The Church-Fitch knowability paradox in the light of structural proof theory

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Abstract

Anti-realist epistemic conceptions of truth imply what is called the knowability principle: All truths are possibly known. The principle can be formalized in a bimodal propositional logic, with an alethic modality $\Diamond$ and an epistemic modality $\mathcal{K}$, by the axiom scheme $A \supset \Diamond\mathcal{K}A$ (**KP**). The use of classical logic and minimal assumptions about the two modalities lead to the paradoxical conclusion that all truths are known, $A \supset \mathcal{K}A$ (**OP**). A Gentzen-style reconstruction of the Church-Fitch paradox is presented following a labelled approach to sequent calculi. First, a cut-free system for classical (resp. intuitionistic) bimodal logic is introduced as the logical basis for the Church-Fitch paradox and the relationships between $\mathcal{K}$ and $\Diamond$ are taken into account. Afterwards, by exploiting the structural properties of the system, in particular cut elimination, the semantic frame conditions that correspond to **KP** are determined and added in the form of a block of nonlogical inference rules. Within this new system for classical and intuitionistic 'knowability logic', it is possibile to give a satisfactory cut-free reconstruction of the Church-Fitch derivation and to confirm that **OP** is only classically derivable, but neither intuitionistically derivable nor intuitionistically admissible. Finally, it is shown that in classical knowability logic, the Church-Fitch derivation is nothing else but a fallacy and does not represent a real threat for anti-realism.

1 Introduction

According to the Dummettian tradition in the philosophy of language, the realism/anti-realism debate can be characterized in terms of the notion of truth involved. Realism takes the notion of truth either as primitive or as defined over the notion of 'fact', whereas anti-realism embraces an *epistemic* conception of truth. One possible version of this epistemic conception is the following:

(1E) A is true if and only if it is possible to exhibit a direct justification for A.

A justification is something connected to linguistic practice, and therefore it is supposed not to transcend our epistemic capacities. This observation leads to:

(2E) If it is possible to exhibit a direct justification for A, then it is possible to know that A.

Putting (1E) and (2E) together we get what is known as the *knowability principle*:

(3E) If A is true, then it is possible to know that A.

What is known as the Fitch or Church-Fitch paradox[1] is an argument that threatens the anti-realist position: In the argument, it is concluded from the knowability principle that all truths are actually known, a paradoxical consequence, known as the principle of *omniscience*, that undermines the epistemic conception of truth.

The force of the argument lies in the fact that it is a formal argument, completely developed in a plainly faultless logical setting. More precisely, the knowability principle is formalized with a scheme that uses two modal operators, $\mathcal{K}$ and $\Diamond$. The first is an epistemic operator to be read as 'it is known that ... ' or 'someone knows that... '. The second is an alethic operator to be read as 'it is possible that... '.

In this formal language, the knowability principle takes the form of the scheme

$$A \supset \Diamond \mathcal{K} A \qquad \textbf{KP}$$

In the same manner, omniscience is formalized by the scheme

$$A \supset \mathcal{K} A \qquad \textbf{OP}$$

The Church-Fitch paradox consists in a formal derivation that starts from **KP**, passes through its instance with the Moore sentence[2] $A \,\&\, \neg \mathcal{K} A$, and then leads to **OP** by using only logical steps. We shall consider here only the definition in which $\mathcal{K}$ is a primitive modal operator, and not the one, alternatively proposed by Fitch at the end of his paper (1963, p. 141), in which $\mathcal{K}$ is defined on the basis of a causal relation that allows to define knowledge in terms of justified true belief.

Many different ways to block the paradox have been proposed. They can be grouped into three categories of intervention:

1. Restriction on the possible instances of **KP** (Dummett 2001, Tennant 1997, 2009, Restall 2009);

2. Reformulation of the formalization of the knowability principle (Edington 1985, Rabinowicz and Segerberg 1994, Martin-Löf 1998, van Benthem 2009, Burgess 2009, Proietti and Sandu 2010, Artemov and Protopopescu 2011, Proietti 2011);

3. Revision of the logical framework in which the derivation is made (Williamson 1982, Beall 2000, 2009, Wansing 2002, Dummett 2009, Giaretta 2009, Priest 2009).

Even if some of the proposed solutions focus on the type of derivability relation that connects **OP** to **KP**, none of them has taken derivations themselves as objects of study or analyzed the *structure* of the derivation of **OP** from **KP**. Our precise aim, instead, is to focus on this analysis.

Before proceeding, it is worth noting that the standard derivation of the Church-Fitch paradox is given in an axiomatic calculus (Beall 2000, Brogaard and Salerno 2009, Wansing 2002). This calculus hides structural operations such as *cut*, *weakening* and *contraction*. For the purposes

[1] The paradox was presented in Fitch (1963) but, as recently discovered by Joe Salerno and Julien Murzi, it was actually suggested by Church in a series of referee's reports dating back to 1945 and now reproduced in Salerno (2009).

[2] We extend here to knowledge the usual notion of Moore sentence, originally conceived for belief.

of an analysis that leaves no inferential passage implicit, it is therefore preferable to move to systems of sequent calculus that make these operations explicit, and, by a suitable design as achieved in the G3 systems, completely eliminable. We begin in Section 2 with a sequent calculus derivation of the Church-Fitch paradox, built by translating a natural deduction derivation. The calculus that is used is contraction free and cut free, thus a good basis for the structural analysis of the paradox. However, the presence of an axiomatic assumption in the derivation results in a non-eliminable cut. The method of conversion of axioms into rules of Negri and von Plato (1998, 2001), briefly recalled in Section 3, is not applicable here because the knowability principle cannot be reduced to its atomic instances. This fact is established syntactically by means of a failed proof search in the given sequent system. We turn therefore to the method of labelled calculi in the style of Negri (2005) and present in Section 3.2 a bimodal extension of a G3-style labelled system for intuitionistic logic. We show in Section 3.2.1 that the system has all the structural rules admissible. The system is equivalent to a standard axiomatic system used in the analysis of the paradox, but the labelled approach allows a stronger completeness result; In Section 3.2.2, we prove completeness in a direct way by showing that, for every sequent in the language of the logic in question, either there is a proof in the calculus or a countermodel in a precisely defined frame class is found.

The completeness result is used in Section 5 for showing that the classical standard form of the Church-Fitch paradox is not derivable intuitionistically (Section 5.2): We consider the classically derivable sequent with **KP** instantiated with the Moore sentence $A \,\&\, \neg\mathcal{K}A$ as an antecedent and **OP** instantiated with A as a succedent; then, by the failed proof search, we extract a countermodel for **OP**. This argument suffices for blocking, within an intuitionistic bimodal system, the specific proof of the paradox, but it is not yet conclusive. To conclude that an intuitionistic system that incorporates **KP** as a derivation principle does not derive **OP**, it is not sufficient to show that **OP** does not follow from a particular instance of **KP**. At the end of Section 5 we make clear, through an example from classical logic, that the notion to be considered when comparing principles of proof should be *admissibility* rather than *derivability*. To clarify the relation between the two principles, it is necessary to make explicit the conditions that characterize their validity. The semantical assumption behind the axiom scheme **KP** is determined in the form of a frame property that involves all the three - preorder, alethic, epistemic - accessibility relations (Section 5.3). The frame condition **KP-Fr** is then made part of the logical system in the form of a block of additional rules of inference, linked by a variable condition. By this addition, a complete contraction- and cut-free proof system for intuitionistic bimodal logic extended by the knowability principle is obtained. We show, using proof search and construction of countermodels, that **OP** is not derivable in the system, therefore not valid. We also discuss how an oversight on the variable condition could lead to an opposite conclusion. We then show how, by just adding symmetry of the preorder, **OP** becomes derivable. The latter is a cut-free derivation of the Church-Fitch paradox that uses **KP** as a derivation principle and that guarantees that the source of the paradox is to be found only in the assumption on which it depends. It is also shown that the same result can be obtained for belief-like notions of knowledge that do not assume factivity among their defining principles.

2 Towards a structural analysis of the Church-Fitch argument

The Church-Fitch paradox was originally presented in Fitch (1963) without using an explicit logical system, and it was later formalized using semantic arguments and various deductive systems for modal logic: linear derivations, natural deduction, sequent calculus. All these formalizations have contributed to single out a minimal logical ground that gives rise to the paradox. It consists in a basic bimodal logic that extends classical propositional logic with an alethic modality $\Diamond$ and an epistemic modality $\mathcal{K}$. No requirement is made on the alethic modality, whereas the epistemic modality is supposed to satisfy *distributivity over conjunction*, $\mathcal{K}(A \,\&\, B) \supset \mathcal{K}A \,\&\, \mathcal{K}B$, and *factivity*, $\mathcal{K}A \supset A$. The former property is derivable for any necessity-like modality in normal modal logic, so the only requirement added to a normal bimodal logic is factivity of $\mathcal{K}$.

A formalization of the Church-Fitch argument is the first step towards its analysis. We start with a derivation in natural deduction:

$$\dfrac{\dfrac{\dfrac{\overset{2}{A \,\&\, \neg\mathcal{K}A} \qquad \overset{\mathbf{KP}}{A \,\&\, \neg\mathcal{K}A \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)}}{\Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)} \qquad \dfrac{\dfrac{\overset{1}{\mathcal{K}(A \,\&\, \neg\mathcal{K}A)}}{\mathcal{K}A}{\scriptstyle \mathcal{K}\&_1} \qquad \dfrac{\dfrac{\overset{1}{\mathcal{K}(A \,\&\, \neg\mathcal{K}A)}}{\mathcal{K}\neg\mathcal{K}A}{\scriptstyle \mathcal{K}\&_2}}{\neg\mathcal{K}A}{\scriptstyle \mathcal{K}E}}{\bot}}{\bot}{\scriptstyle \Diamond E,1}}{\neg(A \,\&\, \neg\mathcal{K}A)}{\scriptstyle \supset I,2}$$

The conclusion is the weaker intuitionistic version of **OP**, and intuitionistically equivalent to $A \supset \neg\neg\mathcal{K}A$. We call both of them **WOP** (for *weak omniscience principle*).

The conclusion $A \supset \mathcal{K}A$ is obtained by classical propositional steps and leads, in conjunction with factivity, to the identification of truth and knowledge.

A closer inspection of the above derivation shows that we used the following rules:

$$\dfrac{\mathcal{K}(A \,\&\, B)}{\mathcal{K}A}{\scriptstyle \mathcal{K}\&_1} \qquad \dfrac{\mathcal{K}(A \,\&\, B)}{\mathcal{K}B}{\scriptstyle \mathcal{K}\&_2}$$

These are derivable in any system of normal epistemic modal logic. Rule $\mathcal{K}E$ corresponds to factivity of knowledge and rule $\Diamond E$ is the dual of the familiar necessitation rule: observe that the latter can be formulated in natural deduction as

$$\dfrac{\Diamond A \qquad \begin{matrix}[A]^1 \\ \vdots \\ \bot\end{matrix}}{\bot}{\scriptstyle \Diamond E\ 1}$$

The minor premiss of the rule is $\bot$ and may depend on A, discharged by the rule, similarly to the rule of existence elimination, with falsity, rather than any formula not containing the eigenvariable, as the minor premiss. With a sequent notation and an empty succedent in place of $\bot$, the rule becomes

$$\dfrac{A \rightarrow}{\Diamond A \rightarrow}$$

This rule is the dual of the rule of necessitation:

$$\dfrac{\rightarrow A}{\rightarrow \Box A}$$

A further step in the analysis of derivations comes from sequent calculus that has several advantages over natural deduction. First, structural steps are explicit and not hidden in vacuous and multiple discharge and in non-normal instances of rules (cf. Negri and von Plato 2001, Chapter 1). Secondly, sequent calculus, contrary to natural deduction, is well suited for classical logic and its modal extensions.

The sequent calculus that we shall use is obtained as an extension of the classical propositional contraction-free sequent calculus **G3c** with the following rules for the alethic and epistemic modalities, where $\mathcal{K}\Gamma$ denotes the multiset of all the $\mathcal{K}A$ for A in Γ:

$$\frac{\Gamma \to A}{\mathcal{K}\Gamma, \Theta \to \Delta, \mathcal{K}A}\, \scriptstyle{LR\text{-}\mathcal{K}} \qquad \frac{A, \mathcal{K}A, \Gamma \to \Delta}{\mathcal{K}A, \Gamma \to \Delta}\, \scriptstyle{L\mathcal{K}}$$

$$\frac{A \to \Delta}{\Diamond A, \Gamma \to \Theta, \Diamond\Delta}\, \scriptstyle{LR\text{-}\Diamond}$$

Modal rules of G3◇K

The resulting system, called **G3◇K**, is an extension of the calculus **G3K** presented in section 4 of Hakli and Negri (2011), and the proof of its structural properties follows the lines of the proof for **G3K**.

Theorem 2.1. *In* **G3◇K** *the following hold:*

1. *Sequents $A, \Gamma \to \Delta, A$ are derivable for arbitrary A.*
2. *Propositional rules are height-preserving invertible.*
3. *The rules of left and right weakening and contraction are height-preserving admissible.*
4. *Cut is admissible.*

Proof. We show here only one extra case that arises in the proof of cut elimination because of the addition of rule $L\mathcal{K}$, with the cut formula principal in both premisses of cut, the right one being $L\mathcal{K}$:

$$\frac{\dfrac{\Gamma \to A}{\Theta, \mathcal{K}\Gamma \to \Delta, \mathcal{K}A}\, \scriptstyle{LR\text{-}\mathcal{K}} \qquad \dfrac{\mathcal{K}A, A, \Gamma' \to \Delta'}{\mathcal{K}A, \Gamma' \to \Delta'}\, \scriptstyle{L\mathcal{K}}}{\Theta, \mathcal{K}\Gamma, \Gamma' \to \Delta, \Delta'}\, \scriptstyle{Cut}$$

The cut is transformed as follows in two consecutive cuts, the upper of decreased height, the lower of decreased cut formula weight. Repeated applications of rule $L\mathcal{K}$ are denoted by $L\mathcal{K}^*$:

$$\frac{\dfrac{\Gamma \to A \qquad \dfrac{\Theta, \mathcal{K}\Gamma \to \Delta, \mathcal{K}A \quad \mathcal{K}A, A, \Gamma' \to \Delta'}{A, \Theta, \mathcal{K}\Gamma, \Gamma' \to \Delta, \Delta'}\, \scriptstyle{Cut}}{\Theta, \Gamma, \mathcal{K}\Gamma, \Gamma' \to \Delta, \Delta'}\, \scriptstyle{Cut}}{\Theta, \mathcal{K}\Gamma, \Gamma' \to \Delta, \Delta'}\, \scriptstyle{L\mathcal{K}^*}$$

The conversion for a cut formula of the form $\Diamond A$ principal in both premisses of cut in LR-$\Diamond$ is symmetric to the conversion of a cut formula of the form $\mathcal{K}A$ principal in both premisses of cut in LR-$\mathcal{K}$ treated in the above mentioned article. QED

The sequent-style reconstruction of the Church-Fitch paradox calls for the following

Lemma 2.2. *The following rules are derivable in* **G3◇K** $+ Cut$*:*

$$\frac{\to A \,\&\, \neg\mathcal{K}A \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)}{A, \neg\mathcal{K}A \to \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)}\, \scriptstyle{Inv} \qquad \frac{\mathcal{K}A, \mathcal{K}\neg\mathcal{K}A \to}{\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to}\, \scriptstyle{Distr}$$

Proof. See Appendix A. QED

A proof of the Church-Fitch paradox can now be obtained as a derivation in system **G3$\Diamond$K** of the sequent $\to A \supset \mathcal{K}A$ from a special instance of the knowability principle **KP**, the sequent $\to (A \,\&\, \neg\mathcal{K}A) \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)$, as follows:

$$
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\dfrac{\mathbf{KP}}{\to A \,\&\, \neg\mathcal{K}A \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)}
}{A, \neg\mathcal{K}A \to \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)}\,{\scriptstyle Inv}
\qquad
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\dfrac{}{\mathcal{K}A \to \mathcal{K}A}
}{\mathcal{K}A, \neg\mathcal{K}A \to}\,{\scriptstyle L\neg}
}{\mathcal{K}A, \mathcal{K}\neg\mathcal{K}A, \neg\mathcal{K}A \to}\,{\scriptstyle L\text{-}Wk}
}{\mathcal{K}A, \mathcal{K}\neg\mathcal{K}A \to}\,{\scriptstyle L\mathcal{K}}
}{\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to}\,{\scriptstyle Distr}
}{\Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to}\,{\scriptstyle LR\text{-}\Diamond}
}{A, \neg\mathcal{K}A \to}\,{\scriptstyle Cut}
}{A \to \neg\neg\mathcal{K}A}\,{\scriptstyle R\neg}
\qquad
\dfrac{}{\neg\neg\mathcal{K}A \to \mathcal{K}A}
}{
\dfrac{A \to \mathcal{K}A}{\to A \supset \mathcal{K}A}\,{\scriptstyle R\supset}
}\,{\scriptstyle Cut}
$$

Observe that the presence of the sequent $\to A \,\&\, \neg\mathcal{K}A \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)$ from which the derivation starts makes the application of cut non-eliminable because, in general, cut elimination fails when cuts depend on proper axioms.[3] A possible way out was found in Negri and von Plato (1998): If axioms are converted into suitable inference rules, the eliminability of cut is maintained. We shall recall the basic properties of the method in the next section, but anticipate here a problem that is met when the method is applied to the knowability principle. First, **KP** should be converted into a rule of the form

$$\dfrac{\Diamond\mathcal{K}A, \Gamma \to \Delta}{A, \Gamma \to \Delta}\,{\scriptstyle Kn}$$

This rule can be easily proved to be equivalent to the sequent $\to A \supset \Diamond\mathcal{K}A$. Then, it should be reduced to a rule that has only formulas devoid of logical structure as principal, i.e., a reduction of the general knowability principle to the knowability principle for only atomic formulas. If such were the case, the rule in the above derivation could be turned into a left rule of sequent calculus with atomic principal formulas, of the form

$$\dfrac{\Diamond\mathcal{K}P, \Gamma \to \Delta}{P, \Gamma \to \Delta}\,{\scriptstyle Kn\text{-}At}$$

However, it can be proved that the knowability principle cannot be reduced to its atomic instances. By the following result, the knowability principle on a conjunction does not follow from knowability on the conjuncts.

Lemma 2.3. *The sequent* $P \supset \Diamond\mathcal{K}P, Q \supset \Diamond\mathcal{K}Q \to P \,\&\, Q \supset \Diamond\mathcal{K}(P \,\&\, Q)$ *is not derivable in* **G3$\Diamond$K**.

Proof. See Appendix B. QED

By Lemma 2.3 and the equivalence of systems with rules and systems with axioms as contexts (cf. Theorem 6.3.2 in Negri and von Plato 2001), we conclude that the rule of knowability on arbitrary formulas does not follow from its restriction to atomic formulas. The method of conversion of axiom into rules, successfully employed elsewhere for extending structural proof

[3]For more details about the failure of cut elimination in the presence of proper axioms see Girard (1987), p. 125, Negri and von Plato (1998), p. 418, and Troelstra and Schwichtenberg (2000), p. 127.

analysis from standard sequent calculi to systems with added axioms (cf. Negri and von Plato 2001, 2011) thus cannot be applied in this case. We shall therefore use the more refined labelled deductive machinery of Negri (2005).

3 The proof-theoretical machinery

For a self-contained presentation of the G3-style calculus that we shall use for dealing with a bimodal system, we need to recall briefly the method of conversion of axioms into sequent rules presented in Negri and von Plato (1998, 2001) for mathematical theories and in Negri (2005, 2008), Negri and von Plato (2011) for modal and non-classical logics. The reader already acquainted with the method can skip this section.

The starting point is the classical propositional sequent calculus **G3c** in which all the rules are invertible and all the structural rules are admissible (cf. Troelstra and Schwichtenberg 2000 or Negri and von Plato 2001 for the rules and the basic properties). In this calculus, weakening and contraction have the stronger property of *height-preserving-* (*hp-*) admissibility, that is, whenever their premisses are derivable, also their conclusion is, with at most the same derivation height (the *height* of a derivation is its height as a tree, that is, the length of its longest branch). Moreover, the calculus enjoys hp-admissibility of substitution of individual variables. Invertibility of the rules of **G3c** is also height-preserving (*hp-invertible*). Negri and von Plato (1998) showed that it is possible to extend **G3c** by suitably formulated rules that correspond to axioms for specific theories while maintaining all the structural properties of the basic **G3c** system. (Detailed proofs can be found in chapters 3, 4, and 6 of Negri and von Plato 2001). Universal axioms are first transformed, through the rules of **G3c**, into a normal form that consists of conjunctions of formulas of the form

$$P_1 \,\&\, \dots \,\&\, P_m \supset Q_1 \vee \dots \vee Q_n$$

All the P_i, Q_j are atomic; If $m = 0$, the implication reduces to the succedent, the latter with the limiting case of $\bot$ if $n = 0$. The universal closure of any such formula is called a *regular formula*. We abbreviate the multiset $P_1, \dots, P_m$ as $\overline{P}$. Each conjunct is then converted into a schematic rule, called the *regular rule scheme*, of the form

$$\frac{Q_1, \overline{P}, \Gamma \to \Delta \quad \cdots \quad Q_n, \overline{P}, \Gamma \to \Delta}{\overline{P}, \Gamma \to \Delta}\,\mathit{Reg}$$

To maintain admissibility of contraction in the extensions with regular rules, the formulas $P_1, \dots, P_m$ in the antecedent of the conclusion of the scheme have to be repeated in the antecedent of each of the premisses. Consider an instantiation of free parameters in atoms that produces a duplication (two identical atoms) in the conclusion of a rule instance, as in

$$P_1, \dots, P, P, \dots, P_m, \Gamma \to \Delta$$

There is a corresponding duplication in each premiss. The *closure condition* imposes the requirement that the rule with the duplication P, P contracted into a single P, both in the premisses and in the conclusion, be added to the system of rules. For each axiom system, there is only a bounded number of possible cases of contracted rules to be added, very often none at all, so that the condition is unproblematic.

By the same method, it is possible to convert into rules also existential axioms, or, more generally, axioms of the form of *geometric implications*. These are universal closures of implications,

$\forall x_1 \dots x_n(A \supset B)$ in which A and B do not contain implications or universal quantifiers. Geometric implications can be turned into a useful normal form that consists in conjunctions of formulas of the form

$$\forall \overline{x}(P_1 \& \dots \& P_m \supset \exists \overline{y}_1 M_1 \vee \dots \vee \exists \overline{y}_n M_n)$$

Each P_i is an atomic formula, each M_j a conjunction of a list of atomic formulas $\overline{Q}_j$, and none of the variables in the vectors $\overline{y}_j$ are free in P_i. In turn, each of these formula can be turned into an inference rule of the following form:

$$\frac{\overline{Q}_1(\overline{z}_1/\overline{y}_1), \overline{P}, \Gamma \to \Delta \quad \dots \quad \overline{Q}_n(\overline{z}_n/\overline{y}_n), \overline{P}, \Gamma \to \Delta}{\overline{P}, \Gamma \to \Delta} \, GRS$$

The variables $\overline{y}_i$ are called the *replaced variables* of the scheme, and the variables $\overline{z}_i$ the *proper variables*, or *eigenvariables*. In what follows, we shall consider for ease of notation the case in which the vectors of variables $\overline{y}_i$ consist of a single variable. All the proofs can be adapted in a straightforward way to the general case.

The geometric rule scheme is subject to the condition that the eigenvariables must not be free in the conclusion of the rule, $\overline{P}, \Gamma, \Delta$. With this condition, the rule expresses in a logic-free way the role of the existential quantifier in a geometric axiom.

All universal and geometric theories can be formulated by this method as contraction- and cut-free systems of sequent calculi, as was shown by the following result (Negri and von Plato 1998, Negri 2003):

Theorem 3.1. *The structural rules of* Weakening*,* Contraction *and* Cut *are admissible in all extensions of* **G3c** *with the regular or geometric rule scheme and satisfying the closure condition.* Weakening *and* Contraction *are hp-admissible.*

3.1 Proof analysis for non-classical logics

The method of extension of sequent calculi can be applied not only *outside* logic, to specific axiomatic theories such as lattice theory, arithmetic, and geometry (cf. Negri and von Plato 2011), but also *inside* logic, and in particular to modal logics, and, more generally, to those non-classical logics that can be characterized in terms of relational semantics. The language is enriched with terms for possible worlds and relations between them, and expressions for the forcing relation between worlds and formulas. The basic modal logic **K** gets formulated as a labelled sequent calculus by prefixing every formula A with a label x that ranges in a set W. A full internalization of the semantics is obtained by allowing expressions for accessibility relations xRy between labels. The rules for the modalities $\Box$ and $\Diamond$ are obtained through a meaning explanation in terms of Kripke semantics and an inversion principle. Logics stronger that **K** are formulated by adding to the basic calculus rules for R: First-order conditions usually imposed on R such as reflexivity, transitivity, symmetry, are considered as axioms and then converted into regular rules.

We shall now recall the determination and the basic structural properties of a labelled G3-style sequent calculus for intuitionistic logic (cf. Dyckhoff and Negri 2012, Negri and von Plato 2011).

It is well known that the semantics of **S4** can be used to provide a direct interpretation of the intuitionistic connectives, the intuitionistic implication being a $\Box$-type modality (see Kripke 1965). In fact, the inductive definition of validity of implicative formulas is:

$$x \Vdash A \supset B \quad \text{if and only if} \quad \text{for all } y,\, x \leqslant y \text{ and } y \Vdash A \text{ implies } y \Vdash B$$

Here the intuitionistic accessibility relation is denoted, as usual, by $\leqslant$ and it is assumed to be reflexive and transitive, i.e., a preorder. Along with the clauses for the other connectives, the definition can be converted into a pair of sequent rules; The arbitrariness of y is expressed by the condition that it must not appear in the conclusion of the right rule for implication. In addition, the forcing relation has to be proved monotone with respect to the relation $\leqslant$. That is, for any arbitrary formula A the following has to hold:

$$x \leqslant y \text{ and } x \Vdash A \quad \text{implies} \quad y \Vdash A$$

It is enough to impose monotonicity of forcing, in the form of an initial sequent, only for atomic formulas. This is not a restriction because full monotonicity is then derivable. Thus, one of the design principles of G3-style calculi, namely the restriction of initial sequents to atomic formulas needed to guarantee the full range of structural properties, is respected.

The following labelled sequent calculus **G3I** for intuitionistic logic is thus obtained (negation is defined in terms of $\bot$ and $\supset$, the formulas P are atomic, and $y \notin \Gamma, \Delta$ in rule $R\supset$).

Initial sequents

$x \leqslant y, x : P, \Gamma \to \Delta, y : P$

Logical Rules

$$\frac{x : A, x : B, \Gamma \to \Delta}{x : A \,\&\, B, \Gamma \to \Delta}\, L\& \qquad \frac{\Gamma \to \Delta, x : A \quad \Gamma \to \Delta, x : B}{\Gamma \to \Delta, x : A \,\&\, B}\, R\&$$

$$\frac{x : A, \Gamma \to \Delta \quad x : B, \Gamma \to \Delta}{x : A \vee B, \Gamma \to \Delta}\, L\vee \qquad \frac{\Gamma \to \Delta, x : A, x : B}{\Gamma \to \Delta, x : A \vee B}\, R\vee$$

$$\frac{x \leqslant y, x : A \supset B, \Gamma \to \Delta, y : A \quad x \leqslant y, x : A \supset B, y : B, \Gamma \to \Delta}{x \leqslant y, x : A \supset B, \Gamma \to \Delta}\, L\supset$$

$$\frac{}{x : \bot, \Gamma \to \Delta}\, L\bot \qquad \frac{x \leqslant y, y : A, \Gamma \to \Delta, y : B}{\Gamma \to \Delta, x : A \supset B}\, R\supset$$

Mathematical Rules

$$\frac{x \leqslant x, \Gamma \to \Delta}{\Gamma \to \Delta}\, Ref_{\leqslant} \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \to \Delta}{x \leqslant y, y \leqslant z, \Gamma \to \Delta}\, Trans_{\leqslant}$$

Full monotonicity of forcing is obtained by the following:

Lemma 3.2. *All sequents of the form*

1. $x \leqslant y, x : A, \Gamma \to \Delta, y : A$
2. $x : A, \Gamma \to \Delta, x : A$

are derivable in **G3I**.

Proof. See the proof of Lemma 12.25 of Negri and von Plato (2011). QED

System **G3I** enjoys all the structural properties usually required of sequent systems and the same holds for each extension **G3I*** with rules that follow the regular or the geometric rule scheme.

Theorem 3.3. *In* **G3I*** *the following hold:*

1. *All logical rules are hp-invertible.*
2. *The rules of left and right weakening and contraction are hp-admissible.*
3. *Cut is admissible.*

Proof. See the proofs of Theorems 12.27–12.29 of Negri and von Plato (2011). QED

For our purposes, the most remarkable extension of **G3I** is obtained by imposing symmetry of the accessibility relation

$$\frac{y \leqslant x, x \leqslant y, \Gamma \rightarrow \Delta}{x \leqslant y, \Gamma \rightarrow \Delta}\,{\scriptstyle Sym_{\leqslant}}$$

This extension gives a system equivalent to classical logic and we shall refer to it as **G3C**. Given that **G3C** is an extension of **G3I** with a rule that follows the regular rule scheme, it admits cut elimination by Theorem 3.3.

3.2 Intuitionistic bimodal logic

Another way to extend **G3I** is to augment the language. As we have said, the formulation of the Church-Fitch paradox requires two modalities, $\mathcal{K}$ and $\Diamond$. The corresponding accessibility relations in Kripke semantics are $R_{\mathcal{K}}$ and $R_{\Diamond}$, and the behaviour of these two modal operators is captured by the following valuation clauses:

$x \Vdash \mathcal{K}A$ if and only if for all y, $xR_{\mathcal{K}}y$ implies $y \Vdash A$

$x \Vdash \Diamond A$ if and only if for some y, $xR_{\Diamond}y$ and $y \Vdash A$

Each definition can be unfolded in the necessary and sufficient conditions and converted into the following sequent rules, with the condition $y \neq x$, $y \notin \Gamma, \Delta$ for $R\mathcal{K}$ and $L\Diamond$:

$$\frac{y : A, xR_{\mathcal{K}}y, x : \mathcal{K}A, \Gamma \rightarrow \Delta}{xR_{\mathcal{K}}y, x : \mathcal{K}A, \Gamma \rightarrow \Delta}\,{\scriptstyle L\mathcal{K}} \qquad \frac{xR_{\mathcal{K}}y, \Gamma \rightarrow \Delta, y : A}{\Gamma \rightarrow \Delta, x : \mathcal{K}A}\,{\scriptstyle R\mathcal{K}}$$

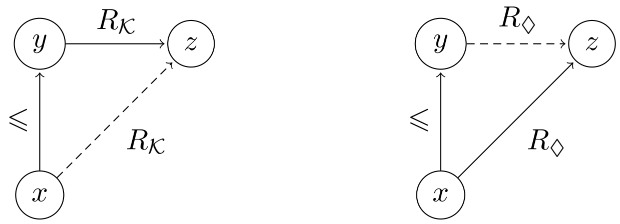
$$\frac{xR_{\Diamond}y, y : A, \Gamma \rightarrow \Delta}{x : \Diamond A, \Gamma \rightarrow \Delta}\,{\scriptstyle L\Diamond} \qquad \frac{xR_{\Diamond}y, \Gamma \rightarrow \Delta, x : \Diamond A, y : A}{xR_{\Diamond}y, \Gamma \rightarrow \Delta, x : \Diamond A}\,{\scriptstyle R\Diamond}$$

Unlike for the extension with $Sym_{\leqslant}$, in the presence of the new rules it is not guaranteed that Theorem 3.3 is still valid. Moreover, we need to prove that the full monotonicity property (Lemma 3.2) extends also to modal formulas. Indeed, it is easy to see that if the standard rules for $\mathcal{K}$ and $\Diamond$ are used, Lemma 3.2 does not hold. A possible way out has been found in Božić and Došen (1984) by requiring that models satisfy the extra conditions

Mon$_\mathcal{K}$ $\forall x \forall y \forall z (x \leqslant y \,\&\, yR_\mathcal{K}z \supset xR_\mathcal{K}z)$

Mon$_\Diamond$ $\forall x \forall y \forall z (x \leqslant y \,\&\, xR_\Diamond z \supset yR_\Diamond z)$

Observe that these conditions state that the following diagrams can be completed (the completing arrows are the dotted ones):



Conditions **Mon**$_\mathcal{K}$ and **Mon**$_\Diamond$ are universal axioms and by applying the method of conversion of axioms into sequent rules they become:

$$\frac{xR_\mathcal{K}z, x \leqslant y, yR_\mathcal{K}z, \Gamma \to \Delta}{x \leqslant y, yR_\mathcal{K}z, \Gamma \to \Delta}{\scriptstyle Mon_\mathcal{K}} \qquad \frac{yR_\Diamond z, x \leqslant y, xR_\Diamond z, \Gamma \to \Delta}{x \leqslant y, xR_\Diamond z, \Gamma \to \Delta}{\scriptstyle Mon_\Diamond}$$

We shall call $\mathbf{G3I}_{\mathcal{K}\Diamond}$ the extension of **G3I** with rules $L\mathcal{K}, R\mathcal{K}, L\Diamond, R\Diamond$, $Mon_\mathcal{K}$, $Mon_\Diamond$.

Monotonicity of forcing can now be extended to cover also modal formulas:

Lemma 3.4. *All sequents of the forms*

$$x \leqslant y, x : \mathcal{K}B, \Gamma \to \Delta, y : \mathcal{K}B \quad \textit{and} \quad x \leqslant y, x : \Diamond B, \Gamma \to \Delta, y : \Diamond B$$

are derivable in $\mathbf{G3I}_{\mathcal{K}\Diamond}$.

Proof. See Appendix C. QED

The above, together with Lemma 3.2, gives monotonicity for arbitrary formulas:

Lemma 3.5. *All sequents of the forms*

1. $x \leqslant y, x : A, \Gamma \to \Delta, y : A$
2. $x : A, \Gamma \to \Delta, x : A$

are derivable in $\mathbf{G3I}_{\mathcal{K}\Diamond}$.

3.2.1 Admissibility of the structural rules

In this section we shall prove admissibility of all the structural rules for system $\mathbf{G3I}_{\mathcal{K}\Diamond}$.

Because of the presence of labels in the language, we need an auxiliary result concerning their substitution:

Lemma 3.6. *The rule of substitution for labels*

$$\frac{\Gamma \to \Delta}{\Gamma(y/x) \to \Delta(y/x)}\,{\scriptstyle Subst}$$

is hp-admissible in $\mathbf{G3I}_{\mathcal{K}\Diamond}$.

Proof. See Appendix C. QED

Proposition 3.7. *The rules of weakening*

$$\frac{\Gamma \to \Delta}{x : A, \Gamma \to \Delta}\,{\scriptstyle L\text{-}Wk} \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, x : A}\,{\scriptstyle R\text{-}Wk} \qquad \frac{\Gamma \to \Delta}{xRy, \Gamma \to \Delta}\,{\scriptstyle L\text{-}Wk_R}$$

are hp-admissible in $\mathbf{G3I}_{\mathcal{K}\Diamond}$.

Proof. See Appendix C. QED

Lemma 3.8. *All the rules of* $\mathbf{G3I}_{\mathcal{K}\Diamond}$ *are hp-invertible.*

Proof. See Appendix C. QED

Now we are in a position to prove the most important structural property of our calculi besides cut-admissibility, namely hp-admissibility of the rules of contraction.

Theorem 3.9. *The following rules of contraction*

$$\frac{x : A, x : A, \Gamma \to \Delta}{x : A, \Gamma \to \Delta}\,{\scriptstyle L\text{-}Ctr} \qquad \frac{\Gamma \to \Delta, x : A, x : A}{\Gamma \to \Delta, x : A}\,{\scriptstyle R\text{-}Ctr} \qquad \frac{xRy, xRy, \Gamma \to \Delta}{xRy, \Gamma \to \Delta}\,{\scriptstyle L\text{-}Ctr_R}$$

are hp-admissible in $\mathbf{G3I}_{\mathcal{K}\Diamond}$.

Proof. See Appendix C. QED

Theorem 3.10. *The rule of cut*

$$\frac{\Gamma \to \Delta, x : A \quad x : A, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'}\,{\scriptstyle Cut}$$

is admissible in $\mathbf{G3I}_{\mathcal{K}\Diamond}$.

Proof. See Appendix C. QED

Observe that all the above structural results that have been established for $\mathbf{G3I}_{\mathcal{K}\Diamond}$ hold also for any of its extensions with frame rules that follow the regular or geometric rule scheme. The details can be easily spelled out following the general pattern of the parallel results for extensions of basic modal logic in Negri (2005).

3.2.2 Completeness

There are three main methods for proving Kripke completeness of a sequent system: One is the indirect method that establishes an equivalence with an axiomatic system known to be complete with respect to a certain class of frames. The second is through Henkin sets with the canonical frame construction, and the third by a direct method that shows how root-first proof search in the sequent system either gives a proof or leads to a countermodel. By the results of the previous section, the sequent system we have presented for intuitionistic bimodal logic is closed under the rules of modus ponens and necessitation and allows to derive the axioms of a standard axiomatic presentation. Instead of going into the details of this completeness proof or of the *tour de force* through Henkin set contructions, we shall sketch the direct completeness proof, along the lines of Negri (2009). It is the method that will make possible proofs of underivability and constructions of countermodels in what follows.

Definition 3.11. *Let K be a frame with the accessibility relations $\leqslant$, $R_{\mathcal{K}}$, and $R_{\Diamond}$ that satisfy the properties $Ref_{\leqslant}$, $Trans_{\leqslant}$, $Mon_{\mathcal{K}}$, $Mon_{\Diamond}$. Let W be the set of variables (labels) used in derivations in* $\mathbf{G3I}_{\mathcal{K}\Diamond}$. *An* interpretation *of the labels W in a frame K is a function $[\![\cdot]\!] : W \to K$. A* valuation *of atomic formulas in frame K is a map $\mathcal{V} : AtFrm \to \mathcal{P}(K)$ that assigns to each atom P the set of nodes of K in which P holds. The standard notation for $k \in \mathcal{V}(P)$ is $k \Vdash P$.*

Valuations are extended to arbitrary formulas by the following inductive clauses:

$k \Vdash \bot$ for no k,

$k \Vdash A\&B$ if $k \Vdash A$ and $k \Vdash B$,

$k \Vdash A \vee B$ if $k \Vdash A$ or $k \Vdash B$,

$k \Vdash A \supset B$ if for all k', from $k \leqslant k'$ and $k' \Vdash A$ follows $k' \Vdash B$,

$k \Vdash \mathcal{K}A$ if for all k', from $kR_{\mathcal{K}}k'$ follows $k' \Vdash A$,

$k \Vdash \Diamond A$ if there exists k' such that $kR_{\Diamond}k'$ and $k' \Vdash A$.

Definition 3.12. *A sequent $\Gamma \to \Delta$ is* valid for an interpretation and a valuation *in K if for all labelled formulas $x : A$ and relational atoms $y \leqslant z$, $y'R_{\mathcal{K}}z'$, $y''R_{\Diamond}z''$ in Γ, whenever $[\![x]\!] \Vdash A$ and $[\![y]\!] \leqslant [\![z]\!]$, $[\![y']\!]R_{\mathcal{K}}[\![z']\!]$, $[\![y'']\!]R_{\Diamond}[\![z'']\!]$ in K, then for some $w : B$ in Δ, $[\![w]\!] \Vdash B$. A sequent is* valid *if it is valid for every interpretation and every valuation in a frame.*

Theorem 3.13 (Soundness)**.** *If the sequent $\Gamma \to \Delta$ is derivable in* $\mathbf{G3I}_{\mathcal{K}\Diamond}$*, it is valid in every frame with the properties $Ref_{\leqslant}$, $Trans_{\leqslant}$, $Mon_{\mathcal{K}}$, $Mon_{\Diamond}$.*

Proof. See Appendix D. QED

Next, we show that derivability of a formula in the calculus is equivalent to validity, that is, validity at an arbitrary world for an arbitrary valuation. The latter is expressed by $x \Vdash A$ where x is arbitrary, and it is translated into a sequent $\to x : A$ in our calculus. The rules of the calculus applied backwards give equivalent conditions until the atomic components of A are reached. It can happen that we find a proof, or that we find that a proof does not exist either because we reach a stage where no rule is applicable, or because we go on with the search forever. In the two latter cases the attempted proof itself gives directly a countermodel.

Theorem 3.14. *Let $\Gamma \to \Delta$ be a sequent in the language of* $\mathbf{G3I}_{\mathcal{K}\Diamond}$*. Then either the sequent is derivable in* $\mathbf{G3I}_{\mathcal{K}\Diamond}$ *or it has a Kripke countermodel with properties $Ref_{\leqslant}$, $Trans_{\leqslant}$, $Mon_{\mathcal{K}}$, $Mon_{\Diamond}$.*

Proof. See Appendix D. QED

The above theorem immediately yields the following

Corollary 3.15 (Completeness)**.** *If a sequent $\Gamma \to \Delta$ is valid in every Kripke model with the frame properties $Ref_{\leqslant}$, $Trans_{\leqslant}$, $Mon_{\mathcal{K}}$, $Mon_{\Diamond}$, it is derivable in* $\mathbf{G3I}_{\mathcal{K}\Diamond}$.

This result can be directly generalized, by an adaptation of the proof in Negri (2009), to every system obtained by extending $\mathbf{G3I}_{\mathcal{K}\Diamond}$ with frame rules that follow the geometric rule scheme. It is easy to check that the addition of frame rules does not change the structure of the proof of Theorem 3.14.

4 Digression: A conceptual analysis of accessibility relations

Before proceeding to the structural analysis of the Church-Fitch paradox by our labelled calculus, we shall outline a conceptual analysis of the accessibility relations introduced in the previous section. This will serve both as an explanation of the notions used, as well as a justification of the formal choices made in defining system $\mathbf{G3I}_{\mathcal{K}\Diamond}$.

First, the relation $\leqslant$ is the standard accessibility relation for the semantics of intuitionistic logic. Its intuitive meaning is clarified in Kripke (1965, pp. 98–99). Because worlds in a model can be identified with the propositions true in them, the relation gets the following intuitive meaning: A world y is $\leqslant$-accessible from a world x if y is a possible development of the information contained in x. Under this interpretation, worlds are recognized as temporal states in a process of acquisition of information. The properties of reflexivity and transitivity of the preorder thus appear obvious, whereas monotonicity of forcing reflects the requirement that the acquisition of information is a cumulative process.

When agents who can gain knowledge are added to the scenario, epistemic operators together with their accessibility relations are needed. Here we have considered just one (impersonal and generic) epistemic attitude, $\mathcal{K}$, with the accessibility relation $R_{\mathcal{K}}$. The question naturally arises of what the relation should be between $R_{\mathcal{K}}$ and $\leqslant$. A minimal requirement is that, in the language extended with formulas as $\mathcal{K}A$, monotonicity of forcing is preserved: The perfect recall should apply to all formulas, not just to the purely propositional ones and this is achieved by imposing the property $\mathbf{Mon}_{\mathcal{K}}$. On the other hand, factivity of knowledge, i.e., axiom $\mathcal{K}A \supset A$, which is explicitly assumed in Fitch's derivation, is ensured by reflexivity of $R_{\mathcal{K}}$. This axiom states that only true formulas can be known and separates knowledge from what is mere belief. As observed by Pierluigi Minari (personal communication) in intuitionistic frames the weaker property $W\text{-}Ref_{\mathcal{K}}$ $\forall x \exists y (x R_{\mathcal{K}} y \,\&\, y \leqslant x)$ suffices to characterize factivity of $\mathcal{K}$. A similar weaker property $W\text{-}Ref_{\Diamond}$ $\forall x \exists y (x R_{\Diamond} y \,\&\, x \leqslant y)$ characterizes $A \supset \Diamond A$. Our results continue to hold with $W\text{-}Ref_{\mathcal{K}}$ and $W\text{-}Ref_{\Diamond}$ in place of $Ref_{\mathcal{K}}$ and $Ref_{\Diamond}$; the latter are however simpler to handle because the corresponding sequent calculus rules do not involve eigenvariables.

Monotonicity and reflexivity of $R_{\mathcal{K}}$ imply that what is temporally accessible is also epistemically accessible, i.e., the condition $\forall x \forall y (x \leqslant y \supset x R_{\mathcal{K}} y)$ holds (see the first part of the proof of Proposition 5.16 below). Notice that this implication does not exclude the possibility of the existence of epistemically accessible states that are not future states. Our analysis will show that if this existence is explicitly imposed, i.e., if $\exists x \exists y (x R_{\mathcal{K}} y \,\&\, x \nleqslant y)$ holds, then the identification of truth and knowledge is avoided (cf. Proietti 2011).

Similar formal requirements apply to the accessibility relation $R_{\Diamond}$, a relation that expresses logical possibility. A state y is $R_{\Diamond}$-accessible from x when y is logically compatible with x, in the sense that y is a state that can in principle be reached from x, even if we cannot specify the nature of this access (temporal, causal, epistemic, etc.). Note that this relation is temporally upward closed: If a state z is possibly reached from x, then z is possibly reached from all the future states of x. We do not want to commit ourselves in any way to assuming more than the necessary properties of $R_{\Diamond}$, in particular we do not identify it with any other of the accessibility relations considered. A different choice is pursued in Proietti (2011) and in Artemov and Protopopescu (2011), where the intuitionistic double negation gets interpreted as a possibility operator, leading to a reformulation of the knowability principle that employs only the epistemic modality.

The above interpretations also allow to capture the temporal flavor ascribed to the knowledge operator in Fitch's original article. Its core result, Theorem 5, is based on the existence of "some true proposition which nobody knows (or has known or will know) to be true" (Fitch 1963, p. 139). The temporal interpretation of $\leqslant$ suggests that the statement $\mathcal{K}A$ has to be evaluated in all situations temporally accessible from x, where x can be considered as the actual world, but also as a past world, or better, x can be considered as any world in which A is true. More generally, the structural reconstruction of Fitch's derivation will reveal that every occurrence of $\mathcal{K}A$ is always in the scope of a negation or of an implication. Therefore, reasoning root first, the application of a $\mathcal{K}$-rule is always preceded by an application of a rule that imposes a temporal-dependent evaluation of $\mathcal{K}A$.

5 Proof-theoretical analysis of the Church-Fitch paradox

We have now all the logical instruments needed for a structural proof analysis of the paradox. We start with the reconstruction of the standard derivation of the paradox that uses the labelled sequent calculus introduced in Section 3.2.

5.1 The Church-Fitch paradox in labelled sequent calculus

The analysis of Section 2 made clear what the ingredients of the Church-Fitch paradox are:

(i) Distributivity of $\mathcal{K}$ over conjunction, $\mathcal{K}(A \,\&\, B) \supset \mathcal{K}A \,\&\, \mathcal{K}B$,

(ii) Factivity of knowledge, $\mathcal{K}A \supset A$.

Property (i) holds for operators that satisfy necessitation and the normality axiom in any system for normal modal logic. Factivity of knowledge is guaranteed by reflexivity of the accessibility relation, i.e., $xR_{\mathcal{K}}x$, for all possible worlds x. Through the method of conversion of axioms into sequent rules we obtain the following:

$$\frac{xR_{\mathcal{K}}x, \Gamma \to \Delta}{\Gamma \to \Delta}\, Ref_{\mathcal{K}}$$

Axiom **T** is shown derivable by this rule.

Properties (i) and (ii) are provable in $\mathbf{G3I}_{\mathcal{K}\Diamond}$ and in $\mathbf{G3I}_{\mathcal{K}\Diamond}$+$Ref_{\mathcal{K}}$, respectively. The following two lemmas single out the special instances of (i) and (ii) that are needed in the proof of the Church-Fitch paradox:

Lemma 5.1. *The following sequents are derivable in* $\mathbf{G3I}_{\mathcal{K}\Diamond}$*:*

1. $x : \mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to x : \mathcal{K}A$
2. $x : \mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to x : \mathcal{K}\neg\mathcal{K}A$

Lemma 5.2. *The sequent $x : \mathcal{K}\neg\mathcal{K}A \to x : \neg\mathcal{K}A$ is derivable in $\mathbf{G3I}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}}$.*

Proof. By the derivation

$$
\begin{prooftree}
\AxiomC{Lemma 3.5}
\UnaryInfC{$x : \neg\mathcal{K}A, xR_{\mathcal{K}}x, x : \mathcal{K}\neg\mathcal{K}A \to x : \neg\mathcal{K}A$}
\RightLabel{$L\mathcal{K}$}
\UnaryInfC{$xR_{\mathcal{K}}x, x : \mathcal{K}\neg\mathcal{K}A \to x : \neg\mathcal{K}A$}
\RightLabel{$Ref_{\mathcal{K}}$}
\UnaryInfC{$x : \mathcal{K}\neg\mathcal{K}A \to x : \neg\mathcal{K}A$}
\end{prooftree}
$$

QED

No further explicit conditions beyond $R_{\Diamond}$, $L_{\Diamond}$, and $Mon_{\Diamond}$ need be imposed on $\Diamond$ to reconstruct the standard proof of the Church-Fitch paradox.

By a proof analogous to that of Lemma 2.2, we have:

Lemma 5.3. *The following rule*

$$
\frac{\to x : A \,\&\, \neg\mathcal{K}A \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)}{x \leqslant y, y : A \,\&\, \neg\mathcal{K}A \to y : \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)}\, Inv
$$

is derivable in $\mathbf{G3I}_{\mathcal{K}\Diamond} + Cut$.

The use of classical logic is a further requirement for obtaining the standard proof of **OP**. In particular, the following lemma has to be proved.

Lemma 5.4. *The sequent $x : \neg(A \,\&\, \neg\mathcal{K}A) \to x : A \supset \mathcal{K}A$ is derivable in $\mathbf{G3C}_{\mathcal{K}\Diamond}$.*

Proof. By root-first proof search from the sequent to be derived. Note that the proof is classical because it makes appeal to a non-eliminable application of rule *Sym*. QED

We have now all the information that is needed to reconstruct the standard derivation of the Church-Fitch paradox.

Theorem 5.5. Fitch's paradox. *The sequent $\to x : A \supset \mathcal{K}A$ is derivable in $\mathbf{G3C}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}} + Cut + \mathbf{KP}$.*

Proof. By the following derivation:

$$
\begin{prooftree}
\AxiomC{$\mathbf{KP}$}
\noLine
\UnaryInfC{$\to x : A \,\&\, \neg\mathcal{K}A \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)$}
\RightLabel{Inv}
\UnaryInfC{$x \leqslant y, y : A \,\&\, \neg\mathcal{K}A \to y : \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)$}
\AxiomC{Lemma 5.1}
\UnaryInfC{$z : \mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to z : \mathcal{K}A$}
\RightLabel{$R\text{-}Wk$}
\UnaryInfC{$yR_{\Diamond}z, z : \mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to z : \mathcal{K}A, y : \bot$}
\AxiomC{Lemma 5.1}
\UnaryInfC{$z : \mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to z : \mathcal{K}\neg\mathcal{K}A$}
\AxiomC{Lemma 5.2}
\UnaryInfC{$z : \mathcal{K}\neg\mathcal{K}A \to z : \neg\mathcal{K}A$}
\RightLabel{Cut}
\BinaryInfC{$z : \mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to z : \neg\mathcal{K}A$}
\RightLabel{$R{\supset}Inv$}
\UnaryInfC{$z \leqslant y, z : \mathcal{K}(A \,\&\, \neg\mathcal{K}A), y : \mathcal{K}A \to y : \bot$}
\AxiomC{}
\RightLabel{$L\bot$}
\UnaryInfC{$y : \bot \to$}
\RightLabel{Cut}
\BinaryInfC{$z \leqslant y, z : \mathcal{K}(A \,\&\, \neg\mathcal{K}A), y : \mathcal{K}A \to$}
\RightLabel{$Subst\ z/y$}
\UnaryInfC{$z \leqslant z, z : \mathcal{K}(A \,\&\, \neg\mathcal{K}A), z : \mathcal{K}A \to$}
\RightLabel{Cut}
\BinaryInfC{$z \leqslant z, yR_{\Diamond}z, z : \mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to y : \bot$}
\RightLabel{$Ref_{\leqslant}$}
\UnaryInfC{$yR_{\Diamond}z, z : \mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to y : \bot$}
\RightLabel{$L\Diamond$}
\UnaryInfC{$y : \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to y : \bot$}
\RightLabel{Cut}
\BinaryInfC{$x \leqslant y, y : A \,\&\, \neg\mathcal{K}A \to y : \bot$}
\RightLabel{$R{\supset}$}
\UnaryInfC{$\to x : \neg(A \,\&\, \neg\mathcal{K}A)$}
\AxiomC{Lemma 5.4}
\UnaryInfC{$x : \neg(A \,\&\, \neg\mathcal{K}A) \to x : A \supset \mathcal{K}A$}
\RightLabel{Cut}
\BinaryInfC{$\to x : A \supset \mathcal{K}A$}
\end{prooftree}
$$

The applications of the rule of weakening are eliminable by pushing them up to the initial sequents of the derivations used for the proof of Lemma 5.1.

QED

Theorem 5.5 states a *derivability* result: There is a derivation of **OP** from **KP** by means of the rules of $\mathbf{G3C}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}}$+*Cut:*

$$\vdash_{\mathbf{G3C}_{\mathcal{K}\Diamond}+Ref_{\mathcal{K}}+Cut+\mathbf{KP}} \rightarrow x : A \supset \mathcal{K}A \tag{1}$$

In this result, **KP** plays the role of a derivation principle, similar to a zero-premiss inference rule; nonetheless, a crucial difference remains. On the one hand, the inference rules are valid in the sense that they respect the deductive *harmony* imposed by the inversion principle, as it is stated in Negri and von Plato (2001, p. 6). On the other hand, the validity of **KP** is fixed by stipulation, because, at the syntactical level, there is nothing that differentiates **KP** from another sentence of the bimodal language under analysis. A crucial step of our work will be to understand which class of relational structures **KP** singles out, so to determine as well in which class of models **KP** can be considered as formally true.

5.2 Structural analysis of the Church-Fitch paradox

There are two special aspects of the proof of Theorem 5.5:

1. The instance of **KP** appears in the derivation in the form of an axiomatic sequent $\rightarrow A$.[4]
2. The proof uses cuts.

The presence of cuts makes it difficult to point out where the paradox arises from, in the first place because the structure of such derivations is not transparent. Secondly, by a thesis of Tennant's, a paradox is a non-normal derivation the normalization of which enters into a loop (Tennant 1982). In sequent calculus, the notion of normalization is replaced by cut elimination that becomes the essential means for analyzing the precise nature of the paradox, and for distinguishing the case of a derivation without eliminability of cut from that of a fallacy, in which latter the assumption and the paradoxical conclusion are equivalent principles. Applying the cut elimination procedure for $\mathbf{G3C}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}}$ to our derivation of **OP**, we obtain the following derivation in which, to save space, we have abbreviated as $KP(A)$ the formula $(A \,\&\, \neg\mathcal{K}A) \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)$:

$$\dfrac{\dfrac{\mathbf{KP}}{\rightarrow x : KP(A)} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ S_1\end{matrix} \qquad \begin{matrix}\vdots \\ S_2\end{matrix}}{x \leqslant y, yR_{\mathcal{K}}z, x : KP(A), y : A \rightarrow z : A}\,L\supset}{x \leqslant y, x : KP(A), y : A \rightarrow y : \mathcal{K}A}\,R\mathcal{K}}{x : KP(A) \rightarrow x : A \supset \mathcal{K}A}\,R\supset}{\rightarrow x : A \supset \mathcal{K}A}\,Cut$$

The right premiss S_2 of $L\supset$ is derivable as follows:

[4] Cf. Definition 6.3.1(a) in Negri and von Plato (2001, p. 134).

$$
\begin{array}{c}
\text{Lemma 3.5}\\ \hline
x \leqslant y, yR_{\mathcal{K}}z, yR_{\Diamond}w, wR_{\mathcal{K}}w, wR_{\mathcal{K}}t, w : A, t : A, t : \neg\mathcal{K}A, w : \mathcal{K}(A \,\&\, \neg\mathcal{K}A), x : KP(A), y : A \rightarrow z : A, t : A \\ \hline
x \leqslant y, yR_{\mathcal{K}}z, yR_{\Diamond}w, wR_{\mathcal{K}}w, wR_{\mathcal{K}}t, w : A, t : A \,\&\, \neg\mathcal{K}A, w : \mathcal{K}(A \,\&\, \neg\mathcal{K}A), x : KP(A), y : A \rightarrow z : A, t : A \quad {\scriptstyle L\&} \\ \hline
x \leqslant y, yR_{\mathcal{K}}z, yR_{\Diamond}w, wR_{\mathcal{K}}w, wR_{\mathcal{K}}t, w : A, w : \mathcal{K}(A \,\&\, \neg\mathcal{K}A), x : KP(A), y : A \rightarrow z : A, t : A \quad {\scriptstyle L\mathcal{K}} \\ \hline
x \leqslant y, yR_{\mathcal{K}}z, yR_{\Diamond}w, wR_{\mathcal{K}}w, w : A, w : \mathcal{K}(A \,\&\, \neg\mathcal{K}A), x : KP(A), y : A \rightarrow z : A, w : \mathcal{K}A \quad {\scriptstyle R\mathcal{K}} \\ \hline
x \leqslant y, yR_{\mathcal{K}}z, yR_{\Diamond}w, wR_{\mathcal{K}}w, w : A, w : \neg\mathcal{K}A, w : \mathcal{K}(A \,\&\, \neg\mathcal{K}A), x : KP(A), y : A \rightarrow z : A \quad {\scriptstyle L\supset} \\ \hline
x \leqslant y, yR_{\mathcal{K}}z, yR_{\Diamond}w, wR_{\mathcal{K}}w, w : A \,\&\, \neg\mathcal{K}A, w : \mathcal{K}(A \,\&\, \neg\mathcal{K}A), x : KP(A), y : A \rightarrow z : A \quad {\scriptstyle L\&} \\ \hline
x \leqslant y, yR_{\mathcal{K}}z, yR_{\Diamond}w, wR_{\mathcal{K}}w, w : \mathcal{K}(A \,\&\, \neg\mathcal{K}A), x : KP(A), y : A \rightarrow z : A \quad {\scriptstyle L\mathcal{K}} \\ \hline
x \leqslant y, yR_{\mathcal{K}}z, yR_{\Diamond}w, w : \mathcal{K}(A \,\&\, \neg\mathcal{K}A), x : KP(A), y : A \rightarrow z : A \quad {\scriptstyle Ref_{\mathcal{K}}} \\ \hline
x \leqslant y, yR_{\mathcal{K}}z, y : \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A), x : KP(A), y : A \rightarrow z : A \quad {\scriptstyle L\Diamond}
\end{array}
$$

The right premiss of $L\supset$ in the derivation of S_2 is derivable because it is an instance of $L\bot$, left unwritten here. The left premiss S_1 is derivable:

$$
\dfrac{
\begin{array}{c}
\text{Lemma 3.5}\\ \hline
x \leqslant y, yR_{\mathcal{K}}z, y : A \rightarrow z : A, y : A
\end{array}
\qquad
\begin{array}{c}
\text{Lemma 3.5}\\ \hline
rR_{\mathcal{K}}z, x \leqslant y, yR_{\mathcal{K}}z, y \leqslant r, r \leqslant y, y : A, r : \mathcal{K}A, z : A \rightarrow z : A, r : \bot \\ \hline
rR_{\mathcal{K}}z, x \leqslant y, yR_{\mathcal{K}}z, y \leqslant r, r \leqslant y, y : A, r : \mathcal{K}A \rightarrow z : A, r : \bot \quad {\scriptstyle L\mathcal{K}} \\ \hline
x \leqslant y, yR_{\mathcal{K}}z, y \leqslant r, r \leqslant y, y : A, r : \mathcal{K}A \rightarrow z : A, r : \bot \quad {\scriptstyle Mon_{\mathcal{K}}} \\ \hline
x \leqslant y, yR_{\mathcal{K}}z, y \leqslant r, y : A, r : \mathcal{K}A \rightarrow z : A, r : \bot \quad {\scriptstyle Sym_{\leqslant}} \\ \hline
x \leqslant y, yR_{\mathcal{K}}z, y : A \rightarrow z : A, y : \neg\mathcal{K}A \quad {\scriptstyle R\supset}
\end{array}
}{
x \leqslant y, yR_{\mathcal{K}}z, y : A \rightarrow z : A, y : A \,\&\, \neg\mathcal{K}A
}\ {\scriptstyle R\&}
$$

There remains one application of *Cut* in the derivation. Unlike the other instances of *Cut*, the last one is not eliminable because it depends on an instance of **KP** that behaves like a proper axiom. We shall discuss this aspect later.

From the previous proof, just by ignoring the last step, we obtain the following result:

Proposition 5.6. *The sequent*

$$x : A \,\&\, \neg\mathcal{K}A \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \rightarrow x : A \supset \mathcal{K}A$$

has a cut-free derivation in $\mathbf{G3C}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}}$.

The result can be stated briefly as follows:

> **OP** *is derivable from the special instance* $KP(A)$ *of* **KP**.

Moreover, we notice that classical logic is used only in the step of symmetry in the right branch of the derivation S_1. Therefore that branch, pruned just before the application of $Sym_{\leqslant}$, suggests a countermodel to the sequent of Proposition 5.6 in the intuitionistic system $\mathbf{G3I}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}}$:

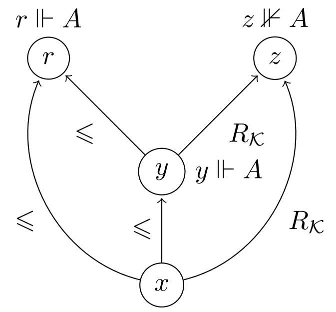
Theorem 5.7. *The sequent*

$$x : A \,\&\, \neg\mathcal{K}A \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \rightarrow x : A \supset \mathcal{K}A$$

is not derivable in $\mathbf{G3I}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}}$.

Proof. Consider the model $\langle W, \leqslant, R_{\mathcal{K}}, R_{\Diamond}, \mathcal{V}\rangle$ where $W = \{x, y, z, r\}$, $x \leqslant y$, $y \leqslant r$, $x \leqslant r$, $yR_{\mathcal{K}}z$, $xR_{\mathcal{K}}z$, all the reflexivities for $\leqslant$ and $R_{\mathcal{K}}$ hold, and A is forced in y and in r but not in z.

A diagrammatic representation, with the omission of the reflexive arrows, takes the form



In this model, we have that $x \Vdash A \,\&\, \neg\mathcal{K}A \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)$ because $y, r \nVdash A \,\&\, \neg\mathcal{K}A$. To see why, just observe that $r \Vdash \mathcal{K}A$ and use the definitions to conclude that r, and therefore also y, does not force $\neg\mathcal{K}A$. On the other hand, $x \nVdash A \supset \mathcal{K}A$ because $y \Vdash A$ but $y \nVdash \mathcal{K}A$. QED

It is well known that one can obtain a derivation of the weak **OP** in the intuitionistic system. More precisely, a cut-free derivation of **WOP** from the assumption $KP(A)$ is obtained in our system as follows:

Theorem 5.8. *The sequent*

$$x : A \,\&\, \neg\mathcal{K}A \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to x : \neg(A \,\&\, \neg\mathcal{K}A)$$

is derivable in $\mathbf{G3I}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}}$.

Proof. See Appendix E. QED

The countermodel of Theorem 5.7 shows, together with the completeness theorem, that the classical version of the Church-Fitch paradox is not derivable in an intuitionistic setting, thus seemingly confirming the thesis that intuitionistic logic saves anti-realism from the threat of the paradox (Williamson 1982).

To say that **KP** implies **OP** does not require that there is a deduction from a special instance of **KP** to the conclusion **OP**, as in Proposition 5.6. In fact, the admission of the knowability principle corresponds to the assumption that **KP** is generally valid, instead of the assumption of just a particular instance. Therefore, the following *admissibility* statement should be put under analysis:

If **KP** *is valid, then also* **OP** *is valid.* (2)

Merely to show that **OP** does not follow intuitionistically from a particular instance of **KP** is not sufficient for establishing that **OP** is not derivable in an intuitionistic system that incorporates **KP** as a *derivation principle*. In other words, the countermodel given in the proof of Theorem 5.7 is not sufficient for showing that (2) does not hold in an intuitionistic setting. An analogy from propositional logic may clarify this point: The law of double negation $\neg\neg A \supset A$ follows from the principle of excluded middle, $A \vee \neg A$, in the sense that there is an intuitionistic derivation of $(A \vee \neg A) \supset (\neg\neg A \supset A)$. The converse $(\neg\neg A \supset A) \supset (A \vee \neg A)$ instead is not intuitionistically derivable even if the two principles give equivalent extensions of intuitionistic logic. However, $A \vee \neg A$ follows from a particular instance of the law of double negation, namely $\neg\neg(A \vee \neg A) \supset (A \vee \neg A)$.

In conclusion, the cut-free analysis we have made suffices to establish intuitionistic underivability of **OP** from a particular instance of **KP**. The latter does not exclude, however, the derivability of **OP** from other instances of **KP**, a question to which a definite answer is given in the next section.

5.3 Proof analysis of KP

We proceed to find a frame property that is necessary and sufficient for the validity of **KP**. First, we use our calculus to single out frame rules that suffice for a derivation of **KP**. Then we extract from these rules a frame property and show that it is necessary and sufficient to validate **KP**.

We start root first from the sequent to be derived. Observe that the only applicable rule is $R\supset$. Next, for the proof search to continue, to be able to apply $R\Diamond$ it is necessary to have an $R_\Diamond$-accessibility. The only rules that make available such a relational atom in the absence of other $R_\Diamond$-atoms are $Ser_\Diamond$ and $Ref_\Diamond$. $Ser_\Diamond$ is derivable from $Ref_\Diamond$, and to make the set of assumptions on the accessibility relations minimal, we choose the former. Notice that $Ser_\Diamond$ has the variable restriction that y must not occur in the conclusion. After that, the only applicable rule is $R\mathcal{K}$. An initial sequent is then obtained if a rule is used that adds the atom $y \leqslant w$, indicated by $\Diamond\mathcal{K}\text{-}Tr$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \leqslant y, y \leqslant w, yR_\Diamond z, zR_\mathcal{K} w, y : A \to y : \Diamond\mathcal{K}A, w : A}{x \leqslant y, yR_\Diamond z, zR_\mathcal{K} w, y : A \to y : \Diamond\mathcal{K}A, w : A}\,\Diamond\mathcal{K}\text{-}Tr}{x \leqslant y, yR_\Diamond z, y : A \to y : \Diamond\mathcal{K}A, z : \mathcal{K}A}\,R\mathcal{K}}{x \leqslant y, yR_\Diamond z, y : A \to y : \Diamond\mathcal{K}A}\,R\Diamond}{x \leqslant y, y : A \to y : \Diamond\mathcal{K}A}\,Ser_\Diamond}{\to x : A \supset \Diamond\mathcal{K}A}\,R\supset
$$

This derivation would seem to suggest that the frame properties needed are those that correspond to the two extra-logical rules used, namely,

$$
\dfrac{xR_\Diamond y, \Gamma \to \Delta}{\Gamma \to \Delta}\,Ser_\Diamond \qquad \dfrac{x \leqslant z, xR_\Diamond y, yR_\mathcal{K} z, \Gamma \to \Delta}{xR_\Diamond y, yR_\mathcal{K} z, \Gamma \to \Delta}\,\Diamond\mathcal{K}\text{-}Tr
$$

Rule $Ser_\Diamond$ has the variable condition that $y \notin \Gamma, \Delta$, which corresponds to an existential condition, whereas rule $\Diamond\mathcal{K}\text{-}Tr$ corresponds to a universal one:

$$
\forall x \exists y \,.\, xR_\Diamond y \qquad \mathbf{Ser}_\Diamond
$$

$$
\forall y \forall z \forall w (yR_\Diamond z \,\&\, zR_\mathcal{K} w \supset y \leqslant w) \qquad \Diamond\mathcal{K}\text{-}\mathbf{Tr}
$$

The universal frame property $\Diamond\mathcal{K}$-**Tr** is, however, too strong: The instance of rule $\Diamond\mathcal{K}\text{-}Tr$ used in the derivation of **KP** is not applied, root first, to an arbitrary sequent, but to one in which the middle term is the eigenvariable introduced by $Ser_\Diamond$. The requirement that $\Diamond\mathcal{K}\text{-}Tr$ has to be applied above $Ser_\Diamond$ and that the middle term of $\Diamond\mathcal{K}\text{-}Tr$ is the eigenvariable of $Ser_\Diamond$, is the side condition of the rule.

Thus the following frame property can be read off from the derivation of **KP**:

$$\forall x \exists y (xR_\Diamond y \,\&\, \forall z(yR_\mathcal{K} z \supset x \leqslant z)) \quad \textbf{KP-Fr}$$

It is easy to show that **KP-Fr** is derivable in a G3-sequent system for intuitionistic first-order logic extended by the two rules $Ser_\Diamond$ and $\Diamond\mathcal{K}\text{-}Tr$:

$$\dfrac{\dfrac{\dfrac{\dfrac{xR_\Diamond y \to xR_\Diamond y \qquad \dfrac{\dfrac{\dfrac{\dfrac{x \leqslant z, xR_\Diamond y, yR_\mathcal{K} z \to x \leqslant z}{xR_\Diamond y, yR_\mathcal{K} z \to x \leqslant z}\,{\scriptstyle \Diamond\mathcal{K}\text{-}Tr}}{xR_\Diamond y \to yR_\mathcal{K} z \supset x \leqslant z}\,{\scriptstyle R\supset}}{xR_\Diamond y \to \forall z(yR_\mathcal{K} z \supset x \leqslant z)}\,{\scriptstyle R\forall}}{xR_\Diamond y \to xR_\Diamond y \,\&\, \forall z(yR_\mathcal{K} z \supset x \leqslant z)}\,{\scriptstyle R\&}}{xR_\Diamond y \to \exists y(xR_\Diamond y \,\&\, \forall z(yR_\mathcal{K} z \supset x \leqslant z))}\,{\scriptstyle R\exists}}{\to \exists y(xR_\Diamond y \,\&\, \forall z(yR_\mathcal{K} z \supset x \leqslant z))}\,{\scriptstyle Ser_\Diamond}}{\to \forall x \exists y(xR_\Diamond y \,\&\, \forall z(yR_\mathcal{K} z \supset x \leqslant z))}\,{\scriptstyle R\forall}$$

Observe that the side condition on the application of $\Diamond\mathcal{K}\text{-}Tr$ is respected. Conversely, any derivation that uses the rules $Ser_\Diamond$ and $\Diamond\mathcal{K}\text{-}Tr$ in compliance with the side condition, can be transformed into a derivation that uses cuts with **KP-Fr**. If rule $\Diamond\mathcal{K}\text{-}Tr$ is used, it is followed by $Ser_\Diamond$ because of the side condition, and the derivation contains a subderivation of the form

$$\dfrac{x \leqslant z, xR_\Diamond y, yR_\mathcal{K} z, \Gamma' \to \Delta'}{xR_\Diamond y, yR_\mathcal{K} z, \Gamma' \to \Delta'}\,{\scriptstyle \Diamond\mathcal{K}\text{-}Tr}$$

$$\vdots\; \mathcal{D}\; \vdots$$

$$\dfrac{xR_\Diamond y, \Gamma \to \Delta}{\Gamma \to \Delta}\,{\scriptstyle Ser_\Diamond}$$

We transform it as follows:

$$\dfrac{\dfrac{\dfrac{\overline{yR_\mathcal{K} z, yR_\mathcal{K} z \supset x \leqslant z \to x \leqslant z} \qquad x \leqslant z, xR_\Diamond y, yR_\mathcal{K} z, \Gamma' \to \Delta'}{xR_\Diamond y, yR_\mathcal{K} z, yR_\mathcal{K} z, yR_\mathcal{K} z \supset x \leqslant z, \Gamma', \to, \Delta'}\,{\scriptstyle Cut}}{xR_\Diamond y, yR_\mathcal{K} z, yR_\mathcal{K} z \supset x \leqslant z, \Gamma', \to, \Delta'}\,{\scriptstyle Ctr}}{xR_\Diamond y, yR_\mathcal{K} z, \forall z(yR_\mathcal{K} z \supset x \leqslant z), \Gamma', \to, \Delta'}\,{\scriptstyle L\forall}$$

$$\vdots\; \mathcal{D}'\; \vdots$$

$$\dfrac{\to \forall x \exists y(xR_\Diamond y \,\&\, \forall z(yR_\mathcal{K} z \supset x \leqslant z)) \qquad \dfrac{\dfrac{\dfrac{xR_\Diamond y, \forall z(yR_\mathcal{K} z \supset x \leqslant z), \Gamma \to \Delta}{xR_\Diamond y \,\&\, \forall z(yR_\mathcal{K} z \supset x \leqslant z), \Gamma \to \Delta}\,{\scriptstyle L\&}}{\exists y(xR_\Diamond y \,\&\, \forall z(yR_\mathcal{K} z \supset x \leqslant z), \Gamma \to \Delta}\,{\scriptstyle L\exists}}{\forall x \exists y(xR_\Diamond y \,\&\, \forall z(yR_\mathcal{K} z \supset x \leqslant z), \Gamma \to \Delta}\,{\scriptstyle L\forall}}{\Gamma \to \Delta}\,{\scriptstyle Cut}$$

Here $\mathcal{D}'$ is obtained by adding $\forall z(yR_\mathcal{K} z \supset x \leqslant z)$ to all the antecedents of the sequents in $\mathcal{D}$. If rule $Ser_\Diamond$ is used alone, namely without occurrences of $\Diamond\mathcal{K}\text{-}Tr$ above it, the conversion is obtained through $L\exists$ applied on the premiss of $Ser_\Diamond$ and a cut with $\to \forall x \exists y xR_\Diamond y$; the latter follows from $\to \forall x \exists y(xR_\Diamond y \,\&\, \forall z(yR_\mathcal{K} z \supset x \leqslant z))$. We therefore conclude:

Proposition 5.9. *The system with rules $\Diamond\mathcal{K}\text{-}Tr$ and $Ser_\Diamond$ that respect the side condition is a cut-free equivalent of the system that uses* **KP-Fr** *as an axiomatic sequent in addition to the structural rules.*

The rules that correspond to **KP-Fr** do not follow the geometric rule scheme. However, all the structural rules are still admissible in the presence of such rules. In particular, cut elimination holds and the proof follows the pattern of 3.10.

Theorem 5.10. *The rule of cut*

$$\frac{\Gamma \to \Delta, x : A \quad x : A, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'}\, Cut$$

is admissible in $\mathbf{G3I}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}}$ *extended by* $Ser_{\Diamond}$ *and* $\Diamond\mathcal{K}\text{-}Tr$.

Proof. Suppose that one of the premisses of cut has been derived by $\Diamond\mathcal{K}\text{-}Tr$ followed by $Ser_{\Diamond}$ and that the middle term of the former disappeared by an application of the latter. We have

$$\dfrac{\Gamma \to \Delta, x : A \qquad \dfrac{\begin{array}{c}\dfrac{y \leqslant z, xR_{\Diamond}y, yR_{\mathcal{K}}z, \Gamma'' \to \Delta''}{xR_{\Diamond}y, yR_{\mathcal{K}}z, \Gamma'' \to \Delta''}\,\Diamond\mathcal{K}\text{-}Tr \\ \vdots \\ x : A, xR_{\Diamond}y, \Gamma' \to \Delta'\end{array}}{x : A, \Gamma' \to \Delta'}\,Ser_{\Diamond}}{\Gamma, \Gamma' \to \Delta, \Delta'}\,Cut$$

Observe that by hp-admissibility of substitution (Lemma 3.6) we can assume without loss of generality that the variable y does not occur in the left premiss of cut. The derivation is transformed into the following in which the application of cut is of lower height and therefore eliminable by the inductive hypothesis:

$$\dfrac{\dfrac{\Gamma \to \Delta, x : A \qquad \begin{array}{c}\dfrac{x \leqslant z, xR_{\Diamond}y, yR_{\mathcal{K}}z, \Gamma'' \to \Delta''}{xR_{\Diamond}y, yR_{\mathcal{K}}z, \Gamma'' \to \Delta''}\,\Diamond\mathcal{K}\text{-}Tr \\ \vdots \\ x : A, xR_{\Diamond}y, \Gamma' \to \Delta'\end{array}}{xR_{\Diamond}y, \Gamma, \Gamma' \to \Delta, \Delta'}\,Cut}{\Gamma, \Gamma' \to \Delta, \Delta'}\,Ser_{\Diamond}$$

QED

It is worth noting that the acceptance **KP** as valid implicitly forces us to accept some properties of the operator $\Diamond$, in particular, the derivability of $A \supset \Diamond A$.

Proposition 5.11. *The sequent* $\to x : A \supset \Diamond A$ *is derivable in* $\mathbf{G3I}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}} + Ser_{\Diamond} + \Diamond\mathcal{K}\text{-}Tr$.

Proof. We have the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y \leqslant z, zR_{\mathcal{K}}z, yR_{\Diamond}z, x \leqslant y, y : A \to y : \Diamond A, z : A}{zR_{\mathcal{K}}z, yR_{\Diamond}z, x \leqslant y, y : A \to y : \Diamond A, z : A}\,\Diamond\mathcal{K}\text{-}Tr}{yR_{\Diamond}z, x \leqslant y, y : A \to y : \Diamond A, z : A}\,Ref_{\mathcal{K}}}{yR_{\Diamond}z, x \leqslant y, y : A \to y : \Diamond A}\,R\Diamond}{x \leqslant y, y : A \to y : \Diamond A}\,Ser_{\Diamond}}{\to x : A \supset \Diamond A}\,R\supset$$

Observe that the side condition on $\Diamond\mathcal{K}\text{-}Tr$ is respected. QED

In monomodal systems, the axiom scheme $A \supset \Diamond A$ is characterized by reflexive frames, i.e., frames in which $\forall x \, . \, x R_{\Diamond} x$ holds. This is not any longer the case in multimodal systems. The above proposition shows, in fact, that the reflexivity of $R_{\Diamond}$ is a sufficient, but not a necessary, condition for the validity of $A \supset \Diamond A$. We have a derivation of a purely alethic property that uses properties of the global system, in particular, of the epistemic accessibility relation, thus a non-conservativity of the whole system with respect to the system without $\mathcal{K}$; To restore conservativity, we add to our set of rules the rule of reflexivity of the alethic accessibility relation:

$$\frac{x R_{\Diamond} x, \Gamma \to \Delta}{\Gamma \to \Delta}\, \mathit{Ref}_{\Diamond}$$

With $\mathit{Ref}_{\Diamond}$ at our disposal, it becomes clear why the unrestricted $\Diamond\mathcal{K}$-Tr is too strong: In fact, together with reflexivity of $R_{\Diamond}$ it would collapse our intuitionistic system into a classical one because it would permit to derive symmetry of $\leqslant$, as in

$$\cfrac{\cfrac{\cfrac{\cfrac{y \leqslant x, y R_{\Diamond} x, x R_{\mathcal{K}} x, x \leqslant y, x R_{\Diamond} x \to y \leqslant x}{y R_{\Diamond} x, x R_{\mathcal{K}} x, x \leqslant y, x R_{\Diamond} x \to y \leqslant x}\, \Diamond\mathcal{K}\text{-}\mathit{Tr}}{y R_{\Diamond} x, x \leqslant y, x R_{\Diamond} x \to y \leqslant x}\, \mathit{Ref}_{\mathcal{K}}}{x \leqslant y, x R_{\Diamond} x \to y \leqslant x}\, \mathit{Mon}_{\Diamond}}{x \leqslant y \to y \leqslant x}\, \mathit{Ref}_{\Diamond}$$

The derivation of the knowability principle in $\mathbf{G3I}_{\mathcal{K}\Diamond} + \mathit{Ref}_{\mathcal{K}} + \mathit{Ser}_{\Diamond} + \Diamond\mathcal{K}\text{-}\mathit{Tr}$ guarantees that the two rules are strong enough to capture the force of **KP** but does not yet permit to conclude that **KP-Fr** is the characterizing frame property of **KP**. This latter is achieved by the following:

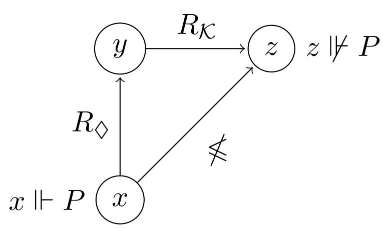
Proposition 5.12. *The property* **KP-Fr** *is necessary and sufficient to validate* **KP** *in intuitionistic bimodal frames.*

Proof. For sufficiency, it is enough to use the standard definition of forcing in Kripke models. Let x be a world in a frame. To show $x \Vdash A \supset \Diamond\mathcal{K}A$, let y be such that $x \leqslant y$, and suppose $y \Vdash A$. By **KP-Fr** and monotonicity of forcing, $y \Vdash \Diamond\mathcal{K}A$.

For necessity, we reason by contraposition. Consider an arbitrary frame and suppose that **KP-Fr** does not hold in it, i.e., that the following holds:

$$\exists x \forall y (x R_{\Diamond} y \supset \exists z (y R_{\mathcal{K}} z \,\&\, x \nleqslant z))$$

Let P be an arbitrary atomic formula. We can define a valuation that respects monotonicity by imposing the forcing of P in x, but not in z:



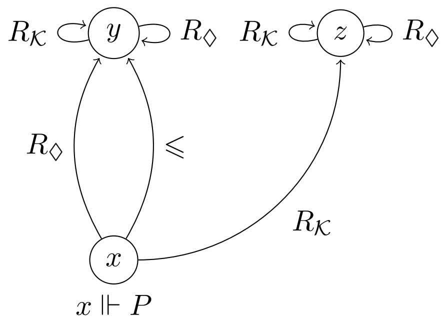
Here $x \Vdash P$ but $x \nVdash \Diamond\mathcal{K}P$, so this is a countermodel to **KP**. QED

We have achieved by our analysis a correspondence between the knowability principle in the form of the bimodal axiom **KP** and the frame property **KP-Fr**. We have also shown that **KP-Fr** is equivalent to the two rules $\Diamond\mathcal{K}\text{-}Tr$ and $\mathit{Ser}_{\Diamond}$ used in compliance with a side condition. By this equivalence, the system obtained by the addition of suitable combinations of these two rules provides a complete contraction- and cut-free system for intuitionistic bimodal logic extended by **KP**. We shall call it *knowability logic*, to be indicated by **G3KP**. Thus, **G3KP** is defined as $\mathbf{G3I}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}} + Ser_{\Diamond} + \Diamond\mathcal{K}\text{-}Tr$ with the side condition added.

By **KP-Fr**, we can establish the following result:

Proposition 5.13. *There exists an intuitionistic frame that validates* **KP***, but not* **OP***.*

Proof. Take the frame with three worlds, x, y, and z such that $xR_{\Diamond}y$, $x \leqslant y$, and $xR_{\mathcal{K}}z$. The only $R_{\mathcal{K}}$ and $R_{\Diamond}$-accessibilities from y and z are the reflexive ones:



The frame respects condition **KP-Fr** and therefore validates **KP** (this can be checked also directly). On the other hand, the valuation defined by $x \Vdash P$ and $z \nVdash P$ shows that $x \nVdash P \supset \mathcal{K}P$.

QED

This result shows that the admissibility statement (2) of Section 5.2 does not hold for intuitionistic logic. Our proof system gives a confirmation for this semantic argument through a syntactic criterion, a failed exhaustive proof search.

To show that **OP** is not derivable in **G3KP**, care is needed with the use of labels. Consider the following attempt:

$$\cfrac{\cfrac{\cfrac{\cfrac{y \leqslant z, x \leqslant y, yR_{\Diamond}y, yR_{\mathcal{K}}z, y : A \to z : A}{x \leqslant y, yR_{\Diamond}y, yR_{\mathcal{K}}z, y : A \to z : A}\;\scriptstyle \Diamond\mathcal{K}\text{-}Tr}{x \leqslant y, yR_{\Diamond}y, y : A \to y : \mathcal{K}A}\;\scriptstyle R\mathcal{K}}{x \leqslant y, y : A \to y : \mathcal{K}A}\;\scriptstyle Ref_{\Diamond}}{\to x : A \supset \mathcal{K}A}\;\scriptstyle R\supset$$

This would seem to be a derivation of **OP**, in contrast to what one would expect from the semantic argument above. Here, similarly to what happened in the derivation of symmetry, the application of $\Diamond\mathcal{K}\text{-}Tr$ is not correct because the eigenvariable (here y) appears also where it shouldn't, namely as a first argument of $R_{\Diamond}$, in the preorder atom, and in two labelled formulas. The variable condition expresses formally that rule $\Diamond\mathcal{K}\text{-}Tr$ should consider the most general $R_{\Diamond}$ accessibility; by admitting only the reflexive one, actuality and possibility are conflated with the "the mystery of the disappearing diamond" (Jenkins 2009) as a consequence. Replacing reflexivity with seriality, the search turns into

$$\cfrac{\cfrac{\cfrac{\vdots}{x \leqslant y, yR_{\Diamond}z, yR_{\mathcal{K}}w, y : A \to w : A}}{\cfrac{x \leqslant y, yR_{\Diamond}z, y : A \to y : \mathcal{K}A}{x \leqslant y, y : A \to y : \mathcal{K}A}\;\scriptstyle Ser_{\Diamond}}\;\scriptstyle R\mathcal{K}}{\to x : A \supset \mathcal{K}A}\;\scriptstyle R\supset$$

Rule $\Diamond\mathcal{K}$-Tr is no longer applicable because the upper sequent in the attempted proof does not match its conclusion. The only applicable rule is $Mon_{\mathcal{K}}$ that adds $xR_{\mathcal{K}}w$. The search is exhaustive and we do not get what we would need to close it, namely the relational atom $y \leqslant w$. The failed search can be used instead to extract a countermodel to **OP**. The accessibilities are $xR_{\mathcal{K}}w$ in addition to those in the antecedent of the upper sequent; A is forced at x and at y but not at w. Clearly $x \nVdash A \supset \mathcal{K}A$. By our analysis, the use of intuitionistic logic blocks the paradox in general, not only the specific derivation that uses a special instance of the knowability principle (see Theorem 5.7).

It is only in classical logic that the paradox may arise. The question remains as to whether the Moore sentence $A \,\&\, \neg\mathcal{K}A$ is an essential ingredient of the paradox in its classical derivation. It is a natural question, because of the attempts at circumventing the paradox through a limitation of **KP** to certain classes of formulas that exclude seemingly pathological ones such as $A \,\&\, \neg\mathcal{K}A$ (as in Dummett 2001).

Whether Moore sentences are indispensable in the derivation of **OP** can be determined by a root-first proof search. The search in our calculus leads to a sufficient condition for the derivation of **OP**, starting with the "compulsory" steps

$$\dfrac{\dfrac{\vdots \atop yR_{\mathcal{K}}z, x \leqslant y, y : A \to z : A}{x \leqslant y, y : A \to y : \mathcal{K}A}\,R\mathcal{K}}{\to x : A \supset \mathcal{K}A}\,R\supset$$

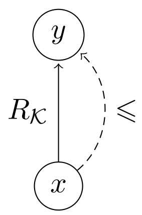
A correct derivation is obtained if the preorder atom $y \leqslant z$ can be added, that is, if $\mathcal{K}$-accessibility implies $\leqslant$-accessibility, or, in other words, if we are allowed to use the following rule:

$$\dfrac{x \leqslant y, xR_{\mathcal{K}}y, \Gamma \to \Delta}{xR_{\mathcal{K}}y, \Gamma \to \Delta}\,Know$$

The rule is the translation of the frame property

$$\forall x \forall y (xR_{\mathcal{K}}y \supset x \leqslant y) \quad \textbf{Know}$$

As a diagram, it takes the form



We then have

Proposition 5.14. *Rule* Know *is admissible in* $\mathbf{G3C}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}} + Ser_{\Diamond} + \Diamond\mathcal{K}$-$Tr$.

Proof. Using admissibility of weakening, we have the derivation

$$
\begin{array}{cl}
x \leqslant y, xR_{\mathcal{K}}y, \Gamma \to \Delta & \\ \cline{1-1}
x \leqslant y, z \leqslant x, x \leqslant z, xR_{\Diamond}z, zR_{\mathcal{K}}y, zR_{\mathcal{K}}z, xR_{\mathcal{K}}y, \Gamma \to \Delta & L\text{-}Wk_R \\ \cline{1-1}
z \leqslant x, x \leqslant z, xR_{\Diamond}z, zR_{\mathcal{K}}y, zR_{\mathcal{K}}z, xR_{\mathcal{K}}y, \Gamma \to \Delta & \Diamond\mathcal{K}\text{-}Tr \\ \cline{1-1}
z \leqslant x, x \leqslant z, xR_{\Diamond}z, zR_{\mathcal{K}}z, xR_{\mathcal{K}}y, \Gamma \to \Delta & Mon_{\mathcal{K}} \\ \cline{1-1}
x \leqslant z, xR_{\Diamond}z, zR_{\mathcal{K}}z, xR_{\mathcal{K}}y, \Gamma \to \Delta & Sym \leqslant \\ \cline{1-1}
xR_{\Diamond}z, zR_{\mathcal{K}}z, xR_{\mathcal{K}}y, \Gamma \to \Delta & \Diamond\mathcal{K}\text{-}Tr \\ \cline{1-1}
xR_{\Diamond}z, xR_{\mathcal{K}}y, \Gamma \to \Delta & Ref_{\mathcal{K}} \\ \cline{1-1}
xR_{\mathcal{K}}y, \Gamma \to \Delta & Ser_{\Diamond}
\end{array}
$$

Observe that two applications of $\Diamond\mathcal{K}$-Tr, with the same eigenvariable z, are used. This is a licit use of the block of rules, because a double use of $\Diamond\mathcal{K}$-Tr, followed by a step of seriality that removes the eigenvariable, corresponds to a multiple discharge of the minor assumption in the rule of elimination of the existential quantifier in natural deduction. QED

The derivation of **OP** can be given also directly in the system $\mathbf{G3C}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}} + Ser_{\Diamond} + \Diamond\mathcal{K}\text{-}Tr$:

$$
\begin{array}{cl}
x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{K}}z, y \leqslant z, z \leqslant y, yR_{\mathcal{K}}w, zR_{\mathcal{K}}w, y \leqslant w, y : A \to w : A & \\ \cline{1-1}
x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{K}}z, y \leqslant z, z \leqslant y, yR_{\mathcal{K}}w, zR_{\mathcal{K}}w, y : A \to w : A & \Diamond\mathcal{K}\text{-}Tr \\ \cline{1-1}
x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{K}}z, y \leqslant z, z \leqslant y, yR_{\mathcal{K}}w, y : A \to w : A & Mon_{\mathcal{K}} \\ \cline{1-1}
x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{K}}z, y \leqslant z, z \leqslant y, y : A \to y : \mathcal{K}A & R\mathcal{K} \\ \cline{1-1}
x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{K}}z, y \leqslant z, y : A \to y : \mathcal{K}A & Sym_{\leqslant} \\ \cline{1-1}
x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{K}}z, y : A \to y : \mathcal{K}A & \Diamond\mathcal{K}\text{-}Tr \\ \cline{1-1}
x \leqslant y, yR_{\Diamond}z, y : A \to y : \mathcal{K}A & Ref_{\mathcal{K}} \\ \cline{1-1}
x \leqslant y, y : A \to y : \mathcal{K}A & Ser_{\Diamond} \\ \cline{1-1}
\to x : A \supset \mathcal{K}A & R\supset
\end{array}
$$

There are no occurrences of Moore sentences in this derivation. Could we then conclude that it is not necessary to appeal to them for the derivation of **OP**? Actually, the absence of Moore sentences in this derivation is only fictitious, because **KP-Fr** has been identified by considering all possible instances of **KP** and so, *a fortiori*, also the instances with Moore sentences. On the contrary, allowing only a limited type of instances of **KP**, it could be the case that we are restricting also the class of frames validating **KP** and that these particular frames would not validate **OP**.

The cut-free derivation indicates that the source of the paradox has to be found in the joint use of **KP-Fr** and classical logic. This means that **KP** is not *per se* paradoxical, but becomes so when used in a classical frame. Moreover, it is possible to show in classical logic that if $Ref_{\Diamond}$ is included from the beginning in the derivation system (cf. the conservativity argument before Proposition 5.12), then **KP** corresponds to the frame condition **Know**.

Proposition 5.15. *The frame property* **Know** *is necessary and sufficient to validate* **KP** *in classical bimodal frames that satisfy* $Ref_{\Diamond}$.

Proof. Necessity can be proved by the following chain of implications: Validity of **KP** implies validity of **KP-Fr** (Proposition 5.12); validity of **KP-Fr** implies admissibility of $Ser_{\Diamond}$ and $\Diamond\mathcal{K}$-Tr with the side condition (Proposition 5.9); $Ser_{\Diamond}$ and $\Diamond\mathcal{K}$-Tr with the side condition imply admissibility of *Know* (Proposition 5.14).

For sufficiency, consider the derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y \leqslant z, yR_{\mathcal{K}}z, yR_{\Diamond}y, x \leqslant y, y : A \to y : \Diamond\mathcal{K}A, z : A}{yR_{\mathcal{K}}z, yR_{\Diamond}y, x \leqslant y, y : A \to y : \Diamond\mathcal{K}A, z : A}\,{\scriptstyle Know}}{yR_{\Diamond}y, x \leqslant y, y : A \to y : \Diamond\mathcal{K}A, y : \mathcal{K}A}\,{\scriptstyle R\mathcal{K}}}{yR_{\Diamond}y, x \leqslant y, y : A \to y : \Diamond\mathcal{K}A}\,{\scriptstyle R\Diamond}}{x \leqslant y, y : A \to y : \Diamond\mathcal{K}A}\,{\scriptstyle Ref_{\Diamond}}}{\to x : A \supset \Diamond\mathcal{K}A}\,{\scriptstyle R\supset}
$$

QED

As we have seen, in classical logic **Know** is sufficient for deriving **OP** and even has the collapse of truth and knowledge as a consequence:

Proposition 5.16. *In* $\mathbf{G3C}_{\mathcal{K}\Diamond} + Ref_{\mathcal{K}} + Ref_{\Diamond} + Know$*, the relations* $\leqslant$ *and* $R_{\mathcal{K}}$ *coincide.*

Proof. To preserve the monotonicity of $\leqslant$ in the presence of $R_{\mathcal{K}}$, we have assumed the validity of $\mathbf{Mon}_{\mathcal{K}}$. By reflexivity of $R_{\mathcal{K}}$, $\mathbf{Mon}_{\mathcal{K}}$ gives $\forall x \forall y(x \leqslant y \supset xR_{\mathcal{K}}y)$, i.e., $\leqslant \subseteq R_{\mathcal{K}}$. The other direction of the inclusion, i.e., $R_{\mathcal{K}} \subseteq \leqslant$, holds by **Know**. QED

We have thus shown that if $R_{\Diamond}$ is reflexive, truth and knowledge coincide in classical logic. Therefore, in the standard classical presentation of Fitch's paradox, the assumption **KP** is semantically equivalent to **OP**.

Finally, let us consider the indispensability of the principle of factivity of knowledge in the derivation of the Church-Fitch paradox. Mackie (1980) and Tennant (1997) have maintained that the principle is not necessary, and that the paradox arises equally for belief-like notions. That such is the case is confirmed by our analysis as follows: First it is seen that a knowability principle for belief imposes the same frame condition as it did for knowledge: The characterization result never employs the rule of reflexivity for epistemic accessibility. Then it can be shown that a "belief omniscience" is derivable when reflexivity for knowledge accessibility is replaced by seriality and transitivity for belief accessibility (the names of the rules are obtained from those for $\mathcal{K}$):

Proposition 5.17. *The sequent* $\to x : A \supset \mathcal{B}A$ *is derivable in* $\mathbf{G3C}_{\mathcal{B}\Diamond} + Ser_{\mathcal{B}} + Trans_{\mathcal{B}} + Ser_{\Diamond} + \Diamond\mathcal{B}\text{-}Tr$.

Proof. By the derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y \leqslant t, w \leqslant y, y \leqslant w, x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{B}}w, wR_{\mathcal{B}}t, zR_{\mathcal{B}}t, yR_{\mathcal{B}}t, y : A \to t : A}{w \leqslant y, y \leqslant w, x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{B}}w, wR_{\mathcal{B}}t, zR_{\mathcal{B}}t, yR_{\mathcal{B}}t, y : A \to t : A}\,{\scriptstyle \Diamond\mathcal{B}\text{-}Tr}}{w \leqslant y, y \leqslant w, x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{B}}w, wR_{\mathcal{B}}t, yR_{\mathcal{B}}t, y : A \to t : A}\,{\scriptstyle Trans_{\mathcal{B}}}}{w \leqslant y, y \leqslant w, x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{B}}w, yR_{\mathcal{B}}t, y : A \to t : A}\,{\scriptstyle Mon_{\mathcal{B}}}}{y \leqslant w, x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{B}}w, yR_{\mathcal{B}}t, y : A \to t : A}\,{\scriptstyle Sym_{\leqslant}}}{y \leqslant w, x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{B}}w, y : A \to y : \mathcal{B}A}\,{\scriptstyle R\mathcal{B}}}{x \leqslant y, yR_{\Diamond}z, zR_{\mathcal{B}}w, y : A \to y : \mathcal{B}A}\,{\scriptstyle \Diamond\mathcal{B}\text{-}Tr}}{x \leqslant y, yR_{\Diamond}z, y : A \to y : \mathcal{B}A}\,{\scriptstyle Ser_{\mathcal{B}}}}{x \leqslant y, y : A \to y : \mathcal{B}A}\,{\scriptstyle Ser_{\Diamond}}}{\to x : A \supset \mathcal{B}A}\,{\scriptstyle R\supset}
$$

QED

6 Final remarks

In Tennant (2009), the following is written about the prospects of a proof theory that covers the Church-Fitch paradox: "We are still a long way, of course, from having a fully adequate proof-theory governing the interaction among [the modalities involved] (let alone a formal semantics, with respect to which one might be able to establish the soundness and completeness of whatever proof system is devised)" (*ibid.*, p. 237).

The proof systems $\mathbf{G3I}_{\mathcal{K}\Diamond}$ and $\mathbf{G3C}_{\mathcal{K}\Diamond}$ developed in this paper, with the analysis of the accessibility relations $\leqslant$, $R_{\mathcal{K}}$, and $R_{\Diamond}$ and the way they interact in formal proofs, offer an answer to the first of Tennant's issues. The completeness theorem with respect to Kripke semantics for these calculi answers Tennant's second issue. The results are here formulated for labelled sequent calculi but can be adapted also to proof systems based on natural deduction.

Our work offers a new methodology for a general theory of knowability and, more broadly, of logical epistemology. We have determined the first-order correspondents of modal axioms on the basis of a root-first proof search in labelled sequent calculi for bimodal logic. The correspondence results have a standing independent of the use of labelled calculi. Extending a general Kripke completeness result, we have shown that the modal logic obtained by the addition of the knowability principle is complete with respect to the class of frames that satisfy the first-order frame condition which was determined by the procedure. The resulting calculi are complete proof systems for *knowability logic*, both in a classical and in an intuitionistic setting. The strong structural properties of these calculi make it possible to draw conclusions not only about questions of derivability, but also about underivability of the paradox in precisely defined formal systems of intuitionistic and classical bimodal logic. The crucial step here is the conversion of a non-geometric axiom, the frame condition that corresponds to **KP**, into a system of rules so as to achieve full control over derivations in intuitionistic bimodal logic extended by the knowability principle.

Exploiting the frame property that corresponds to **KP**, our work goes a step further, namely it shows that the use of intuitionistic logic for blocking the paradox succeeds: Not only is **OP** intuitionistically underivable from **KP** instantiated with the Moore sentence, but **OP** is not even intuitionistically admissible under **KP**. On the other hand, the paradox is indeed derivable in classical logic: the standard proof is reconstructed in our analysis and converted into a cut-free form. Nonetheless, we claim that this derivation is nothing else than a fallacious argument in disguise: The reason is that **KP** and **OP** are semantically equivalent in a classical frame.

We thus have an argument in favor of the anti-realist position, provided that the formalization of the knowability principle corresponds to **KP**. If anti-realism is conceived in a strict Dummettian sense, then intuitionistic logic is already sufficient for blocking Fitch's argument. Otherwise, if a weaker anti-realism is embraced and accordingly classical logic is allowed, the paradox gets reduced to a *petitio principii.*

The conversion of the frame property **KP-Fr** into a combination of rules governed by a side condition follows the methodology of proof analysis in which universal and geometrical axioms have been treated so far. It is a first successful attempt to extract a general method for transforming a much wider type of mathematical axioms into a set of inference rules. From this perspective, the proof-theoretical analysis of **KP** opens up promising possibilities also for a more traditional type of foundational study.

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Appendices

A Proof of Lemma 2.2

By the two derivations

$$\dfrac{\to (A \,\&\, \neg\mathcal{K}A) \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \qquad \dfrac{\dfrac{\overline{A \to A} \quad \overline{\neg\mathcal{K}A \to \neg\mathcal{K}A}}{A, \neg\mathcal{K}A \to A \,\&\, \neg\mathcal{K}A}{\scriptstyle R\&} \quad \overline{\Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)}}{A, \neg\mathcal{K}A, (A \,\&\, \neg\mathcal{K}A) \supset \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)}{\scriptstyle L\supset}}{A, \neg\mathcal{K}A \to \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A)}{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\dfrac{\overline{A, \neg\mathcal{K}A \to \neg\mathcal{K}A}}{A \,\&\, \neg\mathcal{K}A \to \neg\mathcal{K}A}{\scriptstyle L\&}}{\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to \mathcal{K}\neg\mathcal{K}A}{\scriptstyle LR\text{-}\mathcal{K}} \qquad \dfrac{\dfrac{\dfrac{\overline{A, \neg\mathcal{K}A \to A}}{A \,\&\, \neg\mathcal{K}A \to A}{\scriptstyle L\&}}{\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to \mathcal{K}A}{\scriptstyle LR\text{-}\mathcal{K}} \quad \mathcal{K}A, \mathcal{K}\neg\mathcal{K}A \to}{\mathcal{K}(A \,\&\, \neg\mathcal{K}A), \mathcal{K}\neg\mathcal{K}A \to}{\scriptstyle Cut}}{\mathcal{K}(A \,\&\, \neg\mathcal{K}A) \to}{\scriptstyle Cut}$$

The topmost sequents, except the premisses of the rules in question, are derivable by Theorem 2.1.

B Proof of Lemma 2.3

The result is obtained through a failed proof-search procedure: Start a derivation tree with the sequent $P \supset \Diamond\mathcal{K}P, Q \supset \Diamond\mathcal{K}Q \to P \,\&\, Q \supset \Diamond\mathcal{K}(P \,\&\, Q)$ as a root and apply backwards all the propositional rules:

$$\dfrac{\dfrac{\dfrac{Q \supset \Diamond\mathcal{K}Q, P, Q \to \Diamond\mathcal{K}(P \,\&\, Q), P \qquad \dfrac{\Diamond\mathcal{K}P, P, Q \to \Diamond\mathcal{K}(P \,\&\, Q), Q \quad \begin{matrix}\vdots\\ \Diamond\mathcal{K}P, \Diamond\mathcal{K}Q, P, Q \to \Diamond\mathcal{K}(P \,\&\, Q)\end{matrix}}{\Diamond\mathcal{K}P, Q \supset \Diamond\mathcal{K}Q, P, Q \to \Diamond\mathcal{K}(P \,\&\, Q)}{\scriptstyle L\supset}}{P \supset \Diamond\mathcal{K}P, Q \supset \Diamond\mathcal{K}Q, P, Q \to \Diamond\mathcal{K}(P \,\&\, Q)}{\scriptstyle L\supset}}{P \supset \Diamond\mathcal{K}P, Q \supset \Diamond\mathcal{K}Q, P \,\&\, Q \to \Diamond\mathcal{K}(P \,\&\, Q)}{\scriptstyle L\&}}{P \supset \Diamond\mathcal{K}P, Q \supset \Diamond\mathcal{K}Q \to P \,\&\, Q \supset \Diamond\mathcal{K}(P \,\&\, Q)}{\scriptstyle R\supset}$$

Since the rules used are invertible, there is no need of backtracking. The left premisses of the two steps of $L\supset$ are initial sequents, and therefore derivability of the sequent is equivalent to derivability of the rightmost sequent, $\Diamond\mathcal{K}P, \Diamond\mathcal{K}Q, P, Q \to \Diamond\mathcal{K}(P \,\&\, Q)$. Proof search for the latter can be effected in two ways, depending on the choice of principal formula in LR-$\Diamond$, each followed by an application of LR-$\mathcal{K}$. In one case it leads to the sequent $Q \to P \,\&\, Q$, in the other to $P \to P \,\&\, Q$. Since both are underivable, the proof search fails.

C Structural properties of $\mathbf{G3I}_{\mathcal{K}\Diamond}$

C.1 Proof of Lemma 3.4

We have the following derivations for the former and the latter sequent, respectively:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Lemma 3.5}}{z \leqslant z, z : B, xR_{\mathcal{K}}z, x \leqslant y, yR_{\mathcal{K}}z, x : \mathcal{K}B, \Gamma \to \Delta, z : B}}{z : B, xR_{\mathcal{K}}z, x \leqslant y, yR_{\mathcal{K}}z, x : \mathcal{K}B, \Gamma \to \Delta, z : B}\,Ref_{\leqslant}}{xR_{\mathcal{K}}z, x \leqslant y, yR_{\mathcal{K}}z, x : \mathcal{K}B, \Gamma \to \Delta, z : B}\,L\mathcal{K}}{x \leqslant y, yR_{\mathcal{K}}z, x : \mathcal{K}B, \Gamma \to \Delta, z : B}\,Mon_{\mathcal{K}}}{x \leqslant y, x : \mathcal{K}B, \Gamma \to \Delta, y : \mathcal{K}B}\,R\mathcal{K}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Lemma 3.5}}{z \leqslant z, yR_{\Diamond}z, x \leqslant y, xR_{\Diamond}z, z : B, \Gamma \to \Delta, y : \Diamond B, z : B}}{yR_{\Diamond}z, x \leqslant y, xR_{\Diamond}z, z : B, \Gamma \to \Delta, y : \Diamond B, z : B}\,Ref_{\leqslant}}{yR_{\Diamond}z, x \leqslant y, xR_{\Diamond}z, z : B, \Gamma \to \Delta, y : \Diamond B}\,R\Diamond}{x \leqslant y, xR_{\Diamond}z, z : B, \Gamma \to \Delta, y : \Diamond B}\,Mon_{\Diamond}}{x \leqslant y, x : \Diamond B, \Gamma \to \Delta, y : \Diamond B}\,L\Diamond$$

C.2 Proof of Lemma 3.6

By induction on the height h of the derivation of the premiss. If $h = 0$ and the substitution is not vacuous, then $\Gamma \to \Delta$ is $x \leqslant y, x : P, \Gamma' \to \Delta', y : P$ or $x : \bot, \Gamma' \to \Delta$. In each case, by applying *Subst* we obtain an initial sequent or a conclusion of $L\bot$. If $h = n + 1$, suppose by induction hypothesis that we have the conclusion for derivations of height n and consider the last rule applied. If it is a rule without a variable condition, apply the induction hypothesis to the premiss(es) and then the rule. If the premiss of *Subst* is concluded by either $R\supset$, or $R\mathcal{K}$, or $L\Diamond$ we have to consider whether y is the eigenvariable. Consider the case of $L\Diamond$, the others being analogous. If y is the eigenvariable, then the premiss of *Subst* is $xR_{\Diamond}y, y : A, \Gamma' \to \Delta$. We refresh by induction hypothesis y with a new z in order to avoid a variable clash and obtain a derivation of $xR_{\Diamond}z, z : A, \Gamma' \to \Delta$. Again by induction hypothesis, we replace x with y and thus obtain $yR_{\Diamond}z, z : A, \Gamma' \to \Delta$; Next, we are allowed to apply $L\Diamond$ to conclude $y : \Diamond A, \Gamma' \to \Delta$. Note that if the eigenvariable is x, the substitution is vacuous.

C.3 Proof of Proposition 3.7

Consider the case of weakening with xRy. The proof is by induction on the height h of the derivation of the premiss. The inductive step is straightfoward if the premiss is concluded by a rule without a variable condition. If the last rule is a rule with a variable condition, say $R_{\mathcal{K}}$

with $x : \mathcal{K}B$ as principal formula, Lemma 3.6 is applied to its premiss $xR_{\mathcal{K}}y, \Gamma \to \Delta', y : B$ to replace the eigenvariable y with a new z; Then by the induction hypothesis and $R\mathcal{K}$, we obtain the conclusion $xRy, \Gamma \to \Delta', x : \mathcal{K}B$.

C.4 Proof of Lemma 3.8

We prove the result for those rules that are not in common with **G3I**, the others having been already proved admissible by Theorem 3.3. $L\mathcal{K}$ and $R\Diamond$ are clearly invertible by hp-admissibility of weakening (Proposition 3.7). We consider only the case of $L\Diamond$ because $R\mathcal{K}$ is analogous and proceed by induction on the height h of the derivation of $x : \Diamond A, \Gamma \to \Delta$. If $h = 0$ and the sequent is initial or a conclusion of $L\bot$, then so is $xR_{\Diamond}y, y : A, \Gamma \to \Delta$. If $h = n + 1$, $x : \Diamond A, \Gamma \to \Delta$ has been concluded by a certain rule $\mathcal{R}$. If $x : \Diamond A$ is principal, then $\mathcal{R}$ is $L\Diamond$ and its premiss, that is $xR_{\Diamond}y, y : A, \Gamma \to \Delta$, has a derivation with height n. If on the contrary $x : \Diamond A$ is not principal, consider what rule $\mathcal{R}$ is. If it is a rule without a variable condition, apply the induction hypothesis to its premiss(es) and then $\mathcal{R}$ again. If $\mathcal{R}$ is, for instance, $R\mathcal{K}$ with $x : \mathcal{K}B$ as principal formula, its premiss is $xR_{\mathcal{K}}y, x : \Diamond A, \Gamma \to \Delta', y : B$. Apply first Lemma 3.6 with a new z instead of y and obtain $xR_{\mathcal{K}}z, x : \Diamond A, \Gamma \to \Delta', z : B$. Then by induction hypothesis conclude $xR_{\mathcal{K}}z, xR_{\Diamond}y, y : A, \Gamma \to \Delta', z : B$ and by one application of $R\mathcal{K}$ obtain $xR_{\Diamond}z, y : A, \Gamma \to \Delta', x : \mathcal{K}B$.

C.5 Proof of Theorem 3.9

By simultaneous induction on the height h of the derivation. If $h = 0$, the premiss is an initial sequent or has been concluded by $L\bot$. In each case the conclusion of Ctr is initial or $L\bot$. If $h = n + 1$, suppose the claim holds for derivations of height n and consider the rule $\mathcal{R}$ used to derive the premiss of Ctr. If the contraction formula is not principal in $\mathcal{R}$, both occurrences are in the premiss(es) of $\mathcal{R}$ and by induction hypothesis we can contract the two occurrences and obtain the conclusion with a smaller derivation height. If the contraction formula is principal in $\mathcal{R}$, we distinguish two cases: in the first, the premiss of Ctr is concluded by a rule with the repetition of the principal formula, as in $L\supset$, $L\mathcal{K}$, $R\Diamond$, and the mathematical rules. The induction hypothesis is applicable directly to the premiss of $\mathcal{R}$. For instance, if $\mathcal{R}$ is $R\Diamond$, the premiss of Ctr has the following derivation, with $\Gamma \equiv xR_{\Diamond}y, \Gamma'$:

$$\dfrac{\vdots \atop xR_{\Diamond}y, \Gamma' \to \Delta, x : \Diamond A, x : \Diamond A, y : A}{xR_{\Diamond}y, \Gamma' \to \Delta, x : \Diamond A, x : \Diamond A} R\Diamond$$

By induction hypothesis on the premiss, we obtain $xR_{\Diamond}y, \Gamma' \to \Delta, x : \Diamond A, y : A$ and next by $R\Diamond$ again $xR_{\Diamond}y, \Gamma' \to \Delta, x : \Diamond A$. In the second case, when $\mathcal{R}$ is without repetition of principal formulas, we need hp-inversion on the premisses (Lemma 3.8), as in the standard proof for **G3c**. The crucial steps here are the cases in which $\mathcal{R}$ is either $R\supset$, or $R\mathcal{K}$, or $L\Diamond$, that is, rules with a variable condition. Take for instance the case in which $\mathcal{R}$ is $L\Diamond$, the others being analogous. The premiss has the following derivation:

$$\dfrac{\vdots \atop xR_{\Diamond}y, y : A, x : \Diamond A, \Gamma \to \Delta}{x : \Diamond A, x : \Diamond A, \Gamma \to \Delta} L\Diamond$$

By invertibility of $L\Diamond$, we obtain $xR_{\Diamond}y, y : A, xR_{\Diamond}y, y : A, \Gamma \to \Delta$. Then by induction hypothesis, $y : A, xR_{\Diamond}y, \Gamma \to \Delta$ and, by $L\Diamond$ again, we conclude $x : \Diamond A, \Gamma \to \Delta$.

C.6 Proof of Theorem 3.10

By induction on the size of the cut formula with subinduction on the sum of the heights of the derivations of the premisses of cut. The proof has the same structure as the proof of admissibility of cut for sequent calculus extended by the left rule-scheme (Theorem 6.2.3 in Negri and von Plato 2001, pp. 132–134). We consider in detail only the case of cut formula principal in modal rules in both premisses of cut and in mathematical rules. As for the latter, consider the case of left premiss concluded by $Mon_{\mathcal{K}}$. We have the following derivation

$$\dfrac{\dfrac{xR_{\mathcal{K}}z, x \leqslant y, yR_{\mathcal{K}}z, \Gamma'' \to \Delta, x : A}{x \leqslant y, yR_{\mathcal{K}}z, \Gamma'' \to \Delta, x : A}{\scriptstyle Mon_{\mathcal{K}}} \qquad x : A, \Gamma' \to \Delta'}{x \leqslant y, yR_{\mathcal{K}}z, \Gamma'', \Gamma' \to \Delta', \Delta}{\scriptstyle Cut}$$

It converts to

$$\dfrac{\dfrac{xR_{\mathcal{K}}z, x \leqslant y, yR_{\mathcal{K}}z, \Gamma'' \to \Delta, x : A \quad x : A, \Gamma' \to \Delta'}{xR_{\mathcal{K}}z, x \leqslant y, yR_{\mathcal{K}}z, \Gamma'', \Gamma' \to \Delta', \Delta}{\scriptstyle Cut}}{x \leqslant y, yR_{\mathcal{K}}z, \Gamma'', \Gamma' \to \Delta', \Delta}{\scriptstyle Mon_{\mathcal{K}}}$$

Likewise for the other mathematical rules. If the cut formula is principal in a $\mathcal{K}$-rule, it is of the form $x : \mathcal{K}B$ and the derivation is

$$\dfrac{\dfrac{xR_{\mathcal{K}}z, \Gamma \to \Delta, z : B}{\Gamma \to \Delta, x : \mathcal{K}B}{\scriptstyle R\mathcal{K}} \quad \dfrac{y : B, x : \mathcal{K}B, xR_{\mathcal{K}}y, \Gamma'' \to \Delta'}{x : \mathcal{K}B, xR_{\mathcal{K}}y, \Gamma'' \to \Delta'}{\scriptstyle L\mathcal{K}}}{xR_{\mathcal{K}}y, \Gamma'', \Gamma \to \Delta, \Delta'}{\scriptstyle Cut}$$

It can be converted to

$$\dfrac{\dfrac{\dfrac{xR_{\mathcal{K}}z, \Gamma \to \Delta, z : B}{xR_{\mathcal{K}}y, \Gamma \to \Delta, y : B}{\scriptstyle Subst} \quad \dfrac{\dfrac{xR_{\mathcal{K}}z, \Gamma \to \Delta, z : B}{\Gamma \to \Delta, x : \mathcal{K}B}{\scriptstyle R\mathcal{K}} \quad y : B, x : \mathcal{K}B, xR_{\mathcal{K}}y, \Gamma'' \to \Delta'}{y : B, xR_{\mathcal{K}}y, \Gamma'', \Gamma \to \Delta, \Delta'}{\scriptstyle Cut}}{xR_{\mathcal{K}}y, xR_{\mathcal{K}}y, \Gamma'', \Gamma, \Gamma \to \Delta, \Delta, \Delta'}{\scriptstyle Cut}}{xR_{\mathcal{K}}y, \Gamma'', \Gamma \to \Delta, \Delta'}{\scriptstyle Ctr^*}$$

Note that the first cut is of reduced cut-height and the second is on a smaller formula. If the cut formula is principal in a $\Diamond$-rule, it is of the form $x : \Diamond B$. The derivation is

$$\dfrac{\dfrac{xR_{\Diamond}y, \Gamma'' \to \Delta, x : \Diamond B, y : B}{xR_{\Diamond}y, \Gamma'' \to \Delta, x : \Diamond B}{\scriptstyle R\Diamond} \quad \dfrac{xR_{\Diamond}z, z : B, \Gamma' \to \Delta'}{x : \Diamond B, \Gamma' \to \Delta'}{\scriptstyle L\Diamond}}{xR_{\Diamond}y, \Gamma'', \Gamma' \to \Delta', \Delta}{\scriptstyle Cut}$$

It can be converted to

$$\dfrac{\dfrac{\dfrac{xR_{\Diamond}y, \Gamma'' \to \Delta, x : \Diamond B, y : B \quad x : \Diamond B, \Gamma' \to \Delta'}{xR_{\Diamond}y, \Gamma'', \Gamma' \to \Delta', \Delta, y : B}{\scriptstyle Cut} \quad \dfrac{xR_{\Diamond}z, z : B, \Gamma' \to \Delta'}{xR_{\Diamond}y, y : B, \Gamma' \to \Delta'}{\scriptstyle Subst}}{xR_{\Diamond}y, xR_{\Diamond}y, \Gamma'', \Gamma', \Gamma' \to \Delta', \Delta', \Delta}{\scriptstyle Cut}}{xR_{\Diamond}y, \Gamma'', \Gamma' \to \Delta', \Delta}{\scriptstyle Ctr^*}$$

D Completeness of $\mathbf{G3I}_{\mathcal{K}\Diamond}$

D.1 Proof of Theorem 3.13

By induction on the derivation of $\Gamma \to \Delta$ in $\mathbf{G3I}_{\mathcal{K}\Diamond}$. If it is an initial sequent, there is a labelled atom $x : P$ both in Γ and in Δ so the claim is obvious, and similarly if the sequent is a conclusion of $L\bot$, since for no valuation can $\bot$ be forced at any node.

If $\Gamma \to \Delta$ is a conclusion of a propositional rule, suppose that the rule is $L\&$ with premiss $x : A, x : B, \Gamma' \to \Delta$. Assume that for an arbitrary assignment and interpretation, all the formulas in Γ are valid. Since $[\![x]\!] \Vdash A\&B$ is equivalent to $[\![x]\!] \Vdash A$ and $[\![x]\!] \Vdash B$, the inductive hypothesis, i.e., validity of $x : A, x : B, \Gamma' \to \Delta$ for every interpretation, gives the desired conclusion.

If $\Gamma \to \Delta$ is a conclusion of a modal rule, say $L\Diamond$, with the premiss $xR_\Diamond y, y : A, \Gamma' \to \Delta$, assume by the induction hypothesis that the premiss is valid. Let $[\![\cdot]\!]$ be an arbitrary interpretation that validates all the formulas in Γ. We claim that one of the formulas in Δ is valid under this intepretation. Let k be an arbitrary element of K such that $[\![x]\!]R_\Diamond k$ and $k \Vdash A$; Let $[\![\cdot]\!]'$ be the interpretation identical to $[\![\cdot]\!]$ except possibily on y, where we set $[\![y]\!]' \equiv k$. Clearly $[\![\cdot]\!]'$ validates all the formulas in the antecedent of the premiss, so it validates a formula in Δ.

If the sequent is a conclusion of a rule for the accessibility relations, let the rule be for instance $Mon_\Diamond$:

$$\frac{yR_\Diamond z, x \leqslant y, xR_\Diamond z, \Gamma \to \Delta}{x \leqslant y, xR_\Diamond z, \Gamma \to \Delta}\,{\scriptstyle Mon_\Diamond}$$

Let $[\![x]\!] \leqslant [\![y]\!]$ and $[\![y]\!]R_\Diamond[\![z]\!]$. Since $\leqslant$ and $R_\Diamond$ satisfy$Mon_\Diamond$ by assumption, we have $[\![y]\!]R_\Diamond[\![z]\!]$, so validity of the premiss gives validity of the conclusion.

The preservation of soundness is proved in a similar way for all the other rules.

D.2 Proof of Theorem 3.14

We define for an arbitrary sequent $\Gamma \to \Delta$ in the language of $\mathbf{G3I}_{\mathcal{K}\Diamond}$ a reduction tree by applying the rules of $\mathbf{G3I}_{\mathcal{K}\Diamond}$ root first in all possible ways. If the construction terminates, we obtain a proof, else the tree becomes infinite. By König's lemma, an infinite tree has an infinite branch that is used to define a countermodel to the endsequent.

1. Construction of the reduction tree: The reduction tree is defined inductively in stages as follows:

Stage 0 has $\Gamma \to \Delta$ at the root of the tree. Stage $n > 0$ has two cases:

Case I: If every topmost sequent is an initial sequent or a conclusion of $L\bot$, the construction of the tree ends.

Case II: If not every topmost sequent is an initial sequent or a conclusion of $L\bot$, we continue the construction of the tree by writing above those topsequents that are not initial, nor conclusions of $L\bot$ or of a zero-premiss mathematical rule, other sequents that are obtained by applying root first the rules of $\mathbf{G3I}_{\mathcal{K}\Diamond}$ whenever possible, in a given order.

There are 14 different stages, 10 for the rules of the basic modal systems, 4 for the frame rules. At stage $n = 14 + 1$ we repeat stage 1, at stage $n = 14 + 2$ we repeat stage 2, and so on for each n.

We start, for $n = 1$, with $L\&$: For each topmost sequent of the form

$$x_1 : B_1\&C_1, \ldots, x_m : B_m\&C_m, \Gamma' \to \Delta$$

where $B_1\&C_1, \ldots, B_m\&C_m$ are all the formulas in Γ with a conjunction as the outermost logical connective, we write

$$x_1 : B_1, x_1 : C_1, \ldots, x_m : B_m, x_m : C_m, \Gamma' \to \Delta$$

on top of it. This step corresponds to applying root first m times rule $L\&$.

For $n = 2$, we perform a similar decomposition that corresponds to applying $R\&$ root first successively to all formulas in the succedent that have conjunction as outermost connective.

For $n = 3$ and 4 we consider $L\vee$ and $R\vee$ and define the reductions symmetrically to the cases $n = 2$ and $n = 1$, respectively. Stages $n = 5$ and $n = 6$ decompose analogously all implications, in the antecedent and in the succedent respectively.

For $n = 7$, let $x_1 : \Diamond B_1, \ldots, x_m : \Diamond B_m$ be all the formulas with $\Diamond$ as the outermost connective in the antecedent of topsequents of the tree, and let Γ' be the other formulas. Let $y_1, \ldots, y_m$ be fresh variables, and write on top of each the sequent

$$x_1 R_\Diamond y_1, \ldots, x_m R_\Diamond y_m, y_1 : B_1, \ldots, y_m : B_m, \Gamma' \rightarrow \Delta$$

that is, apply m times rule $L\Diamond$.

For $n = 8$, consider all topsequents with modal formulas $x_1 : \Diamond B_1, \ldots,\ x_m : \Diamond B_m$ in the succedent and relational atoms $x_1 R_\Diamond y_1, \ldots, x_m R_\Diamond y_m$ in the antecedent, and write on top of these the sequents

$$x_1 R_\Diamond y_1, \ldots, x_m R_\Diamond y_m, \Gamma \rightarrow \Delta', x_1 : \Diamond B_1, \ldots, x_m : \Diamond B_m, y_1 : B_1, \ldots, y_m : B_m$$

that is, apply m times rule $R\Diamond$.

For $n = 9$, we consider all topsequents with modal formulas $x_1 : \mathcal{K} B_1, \ldots,\ x_m : \mathcal{K} B_m$ and relational atoms $x_1 R_\mathcal{K} y_1, \ldots, x_m R_\mathcal{K} y_m$ in the antecedent, and write on top of these the sequents

$$y_1 : B_1, \ldots, y_m : B_m, x_1 : \mathcal{K} B_1, \ldots, x_m : \mathcal{K} B_m, x_1 R_\mathcal{K} y_1, \ldots, x_m R_\mathcal{K} y_m, \Gamma' \rightarrow \Delta$$

Here we apply m times rule $L\mathcal{K}$.

For $n = 10$, let $x_1 : \mathcal{K} B_1, \ldots, x_m : \mathcal{K} B_m$ be all the formulas with $\mathcal{K}$ as the outermost connective in the succedent of topsequents of the tree, and let Δ' be the other formulas. Let $y_1, \ldots, y_m$ be fresh variables, not yet used in the reduction tree, and write on top of each sequent the sequent

$$x_1 R_\mathcal{K} y_1, \ldots, x_m R_\mathcal{K} y_m, \Gamma \rightarrow \Delta', y_1 : B_1, \ldots, y_m : B_m$$

Here we apply m times rule $R\mathcal{K}$.

Finally, for $n = 10 + j$, we consider the generic case of a frame rule: Whenever a sequent in the tree matches the conclusion of the rule, the corresponding instance of the premiss of the rule is written above it. Rule $Ref_\leqslant$ is instantiated only on labels that appear in the conclusion of the rule.[5]

For any n, for each sequent that is neither initial, nor conclusion of $L\bot$, nor treatable by any one of the above reductions, we write the sequent itself above it.[6]

[5] Completeness guarantees that it is not restrictive to instantiate $Ref_\leqslant$ only on labels that already appear in its conclusion. Alternatively, one can consider an unrestricted rule of reflexivity and prove, without making use of completeness, that its applications can be limited to ones that do not introduce new labels. This result, and the consequent *subterm property*, is proved for systems that extend basic modal logic in Section 6 of Negri (2005) through permutation arguments, and for systems of intermediate logics in Section 8 of Dyckhoff and Negri (2012) through a uniform substitution.

[6] This step is needed to treat uniformly the case of a proof search that would terminate without producing a proof because no rule is applicable and neither an initial sequent nor a conclusion of $L\bot$ or of a zero-premiss rule is reached, and the case that does not produce a proof because new applicable rules always become available and the search goes on for ever.

If the reduction tree is finite, all its leaves are initial or conclusions of $L\bot$, or of zero-premiss mathematical rules, and the tree, read from the leaves to the root, yields a derivation.

2. Construction of the countermodel: If the reduction tree is infinite, it has an infinite branch. Let $\Gamma_0 \to \Delta_0 \equiv \Gamma \to \Delta, \Gamma_1 \to \Delta_1 \ldots, \Gamma_i \to \Delta_i, \ldots$ be one such branch. Consider the sets of labelled formulas and relational atoms

$$\mathbf{\Gamma} \equiv \bigcup_{i>0} \Gamma_i \qquad \mathbf{\Delta} \equiv \bigcup_{i>0} \Delta_i$$

We define a Kripke model that forces all the formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$ and is therefore a countermodel to the sequent $\Gamma \to \Delta$.

Consider the frame K the nodes of which are all the labels that appear in the relational atoms in $\mathbf{\Gamma}$, with their mutual relationships expressed by the relational atoms in $\mathbf{\Gamma}$. In general, the construction of the reduction tree imposes the frame properties of the countermodel, which in this case are $Ref_{\leqslant}$, $Trans_{\leqslant}$, $Mon_{\mathcal{K}}$, $Mon_{\Diamond}$. The model is defined as follows: For all atomic formulas $x : P$ in $\mathbf{\Gamma}$, we stipulate that $x \Vdash P$ in the frame, and for all atomic formulas $y : Q$ in $\mathbf{\Delta}$ we stipulate that $y \nVdash Q$. Since no sequent in the infinite branch is initial, this choice can be coherently made.

It can then be shown inductively on the weight of formulas that A is forced in the model at node x if $x : A$ is in $\mathbf{\Gamma}$ and A is not forced at node x if $x : A$ is in $\mathbf{\Delta}$. Therefore we have a countermodel to the endsequent $\Gamma \to \Delta$. The details are similar to those in Negri (2009).

E Proof of Theorem 5.8

The sequent is derived as follows:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ R_1\end{matrix} \qquad \begin{matrix}\vdots \\ R_2\end{matrix}}{x \leqslant y, x : KP(A), y : A \,\&\, \neg\mathcal{K}A \to y : \bot}\,L\supset}{x : KP(A) \to x : \neg(A \,\&\, \neg\mathcal{K}A)}\,R\supset$$

In the derivation R_1 is

$$\dfrac{\dfrac{\text{Lemma 3.5}}{y \leqslant y, x \leqslant y, x : KP(A), y : A \,\&\, \neg\mathcal{K}A \to y : \bot, y : A \,\&\, \neg\mathcal{K}A}}{x \leqslant y, x : KP(A), y : A \,\&\, \neg\mathcal{K}A \to y : \bot, y : A \,\&\, \neg\mathcal{K}A}\,Ref_{\leqslant}$$

R_2 instead is a derivation of the sequent $y : \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A), x \leqslant y, x : KP(A), y : A \,\&\, \neg\mathcal{K}A \to y : \bot$, with the same order of rules as in the derivation of $x \leqslant y, yR_{\mathcal{K}}z, y : \Diamond\mathcal{K}(A \,\&\, \neg\mathcal{K}A), x : KP(A), y : A \to z : A$ given in Section 5.2, and therefore omitted.

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