INVERSE PROBLEMS FOR HYPERBOLIC PDES

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Contents

1. A problem in the Minkowski space

In this section we consider a problem to determine a compactly supported potential in the Minkowski space. This avoids some technicalities appearing in more usual inverse boundary value problems, but it still allows us to introduce the main techniques.

Let $n \ge 2$, $T > 0$, $q \in C_0^\infty((0,T) \times \mathbb{R}^n)$, and consider the wave equation

$$\begin{aligned} &\Box u + qu = 0, \quad \text{in } (0,T) \times \mathbb{R}^n, \\ &u|_{t=0} = u_0,\ \partial_t u|_{t=0} = u_1. \end{aligned} \tag{1}$$

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Define also the map

$$L_q : (C_0^\infty(\mathbb{R}^n))^2 \to C_0^\infty(\mathbb{R}^n), \quad L_q(u_0, u_1) = u|_{t=T}.$$

We will study how to solve the inverse problem to determine q given L_q.

Exercise 1. Equation (1) arises in a natural way when considering a purely geometric wave equation on a conformal multiple of the Minkowski space. For a Lorentzian metric tensor g on $\mathbb{R}^{1+n}$, the associated wave operator is defined by

$$\Box_g u = |g|^{-1/2} \partial_{x^j} \left(g^{jk} |g|^{1/2} \partial_{x^k} u \right),$$

where $|g|$ is the determinant of the matrix $g = (g_{jk})_{j,k=0}^n$ and g^{jk} is its inverse. Consider now the Minkowski metric g, see (3) below, and let $c(t,x)$ be smooth and strictly positive. Show that $v = c^{(n-1)/4} u$ satisfies $\Box_g v + q_c v = 0$ if the function u satisfies $\Box_{cg} u = 0$, where

$$q_c = c^{-(n-1)/4} \Box_{cg} c^{(n-1)/4}.$$

1.1. **Geometric optics.** Observe that the $1+1$-dimensional wave equation

$$\partial_t^2 u - \partial_x^2 u = 0, \tag{2}$$

can be written as $(\partial_t + \partial_x)(\partial_t - \partial_x) u = 0$ and we see that functions of the form $u(t,x) = \chi(t \pm x)$ are solutions to this. Thus choosing $\chi(s) \approx \delta(s)$ we obtain a solution concentrating on $\beta(s) = (s, \pm s)$. Also the plane waves $u(t,x) = e^{i\sigma(t \pm x)}$, $\sigma \in \mathbb{R}$, are solutions to (2). In the more general case (1) we will construct geometric optics solutions that combine features of these two types of solutions to (2).

In particular, we will construct solutions to (1) that concentrate on light rays, that is, lines of the form

$$\beta(s) = (s, y + sv), \quad s \in \mathbb{R},$$

where y is a point in $\mathbb{R}^n$ and v is a unit vector in $\mathbb{R}^n$. We write

$$S^{n-1} = \{v \in \mathbb{R}^n;\ |v| = 1\}.$$

The name light ray comes from the fact that the tangent vector $\dot\beta = (1, v)$ is light like with respect to the Minkowski metric

$$g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \tag{3}$$

that is, $(\dot\beta, \dot\beta)_g = 0$.

The idea is to find first an approximate solution of the form

$$e^{i\sigma\phi(t,x)}(a_0(t,x) + \sigma^{-1} a_1(t,x) + \sigma^{-2} a_2(t,x) + \dots), \quad \sigma >> 1,$$

and then an actual solution $u = e^{i\sigma\phi}(a_0 + \dots) + r_\sigma$ where the remainder r_σ converges to zero as $\sigma \to \infty$. We will begin with the single term approximation $e^{i\sigma\phi}a_0$ and write $a_0 = a$.

1.1.1. *Single term ansatz.* The equation $(\Box + q)u = 0$ is equivalent with

$$(\Box + q)r_\sigma = -(\Box + q)(e^{i\sigma\phi}a), \tag{4}$$

and we want to choose ϕ and a so that

$$\Box(e^{i\sigma\phi}a) = e^{i\sigma\phi}\Box a. \tag{“C”}$$

The rationale is that in this case the absolute value of the right-hand side of (4) is independent from σ, and therefore r_σ is at least not blowing up as $\sigma \to \infty$.

It is a simple matter to expand the left-hand side of (“C”) but a useful computational technique is to consider the conjugated wave operator

$$e^{-i\sigma\phi}\Box e^{i\sigma\phi} = e^{-i\sigma\phi}\partial_t^2 e^{i\sigma\phi} + \dots = e^{-i\sigma\phi}\partial_t e^{i\sigma\phi} e^{-i\sigma\phi}\partial_t e^{i\sigma\phi} + \dots.$$

Now $e^{-i\sigma\phi}\partial_t e^{i\sigma\phi} = \partial_t + i\sigma(\partial_t\phi)$ and

$$(e^{-i\sigma\phi}\partial_t e^{i\sigma\phi})^2 = \partial_t^2 + 2i\sigma(\partial_t\phi)\partial_t - \sigma^2|\partial_t\phi|^2 + i\sigma(\partial_t^2\phi).$$

Treating the spacial derivatives in the same way we get

$$e^{-i\sigma\phi}\Box e^{i\sigma\phi} = \Box + i\sigma(2(\partial_t\phi)\partial_t - 2(\nabla\phi)\cdot\nabla + (\Box\phi)) - \sigma^2(|\partial_t\phi|^2 - |\nabla\phi|^2). \tag{5}$$

Therefore for $a \neq 0$, (“C”) is equivalent with the following two equations

$$|\partial_t\phi|^2 - |\nabla\phi|^2 = 0, \tag{E}$$

$$2(\partial_t\phi)\partial_t a - 2(\nabla\phi)\cdot\nabla a + (\Box\phi)a = 0. \tag{T}$$

It is natural to normalize ϕ so that (E) becomes $|\partial_t\phi|^2 = |\nabla\phi|^2 = 1$. There is some freedom when choosing a solution to (E), but for our purposes it suffices to use the linear solution $\phi(t,x) = t + v\cdot x$ where v is a unit vector in $\mathbb{R}^n$.

Exercise 2. Write $D\phi = (\partial_t\phi, \nabla\phi)$ and apply the method of characteristic, as described in [3, Section 3.2], to $F(D\phi) = \frac{1}{2}(|\partial_t\phi|^2 - |\nabla\phi|^2)$. Using the notation there we set $z(s) = \phi(x(s))$ and $p(s) = D\phi(x(s))$ where $x(s)$ is a characteristic of F. (Note that here $x(s)$ is a curve in the space time $\mathbb{R}^{1+n}$, not just in space.) Show that $\dot p = 0$, $\dot z = 0$, and that the compatibility condition $F(p(0)) = 0$ implies that $x(s)$ is a light ray. Setting the initial condition $\phi(x) = v\cdot x$ on the plane $t = 0$, show that the solution to $F(D\phi) = 0$ is $\phi(t,x) = t + v\cdot x$.

To simplify the notation, we may assume after a rotation that $v\cdot x = -x^1$. The functions satisfying $|\nabla\phi| = 1$ are often called distance functions, and the particular choice x^1 is of course the signed distance to the plane $x^1 = 0$. Note also that, with this choice, the factor $e^{i\sigma\phi} = e^{i\sigma(t-x^1)}$ coincides with the $1+1$-dimensional plane wave discussed above.

The transport equation (T) simplifies now to

$$\partial_t a + \partial_{x^1} a = 0.$$

The solutions to this are of the form $a(t,x) = \chi(t-x^1)\eta(x')$ where $x' = (x^2, x^3, \dots, x^n)$. Analogously with $1+1$-dimensional case, taking $\chi \approx \delta$ and $\eta \approx \delta$ we obtain a that concentrates on the light ray $\beta(s) = (s, s, 0)$.

Neither ϕ nor a depend on q in the above construction. In order to obtain information on q there are two typical approaches: use the difference of two solutions, corresponding to different potentials, or use a multi-term approximation.

1.1.2. *Multi-term ansatz.* Let us consider the three term approximation,

$$e^{i\sigma\phi}A, \quad A = a_0 + \sigma^{-1}a_1 + \sigma^{-2}a_2,$$

and choose ϕ and $a_0 = a$ as above. As we are using a more complicated amplitude, we can ask for more than ("C"), namely

$$(\Box + q)(e^{i\sigma\phi}A) = \mathcal{O}(\sigma^{-2}), \quad \sigma >> 1.$$

We use the conjugation formula (5), to obtain

$$e^{-i\sigma\phi}(\Box + q)e^{i\sigma\phi}A = (\Box + q)A + 2i(\partial_t + \partial_{x^1})(a_1 + \sigma^{-1}a_2).$$

This is of order σ^{-2} whenever a_1 and a_2 solve the transport equations

$$\partial_t a_j + \partial_{x^1} a_j - \frac{i}{2}(\Box + q)a_{j-1} = 0, \quad j = 1, 2,$$

or after the change of variables,

$$s = \frac{t + x^1}{2}, \quad r = \frac{t - x^1}{2},$$

equivalently $\partial_s a_j = \frac{i}{2}(\Box + q)a_{j-1}$. Therefore we may choose

$$a_j(s, r, x') = \frac{i}{2}\int_{-r}^{s}(\Box + q)a_{j-1}(s', r, x')ds'.$$

Note that $t = 0$ is equivalent with $s = -r$. The choice of the lower limit $-r$ in the integration implies that $a_j = 0$, $j = 1, 2$, when $t = 0$.

1.1.3. *Solving for the remainder.* When $\chi \in C_0^\infty(\mathbb{R})$ and $\eta \in C_0^\infty(\mathbb{R}^{n-1})$, the restrictions of all the amplitudes a_j, $j = 0, 1, 2$, are compactly supported in $[0, T] \times \mathbb{R}^n$. We recall that the wave equation

$$\Box u + qu = F, \quad \text{in } (0, T) \times \mathbb{R}^n,$$
$$u|_{t=0} = \partial_t u|_{t=0} = 0.$$

has a unique solution u satisfying

$$\|u\|_{C(0,T;H^1(\mathbb{R}^n))} + \|u\|_{C^1(0,T;L^2(\mathbb{R}^n))} \le C \|F\|_{L^2((0,T)\times\mathbb{R}^n)},$$

see e.g. [3, Theorem 7.6]. We solve

$$\Box r_\sigma + q r_\sigma = -(\Box + q)(e^{i\sigma\phi} A), \quad \text{in } (0, T) \times \mathbb{R}^n,$$
$$r_\sigma|_{t=0} = \partial_t r_\sigma|_{t=0} = 0.$$

As the right-hand side is pointwise of order σ^{-2} and compactly supported, we see that $r_\sigma|_{t=T} = \mathcal{O}(\sigma^{-2})$ in $H^1(\mathbb{R}^n)$.

1.2. **Reduction to the light ray transform.** Now $u = e^{i\sigma\phi} A + r_\sigma$ solves (1) with

$$u_0 = (e^{i\sigma\phi} A)|_{t=0}, \quad u_1 = \partial_t (e^{i\sigma\phi} A)|_{t=0}.$$

As q vanishes near $t = 0$, we see that u_0 and u_1 are independent from q. This again implies that L_q determines $u|_{t=T}$. For $\psi \in C_0^\infty(\mathbb{R}^n)$ we obtain

$$\sigma((e^{-i\sigma\phi} u - a_0)|_{t=T}, \psi)_{L^2(\mathbb{R}^n)} \to (a_1|_{t=T}, \psi)_{L^2(\mathbb{R}^n)}, \quad \sigma \to \infty.$$

This determines $a_1|_{t=T}$. As $\Box a_0$ is known, we find the integral

$$\int_{-r}^{T-r} q a_0(s', r, x') ds', \quad (s, r, x') \in \Omega.$$

Here we used the fact that $t = T$ is equivalent with $s = T - r$.

In (s, r, x') coordinates $a_0 = \chi(2r)\eta(x')$ and the above integral reduces at $(r, x') = 0$ to

$$\int_0^T q(s', 0, 0) ds' \chi(0)\eta(0).$$

As q vanishes for $t < 0$ and $t > T$, we can recover in (s, r, x') coordinates $\int_{\mathbb{R}} q(s, 0, 0) ds$, or equivalently, in (t, x^1, x') coordinates

$$\int_{\mathbb{R}} q(\beta(s)) ds, \quad \beta(s) = (s, s, 0).$$

Repeating the above argument, after using rotations and translations, we obtain the light ray transform of q,

$$\mathcal{L}q(y, v) = \int_{\mathbb{R}} q(\beta_{y,v}(s)) ds, \quad \beta_{y,v}(s) = (s, y + sv), \ y \in \mathbb{R}^n, \ v \in S^{n-1}.$$

1.3. **Inversion of the light ray transform.** The above reduction works also when $n = 1$, but inversion of $\mathcal{L}$ requires $n \geq 2$. Indeed, if $n = 1$, then $\mathcal{L}q = 0$ for $q(s, r) = q_0(s)q_1(r)$ with q_j, $j = 0, 1$, integrating to zero.

For a fixed $v \in S^{n-1}$, consider the change of coordinates in $\mathbb{R}^{1+n}$,

$$(t, x) = (s, y + sv).$$

Then $y = x - tv$. Using this, we obtain the Fourier slicing

$$\int_{\mathbb{R}^n} e^{-i\eta\cdot y}\mathcal{L}q(y, v)\, dy = \int_{\mathbb{R}^{1+n}} e^{-i\eta\cdot(x-tv)}q(t, x)\, dtdx = \widehat{q}(-\eta \cdot v, \eta).$$

Here $|\eta\cdot v| \leq |\eta|$. Moreover, as $n \geq 2$, we can choose a unit vector w that is orthogonal to η. Then for $a \in [-1, 1]$ and $\eta \neq 0$, we may choose

$$v = -\frac{a}{|\eta|}\eta + \sqrt{1 - a^2}\, w \in S^{n-1}.$$

This gives $-\eta \cdot v = a|\eta|$, and we see that the Fourier slicing allows us to recover $\widehat{q}(a|\eta|, \eta)$ for any $a \in [-1, 1]$.

As q is compactly supported, $\hat{q}$ is analytic. We know $\hat{q}$ in a non-empty open cone (in fact, in the cone of spacelike directions), and therefore everywhere by analytic continuation. This shows that $\mathcal{L}q$ determines q. By the above reduction also L_q determines q.

2. A problem in simple geometry

We will now consider an inverse problem to determine a time-independent potential on a simple Riemannian manifold.

Let (M, g) be a simple Riemannian manifold with boundary and let

$$T > \max\{d(x, y);\ x, y \in M\}, \tag{6}$$

where $d(x, y)$ is the distance on M. Let $q \in C^\infty(M)$ and consider the wave equation

$$\begin{aligned} &\partial_t^2 u - \Delta u + qu = 0, \quad \text{in } (0, T) \times M, \\ &u|_{x\in\partial M} = f, \\ &u|_{t=0} = \partial_t u|_{t=0} = 0. \end{aligned} \tag{7}$$

Here Δ is the Laplace operator on (M, g). Define also the map

$$\Lambda_q : C_0^\infty((0, T) \times \partial M) \to C^\infty((0, T) \times \partial M), \quad \Lambda_q f = \partial_\nu u|_{x\in\partial M}.$$

We will study how to solve the inverse problem to determine q given Λ_q.

2.1. **Some geometric preliminaries.** We denote by $\gamma(r; x, v)$ the geodesic on (M, g) with the initial data $(x, v) \in TM$. Then the exponential map at $x \in M$ is given by $\exp_x(v) = \gamma(1; x, v)$. As (M, g) is simple, $\exp_x$ is a diffeomorphism onto M for all $x \in M$. It is often convenient to consider a slightly larger simple manifold $\tilde{M}$ such that $M \subset \tilde{M}$. In what follows, $\exp_x$ being a diffeomorphism onto $\tilde{M}$, for all $x \in \tilde{M}$, is the only aspect of simplicity that we will need.

2.1.1. *Polar coordinates.* Let $x \in M$ and choose an orthonormal basis $e_1, \dots, e_n$ for T_xM. Then the map $y = (y^1, \dots, y^n) \mapsto \exp_x(y^j e_j)$ gives global coordinates on M that are called normal coordinates. Let us now write $y \in \mathbb{R}^n \setminus 0$ in polar coordinates $y = rv$ where $r > 0$ and $v \in S^{n-1}$.

We will show that in rv coordinates the metric tensor has the form

$$g(rv) = \begin{pmatrix} 1 & 0 \\ 0 & h(rv) \end{pmatrix}, \tag{8}$$

where $h(rv)$ is a smooth family of metric tensors on S^{n-1}. We write $(v, w)_g = v^j g_{jk} w^k$ and $|v|_g$ for the inner product with respect to g and the corresponding norm. As coordinate vectors for coordinates v are tangential to S^{n-1}, the form (8) is equivalent with

$$|\partial_r|_g = 1, \quad (\partial_r, w)_g = 0, \tag{9}$$

for all $w \in T_v S^{n-1}$ and $v \in S^{n-1}$.

We will consider S^{n-1} as a subset of T_xM. Then in rv coordinates

$$\gamma(r; x, v) = rv.$$

In particular, the coordinate vector $\partial_r = v$ at rv coincides with $\dot{\gamma}(r; x, v)$. Therefore $|\partial_r|_g = 1$. Let us now turn to the second equation in (9).

Let ω be a path in S^{n-1}. Then $\gamma(r; x, \omega(s)) = r\omega(s)$ and

$$\partial_s \gamma(r; x, \omega(s)) = r\dot{\omega}(s) \in T_{\omega(s)} S^{n-1}.$$

For any $r > 0$ and $(v, w) \in TS^{n-1}$ we can choose ω so that $\omega(0) = v$ and $r\dot{\omega}(0) = w$. Thus it is enough to show that

$$(\dot{\gamma}(r; x, v), \partial_s \gamma(r; x, \omega(s))|_{s=0})_g = 0,$$

or equivalently,

$$(\dot{\gamma}(r; x, \omega(s)), \partial_s \gamma(r; x, \omega(s)))_g|_{s=0} = 0.$$

We begin by showing that this inner product is constant in r. Using the shorthand notation $\Gamma(r, s) = \gamma(r; x, \omega(s))$, we have

$$\partial_r(\partial_r \Gamma, \partial_s \Gamma)_g = (\partial_r \Gamma, D_r \partial_s \Gamma)_g = (\partial_r \Gamma, D_s \partial_r \Gamma)_g = \frac{1}{2} \partial_s |\partial_r \Gamma|_g^2 = 0,$$

where we used the following three facts:

(1) For fixed s, the path $\Gamma(r, s) = \gamma(r; x, \omega(s))$ is a geodesic, and therefore its acceleration $D_r \partial_r \Gamma$ vanishes.
(2) The symmetry $D_r \partial_s \Gamma = D_s \partial_r \Gamma$, see e.g. [8, Lemma 6.3].
(3) For each s, the geodesic $\Gamma(r, s)$ has unit speed, and therefore $\partial_s |\partial_r \Gamma|^2 = 0$.

To conclude, we observe that $\partial_s \gamma(0; x, \omega(s)) = 0$.

Exercise 3. Consider the boundary ∂M as a submanifold of M and let $x \in \partial M$. Then $T_x(\partial M)$ is a subspace of $T_x M$ and we can choose a unit vector $\nu(x) \in T_x M$ such that $(\nu(x), w)_g = 0$ for all $w \in T_x(\partial M)$. Let us choose $\nu(x)$ so that it is inward pointing in the sense that $\gamma(r; x, \nu(x))$ stays in M for small $r > 0$. Then the choice of $\nu(x)$ is unique and depends smoothly on $x \in \partial M$. We choose local coordinates $x = (x^1, \dots, x^{n-1})$ on ∂M, and define the boundary normal coordinates by the map $(r, x) \mapsto \gamma(r; x, \nu(x))$.

This map gives local coordinates near a point $(0, x)$. Indeed, $\gamma(0; x, \nu(x)) = x$ and thus $\partial_{x^j} \gamma(0; x, \nu(x))$, $j = 1, \dots, n-1$, give a basis of $T_x(\partial M)$. Moreover, $\partial_r \gamma(r; x, \nu(x))|_{r=0} = \nu(x) \notin T_x(\partial M)$, and we conclude that $(r, x) \mapsto \gamma(r; x, \nu(x))$ has surjective differential.

Show that in (r, x) coordinates the metric tensor has the form

$$g(r, x) = \begin{pmatrix} 1 & 0 \\ 0 & h(r, x) \end{pmatrix}, \tag{10}$$

where $h(r, x)$ is a smooth family of metric tensors in the coordinates of ∂M. A proof can be based on considering the family $\Gamma(r, s) = \gamma(r; y(s), \nu(y(s)))$ where y is a path on ∂M.

2.1.2. *Laplace operator and Green's identity.* The Laplace operator is defined in coordinates by

$$\Delta u = |g|^{-1/2} \partial_{x^j} \left(g^{jk} |g|^{1/2} \partial_{x^k} u \right).$$

It can be viewed as the composition of the gradient and divergence

$$(\nabla u)^j = g^{jk} \partial_{x^k} u, \quad \operatorname{div} V = |g|^{-1/2} \partial_{x^j} (|g|^{1/2} V^j).$$

The normal derivative is defined by $\partial_\nu u = (\nabla u, \nu)_g$ where ν is the unit outward pointing normal to ∂M. The volume measure is defined in coordinates by

$$dV = |g|^{1/2} dx^1 \dots dx^n.$$

The metric tensor restricts on the boundary ∂M simply by $(v, w)_g$, $v, w \in T_x(\partial M)$, $x \in \partial M$, where $T_x(\partial M)$ is considered as a subspace of $T_x M$. The surface measure dS on ∂M is then defined in coordinates of ∂M by using the above formula for the restriction of g. We define the L^2-inner products on M and ∂M with respect to these measures. Now we are ready to state Green's identity, see e.g. [11, Proposition 2.4.1],

$$(\Delta u, v)_{L^2(M)} - (u, \Delta v)_{L^2(M)} = (\partial_\nu u, v)_{L^2(\partial M)} - (u, \partial_\nu v)_{L^2(\partial M)}, \quad u, v \in C^\infty(M).$$

Exercise 4. Consider boundary normal coordinates (r, x) in $[0, R) \times X$ where $R > 0$ is small and $X \subset \partial M$ is a small coordinate neighbourhood. Using a partition of unity, the proof of Green's identity can be reduced to case where $u, v \in C_0^\infty([0, R) \times X)$. Using the form (10) show Green's identity in this case.

2.2. **Geometric optics.** We will again use the ansatz $e^{i\sigma\phi}a$. To derive the geometric analogues of the eikonal and transport equations, observe first that

$$e^{-i\sigma\phi} \nabla e^{i\sigma\phi} = \nabla + i\sigma\nabla\phi.$$

In order to compute $e^{-i\sigma\phi} \operatorname{div} e^{i\sigma\phi}$, note that in general

$$\operatorname{div}(uV) = u \operatorname{div} V + V^j \partial_{x^j} u = u \operatorname{div} V + (\nabla u, V)_g.$$

Thus $e^{-i\sigma\phi} \operatorname{div} e^{i\sigma\phi} = \operatorname{div} + i\sigma(\nabla\phi, \cdot)_g$.

Now for $a \neq 0$, ("C") is equivalent with the following two equations

$$\text{(E')} \qquad |\partial_t \phi|^2 - |\nabla\phi|_g^2 = 0,$$

$$\text{(T')} \qquad 2(\partial_t\phi)\partial_t a - 2(\nabla\phi, \nabla a)_g + (\Box\phi)a = 0.$$

Recall that in polar coordinates rv the metric tensor g has the form (8). This form implies that $\nabla r = \partial_r$ and $|\nabla r|_g = 1$. In particular $\phi = t - r$ is a solution to (E').

Note that $\Box\phi = |h|^{-1/2}\partial_r(|h|^{1/2}) = \frac{1}{2}\partial_r \ln|h|$. Thus (T') reduces to

$$\partial_t a + \partial_r a + (\partial_r\mu)a = 0, \quad \mu = \ln(|h|^{1/4}).$$

We use the integrating factor $e^\mu = |h|^{1/4}$. That is, writing $e^{-\mu}\partial_r e^\mu = \partial_r + (\partial_r\mu)$, we see that (T') is equivalent with

$$\partial_t b + \partial_r b = 0, \quad b = e^\mu a.$$

As in the Minkowski case, we see that the solutions are $b = \chi(t - r)\eta(v)$, and hence $a = \chi(t - r)\eta(v)|h|^{-1/4}$.

2.3. **Sharper analysis of the remainder term.** It follows again from [3, Theorem 7.6] that the wave equation

$$\begin{aligned} \Box u + qu &= F, \quad \text{in } (0, T) \times M, \\ u|_{x \in \partial M} &= 0, \\ u|_{t=0} &= \partial_t u|_{t=0} = 0. \end{aligned} \tag{11}$$

has a unique solution u satisfying

$$\|u\|_{C(0,T;H^1(M))} + \|u\|_{C^1(0,T;L^2(M))} \le C \|F\|_{L^2((0,T)\times M)}.$$

Analogously with the Minkowski case, we obtain r_σ by solving (11) with

$$F = -(\Box + q)(e^{i\sigma\phi} a) = -e^{i\sigma\phi}(\Box + q)a.$$

Then $u = e^{i\sigma\phi}a + r_\sigma$ is a solution to $\Box u + qu = 0$ but some work is still needed to show that r_σ decays as $\sigma \to \infty$.

Exercise 5. Define

$$R_\sigma(t,x) = \int_0^t r_\sigma(t',x)dt',$$

and show that R_σ solves (11) with $F = F_\sigma$ where

$$F_\sigma = e^{-i\sigma r}\int_0^t e^{i\sigma t'} H dt', \quad H = -(\Box + q)a.$$

Furthermore, show that $\int_0^t e^{i\sigma t'} H dt' = \mathcal{O}(\sigma^{-1})$ as $\sigma \to \infty$, and conclude that

$$\|r_\sigma\|_{C(0,T;L^2(M))} \le \|R_\sigma\|_{C^1(0,T;L^2(M))} \le C\,\|F_\sigma\|_{L^2((0,T)\times M)} = \mathcal{O}(\sigma^{-1}).$$

2.4. **Reduction to the geodesic ray transform.** Let us now consider two potentials $q_1, q_2 \in C^\infty(M)$ and the corresponding solutions u_1 and u_2 to

$$\begin{aligned}
&\Box u_j + q_j u_j = 0, \quad \text{in } (0,T)\times M,\\
&u_j|_{x\in\partial M} = f,\\
&u_j|_{t=0} = \partial_t u_j|_{t=0} = 0.
\end{aligned}$$

Let $x \in \tilde M \setminus M$, consider polar coordinates centred at x, and fix $v \in S^{n-1}$. We assume that x and v are chosen so that the geodesic $\gamma(r;x,v)$ enters M at $r = r_0$ and then exits M at $r = r_1$ where $r_1 < T$. Note that the exit condition follows from simplicity and the assumption (6) whenever x is close enough to M.

We choose $\chi \in C_0^\infty(\mathbb{R})$ and $\eta \in C^\infty(S^{n-1})$ with small supports. Then

$$a = \chi(t-r)\eta(v)|h|^{-1/4}$$

concentrates near the light ray $\beta(t) = (t, \gamma(t;x,v)) = (t, tv)$. When $\operatorname{supp}(\chi)$ and $\operatorname{supp}(\eta)$ are small enough, the intersection of $\operatorname{supp}(a)$ with the boundary of the cylinder $(0,T)\times M$ is contained in the lateral part $(0,T)\times\partial M$.

We choose $f = (e^{i\sigma\phi}a)|_{x\in\partial M}$ in (7). Then f does not depend on the potentials q_1 and q_2, and the solutions u_1 and u_2 satisfy $u_j = e^{i\sigma\phi}a + r_{\sigma,j}$. Let us now integrate by parts

$$\begin{aligned}
0 &= ((\Box + q_1)u_1, u_2)_{L^2((0,T)\times M)} - (u_1, (\Box + q_2)u_2)_{L^2((0,T)\times M)}\\
&= ((q_1 - q_2)u_1, u_2)_{L^2((0,T)\times M)} - (\partial_\nu u_1, u_2)_{L^2((0,T)\times\partial M)} + (u_1, \partial_\nu u_2)_{L^2((0,T)\times\partial M)}\\
&\quad + (\partial_t u_1|_{t=T}, u_2|_{t=T})_{L^2(M)} - (u_1|_{t=T}, \partial_t u_2|_{t=T})_{L^2(M)}.
\end{aligned}$$

We have $u_j = e^{i\sigma\phi}a + \mathcal{O}(\sigma^{-1})$ in $L^2((0,T)\times M)$. Also $u_j|_{t=T} = r_{\sigma,j}|_{t=T} = \mathcal{O}(\sigma^{-1})$ and $\partial_t u_j|_{t=T} = \partial_t r_{\sigma,j}|_{t=T} = \mathcal{O}(1)$ in $L^2(M)$. Furthermore if $\Lambda_{q_1} = \Lambda_{q_2}$ then

$$\begin{aligned}&(\partial_\nu u_1, u_2)_{L^2((0,T)\times\partial M)} - (u_1, \partial_\nu u_2)_{L^2((0,T)\times\partial M)}\\ &\quad= (\partial_\nu u_1, u_1)_{L^2((0,T)\times\partial M)} - (u_1, \partial_\nu u_1)_{L^2((0,T)\times\partial M)}.\end{aligned}$$

The inner products here are complex valued, but replacing u_2 and q_2 with u_1 and q_1 in the above computation, we see that these boundary terms are of order σ^{-1}. Finally using $e^{i\sigma\phi}\overline{e^{i\sigma\phi}} = 1$ we obtain

$$((q_1 - q_2)a, a)_{L^2((0,T)\times M)} = \mathcal{O}(\sigma^{-1}).$$

Taking $\sigma \to \infty$ we have

$$0 = \int_0^T \int_M (q_1 - q_2)\chi^2(t-r)\eta^2(v)|h|^{-1/2} dV dt.$$

As $dV = |h|^{1/2} dr dv$, we obtain after letting $\chi^2 \to \delta$ and $\eta^2 \to \delta(v - v_0)$,

$$0 = \int_{r_0}^{r_1} (q_1 - q_2)(tv_0) dt.$$

Repeating this construction for all $x \in \tilde{M} \setminus M$ close to M and all $v \in S^{n-1}$, we see that the geodesic ray transform of $q_1 - q_2$ vanishes.

3. Further reading

An analogue of the inverse problem in Section 2, but with time dependent q, and data also on the top and bottom of the cylinder $(0,T)\times M$, is solved in [7]. The problem is open if $M \subset \mathbb{R}^n$ is a convex set and $\partial_t^2 - \Delta$ in (7) is replaced by $\Box_g$ with g near the Minkowski metric. By combining the results in [9] and [10], this problem can be solved under the additional assumption that g is real analytic.

For stability results related to the inverse problems in Sections 1 and 2, see [1] and [2], respectively. For a result similar to that in Section 1, but with non-smooth, unbounded q see [4].

The inverse problem in Section 2 can also be solved by using the Boundary Control method. This approach is not based on reduction to a ray transform, and it works when (M,g) is any compact, smooth Riemannian manifold with boundary. The monograph [5] is a good introduction to the Boundary Control method, and [6] discusses the method in the case of Minkowski geometry.

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