Inverse problems for non-linear hyperbolic and elliptic equations

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2018-2025

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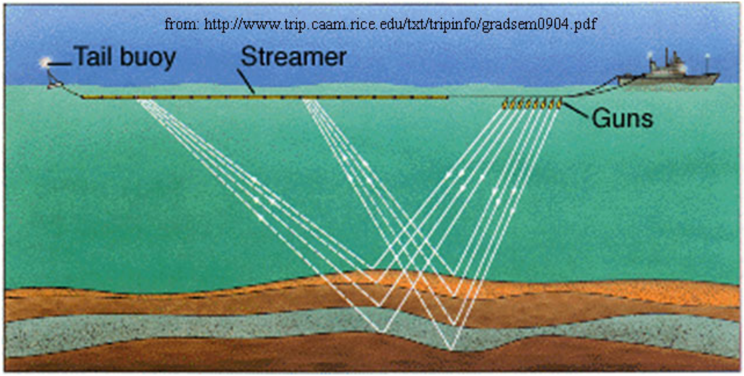
Outline:

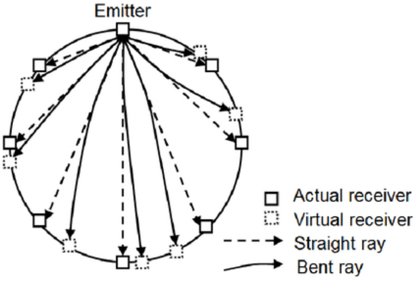
- Inverse problem for linear wave equation
- Inverse problems for non-linear wave equations
- Inverse problems for non-linear wave equations with low resolution observations
- Inverse problems for non-linear elliptic equations



Inverse problem for an anisotropic wave operator

Let $M \subset \mathbb{R}^n$, $n \geq 2$ and ν be the unit normal of the boundary ∂M.
Let $g(x)$ be a matrix valued function and $u(x,t) = u^f(x,t)$ solve

$$(\partial_t^2 - \nabla \cdot g(x)\nabla)u(x,t) = 0 \quad \text{on } (x,t) \in M \times \mathbb{R}_+,$$
$$\nu \cdot g\nabla u(x,t)|_{\partial M \times \mathbb{R}_+} = f(x,t),$$
$$u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0.$$



(Image credit Xiaolei Qu)

The Neumann-to-Dirichlet map is defined by

$$\Lambda : f \to u^f(x,t)|_{(x,t) \in \partial M \times \mathbb{R}_+}.$$

Inverse problem:

Assume that Λ is given.

Can we determine g on local coordinate charts?

To study this problem, we consider (M, g) as a manifold.

Let (M, g) be a Riemannian manifold, $\dim(M) = n \geq 2$, $g = (g_{jk}(x))_{j,k=1}^n$

Let $u(x,t) = u^f(x,t)$ solve the wave equation

$$\begin{aligned} &(\partial_t^2 - \Delta_g) u(x,t) = 0 \quad \text{on } (x,t) \in M \times \mathbb{R}_+, \\ &\partial_\nu u(x,t)|_{\partial M \times \mathbb{R}_+} = f(x,t), \\ &u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0, \end{aligned}$$

where ν is the unit normal vector of the boundary, $(g^{jk}) = (g_{jk})^{-1}$, $|g| = \det(g)$, and

$$\Delta_g u = \sum_{j,k=1}^n |g(x)|^{-1/2} \frac{\partial}{\partial x^j} (|g(x)|^{1/2} g^{jk}(x) \frac{\partial}{\partial x^k} u(x)) = \sum_{j,k=1}^n g^{jk} \frac{\partial^2 u}{\partial x^j \partial x^k} + l.o.t.$$

The Neumann-to-Dirichlet map is $\Lambda f = u^f(x,t)\big|_{(x,t) \in \partial M \times \mathbb{R}_+}$.

For $(\partial_t^2 - c(x)^2 \Delta) u = 0$ the metric is $g_{jk}(x) = c(x)^{-2} \delta_{jk}$

Inverse problem:

Assume that ∂M and Λ are given. Can we determine (M, g) up to an isometry?

Some results on inverse problems for linear hyperbolic equations

- Uniqueness for inverse problem for $(\partial_t^2 - c(x)^2\Delta)u = 0$ in $\Omega \subset \mathbb{R}^n$ by combining the Boundary Control method by Belishev '87, Belishev-Kurylev '87 and Tataru's unique continuation result '95.
- Belishev-Kurylev 1992: Spectral problem for Δ_g on manifold.
- Bingham-Kurylev-L.-Siltanen 2008: Solution for the inverse problem for the wave equation by focusing of waves.
- de Hoop-Kepley-Oksanen 2016: Numerical methods for focusing of waves.

All these results are based on Tataru's unique continuation result and require that the metric is time-independent, or real-analytic in the time variable [Alinhac 1983].

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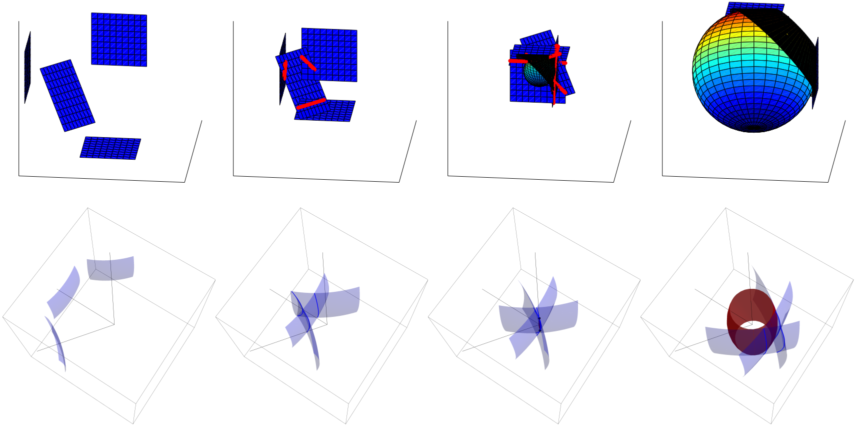
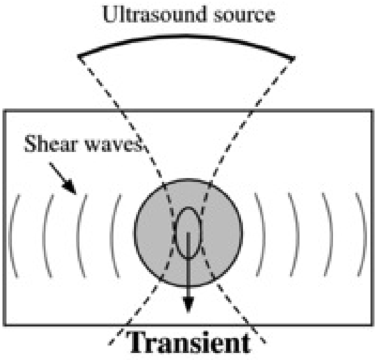
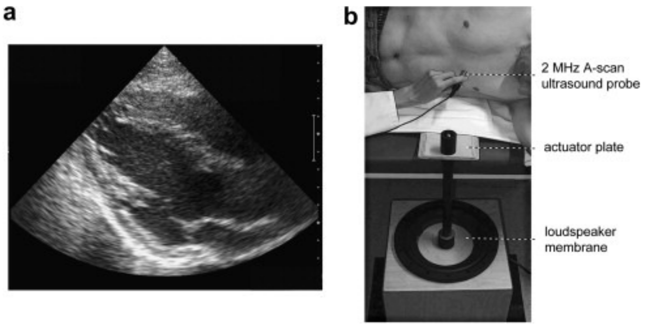


Imaging in non-linear elasticity: Quantitative elastography





Figures: Doyley (Phys. Med. Biol. 2012) and Tzschätzsch (Phys. Med. Biol. 2014)

Inverse problems for non-linear elastic medium: de Hoop-Uhlmann-Wang (2018).

Non-linear wave equation in space-time

Let $M = \mathbb{R} \times N$ be a Lorentzian manifold with time-depending metric g, $\dim(M) = 1 + n$, $n \geq 2$. Let $m \geq 2$ and

$$\begin{aligned} \square_g u(x) + u(x)^m &= f(x), \quad (x^0, x^1, \ldots, x^n) \in (-\infty, T] \times N, \\ u(x) &= 0 \quad \text{for } t = x^0 < 0, \end{aligned}$$

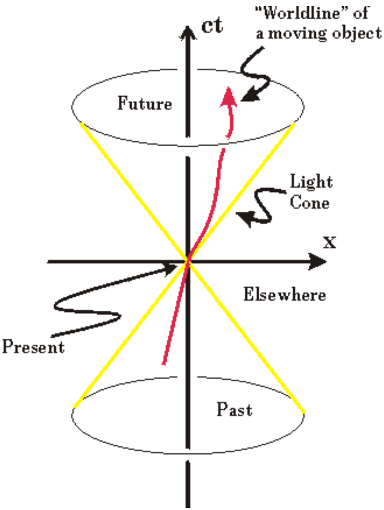
where

$$\square_g u = \sum_{j,k=0}^{n} |\det(g(x))|^{-\frac{1}{2}} \frac{\partial}{\partial x^j} \left(|\det(g(x))|^{\frac{1}{2}} g^{jk}(x) \frac{\partial}{\partial x^k} u(x) \right).$$

An alternative model is

$$\frac{\partial^2}{\partial t^2} u(t,y) - c(t,y)^2 \Delta u(t,y) + a(t,y) u(t,y)^m = f(t,y), \quad x = (t,y) \in \mathbb{R}^{1+3}.$$

This corresponds to the metric $g = (-1, c^{-2}, c^{-2}, c^{-2})$, $c = c(t,y)$.



Definitions

Let (M, g) be a Lorentzian manifold,

where the metric $g = (g_{jk})_{j,k=0}^{n}$ is semi-definite.

$T_x M$ is the space of tangent vectors at x.

$\xi \in T_x M$ is light-like if $g(\xi, \xi) = 0$, $\xi \neq 0$.

$\xi \in T_x M$ is time-like if $g(\xi, \xi) < 0$.

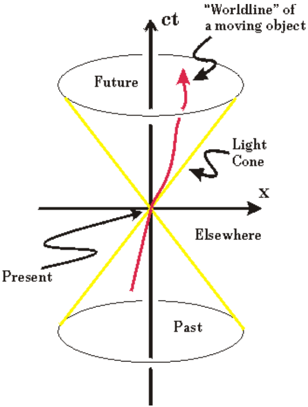
A curve $\mu(s)$ is time-like if $\dot{\mu}(s)$ is time-like.

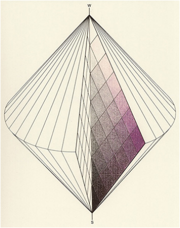
$L_x^+ M = \{\xi \in T_x M \setminus 0;\ g(\xi, \xi) = 0,\ \xi \text{ future pointing}\}$,

Example: Minkowski space $\mathbb{R}^{1+3}$.

Coordinates $(x^0, x^1, x^2, x^3) \in \mathbb{R}^{1+3}$, $x^0 = t$

$g = \text{diag}\,(-1, 1, 1, 1)$.





Definitions

$\gamma_{x,\xi}(t)$ is a geodesic with the initial point (x,ξ),

$J^+(p) = \{x \in M;\ x \text{ is in causal future of } p\}$,

$J^-(p) = \{x \in M;\ x \text{ is in causal past of } p\}$,

(M,g) is globally hyperbolic if

there are no closed causal curves and the set

$J^+(p_1) \cap J^-(p_2)$ is compact for all $p_1, p_2 \in M$.

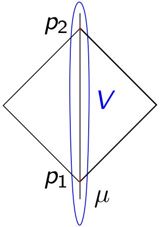
Then M can be represented as $M = \mathbb{R} \times N$.

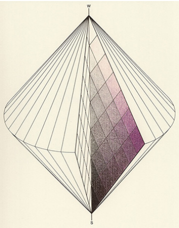
Theorem (Kurylev-L.-Uhlmann 2018 and L.-Uhlmann-Wang 2017)

Let (M, g) be a globally hyperbolic Lorentzian manifold, $\dim(M) = 4$, *$\mu \subset M$ be a time-like curve, $p_1, p_2 \in \mu$ and V be a neighbourhood of μ. Let $L_V : f \mapsto u|_V$ be the source-to-solution map for*

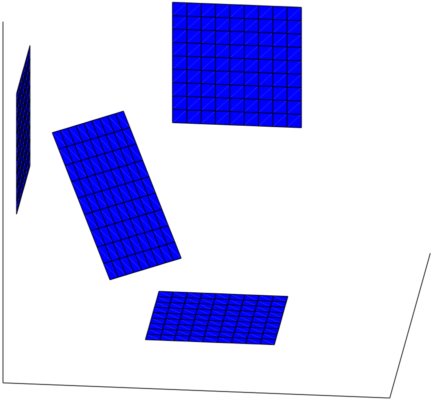
$$\square_g u + u^2 = f \quad \text{in } (-\infty, T) \times N \subset M,$$
$$u = 0 \quad \text{in } t = x^0 < 0.$$

L_V is defined for small sources f, $supp(f) \subset V$. Then V and L_V determine $J^+(p_1) \cap J^-(p_2)$ and the metric g on it (up to change of coordinates).





For the equation $\square_g u + u^2 = f$ in a 4-dimensional space-time, the fourth order non-linear interaction produces artificial microlocal point sources in space-time.



-The non-linear interaction of distorted plane waves creates artificial microlocal point sources.

-Observations of waves from the point sources determine the metric g in the causal diamond $J^+(p_1) \cap J^-(p_2)$.

For the equation $\square_g u + u^2 = f$ in a 4-dimensional space-time, the fourth order non-linear interaction produces artificial microlocal point sources in space-time.

Movie

-The non-linear interaction of distorted plane waves creates artificial microlocal point sources.

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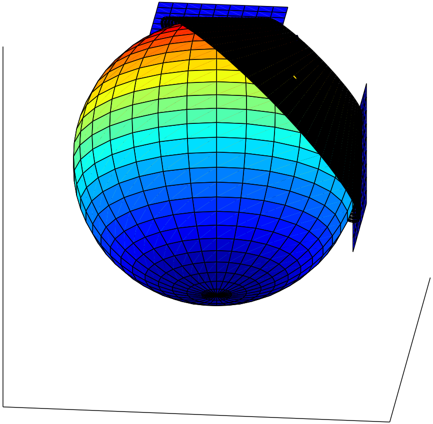

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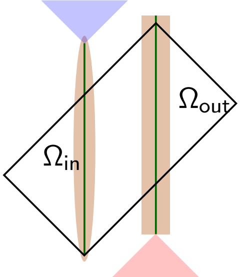
New results based on interaction of three waves in $1+n$ dimensions

Theorem (Feizmohammadi-L.-Oksanen 2020)

Let (M,g) be a globally hyperbolic Lorentzian manifold of dimension $1+n$, $n\geq 2$. Let μ be a time-like curve from p_1 to p_2 and V be a neighborhood of μ and $m\geq 2$. Consider the non-linear wave equation

$$\Box_g u + u^m = f \quad \text{in } N\times(-\infty,T),$$
$$u(x,t) = 0 \quad \text{in } t<0,$$

where $supp(f)\subset V$ is sufficiently small.
Assume that V and the source-to-solution operator $L_V : f\mapsto u|_V$ are given.
If $(n,m)\neq(3,3)$, these data determine the manifold $J^+(p_1)\cap J^-(p_2)$ and the metric $g_{jk}(x)$ on it.
If $(n,m)=(3,3)$, the conformal class of metric of g is determined.



A similar result are valid for separated sources and observations: Assume that sources are supported in Ω_{in} and the waves are observed in Ω_{out}.
When $(n, m) \neq (3, 3)$, the metric is determined in the set R enclosed by the black rectangle.

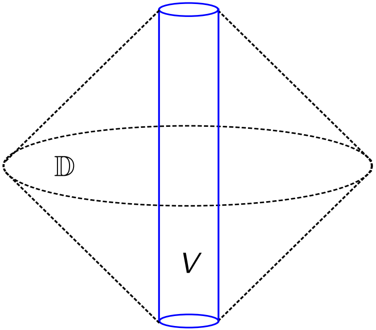
Theorem (Feizmohammadi-L.-Oksanen 2020)

Let $G(x, s)$ be a Lorentzian metric tensor depending on s and $\partial_s^2 G(x, s)|_{s=0} > 0$. In the above geometric setting, the source-to-solution map $L_G : C_0^\infty(\Omega_{in}) \to C^\infty(\Omega_{out})$ for

$$\sum_{j,k=0}^{n} G^{jk}(x, u(x)) \frac{\partial^2 u}{\partial x^j \partial x^k}(x) = f$$

determines the conformal class of $g = G(x, 0)$ in R

Inverse problem for the connection A in the Higgs field equation



Let $\nabla_A = d + A$ be a connection on the trivial vector bundle $\mathbb{C}^n$ over the Minkowski space $\mathbb{R}^{1+3}$.

Let V be a cylinder in $\mathbb{R}^{1+3}$, and let $\mathbb{D}$ be the optimal causal diamond associated to V.

Theorem (Chen-L.-Oksanen-Paternain 2019)

For any $\kappa \in \mathbb{R}_+$, $b \in \mathbb{R}$, and sufficiently small $\rho > 0$ the map

$$L_A : f \to u|_V; \quad (\nabla_A)^* \nabla_A u + \kappa(|u|^2 - b)u = f,\ u|_{t<0} = 0,\ \|f\|_{C_0^4(V)} < \rho,$$

determines A in $\mathbb{D}$ up to the natural gauge transformation.

The linear case $\kappa = 0$ is open as coefficients $A_j(x^0, x')$ are time-depending functions.

Idea of the proof with a non-linear equation $\square_g u + u^3 = f$ in $\mathbb{R}^{1+3}$.

Consider in Minkowski space $\mathbb{R}^{1+3}$ the solutions $u_{\vec{\varepsilon}}(x)$ of

$$\square u_{\vec{\varepsilon}} + (u_{\vec{\varepsilon}})^3 = 0,$$

that depend on parameters $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{R}^3$.
When $u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} = 0$, the linearized waves

$$u_j(x) = \partial_{\varepsilon_j} u_{\vec{\varepsilon}}\big|_{\vec{\varepsilon}=0}, \quad j = 1, 2, 3$$

satisfy $\square u_j = 0$. Then, $w = \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3} u_{\vec{\varepsilon}}(x)\big|_{\vec{\varepsilon}=0}$ satisfies

$$\square w = -6 u_1 u_2 u_3.$$

The function $6u_1 u_2 u_3$ can be considered as an artificial source produced by the non-linear interaction.

We use coordinates $x = (t, y_1, y_2, y_3) \in \mathbb{R}^{1+3}$.
As a motivation, consider we linearized waves

$$\begin{aligned} u_1(t,y) &= \delta(t-y_1), \\ u_2(t,y) &= \delta(t-y_2), \\ u_3(t,y) &= \delta(t-y_3). \end{aligned}$$

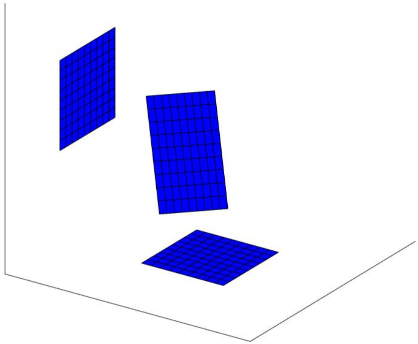
Then

$$\begin{aligned} &u_1 u_2 u_3 = \tfrac{1}{2}\, \delta_L(t,y), \\ &L = \{(t, y_1, y_2, y_3) : \ y_1 = y_2 = y_3 = t\} \subset \mathbb{R}^{1+3} \end{aligned}$$

Let w be the solution of the wave equation

$$\Box w = S, \quad S = -6 u_1 u_2 u_3$$

Physically, S is a moving point source that at the time t is located at the point $y(t) = (t,t,t) \in \mathbb{R}^3$. The line L is the path of the point source in the space-time.



Three plane waves with directions ξ_1, ξ_2, ξ_3 interact and produce a conic wave.
By varying the directions ξ_1 and ξ_2 of the incoming waves near ξ_3, the interaction can produce a wave front to an arbitrary direction [Chen-L.-Oksanen-Paternain 2019].

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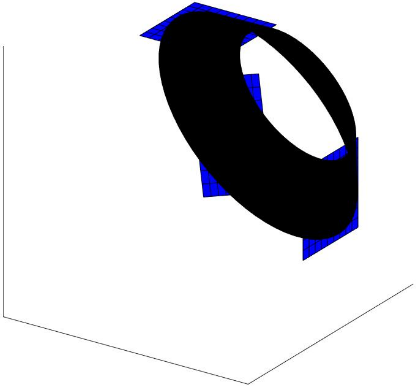

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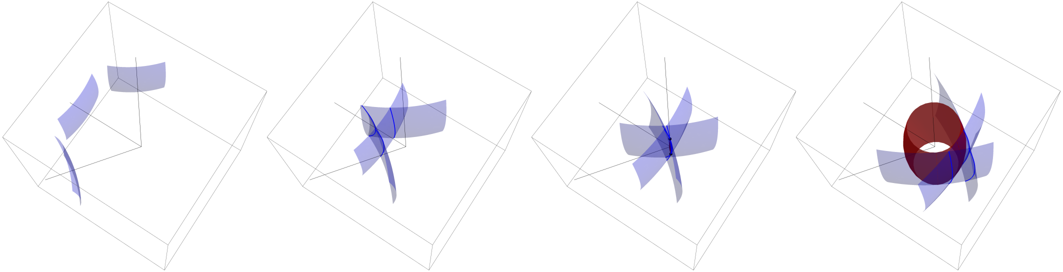
Interaction of three spherical waves in the Euclidean space



When 3 spherical waves (blue) interact they produce a new wave front (red) that propagates to the direction where the spherical waves came from.
The figure shows the wave fronts at four times t_1, t_2, t_3, t_4.
[Chen-L.-Oksanen-Paternain 2019].

Next we return to consider general Lorentzian manifolds.

Reconstruction of a space-time with conjugate points [Feiz.-L.-O. 2020]

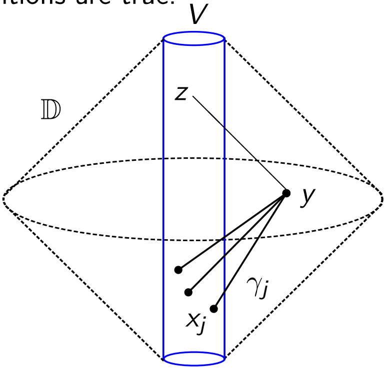
Consider wave fronts that are sent from the points $x_1, x_2, x_3 \in V$ along the light-like geodesics $\gamma_1, \gamma_2, \gamma_3$. For $\Box_g u + u^3 = f$ the following conditions are true:

(A) If $\gamma_1, \gamma_2, \gamma_3$ do not intersect, then we do not observe wave fronts at z.

(B) If y is the first intersection point of geodesics and

$$\gamma_1(s_1) = \gamma_2(s_2) = \gamma_3(s_3) = y,$$
$$\xi \in \text{span}\{\dot\gamma_j(s_j),\ j = 1,2,3\} \cap L_y^+ M,$$
$$z = \gamma_{y,\xi}(s) \in V,$$

then we observe a wave front at z.



Lemma for 3-to-1 scattering relation: We say that a 4-tuple $(\gamma_1, \gamma_2, \gamma_3, z)$ satisfies relation R if we observe a wave front at z. When (A) and (B) are valid, the relation R determines the conformal class of $(\mathbb{D}, g)$.

This lemma can be applied for any non-linear hyperbolic equation of 2nd order.

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Inverse problem in Minkowski space with a 1-dimensional measurement

Theorem (L., Liimatainen, Potenciano-Machado, Tyni (2020))

Let $\Omega \subset \mathbb{R}^n$, diam $(\Omega) < D$, and $m \geq 2$. Let $t_2 > t_1 > D$ and $T > \ t_2 + D$.
There is a measurement function $\psi \in C_0^\infty(\partial\Omega \times [0, T])$ such that the following is true:
Let supp $(q) \subset \Omega \times [t_1, t_2]$. For f small enough, let $u = u^f$ satisfy

$$(\partial_t^2 - \Delta)u(x,t) + q(x,t)u(x,t)^m = 0 \quad \text{in } \Omega \times [0, T],$$
$$u|_{\partial\Omega\times[0,T]} = f, \quad u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0.$$

Then the real-valued non-linear map

$$\lambda_\psi : f \to \langle \psi, \partial_\nu u^f|_{\partial\Omega\times[0,T]}\rangle_{L^2(\partial\Omega\times[0,T])} \in \mathbb{R}$$

determines $q(x,t)$ uniquely. Moreover, when $\|q\|_{C^{n+1}} < C_0$, the reconstruction is Hölder stable.

This means that, $q(x,t)$ can be stably reconstructed from low resolution observations if we can control the source f.

Idea of the proof with one-dimensional measurement.

The m:th Frechet derivative of $\lambda_\psi(f) = \langle \psi, \partial_\nu u^f|_{\partial\Omega\times[0,T]}\rangle_{L^2(\partial\Omega\times[0,T])}$ at $f = 0$ is

$$(D^m\lambda_\psi)_0[f_1, f_2, \dots, f_m] = -m! \int_{\Omega\times[0,T]} v_\psi \cdot q v_1 v_2 \dots v_m dxdt,$$

where v_j are solutions of the linear wave equation

$$(\partial_t^2 - \Delta)v_j(x,t) = 0 \quad \text{in } \Omega \times [0,T],$$
$$v_j|_{\partial\Omega\times[0,T]} = f_j, \quad v_j|_{t=0} = 0, \quad \partial_t v_j|_{t=0} = 0$$

and $(\partial_t^2 - \Delta)v_\psi = 0$, $v_\psi|_{\partial\Omega\times[0,T]} = \psi$ is such that $v_\psi = 1$ in the set $\Omega \times [t_1, t_2]$.
By varying boundary values f_j we find the partial Radon transform of $q(x,t)$.
This determines the function $q(x,t)$.

Related inverse problem with a varying wave speed is studied in Hintz-Uhlmann-Zhai 2020 using the Dirichlet-to-Neumann map.

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Higher order linearization and non-linear interaction of solutions can be applied also for elliptic equations.

Theorem (L.,Liimatainen,Yi-Hsuan Lin, Salo 2019)

Let (M,g) *be a compact connected Riemannian manifold with boundary,* $\dim(M)=2$ *and* $m\geq 2$*. For the equation*

$$\Delta_g u(x)+q(x)u(x)^m=0 \text{ in } M,\quad u|_{\partial M}=f,$$

we define the Dirichlet-to-Neumann map

$$\Lambda_{g,q}: f\to \partial_\nu u|_{\partial M},$$

for small $f\in C^3(\partial M)$*. Then* ∂M *and* $\Lambda_{g,q}$ *determine the conformal class of* (M,g) *and the potential* q *up to a gauge transformation.*

Related results in dimensions $n\geq 3$ are studied in L.-Liimatainen-Lin-Salo 2019, Feizmohammadi-Oksanen 2019, Krupchyk-Uhlmann 2019

The idea of the proof

The Frechet derivative $(D\Lambda_{g,q})_0$ determines the Dirichlet-to-Neumann map for the linear equation $\Delta_g u = 0$. By L.-Uhlmann 2001, this determines the conformal class of the two-dimensional manifold (M, g).
Let us choose $\hat{g} = hg$, $h : M \to \mathbb{R}_+$ that is conformal to g.
The higher order derivatives of $\Lambda_{\hat{g},q}$ are

$$\int_{\partial M} (D^m \Lambda_{\hat{g},q})_0[f_1, f_2, f_3, \ldots, f_m] \cdot f_{m+1}\, dS = -(m!) \int_M q v_1 v_2 v_3 \cdots v_{m+1}\, dV$$

where v_k, $k = 1, \ldots, m+1$, satisfy $\Delta_{\hat{g}} v_k = 0$ with boundary value f_k.
Let $v_3 = v_4 = \cdots = v_{m+1} = 1$.
By Guillarmou-Tzou 2011, the inner products $\langle q, v_1 v_2 \rangle$ determine q.
Note that it is enough to study only the solutions satisfying $\Delta_{\hat{g}} v = 0$, that is we can consider the case when $q = 0$. Roughly speaking, we need to analyze only the linearized inverse problem à la Calderon and do not require Sylverster-Uhlmann type analysis.

Thank you for your attention!

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