# Inverse problems for non-linear partial differential equations

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fame MATHEMATICS PHYSICS COMPUTING

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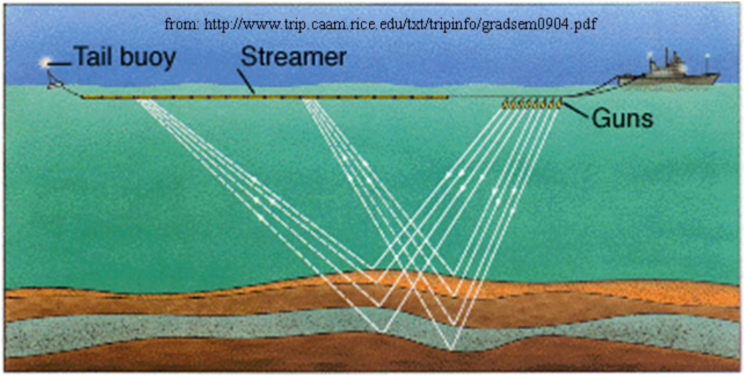
- An inverse problem for a linear wave equation
- Inverse problems for passive observations
- Inverse problems for non-linear wave equations
- Inverse problems for non-linear elliptic equations

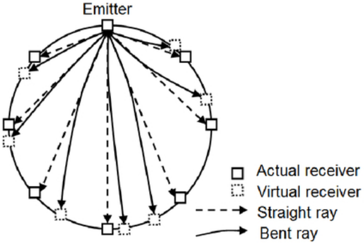


# Wave imaging and Riemannian geometry

Consider a body $M \subset \mathbb{R}^3$ where the wave speed is varying. Travel time of waves between points determine a non-Euclidean metric

$$ds^2 = \sum_{j,k=1}^{n} g_{jk}(x)dx^j dx^k, \quad \text{or with an isotropic wave speed, } ds^2 = c(x)^{-2}dx^2$$

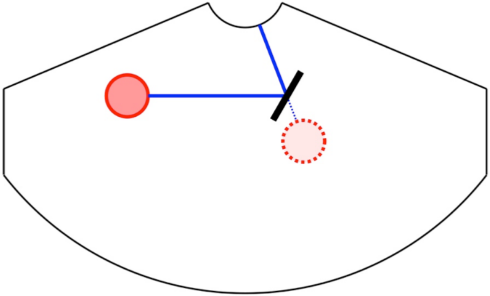
The inverse problem for the linear wave equation is to determine the wave speed, or the metric $(g_{jk}(x))_{j,k=1}^n$, from boundary measurements.




Figures by C. Ammon (Proc. SPIE. 2015)

# Why more mathematics is needed in imaging?

Traditional algorithms in medical imaging based on ray tracing cause artifacts.
The ultrasound image taken during pregnancy (on left), has a mirror image artifact.
This led to a wrong diagnosis of an extra-uterine pregnancy.



By using wave equation such ghost images do not appear.

Figure: Malhotra et al., West J Emerg Med. 2014

# Geometric inverse problem for a linear wave equation

Let $(M,g)$ be a Riemannian manifold with boundary, $\dim(M)=n\geq 2$.
Let $u(x,t)=u^f(x,t)$ solve the wave equation

$$\begin{aligned}
&(\partial_t^2-\Delta_g)u(x,t)=0 \quad \text{on } (x,t)\in M\times\mathbb{R}_+,\\
&\partial_\nu u(x,t)|_{\partial M\times\mathbb{R}_+}=f(x,t),\\
&u|_{t=0}=0, \quad \partial_t u|_{t=0}=0,
\end{aligned}$$

where $\nu$ is the unit normal of the boundary $\partial M$ and

$$\Delta_g u=\sum_{j,k=1}^n |g(x)|^{-1/2}\frac{\partial}{\partial x^j}(|g(x)|^{1/2}g^{jk}(x)\frac{\partial}{\partial x^k}u(x)).$$

The measurements on the boundary are modelled by the Neumann-to-Dirichlet map

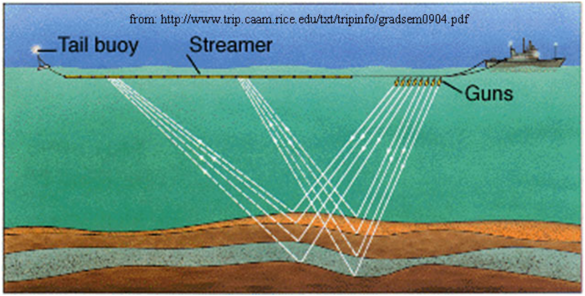
$$\Lambda f=u^f(x,t)\big|_{(x,t)\in\partial M\times\mathbb{R}_+}.$$

**Inverse problem:** Let $(M, g)$ be a Riemannian manifold with boundary. Let the boundary $\partial M$ and the map

$$\Lambda : \partial_\nu u|_{\partial M \times \mathbb{R}_+} \to u|_{\partial M \times \mathbb{R}_+}$$

be given. Can we construct a manifold $(M', g')$ that is isometric to $(M, g)$?

**Example.** Let $M = \mathbb{R}^2 \times \mathbb{R}_+$. Can we construct the metric tensor $g$ in local coordinate charts, or the wave speed function $c(x)$, when the measurements are done on the boundary?



# Inverse problems for linear hyperbolic equations

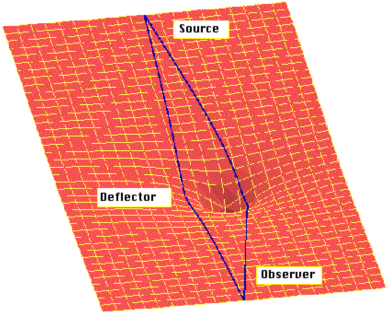
1. Unique solvability of the inverse problem for $(\partial_t^2 - c(x)^2\Delta)u = 0$ in $\Omega \subset \mathbb{R}^n$ was proven by combining the boundary control method by Belishev 1987 and the unique continuation theorem by Tataru 1995.
2. Belishev-Kurylev 1992: Spectral problem for $\Delta_g$ on a manifold.
3. Burago-Ivanov-L.-Lu 2020: Stability of the reconstruction of a Riemannian manifold from the boundary data.
4. Alexakis-Feizmohammadi-Oksanen 2022 (a recent breakthrough!): Determination of the lower order terms in wave equation when the metric depends on time and is close to the Euclidean metric.

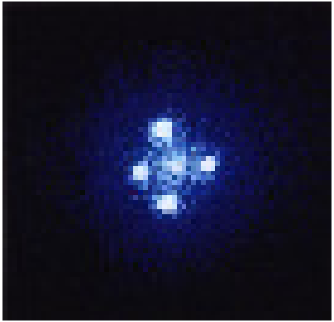
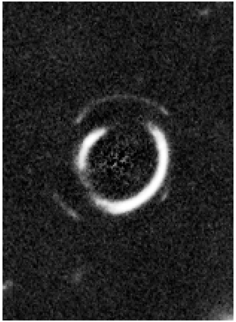
All results 1-3 are based on Tataru's unique continuation theorem and require that the metric is time-independent.

Below, we will show how the non-linearity helps in solving inverse problems and consider a time-depending metric and the new stability results.

Outline:

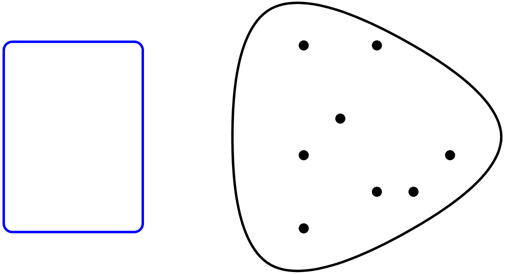
- An inverse problem for a linear wave equation
- Inverse problems for passive observations
- Inverse problems for non-linear wave equations
- Inverse problems for non-linear elliptic equations



Figures: J. Cohn and HST/NASA/ESA

# Inverse problem for passive measurements



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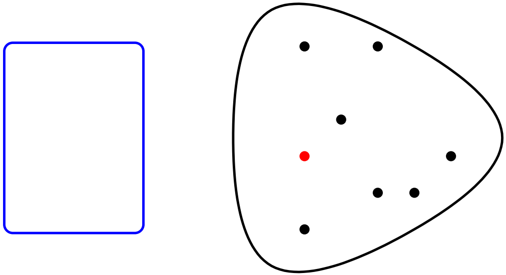

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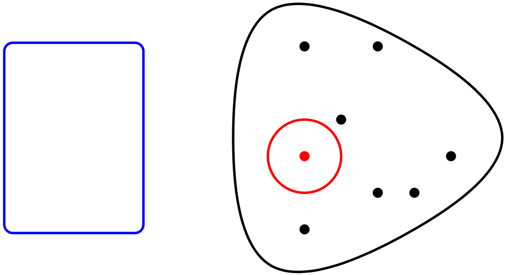

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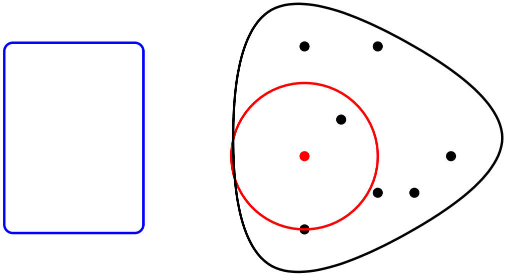

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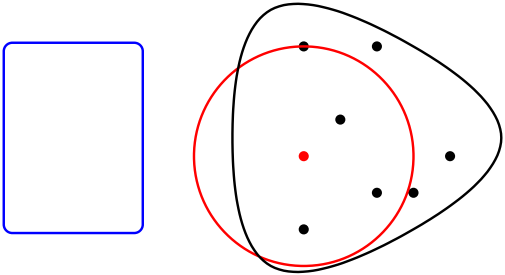

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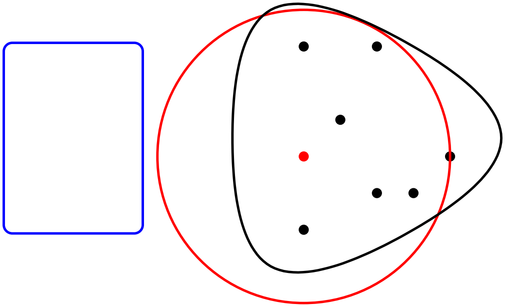

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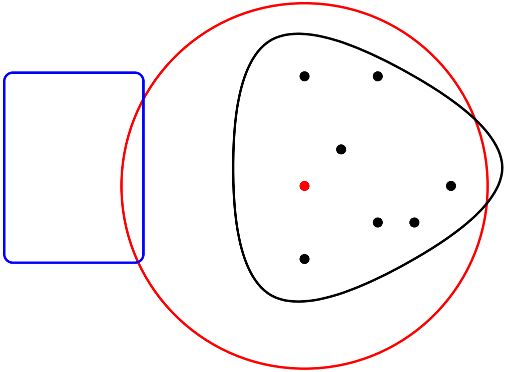

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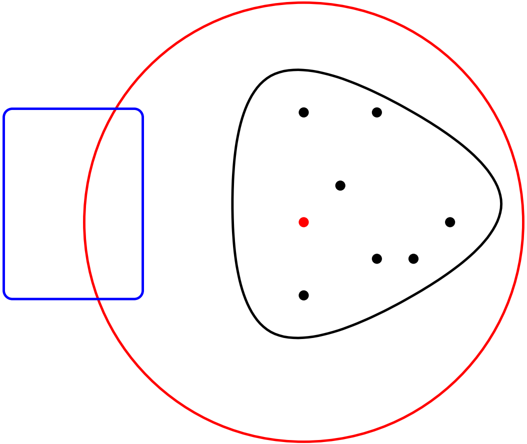

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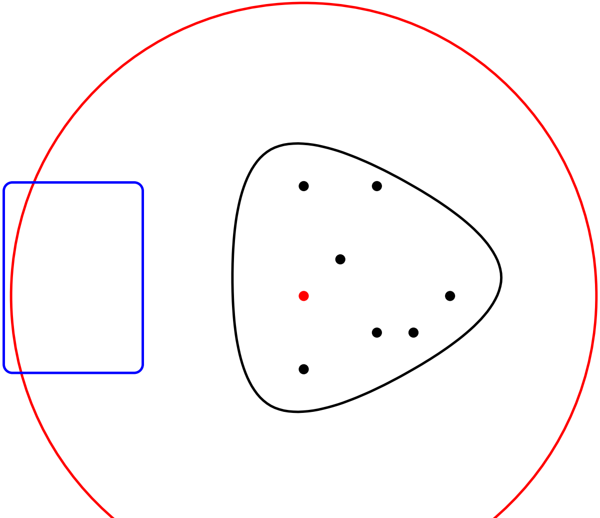

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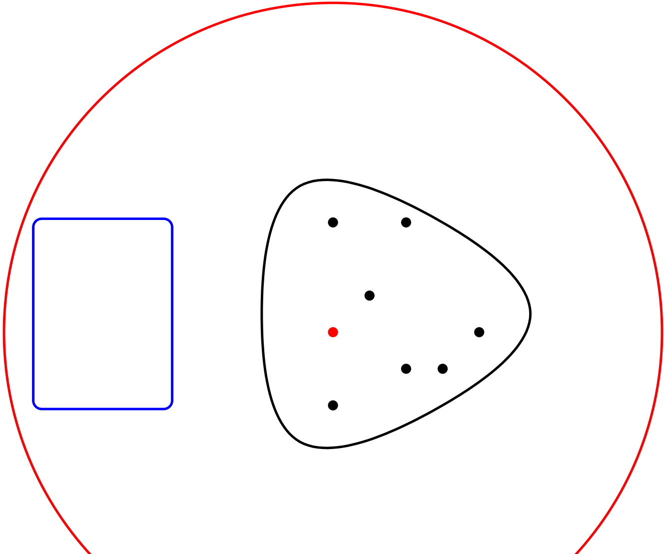

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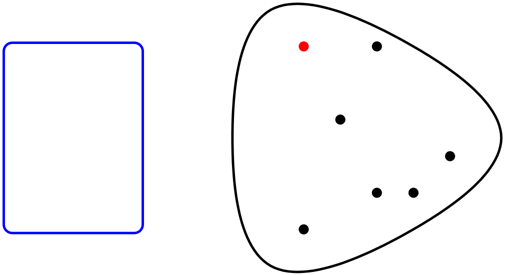

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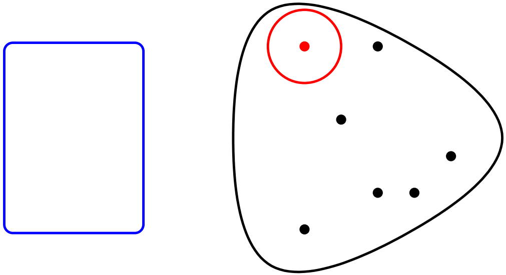

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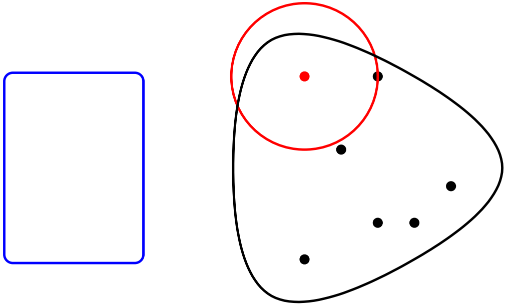

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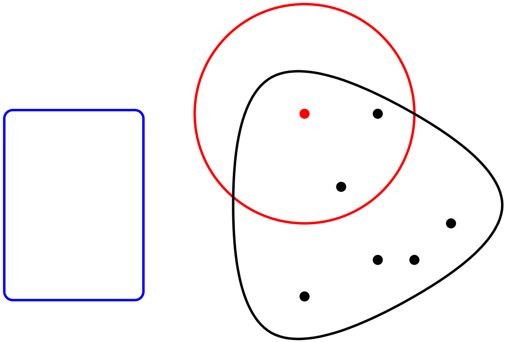

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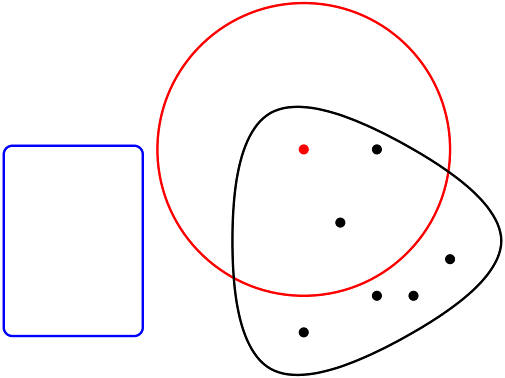

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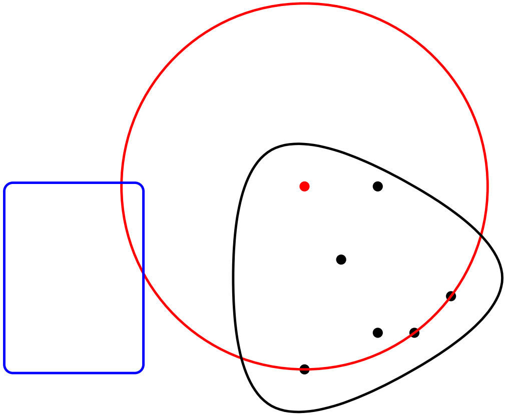

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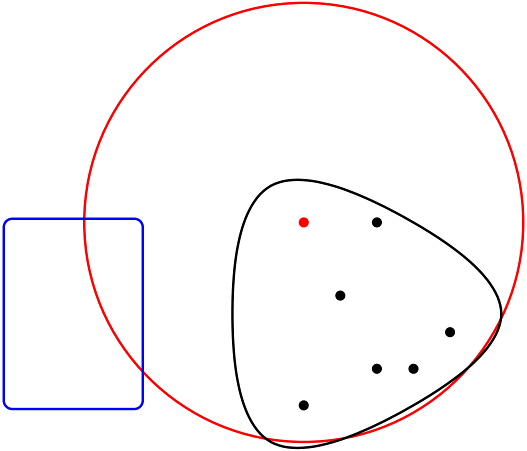

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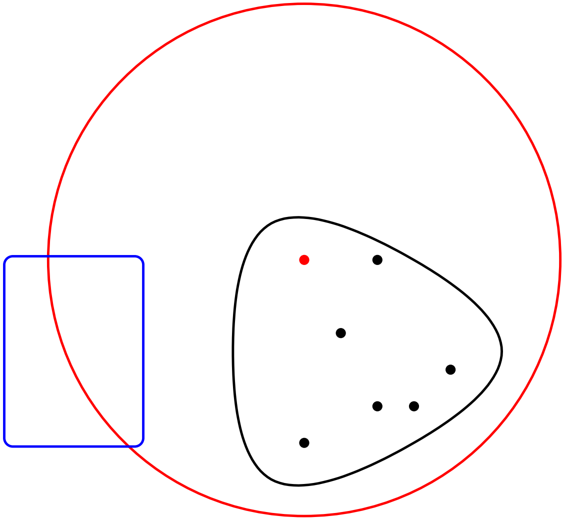
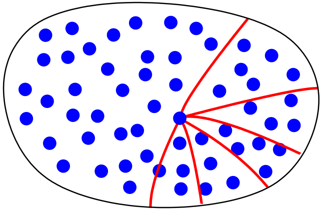


# Inverse problem for point sources on a Riemannian manifold with boundary.

Assume that for all points $x$ in $M$ we know the boundary values of the wave produced by a point source that goes off at $x$ at a known time.
These data give us the boundary distance functions

$$r_x(z) = \mathrm{dist}_M(x,z),\ z \in \partial M \text{ for points } x \in M.$$

Can we reconstruct $(M,g)$ if we know $\partial M$ and all boundary distance functions, $\{r_x \in C(\partial M):\ x \in M\}$?



When the times when the point sources goes off are unknown, a similar question for distance difference function is studied in L.-Saksala 2020 and Ivanov 2021.

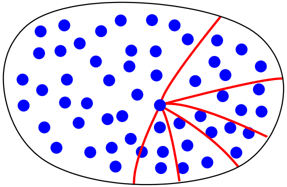
# Reconstruction of the manifold from boundary distance functions

Assume that $(M, g)$ is a compact Riemannian manifold with boundary.
Recall that $r_x(z) = \text{dist}_M(x, z)$, $z \in \partial M$.

## Theorem (Kurylev 1997, Katcalov-Kurylev-L. 2001)

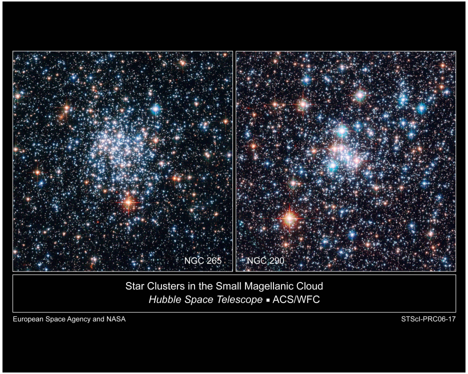
*When $\partial M$ and $R(M) = \{r_x \in C(\partial M) \ : \ x \in M\}$ are given, we can uniquely determine the topological and differentiable structures of $M$ and the metric $g$, up to an isometry.*

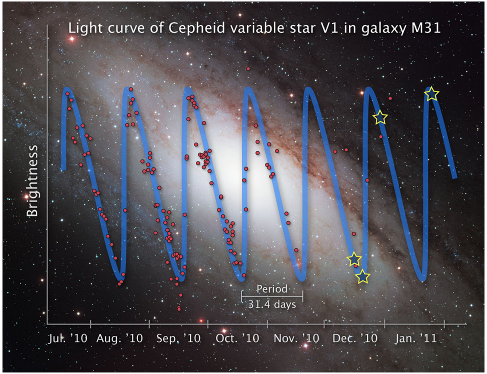
A manifold $M$ is called simple if it is diffeomorphic to a Euclidean ball, has no conjugate points, and the boundary is strictly convex.
For simple manifolds the map $R : x \to r_x$ is an isometry, $\|r_x - r_y\|_\infty = d_M(x, y)$. Then, $R(M)$ is a submanifold of $C(\partial M)$ that is isometric to $M$.



Stability of the reconstruction is analyzed in [Fefferman-Ivanov-L.-Lu-Narayanan 2024].

# Inverse problems in space-time: Passive measurements

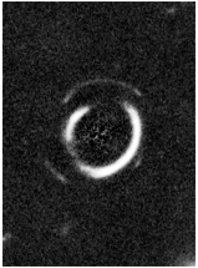


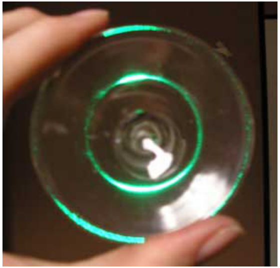




Can we determine the structure of the space-time when we see light coming from several point sources that vary in time?

# Determination of the metric of the space-time

There may exist several light rays between two points (i.e., there are conjugate points or caustics).
This causes difficulties in solving inverse problems in general space-times.



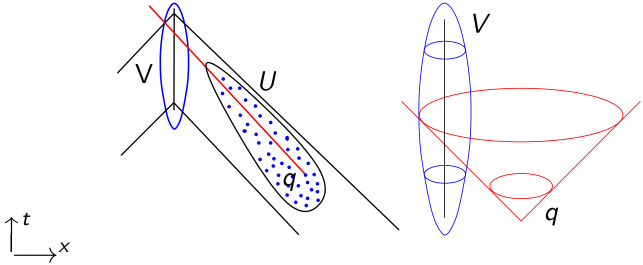


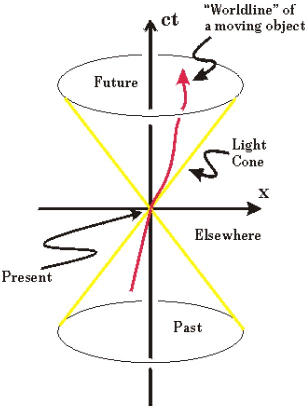
Figures: Einstein's ring by R. Gavazzi and T. Treu and a light source behind a wineglass by P. Doherty.

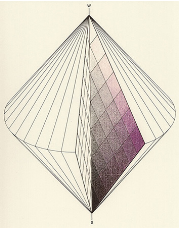
# Inverse problem with passive observations

Passive imaging problem:

Assume that there are a large number of point sources in a subset $U$ of a spacetime $M$.
Light from these point sources are observed in a set $V$.
Do these observations determine the structure of the space-time in $U$?





# Definitions

Let $(M, g)$ be a Lorentzian manifold, e.g. $\mathbb{R}^4$, $g = -dt^2 + dx^2$.

$\gamma_{x,\xi}(s)$ is a geodesic with the initial point $(x, \xi)$

$\xi \in T_xM$ is time-like if $g(\xi, \xi) < 0$,

$\xi \in T_xM$ is light-like if $g(\xi, \xi) = 0$, $\xi \neq 0$.

$J^+(p) = \{x \in M |\ x$ is in causal future of $p\}$,

$J^-(p) = \{x \in M |\ x$ is in causal past of $p\}$.

$(M, g)$ is globally hyperbolic if

there are no closed causal curves and

the set $J^+(p_1) \cap J^-(p_2)$ is compact for all $p_1, p_2 \in M$.

Then $M$ can be represented as $M = \mathbb{R} \times N$.

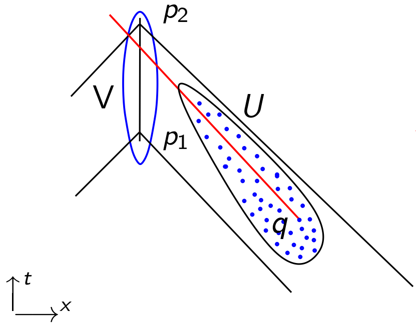
We consider observations in an open set $V \subset M$. Assume that $V$ is a union of time-like curves $\mu_a : (-1,1) \to M$, $a \in A \subset \mathbb{R}^k$. Let $p_1, p_2 \in \mu_{a_0}$.

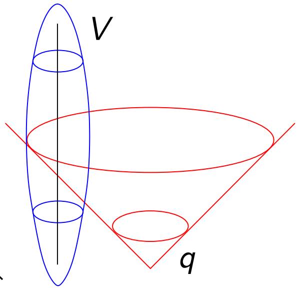
Let $U \subset J^-(p_2) \setminus J^-(p_1)$ be an open, relatively compact set.

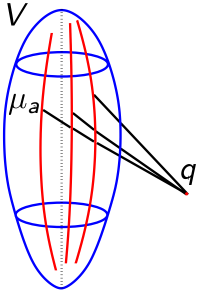
The observation time function $F_q : A \to \mathbb{R}$ for a point $q \in U$ is

$$F_q(a) = \min\{s \in \mathbb{R} \ : \ \text{there is a future-directed light-like geodesic from } q \text{ to } \mu_a(s)\}.$$

That is, $F_q(a)$ is the first time when we observe on $\mu_a$ light that comes from $q$.



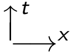



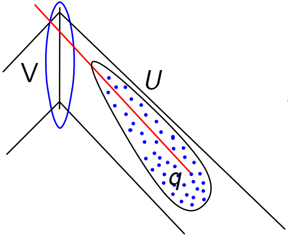
## Theorem (Kurylev-L.-Uhlmann 2018)

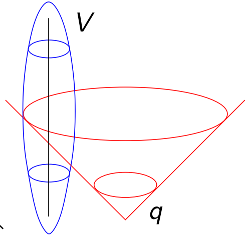
*Let $(M, g)$ be a globally hyperbolic Lorentzian manifold of dimension $n \geq 3$. Assume that $\mu_a(-1, 1) \subset M$, $a \in A \subset \mathbb{R}^m$ are time-like paths, $V = \cup_{a \in A} \mu_a$ is open, and $p_1, p_2 \in \mu_{a_0}$. Let $U \subset J^-(p_2) \setminus J^-(p_1)$ be a relatively compact open set and $U' \subset U$ be dense. Then $(V, g|_V)$ and the collection of the observation time functions,*

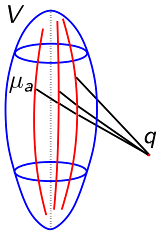
$$\left\{ F_q : A \to \mathbb{R} \;\middle|\; q \in U' \right\} \subset C(A),$$

*determine the set $U$, up to a change of coordinates, and the conformal class of the metric $g$ in $U$.*




The imaging of the space-time is similar to triangulation used in map making.

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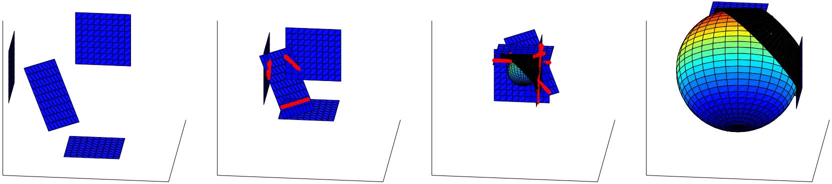


# Non-linear wave equation in space-time

Let $M = \mathbb{R} \times N$ be a Lorentzian manifold with the metric $g$, $\dim(M) = 1 + n$, $n \geq 2$.
Let $m \geq 2$ and

$$\begin{aligned}\square_g u(x) + u(x)^m = f(x), &\quad (x^0, x^1, \ldots, x^n) \in (-\infty, T] \times N,\\ u(x) = 0 &\quad \text{for } t = x^0 < 0,\end{aligned}$$

where

$$\square_g u = \sum_{p,q=0}^{n} |\det(g(x))|^{-\frac{1}{2}} \frac{\partial}{\partial x^p} \left( |\det(g(x))|^{\frac{1}{2}} g^{pq}(x) \frac{\partial}{\partial x^q} u(x) \right).$$

where $g = (g_{jk}(x))_{j,k=0}^{n}$ and $(g^{jk}) = (g_{jk})^{-1}$.
An alternative model is

$$\frac{\partial^2}{\partial t^2} u(t,y) - c(t,y)^2 \Delta u(t,y) + a(t,y) u(t,y)^m = f(t,y), \quad x = (t,y) \in \mathbb{R}^{1+n}.$$

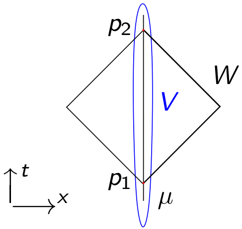
This corresponds to the Lorentzian metric $g = -dt^2 + \sum_{j=1}^{n} c(t,y)^{-2} (dy^j)^2$

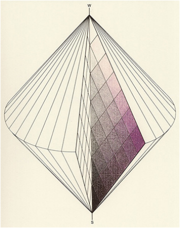
## Theorem (Kurylev-L.-Uhlmann 2018 and L.-Uhlmann-Wang 2017)

*Let $(M,g)$ be a globally hyperbolic Lorentzian manifold,* $\dim(M)=4$*,*
*$\mu \subset M$ be a time-like curve, $p_1, p_2 \in \mu$ and $V$ be a neighbourhood of $\mu$.*
*Let $L_V : f \mapsto u|_V$ be the source-to-solution map for*

$$\Box_g u + u^2 = f \quad in\ (-\infty, T) \times N \subset M,$$
$$u = 0 \quad in\ t = x^0 < 0.$$

*$L_V : f \to u|_V$ is defined for small sources $f$, $supp(f) \subset V$. Then $V$ and $L_V$ determine $W = J^+(p_1) \cap J^-(p_2)$ and the metric $g|_W$ on it (up to change of coordinates).*



Using non-linear interaction of waves
we can solve inverse problems
for non-linear equations
that are unsolved for linear equations.

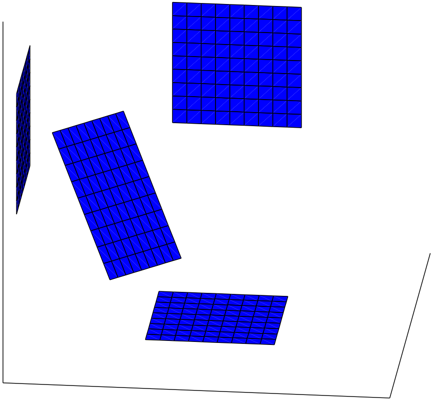
For the equation $\Box_g u + u^2 = f$ in a 4-dimensional space-time, the fourth order non-linear interaction produces artificial microlocal point sources in space-time.

Movie

- The non-linear interaction of distorted plane waves creates artificial microlocal point sources.

- Observations for the point sources determine the conformal class of the metric $g$ in the space-time $J^+(p_1) \cap J^-(p_2)$.

For the equation $\square_g u + u^2 = f$ in a 4-dimensional space-time, the fourth order non-linear interaction produces artificial microlocal point sources in space-time.



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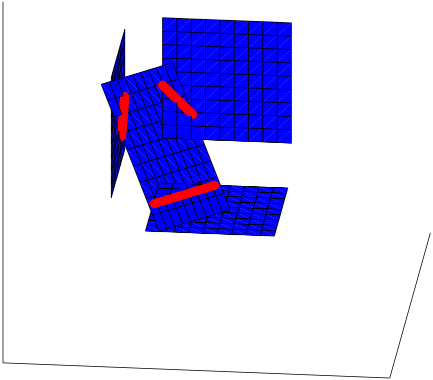

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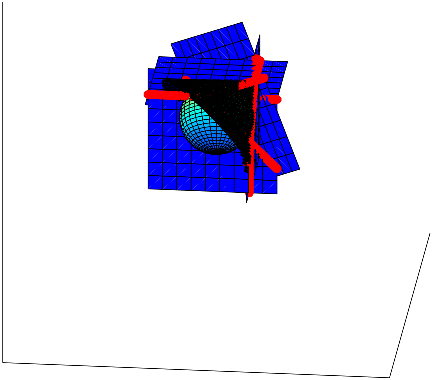

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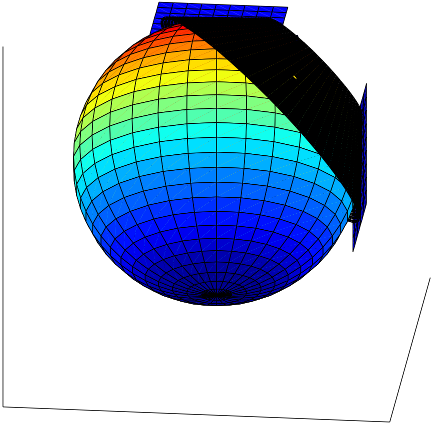

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## Idea of the proof with a 4th order non-linear term

Consider in the Minkowski space $\mathbb{R}^4$ the wave equation

$$\square_g u_{\vec{\varepsilon}}(x) + a(x)\,(u_{\vec{\varepsilon}}(x))^4 = 0,$$

where $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$. Assume $u_{\vec{\varepsilon}}\big|_{\vec{\varepsilon}=0} = 0$. Then, the linearized waves

$$u_j(x) = \partial_{\varepsilon_j} u_{\vec{\varepsilon}}\big|_{\vec{\varepsilon}=0}, \quad j = 1, 2, 3, 4$$

solve the linear wave equation $\square_g u_j = 0$ and

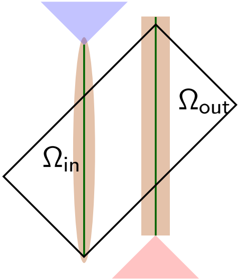
$$w \;=\; \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}\partial_{\varepsilon_4} u_{\vec{\varepsilon}}(x)\big|_{\vec{\varepsilon}=0}$$

satisfies

$$\square_g w = -24 a u_1 u_2 u_3 u_4.$$

Assume $u_j$ that depend on a parameter $s > 0$ and $u_j^{(s)}(x) \to \delta(x \cdot \eta_j)$ as $s \to 0$. That is, $u_j$ are close to plane waves supported on hyperplanes $K_j = \{x \in \mathbb{R}^4 : \; x \cdot \eta_j = 0\}$. Then, we can consider $a u_1 u_2 u_3 u_4$ as a source that is supported on

$$K_1 \cap K_2 \cap K_3 \cap K_4 = \{q\} = \text{one point}.$$




A similar result are valid for separated sources and observations:
Assume that sources are supported in $\Omega_{in}$ and
the waves are observed in $\Omega_{out}$.
The metric is determined in
the set $K$ enclosed by the black rectangle.

**Theorem (Feizmohammadi-L.-Oksanen 2021)**

*Let $G(x,s)$ be a Lorentzian metric tensor depending on $s$, $\partial_s G(x,s)|_{s=0} = 0$ and $\partial_s^2 G(x,s)|_{s=0} > 0$. In the above geometric setting, the source-to-solution map $L_G : C_0^\infty(\Omega_{in}) \to C^\infty(\Omega_{out})$ for*

$$\sum_{p,q=0}^{n} G^{pq}(x,u(x)) \frac{\partial^2 u}{\partial x^p \partial x^q}(x) = f, \quad supp\,(u) \subset J^+(supp\,(f))$$

*determines the conformal class of $g = G(x,0)$ in $K$.*

# Interaction of three waves

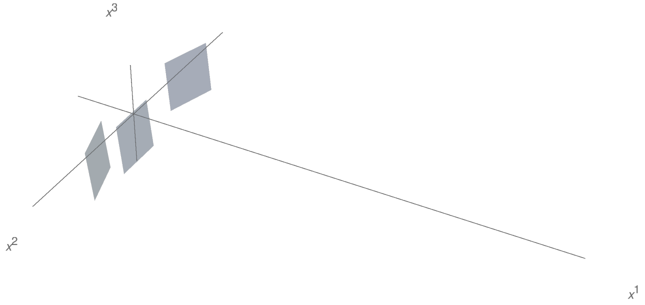



Figure shows the 3-interaction produced by three plane waves that travel closely to each other.
By slightly changing the directions of the plane waves, the 3-interaction sends waves to an arbitrary direction. These waves can be used for the reconstruction of an unknown Lorentzian metric in [Feizmohammadi-L.-Oksanen 2021].

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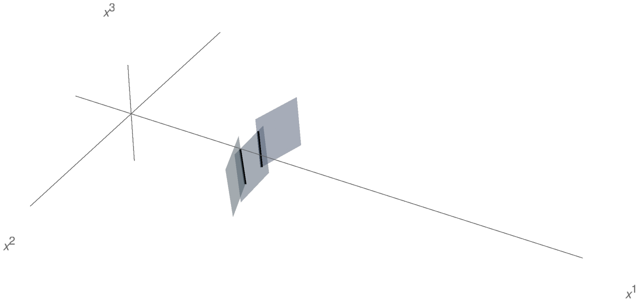


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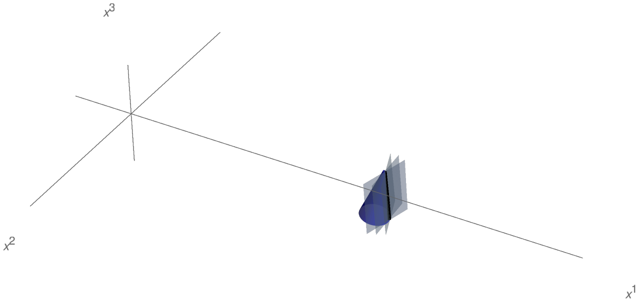


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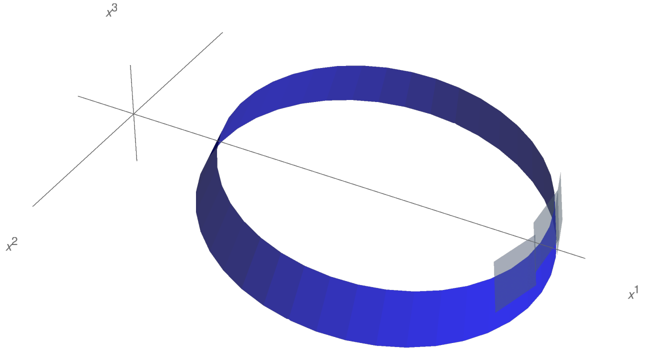


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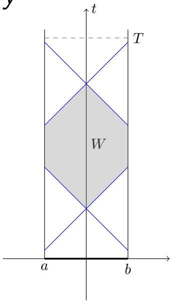
# Inverse problem in Minkowski space with a 1-dimensional measurement

## Theorem (L., Liimatainen, Potenciano-Machado, Tyni 2024)

*Let $\Omega \subset \mathbb{R}^n$, diam $(\Omega) < D$, and $m \geq 2$.*
*There is a measurement function $\psi \in L^2(\partial\Omega \times [0,T])$ such that the following is true:*
*Let supp $(q) \subset \Omega \times [T-D, D]$. For $f$ small enough, let $u = u^f$ satisfy*

$$(\partial_t^2 - \Delta)u(x,t) + q(x,t)u(x,t)^m = 0 \quad \text{in } \Omega \times [0,T],$$
$$u|_{\partial\Omega\times[0,T]} = f, \quad u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0.$$



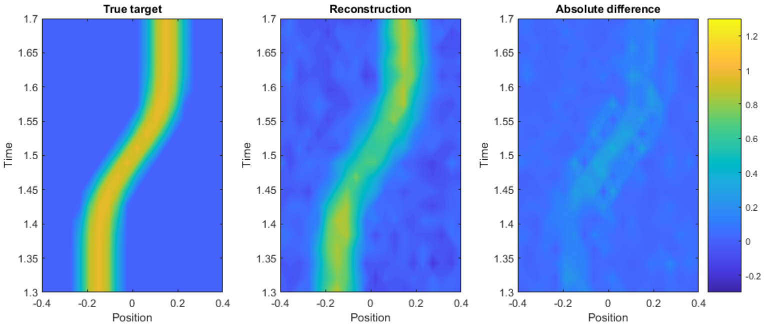

*Then the real-valued non-linear map*

$$\lambda_{q,\psi} : f \to \int_{\partial\Omega}\int_{[0,T]} \psi(x,t)\,\partial_\nu u^f(x,t)\,dS(x)dt$$

*determines $q(x,t)$ uniquely. Moreover, when $\|q\|_{C^{n+1}} < C_0$, the reconstruction is Hölder stable, that is, $\|q_1 - q_2\|_{L^\infty} \leq C\|\lambda_{q_1,\psi} - \lambda_{q_2,\psi}\|^\alpha_{C(B_{H^2}(0,\rho))}$.*

Thus $q$ can be stably reconstructed from low resolution observations by varying $f$.

# Numerical results




Left: $q(x,t)$. Center: Numerical reconstruction. Right: Error in reconstruction.

Reconstruction of the coefficient $q(x,t)$ in equation

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) - q(x,t)u(x,t)^2 = 0, \quad x \in [0,1],\ t \in [0,T],$$

from measurement operator $\lambda_\psi$. Above, data has additive Gaussian noise and the signal-to-noise ratio is 12 dB.

# The Einstein-scalar field equations

Similar techniques can be used for the inverse problem for the Einstein-scalar field equations for the metric $g$ and the scalar fields $\phi = (\phi_l)_{l=1}^L$,

$$\mathrm{Ein}(g) = \mathbb{T}(g, \phi) + \mathcal{F}^1, \tag{1}$$

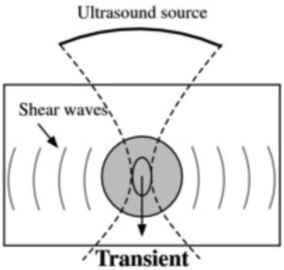
$$\mathbb{T}_{jk}(g, \phi) = \sum_{l=1}^{L} \left( \partial_j \phi_l \partial_k \phi_l - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_l \partial_q \phi_l \right) - \mathcal{V}(\phi) g_{jk}, \tag{2}$$

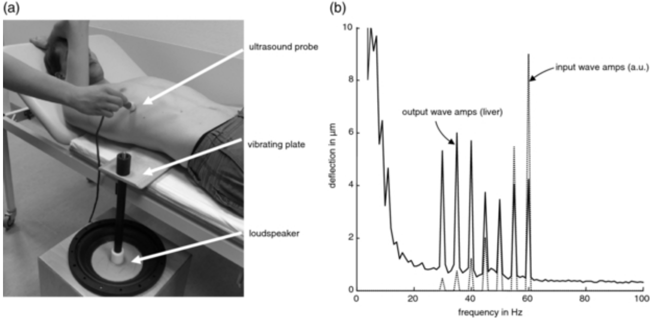
$$\square_g \phi_l - \mathcal{V}_l'(\phi) = \mathcal{F}_l^2, \quad l = 1, 2, \dots, L. \tag{3}$$

Then the source-to-solution map $\mathcal{F} \mapsto (g|_V, \phi|_V)$, defined for sources that are supported in a neighborhood $V$ of a time-like path from $p_1$ to $p_2$ and satisfy the physical conservation law, determines the conformal type of the space time $J^+(p_1) \cap J^-(p_2)$. [Kurylev-L.-Oksanen-Uhlmann 2022, Uhlmann-Wang 2020].

# Can we use methods of General Relativity in medical imaging?

Example on medical imaging with non-linear equations: Elastography




Figures on Elastography:

M. Doyley, Phys. Med. Biol. 2012 , H. Tzschätzsch, Phys. Med. Biol. 2014.

Non-linearity can be used as a beneficial effect in medical imaging. For this, mathematics of general relativity is essential, see [de Hoop-Uhlmann-Wang 2020].

Outline:

- An inverse problem for a linear wave equation
- Inverse problems for passive observations
- Inverse problems for non-linear wave equations
- Inverse problems for non-linear elliptic equations

Simultaneous recovery in 2D: metric and potential.

**Theorem (L.,Liimatainen,Yi-Hsuan Lin, Salo 2019)**

*Let $(M,g)$ be a compact connected manifold with boundary,* $\dim(M)=2$ *and* $m\geq 2$*. Let $q(x)>0$ be a smooth function on $M$. For the equation*

$$\Delta_g u(x) + q(x)u(x)^m = 0 \text{ in } M$$
$$u|_{\partial M} = f,$$

*we define the Dirichlet-to-Neumann map*

$$\Lambda_{M,g,q} : f \to \partial_\nu u|_{\partial M},$$

*for small $f\in C^3(\partial M)$. Then $\partial M$ and $\Lambda_{M,g,q}$ determine the conformal type of $(M,g)$ and the potential $q$ and $g$ up to a gauge transformation.*

The linear problem with $m=1$ is solved in Carstea-Liimatainen-Tzou, Arxiv 2024. Similar results are obtained for the minimal surface equation and for parabolic equations.

# The idea of the proof

The Frechet derivative $(D\Lambda_{M,g,q})_0$ determines the Dirichlet-to-Neumann map for the linear equation $\Delta_g u = 0$. By L.-Uhlmann 2001, this determines the conformal class of the two-dimensional manifold $(M,g)$.
Let us choose a metric $\hat{g} = hg$, $h : M \to \mathbb{R}_+$, that is conformal to $g$.
The higher order derivatives of $\Lambda_{M,\hat{g},q}$ satisfy

$$\int_{\partial M} (D^m \Lambda_{M,\hat{g},q})|_0 [f_1, f_2, f_3, \dots, f_m] \cdot f_{m+1} \, dS = -(m!) \int_M q v_1 v_2 v_3 \cdots v_{m+1} \, dV$$

where $v_k$, $k = 1, \dots, m+1$, satisfy $\Delta_{\hat{g}} v_k = 0$ with the boundary value $f_k$.
Let $v_3 = v_4 = \cdots = v_{m+1} = 1$.
By Guillarmou-Tzou 2011, the inner products $\langle q, v_1 v_2 \rangle_{L^2(M)}$ determine $q$.
Note that it is enough to study only the solutions satisfying $\Delta_{\hat{g}} v = 0$. Roughly speaking, we need to analyze only the linearized inverse problem and do not require analysis of the non-linear problem.

# Thank you for your attention

Supported by the PDE-Inverse project of the European Research Council of the European Union.[1]

[1] Views and opinions expressed are those of the authors only and do not necessarily reflect those of the European Union or the other funding organizations. The European Union can not be held responsible for them.