : [A][B]C \Rightarrow w :^{A,B} C}\;{\scriptstyle L[\,]}}{w : A, w : [A]B, w : [A][B]C \Rightarrow w :^{A,B} C}\;{\scriptstyle L[\,]}}{w : A \,\&\, [A]B, w : [A][B]C \Rightarrow w :^{A,B} C}\;{\scriptstyle L\&}}{w : A \,\&\, [A]B, w : [A][B]C \Rightarrow w :^{A \,\&\, [A]B} C}\;{\scriptstyle R_{cmp}}}{w : [A][B]C \Rightarrow w : [A \,\&\, [A]B]C}\;{\scriptstyle R[\,]}}{\Rightarrow w : [A][B]C \supset [A \,\&\, [A]B]C}\;{\scriptstyle R\supset}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{w :^{A,B} C, w : A \,\&\, [A]B, w : [A \,\&\, [A]B]C \Rightarrow w :^{A,B} C}{w :^{A \,\&\, [A]B} C, w : A \,\&\, [A]B, w : [A \,\&\, [A]B]C \Rightarrow w :^{A,B} C}\, L_{cmp}}{w : A \,\&\, [A]B, w : [A \,\&\, [A]B]C \Rightarrow w :^{A,B} C}\, L[\,]}{w :^{A} B, w : A, w : [A \,\&\, [A]B]C \Rightarrow w :^{A,B} C}\, \star}{w : A, w : [A \,\&\, [A]B]C \Rightarrow w :^{A} [B]C}\, R[\,]{:}^{A}}{w : [A \,\&\, [A]B]C \Rightarrow w : [A][B]C}\, R[\,]}{\Rightarrow w : [A \,\&\, [A]B]C \supset [A][B]C}\, R{\supset}$$

where the uppermost sequents are derivable by Lemma 4.4. □

Admissibility of R1 and R2 is established by the following

Lemma 5.3. *The rule of* modus ponens

$$\frac{\Gamma \Rightarrow w : A \supset B \quad \Delta \Rightarrow w : A}{\Gamma, \Delta \Rightarrow w : B}$$

is admissible in **G3PAL**.

Proof. By invertibility of $R\supset$ (Lemma 4.6) applied to the left premiss we have $w : A, \Gamma \Rightarrow w : B$ and we obtain the conclusion by an admissible cut (Theorem 4.8) with the right premiss. □

Lemma 5.4. *The rule of* necessitation

$$\frac{\Rightarrow w : A}{\Rightarrow w : \mathcal{K}_a A}$$

is admissible in **G3PAL**.

Proof. Suppose we have a derivation of $\Rightarrow w : A$. By Lemma 4.2, we obtain $\Rightarrow v : A$ and by Lemma 4.5 $wR_av \Rightarrow v : A$. Now, by $R\mathcal{K}_a$, we conclude $\Rightarrow w : \mathcal{K}_a A$. □

It is possible to give also a direct completeness proof for **G3PAL** following the pattern of Theorem 6.4 of Negri (2009). The idea pursued by labelled sequent system is the same as in Kripke's original proof (see Kripke 1963), but instead of looking for a failed search of a countermodel, we look directly for a proof: To see whether a formula is derivable, we check if it is universally valid, that is, valid at an arbitrary world for an arbitrary valuation, $w \Vdash A$. This is translated to a sequent $\Rightarrow w : A$. The rules of **G3PAL** applied backwards give equivalent conditions until the atomic components of A are reached. It can happen that we find a proof, or that we find that a proof does not exist either because we reach a stage where no rule is applicable, or because we go on with the search forever. In the two latter cases the attempt proof itself gives a countermodel.

Theorem 5.5. *For all* $\Gamma \Rightarrow \Delta$ *in* **G3PAL** *either* $\Gamma \Rightarrow \Delta$ *is derivable or it has a countermodel.*

Proof. We define for an arbitrary $\Gamma \Rightarrow \Delta$ of **G3PAL** a reduction tree by applying the rules of **G3PAL** root first in all possible ways. If the construction terminates we obtain a proof, else the tree becomes infinite. By König's lemma an infinite tree has an infinite branch that is used to define a countermodel to the end-sequent.

Construction of the reduction tree

The reduction tree is defined inductively in stages as follows: Stage 0 has $\Gamma \Rightarrow \Delta$ at the root of the tree. Stage $n > 0$ has two cases:

CASE I: If every topmost sequent is initial or a conclusion of $L\bot :^{\varphi}$ the construction of the tree ends.

CASE II: If not every topmost sequent is initial or a conclusion of $L\bot :^{\varphi}$, we continue the construction of the tree by writing above those sequents that are not initial nor a conclusion of $L\bot :^{\varphi}$, other sequents that are obtained by applying root first the rules of **G3PAL** whenever possible, in a give order.

There are 14 different stages, 8 for the rules At, $\&$, $\vee$, $\supset$, 2 for the epistemic rules for each $\mathcal{K}_a$, 2 for the announcement rules, and 2 for the composition of announcements. At stage $n = 15$ we repeat stage 1, at stage $n = 16$ we repeat stage 2, and so on for every n. Note that, by Definition 3.1, we do not assume that all R_a^{φ} are equivalence relations and then here we can leave out the mathematical rules for reflexivity, transitivity and symmetry of R_a^{φ}. We will not take into account the details of the proof when the topmost sequents have either a conjunction, or a disjunction, or an implication, or else an epistemic formula as principal formula, the proof being similar to the proof given in Negri (2009).

We start, for $n = 1$, with $LAt :^{\varphi,A}$. For each topmost sequent of the form

$$w_1 :^{\varphi,A} P_1, \ldots, w_m :^{\varphi,A} P_m\Gamma' \Rightarrow \Delta$$

where $P_1, \ldots P_m$ are all the formulas in Γ with an atom as the principal formula, we write

$$w_1 :^{\varphi} A, w_1 :^{\varphi} P_1, \ldots, w_m :^{\varphi} A, w_m :^{\varphi} P_m\Gamma' \Rightarrow \Delta$$

on top of it. This corresponds to applying m times rule $LAt :^{\varphi,A}$.

For $n = 2$, with $RAt :^{\varphi,A}$. For each topmost sequent of the form

$$\Gamma \Rightarrow \Delta', w_1 :^{\varphi,A} P_1, \ldots, w_m :^{\varphi,A} P_m$$

where $P_1, \ldots P_m$ are all the formulas in Δ with an atom as the principal formula, we write

$$\Gamma \Rightarrow \Delta', w_1 :^{\varphi} D_1, \ldots, w_m :^{\varphi} D_m$$

on top of it, where D_i is either A or P_i and all the choices are taken. This corresponds to applying 2^m times rule $RAt :^{\varphi,A}$. For the stages from $n = 3$ to $n = 10$, corresponding to propositional and epistemic cases, the proof is analogous to Negri (2009).

For $n = 11$, take all the topmost sequents with $w_1 :^{\varphi} [B_1]C_1, \ldots, w_m :^{\varphi} [B_m]C_m$ and $w_1 :^{\varphi} B_1, \ldots, w_m :^{\varphi} B_m$ in the antecedent, and write on top of these sequents

$$w_1 :^{\varphi,B_1} C_1, \ldots, w_m :^{\varphi,B_m} C_m, w_1 :^{\varphi} [B_1]C_1, \ldots, w_m :^{\varphi} [B_m]C_m, \ldots,$$
$$w_1 :^{\varphi} B_1, \ldots, w_m :^{\varphi} B_m, \Gamma' \Rightarrow \Delta$$

that is, apply m times rule $L[\,] :^{\varphi}$.

For $n = 12$, take all the topmost sequents with $w_1 :^{\varphi} [B_1]C_1, \ldots, w_m :^{\varphi} [B_m]C_m$ in the succedent, and write on top of these sequents

$$w_1 :^{\varphi} B_1, \ldots, w_m :^{\varphi} B_m\Gamma \Rightarrow \Delta', w_1 :^{\varphi,B_1} C_1, \ldots, w_m :^{\varphi,B_m} C_m$$

that is, apply m times rule $R[\,] :^{\varphi}$.

For $n = 13$, we consider all the topmost sequents with the multiset of formulas $w_1 :^{\varphi,A\,\&\,[A]B} C_1, \ldots, w_m :^{\varphi,A\,\&\,[A]B} C_m$ in the antecedent, and write on top of these sequents

$$w_1 :^{\varphi,A,B} C_1, \ldots, w_m :^{\varphi,A,B} C_m, \Gamma' \Rightarrow \Delta$$

that is, apply m times L_{cmp}.

Likewise, for $n = 14$, take all the topmost sequents with the multiset of formulas $w_1 :^{\varphi,A\,\&\,[A]B} C_1, \ldots, w_m :^{\varphi,A\,\&\,[A]B} C_m$ in the succedent, and write on top of these sequents

$$\Gamma \Rightarrow \Delta', w_1 :^{\varphi,A,B} C_1, \ldots, w_m :^{\varphi,A,B} C_m$$

that is, apply m times R_{cmp}.

For any n, for each sequent that is neither initial, nor conclusion of $L\bot :^{\varphi}$, nor treatable by any one of the above reductions, we write the sequent itself above it. If the reduction tree is finite, all its leaves are initial or conclusions of $L\bot :^{\varphi}$, and the tree, read from the leaves to the root, yields a derivation.

Construction of the countermodel

By König's lemma, if the reduction tree is infinite, it has an infinite branch. Let $\Gamma_0 \Rightarrow \Delta_0 \equiv \Gamma \Rightarrow \Delta, \Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_i \Rightarrow \Delta_i, \ldots$ be one such branch. Consider the set of labelled formulas and relational atoms

$$\mathbf{\Gamma} \equiv \bigcup_{i \geq 0} \Gamma_i \quad \text{and} \quad \mathbf{\Delta} \equiv \bigcup_{i \geq 0} \Delta_i$$

We define a restricted Kripke model that forces all formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$ and is therefore a countermodel to the sequent $\Gamma \Rightarrow \Delta$.

Consider the frame K the nodes of which are all the labels that appear in the relational atoms in $\mathbf{\Gamma}$, with their mutual relationship expressed by the $wR^{\varphi}v$'s in $\mathbf{\Gamma}$. The model is defined as follows: For all atomic formulas $w : P$ in $\mathbf{\Gamma}$, we stipulate that $w \Vdash P$ in the frame K, and for all atomic formulas $v : Q$ in $\mathbf{\Delta}$, we stipulate that $v \nVdash Q$ in K. Since no sequent in the infinite branch is initial, this choice can be coherently made, for if there were the same labelled atom in $\mathbf{\Gamma}$ and in $\mathbf{\Delta}$, then, since the sequents in the reduction tree are defined in a cumulative way, for some i there would be a labelled atom $w : P$ both in the antecedent and in the succedent of $\Gamma_i \Rightarrow \Delta_i$.

We then show inductively on the structure of formulas that B is forced in the model at node w if $w :^{\varphi} B$ is in $\mathbf{\Gamma}$ and B is not forced at node w if $w :^{\varphi} B$ is in $\mathbf{\Delta}$. Therefore we have a countermodel to the end-sequent $\Gamma \Rightarrow \Delta$.

If B is $\perp$, it cannot be in $\mathbf{\Gamma}$ because no sequent in the branch contains $w :^{\varphi} \perp$ in the antecedent, so it is not forced at any node of the model.

If $w :^{\varphi,A} P$, for P atomic, is in $\mathbf{\Gamma}$, there exists i such that $w :^{\varphi,A} P$ is in Γ_i, and therefore, for some $l \geq 0$, $w :^{\varphi} A$ and $w :^{\varphi} P$ is in Γ_{l+i}. By inductive hypothesis (IH), $w \Vdash^{\varphi} A$ and $w \Vdash^{\varphi} P$, and therefore $w \Vdash^{\varphi,A} P$.

If $w :^{\varphi,A} P$ is in $\mathbf{\Delta}$, consider the step i in which the reduction for $w :^{\varphi,A} P$ applies. This gives a branching, and one of the two branches belongs to the infinite branch, so either $w :^{\varphi} A$ or $w :^{\varphi} P$ is in $\mathbf{\Delta}$, and therefore by IH, $w \nVdash^{\varphi} A$ or $w \nVdash^{\varphi} A$, and therefore $w \nVdash^{\varphi,A} P$.

For the cases in which a propositional or modal formula is either in the antecedent or in the succedent, see Negri (2009).

If $w :^{\varphi} [B]C$ is in $\mathbf{\Gamma}$, we check whether any $w :^{\varphi} B$ is in $\mathbf{\Gamma}$. If there is no such $w :^{\varphi} B$ then the condition that $w :^{\varphi} B$ implies $w :^{\varphi,B} C$ is vacuously satisfied, and therefore $w \Vdash^{\varphi} [B]C$ in the model. Else, we find $w :^{\varphi} B$ in $\mathbf{\Gamma}$. By IH $w \Vdash^{\varphi} B$, and therefore $w \Vdash^{\varphi} [B]C$ in the model.

If $w :^{\varphi} [B]C$ is in $\mathbf{\Delta}$, consider the step at which the reduction for $w :^{\varphi} [B]C$ applies. We find $w :^{\varphi} B$ in $\mathbf{\Gamma}$ and $w :^{\varphi,B} C$ in $\mathbf{\Delta}$. By IH $w \Vdash^{\varphi} B$ and $w \nVdash^{\varphi,B} C$, and by Definition 3.1 $w \nVdash^{\varphi} [B]C$.

If $w :^{\varphi,A\,\&\,[A]B} C$ is in $\mathbf{\Gamma}$, for some i, $w :^{\varphi,A,B} C$ is in Γ_i. By IH $w \Vdash^{\varphi,A,B} C$ and by Lemma 3.2 we conclude $w \Vdash^{\varphi,A\,\&\,[A]B} C$.

If $w :^{\varphi,A\,\&\,[A]B} C$ is in $\mathbf{\Delta}$, for some i, $w :^{\varphi,A,B} C$ is in Δ_i. By IH $w \nVdash^{\varphi,A,B} C$ and by Lemma 3.2 we conclude $w \nVdash^{\varphi,A\,\&\,[A]B} C$. □

Corollary 5.6. *If a sequent $\Gamma \Rightarrow \Delta$ is valid in every restricted Kripke model then it is derivable in* **G3PAL**.

6 Conclusion and further work

In this work, we introduced a Gentzen system for the logic of public announcements and proved that all the structural properties are satisfied; moreover, we proved both indirectly, through equivalence with the axiomatic system, and directly, through the method of reduction trees, its completeness with respect to the semantics of restricted Kripke models. As we pointed out, **G3PAL** is not only a different formalism, alternative to the standard axiom systems: It is designed for making explicit the structure of proofs in PAL. In this sense, **G3PAL** is an application of the method developed in Negri (2005) for modal logic, and an extension of the results in Hakli and Negri (2008) in the field of (static) epistemic logic. The novelty here is that the rules of **G3PAL** incorporate the notion of model change and the dynamics of information update through the internalization of semantics of restricted forcing into the syntax of the calculus. The next step should be that of adding rules to deal with the common knowledge operator (cf. van Ditmarsch et al. 2007) in order to formalize sentences such as: "After it is announced that A, it is a common knowledge among the agents that A." However, the proof theory of the logic of common knowledge (with or without public announcements) is problematic and requires a rule with a infinite number of premisses; thus, the possibility of mechanizing proofs is definitely lost. The problem is due to the iterative interpretation of common knowledge like an infinite conjunction or, equivalently, to the presence of an accessibility relation defined as the (reflexive and) transitive closure of each R_a. The same question arises for other logics like **LTL** (Linear Time Logic) and the results of finitization given in Boretti and Negri (2007) should lead the further research in this direction.
A closely related approach is presented in Balbiani et al. (2010) in which a tableau system for PAL is given. From the point of view of sequent systems, a tableau proof can be regarded as a single-sided sequent calculus proof, with formulas only in the antecedent, that aims at a check for satisfiability, whereas a sequent proof in a labelled system is a check for validity. By the duality in a classical framework between the unsatisfiability of a formula and the validity of its negation, the two approaches are duals to each other. The tableau system of Balbiani et al. (2010) operates on labelled formulas and accessibility relations; It has labels that range over natural numbers, which would seem to impose a restriction to linear orders, whereas our system does not assume any underlying implicit structure on the set of labels, but imposes it with suitable properties of the explicit accessibility relation. A closed tableau corresponds to a proof in our system, whereas an open tableau gives a countermodel. Also Balbiani et al. present a direct completeness proof (very similar in spirit and general methodology to the one in Negri 2009).

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