Inverse problems for wave equation and inverse boundary spectral problems

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Motivation

Let $\Omega \subset \mathbb{R}^n$,

$$u(x, t) \text{ satisfy a wave equation in } \Omega \times \mathbb{R}$$

Inverse problem:

Can we determine the coefficients of the wave equation, i.e., physical model in Ω by observing

$$u(x, t) \text{ near } \partial\Omega \times \mathbb{R}$$

for all possible solutions $u(x, t)$?

The inverse problem has no unique solution as

- We can change definition of x-coordinate: Let

$$v(x,t) = u(\phi(x),t)$$

where

$$\phi : \Omega \to \Omega, \quad \phi|_{\partial\Omega} = id$$

- We can change scale of u-coordinate: Let

$$w(x,t) = \kappa(x)u(x,t)$$

where $\kappa(x) > 0$.

All functions u, v and w model the same physical process.

Let us consider Ω as Riemannian manifold

$$d_g(x, y) = \text{travel time between } x \text{ and } y.$$

Let us identify all isometric Riemannian manifolds, that is, we ask following question

Do the boundary measurements determine uniquely the isometry type of the Riemannian manifold?

Setting of the problem in different cases

Let (M, g) be a Riemannian manifold. Let us consider the wave equation

$$
\begin{aligned}
u_{tt}(x,t) + Au(x,t) &= 0, \quad \text{in} \quad M \times \mathbb{R}_+, \\
u|_{t=0} &= 0, \quad u_t|_{t=0} = 0, \\
u|_{\partial M \times \mathbb{R}^+} &= f
\end{aligned}
$$

where M is a n-dimensional manifold and local coordinates

$$
Au = -\sum_{j,k=1}^{n} a^{jk} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{j=1}^{n} b^j \frac{\partial u}{\partial x^j} + cu,
$$

where a^{jk}, b^j, c are real, smooth, $a^{jk}(x) = g^{jk}(x)$. We write below $A = a(x, D) = -\Delta_g + P + q$.
In addition ...

Assume that there is $dV = m(x)dV_g$ such that A is selfadjoint in $L^2(M, dV)$ with

$$\mathcal{D}(A) = H^2(M) \cap H_0^1(M).$$

Then we can write

$$Au = -m^{-1}\mathrm{div}_g(m\,\mathrm{grad}_g u) + qu.$$

Let u satisfy the wave equation on a manifold (M, g),

$$u_{tt} + a(x, D)u = 0.$$

Then the gauge transformation of u,

$$w(x, t) = \kappa(x)u(x, t)$$

satisfy

$$w_{tt} + a_\kappa(x, D)w = 0,$$

where

$$a_\kappa(x, D)w = \kappa a(x, D)(\kappa^{-1} w)$$

We say that the gauge equivalence class of $a(x, D)$ is

$$[a(x, D)] = \{a_\kappa(x, D) \ : \quad \kappa > 0\}$$

Can the equivalence class be uniquely determined?

Invariant inverse problem

The Dirichlet-to-Robin map is

$$\Lambda : u|_{\partial M\times\mathbb{R}_+} \mapsto (\partial_\nu u + \sigma u)|_{\partial M\times\mathbb{R}_+}.$$

Dynamical inverse problem:

Let ∂M and the map Λ be given. Can we determine

$$(M, g) \text{ and } [A(x, D)]?$$

Energy flux through boundary The energy of the wave at time t is

$$E(u,t) = \int_M \left(|\partial_t u(t)|^2 + |\operatorname{Grad} u(t)|_g^2 + q|u(t)|^2\right) dV + \\ + \int_{\partial M} \sigma |u(t)|^2 dS.$$

For $h = u|_{\partial M \times \mathbb{R}_+} \in C_0^\infty(\partial M \times \mathbb{R}_+)$ let

$$\Pi(h) = \lim_{t\to\infty} E(u,t).$$

Inverse problem for energy flux:

Let ∂M and map Π be given. Can we determine

$$(M,g) \text{ and } [A(x,D)]?$$

Inverse boundary spectral problem:
Operator A has in $L^2(M, dV)$ orthonormal eigenfunctions φ_j,

$$\begin{aligned} (A-\lambda_j)\varphi_j &= 0, \\ \varphi_j|_{\partial M} = 0. \end{aligned}$$

Let boundary spectral data

$$\{\partial M, \quad \lambda_j, \quad \partial_\nu \varphi_j|_{\partial M}, \quad j = 1, 2, \dots\}$$

be given. Can we determine

$$(M, g) \text{ and } [A(x, D)]?$$

- Let $\kappa(x) > 0$ be smooth and define $G_\kappa u = \kappa\, u$,

$$A_\kappa = \kappa\, A\, \kappa^{-1}, \quad u_\kappa = \kappa\, u,$$
$$dV_\kappa = \kappa^{-2} dV, \quad \sigma_\kappa = \sigma_\kappa + \kappa^{-1}\partial_\nu \kappa.$$

Then $(\partial_\nu + \sigma_\kappa)u_\kappa = G_\kappa(\partial_\nu + \sigma)u$ and

$$\int_M u \cdot Au\, dV = \int_M u_\kappa \cdot A_\kappa u_\kappa\, dV_\kappa$$

- The Dirichlet-to-Neumann map is the same for all operators in the gauge equivalence class $[A(x, D)]$ of $A(x, D)$. Then there is a unique Schrödinger operator

$$-\Delta_g + q \in [A(x, D)].$$

Because of this we next restrict ourselves to the case $A = -\Delta_g + q$.

Setting of the problem for the Schrödinger equation

Denote by

$$u^f = u^f(x,t)$$

the solutions of

$$\begin{aligned} & u_{tt} - \Delta_g u + qu = 0 \quad \text{on } M \times \mathbb{R}_+, \\ & u|_{\partial M \times \mathbb{R}_+} = f, \\ & u|_{t=0} = 0, \quad u_t|_{t=0} = 0, \end{aligned}$$

where ν is unit interior normal of ∂M. Define

$$\Lambda_T f = (\partial_\nu + \sigma) u^f|_{\partial M \times (0,T)}.$$

We denote $\Lambda = \Lambda_\infty$. Assume that we are given the **boundary data** $(\partial M, g_{\partial M}, \Lambda)$, where $g_{\partial M}$ is the metric on ∂M.

Results on the problem:

- First global result for $\Delta + q$ in $\mathbb{R}^n$, by using exponentially growing solutions, Nachman-Sylvester-Uhlmann '88, Novikov '88.
- $c(x)^2\Delta$ in $\mathbb{R}^n$ by boundary control method, Belishev '87 , Belishev-Kurylev '87, using the local controllability by Tataru '95.
- Δ_g on manifold, Belishev-Kurylev '92.
- Equivalence of above inverse problems Katchalov-Kurylev-L.-Mandache 2004
- Maxwell's equations Kurylev-L.-Somersalo 2006.
- Dirac system Kurylev-L. 2009.
- Reconstruction based on iterated time reversal Bingham-Kurylev-L.-Siltanen 2007.

Next we present the reconstruction of (M, g) from the boundary data using the geometric version of the Belishev-Kurylev-Tataru method.

Direct problem: If $u = u^f(x, t)$ satisfies

$$
\begin{aligned}
& u_{tt} - \Delta_g u + qu = 0 \quad \text{on } M \times \mathbb{R}_+, \\
& u|_{\partial M \times \mathbb{R}_+} = f, \\
& u|_{t=0} = 0, \quad u_t|_{t=0} = 0,
\end{aligned}
$$

then (Lasiesca-Lions-Triggiani 1986) $\mathcal{U} : f \mapsto u$ is a bounded map

$$
\begin{aligned}
& \mathcal{U} : L^2(\partial M \times (0, T)) \to C([0, T]; L^2(M)), \\
& \mathcal{U} : H^1_0(\partial M \times (0, T)) \to C([0, T]; H^1(M)), \\
& \|\partial_\nu u|_{\partial M \times (0,T)}\|_{L^2} \leq C \|f\|_{H^1_0(\partial M \times (0,T))}.
\end{aligned}
$$

Sometimes below we omit the x-variable and denote $u^f(t) = u^f(\,\cdot\,, t) \in C([0, T]; L^2(M))$.

Blagovestchenskii identity

Lemma

Let $f, h \in C_0^\infty(\partial M \times [0, 2T])$. *Then*

$$\int_M u^f(x,T) u^h(x,T)\, dV_g(x) =$$

$$\int_{[0,2T]^2} \int_{\partial M} J(t,s)\big[f(t)(\Lambda_{2T} h)(s) - (\Lambda_{2T} f)(t) h(s)\big]\, dS_g(x) dt ds,$$

where $J(t,s) = \frac{1}{2}\chi_L(s,t)$ *and* χ_L *being the characteristic function of the triangle*

$$L = \{(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+ :\ t + s \leq 2T, \quad s < t\}.$$

Proof. Let $w(t,s) = \int_M u^f(t) u^h(s)\, dV_g$. Integrating by parts, we see that

$$
\begin{aligned}
(\partial_t^2 - \partial_s^2) w(t,s) = - \int_M \left[A u^f(t) u^h(s) - u^f(t) A u^h(s) \right] dV_g(x) \\
= \quad - \int_{\partial M} \left[(\partial_\nu + \sigma) u^f(t) u^h(s) - u^f(t) (\partial_\nu + \sigma) u^h(s) \right] dS_g \\
= \quad - \int_{\partial M} \left[\Lambda f(t)\, h(s) - f(t)\, \Lambda h(s) \right] dS_g .
\end{aligned}
$$

Moreover,

$$
w\big|_{t=0} = w\big|_{s=0} = 0, \quad \partial_t w\big|_{t=0} = \partial_s w\big|_{s=0} = 0.
$$

Thus we can find $w(s,t)$ by solving a wave equation with known initial data and right side. $\square$

Domains of influence

Let $\Gamma \subset \partial M$ be a non-empty open set. We denote by $L^2(\Gamma \times [0,T])$ the subspace of $L^2(\partial M \times [0,T])$ that consists of the functions f with supp $(f) \subset \overline{\Gamma} \times [0,T]$.

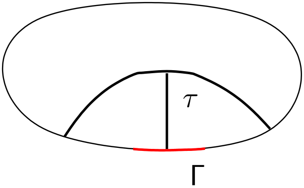
Definition

The subset $M(\Gamma,\tau) \subset M$, $\tau > 0$,

$$M(\Gamma,\tau) = \{x \in M : \ d(x,\Gamma) < \tau\}$$

is called the domain of influence of Γ at time τ.

Observe that we use open domains of influence. By Oksanen (2011), $\overline{M(\Gamma,\tau)} \setminus M(\Gamma,\tau)$ has measure zero.

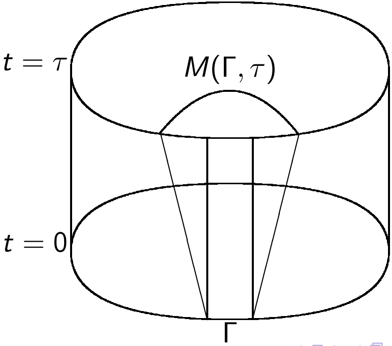


Lemma

Let $\Gamma \subset \partial M$ be open and $f \in L^2(\partial M \times [0,T])$, supp $(f) \subset \Gamma \times (0,T]$. Then

$$\mathit{supp}\ (u^f(\tau)) \subset M(\Gamma, \tau).$$

Proof. The result follows finite velocity of wave propagation. $\square$



We denote by $L^2(\Omega)$, $\Omega \subset M$, the subspace of $L^2(M)$, which consists of all functions $f \in L^2(M)$ that are equal to zero in $M \setminus \Omega$. We prove following Tataru-type controllability type theorem.

Theorem

Let $\tau > 0$. The linear subspace,

$$\{u^f(\tau) \in L^2(M(\Gamma,\tau)) : \ f \in C_0^\infty(\Gamma \times (0,\tau))\},$$

is dense in $L^2(M(\Gamma,\tau))$.

Proof. Let $\psi \in L^2(M(\Gamma,\tau))$ be such that

$$\langle u^f(\cdot\,,\tau),\psi\rangle_{L^2(M)} = 0$$

for all $f \in C_0^\infty(\Gamma \times [0,\tau])$.
To prove the claim, it is sufficient to show that $\psi = 0$.

We consider the wave equation,

$$\begin{aligned}
&(\partial_t^2 - \Delta_g + q)e = 0, \quad \text{in} \quad M \times (0,\tau),\\
&e|_{\partial M \times (0,\tau)} = 0, \quad e|_{t=\tau} = 0, \quad \partial_t e|_{t=\tau} = \psi.
\end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned}
0 &= \int_{M\times(0,\tau)} [u^f (\partial_t^2 - \Delta_g + q)e - ((\partial_t^2 - \Delta_g + q)u^f)e] \, dV_g \, dt \\
&= \int_M u^f(\tau)\, \psi \, dV_g + \int_{\partial M \times (0,\tau)} f \, \partial_\nu e \, dS_g \, dt \\
&= \int_{\partial M \times (0,\tau)} f \, \partial_\nu e \, dS_g \, dt,
\end{aligned}$$

for all $f \in C_0^\infty(\Gamma \times [0,\tau])$.
This yields that the Cauchy data of e vanish on $\Gamma \times (0,\tau)$.

Recall that $e(x,\tau)=0$. We continue e onto $t\in[\tau,2\tau]$ as

$$E(x,t)=\begin{cases} e(x,t), & \text{for } t\le\tau, \\ -e(x,2\tau-t), & \text{for } t>\tau. \end{cases}$$

Then $E\in C([0,2\tau];H^1(M))\cap C^1([0,2\tau];L^2(M))$ and

$$(\partial_t^2-\Delta_g+q)E=0 \quad \text{in } M\times(0,\tau).$$

The Cauchy data of E vanish on $\Gamma\times([0,2\tau]\setminus\{\tau\})$. Since $\partial_\nu E\in L^2(\partial M\times(0,2\tau))$, we see that

$$E|_{\Gamma\times(0,2\tau)}=0,\quad \partial_\nu E|_{\Gamma\times(0,2\tau)}=0.$$

Then $\psi=0$ by the following Tataru-Holmgren-John theorem.

Theorem

Let u be a solution in $M \times (0, 2\tau)$ of the wave equation

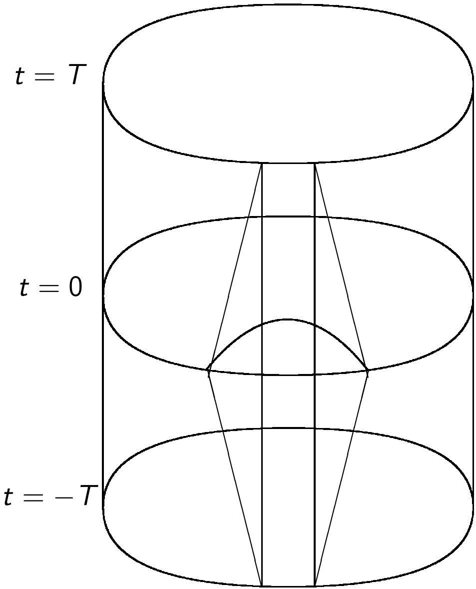
$$(\partial_t^2 - \Delta_g + q)u = 0 \quad \text{in } M \times (0, 2\tau).$$

such that for an open set $\Gamma \subset \partial M$,

$$u|_{\Gamma \times [0,2\tau]} = 0, \ \partial_\nu u|_{\Gamma \times (0,2\tau)} = 0.$$

Then, at $t = \tau$, the function u and its derivative $\partial_t u$ vanish in the domain of influence of Γ,

$$u(x, \tau) = 0, \ \partial_t u(x, \tau) = 0 \quad \text{for } x \in M(\Gamma, \tau).$$


$t = T$
$t = 0$
$t = -T$

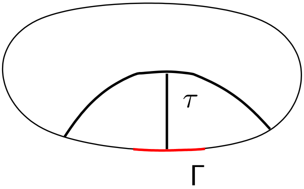
Wave basis

The set

$$\{u^f(\tau) \in L^2(M(\Gamma,\tau)) : \; f \in C_0^\infty(\Gamma \times (0,\tau))\}$$

is dense in $L^2(M(\Gamma,\tau))$. Thus, there are functions f_j, $j = 1, 2, \dots,$ such that $\{u^{f_j}(\tau)\}_{j=1}^\infty$ form an orthonormal basis in the space $L^2(M(\Gamma,\tau))$.

We will construct such functions $f_j \in C_0^\infty(\Gamma \times (0,\tau))$ from the boundary data. The corresponding basis $\{u^{f_j}(\tau)\}_{j=1}^\infty$ is called the wave basis.

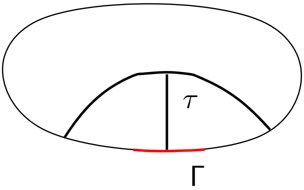


Lemma

Let $\tau > 0$. Given the boundary data it is possible to construct boundary sources $f_j \in C_0^\infty(\Gamma \times (0, \tau))$ such that

$$v_j = u^{f_j}(\tau),\ j = 1, 2, \dots,$$

form an orthonormal basis of $L^2(M(\Gamma, \tau))$.



Proof. Let $\{h_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\Gamma \times (0, \tau))$ be a complete set in $L^2(\Gamma \times [0, \tau])$.
We can compute that inner products

$$c_{jk} = \langle u^{h_j}(\tau), u^{h_k}(\tau)\rangle_{L^2(M)}.$$

Next we use the Gram-Schmidt orthogonalization procedure to construct f_j. More precisely, we define $f_j \in C_0^{\infty}(\Gamma \times (0, \tau))$ recursively by

$$g_j = h_j - \sum_{k=1}^{j-1} \langle u^{h_j}(\tau), u^{f_k}(\tau)\rangle_{L^2(M)} f_k,$$

$$f_j = \frac{g_j}{\langle u^{g_j}(\tau), u^{g_j}(\tau)\rangle_{L^2(M)}^{1/2}}.$$

When $g_j = 0$, we remove the corresponding h_j from the original sequence and continue the procedure. $\square$

Since $\{h_j\} \subset C_0^\infty(\Gamma \times (0,\tau))$, we have $f_j \in C_0^\infty(\Gamma \times (0,\tau))$. Thus $u^{f_j}(\tau) \in C^\infty(M)$.

Let $T > \operatorname{diam}(M)$. Then $M(\partial M, T) = M$, and the corresponding wave basis

$$\{u^{f_j}(\,\cdot\,, T)\}_{j=1}^{\infty}$$

is the orthonormal basis in $L^2(M)$. Next we reserve the notation $\eta_j \in C^\infty(\partial M \times (0, T))$ for the functions f_j for which

$$\{u^{\eta_j}(\,\cdot\,, T)\}_{j=1}^{\infty} \text{ is an orthonormal basis of } L^2(M).$$

We denote below $\psi_j(x) = u^{\eta_j}(x, T)$.

Projectors

Denote by $P_{\Gamma,\tau}$ the orthogonal projector in $L^2(M)$ onto the space $L^2(M(\Gamma,\tau))$,

$$P_{\Gamma,\tau} : L^2(M) \to L^2(M(\Gamma,\tau)),$$

$$(P_{\Gamma,\tau} a)(x) = \chi_{M(\Gamma,\tau)}(x) a(x),$$

where $\chi_{M(\Gamma,\tau)}$ is the characteristic function of the domain of influence $M(\Gamma,\tau)$,

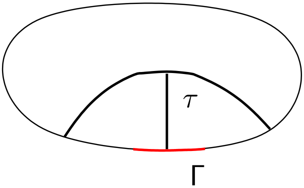
$$\chi_{M(\Gamma,\tau)}(x) = \begin{cases} 1, & \text{for } x \in M(\Gamma,\tau), \\ 0, & \text{for } x \notin M(\Gamma,\tau). \end{cases}$$

Lemma

Let $f, h \in C_0^\infty(\Gamma \times (0,\tau))$ and $\Gamma \subset \partial M$ be an open set. Then, given the the map Λ, it is possible to find the inner product

$$\langle P_{\Gamma,\tau} u^f(t), u^h(s) \rangle_{L^2(M)} = \int_{M(\Gamma,\tau)} u^f(x,t)\, u^h(x,s)\, dV_g$$

for any $0 \leq t, s, \tau \leq T$.



Proof. We can find $f_j \in C_0^\infty(\Gamma \times (0,\tau))$ such that $v_j = u^{f_j}(\tau)$ is an orthonormal basis in $L^2(M(\Gamma,\tau))$,
Then, for any $a \in L^2(M(\Gamma,\tau))$,

$$a = \sum_{j=1}^{\infty} \langle a, v_j \rangle_{L^2(M)}\, v_j.$$

As $\langle P_{\Gamma,\tau} u^f(t), v_j \rangle_{L^2(M)} = \langle u^f(t), v_j \rangle_{L^2(M)}$, we have

$$\langle P_{\Gamma,\tau} u^f(t), u^h(s) \rangle_{L^2(M)} = \sum_{j=1}^{\infty} \langle u^f(t), v_j \rangle_{L^2(M)} \langle u^h(s), v_j \rangle_{L^2(M)}.$$

Here $\langle u^f(t), v_j \rangle_{L^2(M)}$ and $\langle u^h(s), v_j \rangle_{L^2(M)}$ can be computed using boundary data. $\square$

Denote by $M(y,\tau)$ the domain of influence of a point $y \in \partial M$,

$$M(y,\tau) = \{x \in M : \; d(x,y) \leq \tau\},$$

and by $P_{y,\tau}$ the orthoprojector

$$P_{y,\tau} : L^2(M) \to L^2(M(y,\tau)).$$

Corollary

Let $f, h \in L^2(\partial M \times [0, T])$ *and* $y \in \partial M$ *be given. Then the boundary data determine the inner product*

$$\langle P_{y,\tau} u^f(t), u^h(s)\rangle_{L^2(M)} = \int_{M(y,\tau)} u^f(x,t)\, u^h(x,s)\, dV_g$$

for any $0 \le t, s, \tau \le T$.

In particular, we can find $\langle P_{y,\tau} u^{\eta_k}(T), u^{\eta_l}(T)\rangle_{L^2(M)}$ *where* $\{u^{\eta_k}(\tau)\}_{k=1}^{\infty}$ *form an orthonormal basis in* $L^2(M)$.

Proof. Let $\Gamma_l,\ l = 1, 2, \dots$ be open sets such that

$$\Gamma_{l+1} \subset \Gamma_l, \quad \bigcap_{l=1}^{\infty} \Gamma_l = \{y\}.$$

Then,

$$\lim_{l\to\infty} \chi_{M(\Gamma_l,\tau)}(x) = \chi_{M(y,\tau)}(x)$$

pointwise. By the Lebesgue dominated convergence theorem,

$$\lim_{l\to\infty} \langle P_{\Gamma_l,\tau} u^f(t), u^h(s)\rangle_{L^2(M)} = \langle P_{y,\tau} u^f(t), u^h(s)\rangle_{L^2(M)}.$$

$\square$

Corollary

Let $y_j \in \partial M$, $\tau_j > 0$, $k, l \in \mathbb{Z}_+$. *Then the boundary data determine the inner product*

$$\langle Q_N u^{\eta_k}(T), u^{\eta_l}(T) \rangle_{L^2(M)}$$

where

$$Q_N = \prod_{j=1}^{N} P_{y_j, \tau_j}$$

and $\{u^{\eta_j}(\tau)\}_{j=1}^{\infty}$ *form an orthonormal basis in* $L^2(M)$.

Proof. For $N = 1$ the claim follows from Corollary 8. Assume now that it is valid for $N - 1$.
We can write for $f = \eta_p$

$$Q_{N-1}u^f(s) = \sum_{k=1}^{\infty} \langle Q_{N-1}u^f(s), u^{\eta_k}(T)\rangle_{L^2(M)} u^{\eta_k}(T)$$

and

$$\begin{aligned} & \langle Q_N u^f(T), u^{\eta_l}(T)\rangle_{L^2(M)} = \langle P_{y_N,\tau_N} Q_{N-1}u^f(T), u^{\eta_l}(T)\rangle_{L^2(M)} \\ = & \sum_{k=1}^{\infty} \langle P_{y_N,\tau_N} u^{\eta_k}(T), u^{\eta_l}(T)\rangle_{L^2(M)} \langle Q_{N-1}u^f(s), u^{\eta_k}(T)\rangle_{L^2(M)}. \end{aligned}$$

Thus we find the matrix of Q_N in the basis $(u^{\eta_j}(T))_{j=1}^{\infty}$ of $L^2(M)$. From this the claim follows by induction. $\square$

Observations:

- We can compute the Gram matrix $[q_{jk}]_{j,k=1}^{\infty}$,

$$q_{jk} = \langle Q u^{\eta_j}(T), u^{\eta_k}(T) \rangle_{L^2(M)}$$

  where $\{u^{\eta_j}(T)\}_{j=1}^{\infty}$ is an orthonormal basis in $L^2(M)$ and

$$Q = \left(\prod_{j=1}^{N} P_{y_j,\tau_j^+} \right) \left(\prod_{j=1}^{N} (1 - P_{y_j,\tau_j^-}) \right)$$

- The projector $Q : L^2(M) \to L^2(M)$ is

$$Qv(x) = \chi_I(x)\, v(x), \quad I = \bigcap_{j=1}^{N} (M(y_j, \tau_j^+) \setminus M(y_j, \tau_j^-)).$$

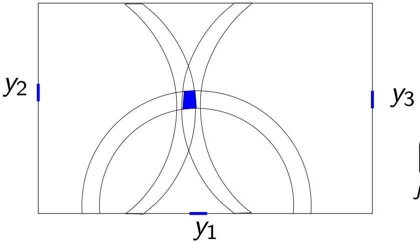
- The projector $Q : L^2(M) \to L^2(M)$ is

$$Qv(x) = \chi_I(x)\, v(x), \quad I = \bigcap_{j=1}^{N} (M(y_j, \tau_j^+) \setminus M(y_j, \tau_j^-)).$$

- The projector $Q : L^2(M) \to L^2(M)$ vanishes, that is, its Gram matrix is zero if and only if

$$m(I) = 0, \quad I = \bigcap_{j=1}^{N} (M(y_j, \tau_j^+) \setminus M(y_j, \tau_j^-)).$$

Thus we can check using boundary data if $m(I) = 0$.



$$\bigcap_{j=1}^{N} (M(y_j, \tau_j^+) \setminus M(y_j, \tau_j^-))$$

Boundary distance functions. For $x \in M$ define

$$r_x(y) = d(x,y), \;\; y \in \partial M.$$

Let

$$R : M \to C(\partial M), \quad R(x) = r_x.$$

Next we consider $R(M)$ as a submanifold on $C(\partial M)$.

Theorem

Using boundary data we can determine

$$R(M) = \{r_x \in C(\partial M) : \quad x \in M\}.$$

Thus the constructed set $R(M)$ can be identified with M.

By previous observations, it is enough to prove the following result:

Lemma

Let $\{z_n\}_{n=1}^{\infty}$ be a dense set on ∂M. Then $r(\cdot) \in C(\partial M)$ lies in $R(M)$ if and only if, for any $N > 0$,

$$I_N = \bigcap_{n=1}^{N} M(z_n, r(z_n) + \frac{1}{N}) \cap \bigcap_{n=1}^{N} (M(z_n, r(z_n) - \frac{1}{N}))^c.$$

satisfies

$$m(I_N) \neq 0 \tag{1}$$

Moreover, condition (1) can be verified using the boundary data. Thus boundary data determines $R(M) \subset C(\partial M)$.

Proof *"If"–part.* Assume that $r(\cdot) = r_x(\cdot)$ with some $x \in M$. Consider a ball $B_{1/N}(x)$. Then,

$$B_{1/N}(x) \subset M(z, r(z) + \frac{1}{N}) \setminus M(z, r(z) - \frac{1}{N}).$$

Thus if $B_{1/N}(x) \subset I_N$ and $m(I_N) \neq 0$.

"Only if"–part. Assume that $m(I_N) \neq 0$. Then there exists

$$x_N \in \bigcap_{n=1}^{N} \left(M(z_n, r(z_n) + \frac{1}{N}) \setminus M(z_n, r(z_n) - \frac{1}{N}) \right).$$

Since M is compact, we can choose a subsequence of x_N (denoted also by x_N), so that there exists a limit

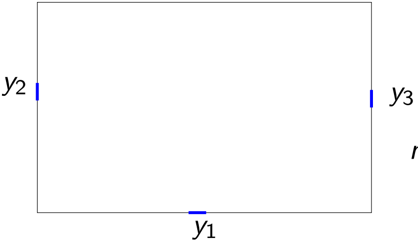
$$x = \lim_{n \to \infty} x_N.$$

By continuity of the distance function, it follows from (2) that

$$d(x, z_n) = r(z_n), \quad n = 1, 2, \ldots.$$

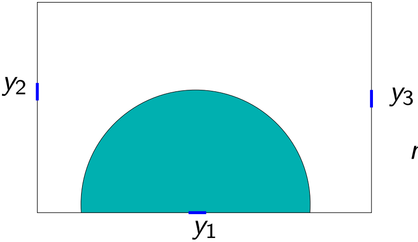
Since $\{z_n\}$ are dense in ∂M, we see that $r(z) = d(x, z)$ for all $z \in \partial M$. Thus $r = r_x$. $\square$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



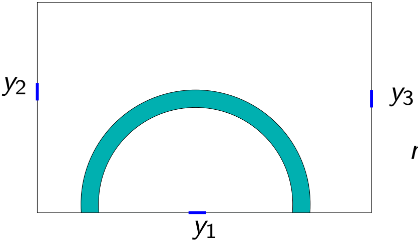
$$m\left(\bigcap_{j=1}^{N}(M(y_j,\tau_j^+)\setminus M(y_j,\tau_j^-))\right)=0?$$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



$$m\left(\bigcap_{j=1}^{N}\left(M(y_j,\tau_j^{+})\setminus M(y_j,\tau_j^{-})\right)\right)=0?$$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



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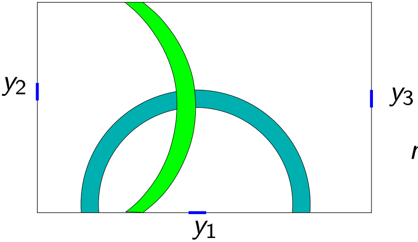

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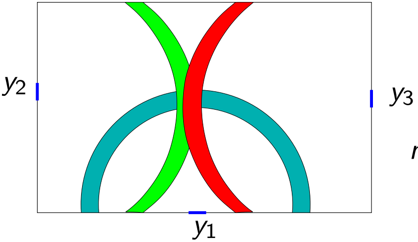
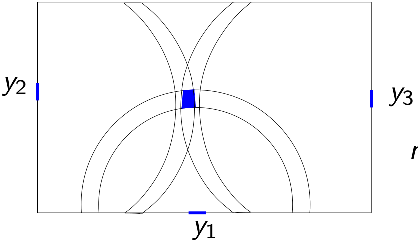
Visualization how to check if $r(\cdot)$ is in $R(M)$.



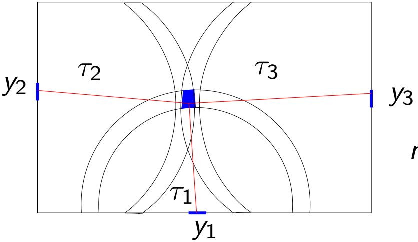
$$m\left(\bigcap_{j=1}^{N}\left(M(y_j, \tau_j^+) \setminus M(y_j, \tau_j^-)\right)\right) = 0?$$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



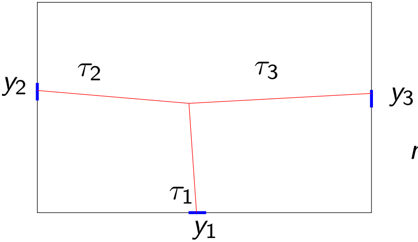
$$m\left(\bigcap_{j=1}^{N}\left(M(y_j,\tau_j^+) \setminus M(y_j,\tau_j^-)\right)\right) = 0?$$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



$$m\left(\bigcap_{j=1}^{N}\left(M(y_j,\tau_j^{+}) \setminus M(y_j,\tau_j^{-})\right)\right) = 0?$$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



$$m\left(\bigcap_{j=1}^{N}\left(M(y_j,\tau_j^+)\setminus M(y_j,\tau_j^-)\right)\right)=0?$$

Reconstruction of (M, g) from $R(M)$.

Next, we prove Kurylev's theorem:

Theorem

The set $R(M)$ has a Riemannian manifold structure which is isometric to (M, g). Moreover, when $C(\partial M)$ and $R(M)$ are given, this Riemannian manifold structure is uniquely determined.

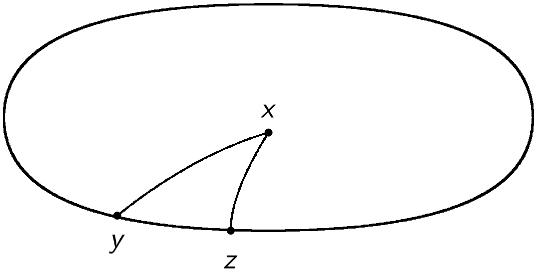
Recall that for $x \in M$

$$r_x(z) = d(x, z), \;\; z \in \partial M$$

and that

$$R : M \to C(\partial M), \quad R(x) = r_x.$$

Next we consider $R(M)$ as a submanifold on $C(\partial M)$.



By triangular inequality we have

$$\|r_x - r_y\|_{C(\partial M)} \leq d(x,y), \quad x,y \in M.$$

Example: Consider that case when all geodesics of a compact manifold (M,g) are the shortest curves between their endpoints and all geodesics can be continued to geodesics that hit the boundary. Then for any $x,y \in M$ the geodesic from x to y hits later to $z \in \partial M$. Then

$$\|r_x - r_y\|_{C(\partial M)} \geq |r_x(z) - r_y(z)| = d(x,y)$$

Then (M,d) is isometric to $(R(M), \|\cdot\|_\infty)$.

Lemma

The set $R(M)$ is homeomorphic to (M, g).

Proof.

Recall the following simple result from topology:

Assume that X and Y are Hausdorff spaces, X is compact and $F : X \to Y$ is a continuous, bijective map from X to Y. Then F is a homeomorphism.

Clearly, $R : M \to R(M)$ is surjective and continuous.
Next we prove that it is one-to-one. Assume that $r_x(\cdot) = r_y(\cdot)$.
Denote by z_0 any point where

$$d(x, \partial M) = \min_{z \in \partial M} r_x(z) = r_x(z_0) \quad \text{or}$$
$$d(y, \partial M) = \min_{z \in \partial M} r_y(z) = r_y(z_0).$$

Then z_0 is a nearest boundary point to x implying that the shortest geodesic from z_0 to x is normal to ∂M. The same is true for y with the same point z_0.
Thus $x = \gamma_{z_0}(s) = y$ for $s = d(x, z_0)$. $\square$

Boundary normal coordinates.

Consider a normal geodesic $\gamma_z(s) = \gamma_{z,\nu}(s)$ starting from z. For small s,

$$d(\gamma_z(s), \partial M) = s, \tag{2}$$

and z is the unique nearest point to $\gamma_z(s)$ on ∂M. Let $\tau(z)$ be the largest value for which (2) is valid. Then for $s > \tau(z)$,

$$d(\gamma_z(s), \partial M) < s,$$

and z is no more the nearest boundary point.

$\tau(z) \in C(\partial M)$ is the cut locus distance function,

$$\tau(z) = \sup\{s > 0;\ d(\gamma_z(s), \partial M) = s\}.$$

The cut locus is

$$\omega = \{x_z : z \in \partial M,\ x_z = \gamma_z(\tau(z))\}.$$

In domain $M \setminus \omega$ we can use the $\partial M \times [0, \infty)$ valued coordinates

$$x \mapsto (z(x), t(x)),$$

where $z(x) \in \partial M$ is the unique nearest point to x and $t(x) = d(x, \partial M)$. More precisely, when $x_0 \in M \setminus \omega$ and $W \subset \partial M$ and $Y : W \to \mathbb{R}^{n-1}$ are some local coordinates near $z(x_0)$,, $x \mapsto (Y(z(x)), t(x))$ are the boundary normal coordinates near x_0.

We will now use boundary normal coordinates to introduce a differential structure and metric tensor, g_R, on $R(M)$ to have an isometry

$$R : (M, g) \to (R(M), g_R).$$

We will concentrate mainly on doing so for $R(M) \setminus R(\omega)$.

First, observe that we can identify those $r = r_x \in R(M)$ with $x \in M \setminus \omega$.

Indeed, $r = r_x$ with $x = \gamma_z(s)$, $s < \tau(z)$ if and only if

i. $r(\cdot)$ has a unique global minimum at some point $z \in \partial M$.

ii. there is $\widetilde{r} \in R(M)$ having a unique global minimum at the same z and $r(z) < \widetilde{r}(z)$.

A differential structure on $R(M \setminus \omega)$ can be defined by introducing coordinates near each $r^0 \in R(M \setminus \omega)$.
In a sufficiently small neighbourhood $V \subset R(M)$ of r^0 the coordinates

$$r \mapsto (Z(r), T(r)) = (Y(\operatorname*{argmin}_{z \in \partial M} r), \min_{z \in \partial M} r)$$

are well defined. The

$$x \mapsto (Z(r_x), T(r_x))$$

coincides with the boundary normal coordinates
$x \mapsto (Y(z(x)), t(x))$ on (M, g).
These coordinate determine the differential structure on $R(M \setminus \omega)$.

Construction the differential structure on $R(M)$ near $R(\omega)$.
Let $z_0 \in \partial M$ and $t_0 = \tau(z_0)$. The first conjugate point $\gamma_{z_0,\nu}(t_1)$ on $\gamma_{z_0,\nu}$ satisfies $t_1 > t_0$.
Let $x_0 = \gamma_{z_0,\nu}(t_0)$. Then if $z_1, \dots, z_{n-1} \in \partial M$ are such points near z_0 that $w_j = \mathrm{Grad}\, d(\,\cdot\,, z_j)|_{x_0}$, $j = 0, 1, 2, \dots, n-1$ are linearly independent, there is a small neighbourhood $V \subset R(M)$ of $r^0 = R(x_0)$ in which the coordinates

$$r \mapsto (r(z_j))_{j=0}^{n-1}$$

are well defined. The

$$x \mapsto (r(z_j))_{j=0}^{n-1}$$

coincides with the distance normal coordinates $x \mapsto (d(x, z_j))_{j=0}^{n-1}$ on (M, g).
These coordinate determine the differential structure near $R(\omega)$.

Construction of the metric g_R on $R(M)$.

Let $r^0 \in R(M \setminus \omega)$, $V \subset R(M)$ be its neighbourhood, and $X : V \to U \subset \mathbb{R}^n$ be local coordinates, $X(r^0) = 0$

For $z \in \partial M$ we define an evaluation function

$$K_z : V \to \mathbb{R}, \quad K_z(r) = r(z).$$

The function $E_z = K_z \circ X^{-1} : U \to \mathbb{R}$ satisfies

$$E_z(y) := d(z, X^{-1}(y)), \quad y \in U.$$

Consider the function $E_z(y)$ as a function of y with a fixed z.
The differential dE_z at point 0 is a covector in T_0^*U.
Since the gradient of a distance function has length one, we see that

$$\|dE_z\|_{g_R}^2 := (g_R)^{jk}\frac{\partial E_z}{\partial y^j}\frac{\partial E_z}{\partial y^k} = 1, \quad j,k = 1,\dots,n.$$

Varying $z \in \partial M$ we obtain a set of covectors $dE_z(0)$ in the unit ball of (T_0^*U, g_R) which contains an open set.

This determines uniquely the tensor g_R.

Hence we have proven the following result (originally proven by Belishev-Kurylev 1992).

Theorem

The boundary data $(\partial M, \Lambda)$ *determine the manifold* (M, g) *upto isometry.*

Also the potential $q(x)$ of the operator $-\Delta_g + q$ can be uniquely determined.

Alternative point of view: Time reversal

Let us next consider the Neumann problem: We denote by

$$u^f = u^f(x,t)$$

the solutions of

$$\begin{aligned} &u_{tt} - \Delta_g u + qu = 0 \quad \text{on } M \times \mathbb{R}_+, \\ &-\partial_\nu u|_{\partial M \times \mathbb{R}_+} = f, \\ &u|_{t=0} = 0, \quad u_t|_{t=0} = 0, \end{aligned}$$

where ν is unit interior normal of ∂M. Define

$$\Lambda_T^{(N)} f = u^f|_{\partial M \times (0,T)}.$$

We denote $\Lambda_T^{(N)} = \Lambda_\infty^{(N)}$. Assume that we are given the **boundary data** $(\partial M, \Lambda^{(N)})$.

Direct problem: If $u = u^f(x,t)$ satisfies

$$\begin{aligned}
&u_{tt} - \Delta_g u + qu = 0 \quad \text{on } M \times \mathbb{R}_+,\\
&-\partial_\nu u|_{\partial M \times \mathbb{R}_+} = f,\\
&u|_{t=0} = 0, \quad u_t|_{t=0} = 0,
\end{aligned}$$

then (Lasieska-Triggiani 1990) $\mathcal{U} : f \mapsto u$ is a bounded map

$$\mathcal{U} : L^2(\partial M \times (0,T)) \to C([0,T]; H^\alpha(M)), \quad \alpha < 3/5.$$

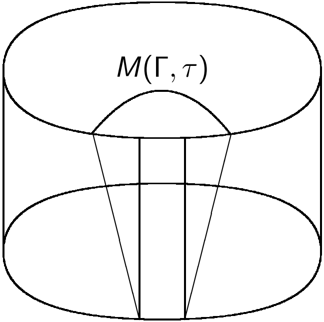
We are interested in the case $\alpha = 0$.

On formal level, the the previous algorithm is based on the following task: Let f be given. Can we find h such that

$$u^h(x, T) = \chi_{M(\Gamma,\tau)}(x)u^f(x, T).$$

This is equivalent of the minimization of

$$\|u^f(T) - u^h(T)\|_{L^2(M)} : \quad h \in L^2(\Gamma \times [0, \tau]).$$



Generally, the minimization problem has no solution and is ill-posed. We consider the regularized minimization problem

$$\min_{h \in L^2(\partial M \times [0,2T])} F(h, \alpha)$$

where $\alpha \in (0,1)$ and

$$F(h, \alpha) = \langle K(Ph - f), Ph - f \rangle_{L^2(\partial M \times [0,2T], dS_g)} + \alpha \|h\|^2_{L^2}.$$

Let us recall the Blagovestchenskii identity

$$\begin{aligned} & \int_M u^f(x,T)u^h(x,T)\,dV_g(x) \\ = & \int_{[0,2T]^2}\int_{\partial M} J(t,s)[f(t)(\Lambda_{2T}^{(N)}h)(s) - (\Lambda_{2T}^{(N)}f)(t)h(s)]dS_g dtds \\ = & \int_{\partial M\times[0,2T]} (Kf)(x,t)h(x,t)\,dS_g(x)dt, \end{aligned}$$

where $J(t,s) = \frac{1}{2}\chi_L(s,t)$ and

$$L = \{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+ :\ t+s\le 2T,\quad s>t\}.$$

Here

$$K = R_{2T}\Lambda_{2T}^{(N)}R_{2T}J - J\Lambda_{2T}^{(N)},$$

where

$$Rf(x,t) = f(x, 2T - t),$$

is the **time reversal operator** and

$$Jf(x,t) = \frac{1}{2}\int_0^{\min(2T-t,t)} f(x,s)ds,$$

is the **time filter**. Note that

$$(\Lambda_{2T}^{(N)})^* = R_{2T}\Lambda_{2T}^{(N)}R_{2T} \quad \text{as} \quad G(x,x',t'-t) = G(x',x,-(t)-(-t')).$$

We also use the **restriction operator**

$$P_Bf(x,t) = \chi_B(x,t)u(x,t),$$

The *processed time reversal iteration* is

$$\begin{aligned} F &:= \frac{1}{\omega} P(R\Lambda_{2T}^{(N)} RJ - J\Lambda_{2T}^{(N)})f, \\ a_n &:= \Lambda_{2T}^{(N)}(h_n), \\ b_n &:= \Lambda_{2T}^{(N)}(RJh_n), \\ h_{n+1} &:= (1 - \frac{\alpha}{\omega})h_n - \frac{1}{\omega}(PRb_n - PJa_n) + F, \end{aligned}$$

where $f \in L^2(\partial M \times [0, 2T])$ and $\alpha, \omega > 0$ are parameters. Iteration starts at $h_0 = 0$.

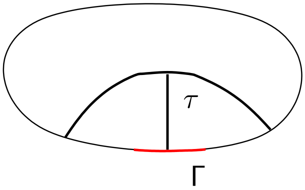
Theorem (Bingham-Kurylev-L.-Siltanen 2007)

Let $\Gamma_1 \subset \partial M$, $0 \le T_1 \le T$, and $B = \Gamma_1 \times [T - T_1, T]$. Let $f \in L^2(\partial M \times \mathbb{R}_+)$ and $h_n = h_n(\alpha)$ be defined by the processed time reversal iteration. Then

$$h(\alpha) = \lim_{n \to \infty} h_n(\alpha)$$

and the limits satisfy in $L^2(M)$

$$\lim_{\alpha \to 0} u^{h(\alpha)}(x, T) = \chi_{M(\Gamma_1, T_1)}(x) u^f(x, T).$$



$M(\Gamma, \tau) = \{x \in M : \ d(x, \Gamma) \le \tau\}.$

Proof. The minimization problem

$$\min_{h \in L^2(\partial M \times [0,2T])} F(h,\alpha)$$

with $\alpha \in (0,1)$ and

$$\begin{aligned} F(h,\alpha) &= \langle K(Ph-f), Ph-f \rangle_{L^2(\partial M \times [0,2T], dS_g)} \\ &\quad + \alpha \|h\|^2_{L^2} \end{aligned}$$

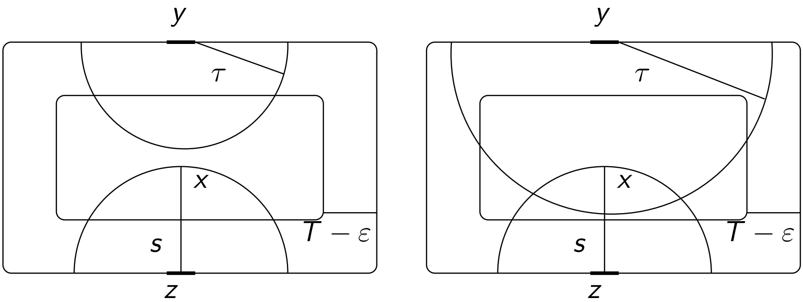
leads to a linear equation

$$(PKP + \alpha)h = PKf.$$

This can be solved using iteration. $\square$

Corollary

Assume we are given the boundary ∂M and the response operator $\Lambda^{(N)}$. Then using the the processed time reversal iteration we can find constructively the manifold (M, g) upto an isometry and on it the operator A uniquely.



Let $x = \gamma_{z,\nu}(s)$. The distance $\mathrm{dist}(x, z)$ is the infimum of all τ that satisfy the condition

(A) The set

$$(M(z,s) \cap M(y,\tau)) \setminus M(\partial M, s-\varepsilon)$$

is non-empty for all $\varepsilon > 0$.

Denote $A_1 = M(z,s)$, $A_2 = M(y,\tau))$, and $B = M(\partial M, s-\varepsilon)$ and $f \in L^2(\partial M \times [0,T])$,

$$\begin{aligned} v &= \chi_{A_1\cap A_2} u^f(\,\cdot\,,T), \\ &= (\chi_{A_1} + \chi_{A_2} - \chi_{A_1\cup A_2})\, u^f(\,\cdot\,,T), \\ w &= \chi_{(A_1\cap A_2)\setminus B}\, u^f(\,\cdot\,,T) \\ &= v - \chi_B v. \end{aligned}$$

Using time reversal we can find for given $\tau > 0$ if there exists f such that where w is non-vanishing. Taking the infimum of such τ we find $\mathrm{dist}(x,z)$.
Thus $R(M)$ can be found using modified time-reversal method.
This determines (M,g).

How to solve inverse spectral problems?

Operator $A = -\Delta_g + q$ has in $L^2(M, dV_g)$ orthonormal eigenfunctions φ_j,

$$\begin{aligned}(-\Delta_g + q - \lambda_j)\varphi_j &= 0,\\ \partial_\nu \varphi_j|_{\partial M} &= 0.\end{aligned}$$

Let boundary spectral data

$$\{\partial M, \quad \lambda_j, \quad \varphi_j|_{\partial M}, \quad j = 1, 2, \dots\}$$

be given. Can we determine

$$(M, g) \text{ and } q?$$

Main points of previous construction for wave equation were:
-Control theory
-Computing inner products $\langle u^f(T), u^h(T)\rangle_{L^2(M)}$ using boundary data.

We consider the Fourier coefficients of $u^f(x,t)$ w.r.t. basis φ_k,

$$u^f(x,t) = \sum_{k=1}^{\infty} u_k^f(t)\varphi_k(x), \quad u_k^f(t) = \langle u^f(t), \varphi_k \rangle_{L^2(M)}.$$

Lemma

Fourier coefficients can be written in terms of boundary spectral data as

$$u_k^f(t) = \int_0^t \int_{\partial M} f(z,t') s_k(t-t')\varphi_k(z)\, dS_g(z) dt'. \tag{3}$$

Here

$$s_k(t) = \begin{cases} \frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}, & \lambda_k > 0, \\ t, & \lambda_k = 0, \\ \frac{\sinh\sqrt{|\lambda_k|}t}{\sqrt{|\lambda_k|}}, & \lambda_k < 0. \end{cases}$$

Proof. We see that $u^f(x,t)|_{t=0} = 0$ and $\partial_t u_k^f(t)|_{t=0} = 0$. Therefore,

$$\begin{aligned}
& \frac{d^2}{dt^2} u_k^f(t) = \int_M \partial_t^2 u^f(x,t) \varphi_k(x) dV_g \\
= & \int_M \big(\Delta_g u^f(x,t) - q(x) u^f(x,t)\big) \varphi_k(x) dV_g \\
= & -\int_{\partial M} (\partial_\nu u^f(x,t)\varphi_k(x) - u^f(x,t)\partial_\nu \varphi_k(x)) dS_g \\
& -\lambda_k \int_M u^f(x,t)\varphi_k(x) dV_g \\
= & -\int_{\partial M} f(x,t)\varphi_k(x) dS_g - \lambda_k u_k^f(t).
\end{aligned}$$

Solving this ordinary differential equation with the initial conditions $u_k^f(0) = \partial_t u_k^f(0) = 0$ we obtain the equation (3). $\square$