A review on invisibility, cloaking, and inverse problems

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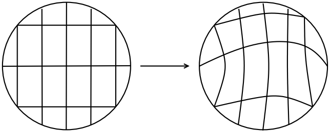
Outline:

- **Inverse conductivity problem and transformation optics**
- Metamaterials
- Inverse conductivity problem
- Perfect invisibility cloaks
- Approximate invisibility cloaks
- Topology and cloaking: Wormholes



Consider a body $\Omega \subset \mathbb{R}^n$. An electric potential $u(x)$ satisfies the conductivity equation

$$\begin{aligned} \nabla \cdot \sigma(x) \nabla u(x) &= 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= f. \end{aligned}$$

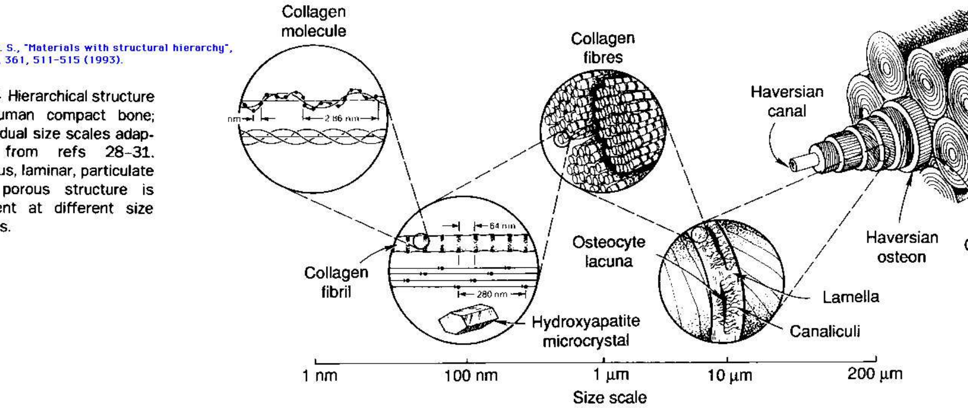
Here the conductivity $\sigma(x)$ can be an **isotropic**, that is, scalar, or an **anisotropic**, that is, matrix valued function.

Calderón's inverse problem: Do the measurements made on the boundary determine the conductivity, that is, does $\partial\Omega$ and the **Dirichlet-to-Neumann map** Λ_σ, i.e., the voltage-to-current map

$$\Lambda_\sigma(f) = \nu \cdot \sigma \nabla u|_{\partial\Omega}$$

determine the conductivity $\sigma(x)$ in Ω?

Anisotropy appears in homogenization of isotropic material

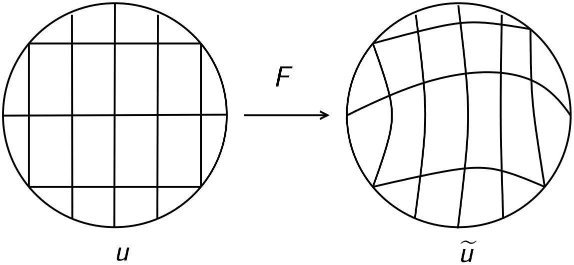


Structure of human bone.

Deformation of the domain. Assume that $\sigma(x) = (\sigma^{jk}(x)) \in \mathbb{R}^{n\times n}$. Let F be diffeomorphism

$$F : \Omega \to \Omega, \quad F|_{\partial\Omega} = Id.$$

Consider $\widetilde{u} \in C^1(\Omega)$ and $u(x) = \widetilde{u}(F(x))$.



Next, let us consider $F : \Omega \to \mathbb{R}^n$ as curvilinear coordinates on Ω, $\widetilde{x} = F(x)$. Then $\widetilde{u}(\widetilde{x})$ is the voltage function $u(x)$ represented in the $\widetilde{x}$ coordinates.

Deformation of the domain and solutions. Assume that $\sigma(x) = (\sigma^{jk}(x)) \in \mathbb{R}^{n\times n}$,

$$\nabla\cdot \sigma \nabla u = 0 \quad \text{in } \Omega.$$

Let F be diffeomorphism

$$F : \Omega \to \Omega, \quad F|_{\partial\Omega} = Id.$$

Then, using chain rule, we see that

$$\nabla\cdot \widetilde{\sigma}\nabla \widetilde{u} = 0 \quad \text{in } \Omega, \quad \text{where}$$
$$\widetilde{u}(x) = u(F^{-1}(x)), \quad \widetilde{\sigma}(y) = F_*\sigma(y) = \left.\frac{(DF)\cdot\sigma\cdot(DF)^t}{\det(DF)}\right|_{x=F^{-1}(y)}$$

Moreover, $\Lambda_{\widetilde{\sigma}} = \Lambda_\sigma$.
This observation is the basis of the transformation optics.

Transformation optics

Formula $\Lambda_{\widetilde{\sigma}} = \Lambda_\sigma$ with

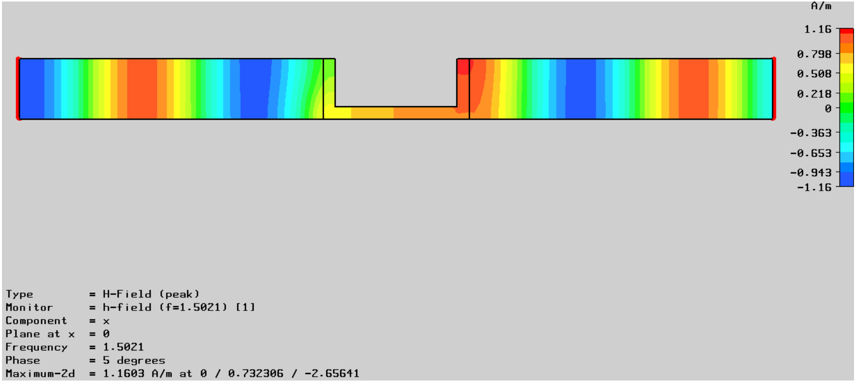
$$\widetilde{\sigma}(y) = F_*\sigma(y) = \left.\frac{(DF)\cdot\sigma\cdot(DF)^t}{\det(DF)}\right|_{x=F^{-1}(y)}$$

has two interpretations:

- A given physical model can be written in two different coordinates, or
- In given coordinate system we have two physical models for which all measurements coincide.

In other words, we can keep either the physical model or the coordinate system fixed and vary the other.

Example on transformation optics: Thin optical cables



Transformaation optics makes possible optical cables with varying thickness (Alu et al 2011).

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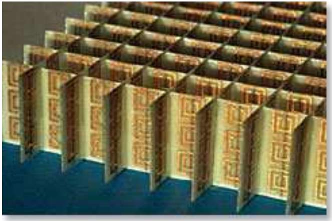
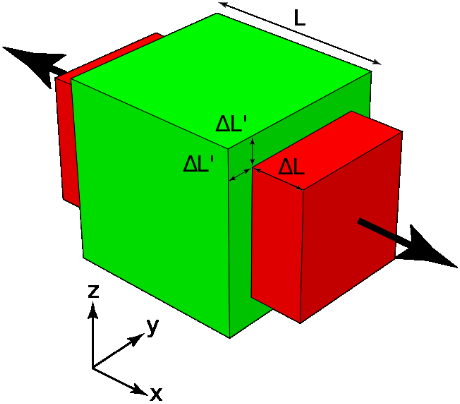
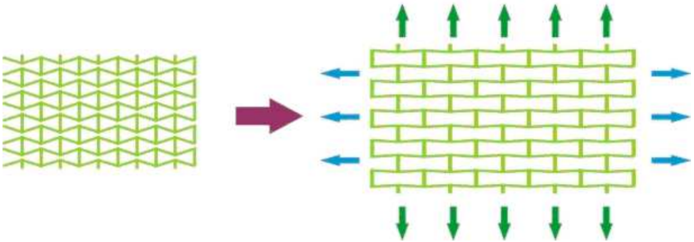


Mechanical metamaterials.
Metamaterial is an artificial material which geometry is essential for its properties



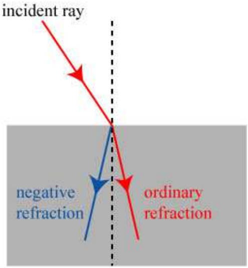
Usual material has positive Poisson ratio: When a block of material is stretched, it becomes thiner.

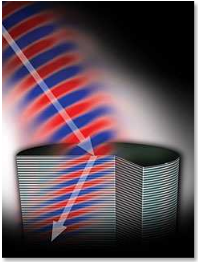


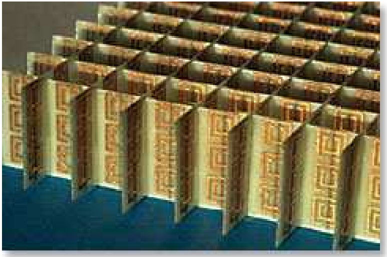
Milton's metamaterial with negative Poisson ration:
When a block of material is stretched, it becomes thicker.

Examples: Paper, Gore-Tex, new polymers.

Metamaterials producing negative refraction

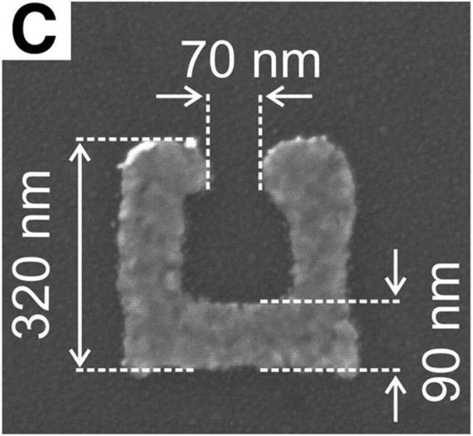


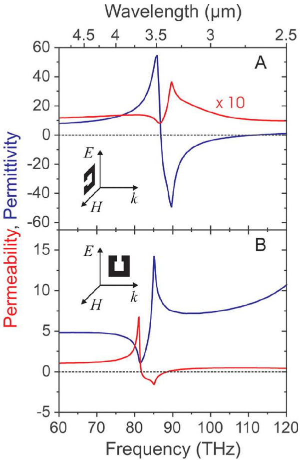


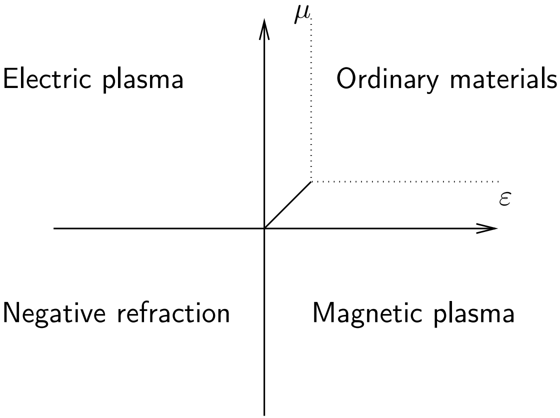


Figures: D. Smith's electromagnetic metamaterial (2006) operating with a single microwave frequency.

Example:
Magnetic Response of Metamaterials at 100 Terahertz by S. Linden et al, Science 2004.







Materials needed for invisibility cloaking are between the ordinary and the negative materials.

On the optical frequencies the permittivity ε for silver and gold is negative, and the permeability μ is 1. Note that the conductivity is small but positive.

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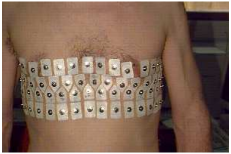


Conductivity equation

$$\nabla\cdot\sigma(x)\nabla u(x) = 0 \quad \text{on } \Omega \subset \mathbb{R}^d.$$

Inverse problem: Do the measurements made on the boundary determine the conductivity, that is,
does the **set of the Dirichlet-Neumann pairs**,

$$C(\sigma) = \{(u|_{\partial\Omega}\,,\, \nu\cdot\sigma\nabla u|_{\partial\Omega})\,;\ \nabla\cdot\sigma(x)\nabla u(x) = 0\}$$

determine the conductivity $\sigma(x)$ in Ω?

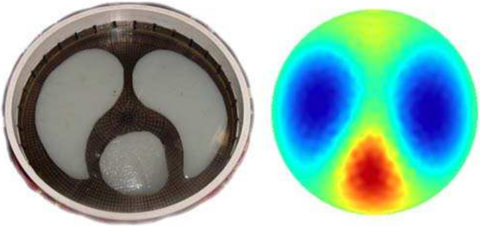


Figure: EIT by Isaacson, Mueller, Newell and Siltanen.

Invariant formulation of inverse conductivity problem. Assume $n \geq 3$. Consider Ω as a Riemannian manifold with

$$g^{jk}(x) = (\det \sigma(x))^{-1/(n-2)} \sigma^{jk}(x).$$

Then conductivity equation is the Laplace-Beltrami equation

$$\begin{aligned} \Delta_g u &= 0 \quad \text{in } \Omega, \\ \Delta_g u &= \sum_{j,k=1}^{n} g^{-1/2} \frac{\partial}{\partial x^j} (g^{1/2} g^{jk} \frac{\partial}{\partial x^k} u), \\ g &= \det(g_{ij}), \quad [g_{ij}] = [g^{jk}]^{-1}. \end{aligned}$$

We denote

$$\widetilde{g} = F_* g \quad \text{when} \quad \widetilde{g}_{jk}(y) = \sum_{p,q=1}^{n} \frac{\partial x^p}{\partial y^j} \frac{\partial x^q}{\partial y^k} g_{pq}(x).$$

Inverse problem: Can we determine the Riemannian manifold (M, g) by knowing ∂M and

$$\Lambda_{M,g} : u|_{\partial M} \mapsto \partial_\nu u|_{\partial M}.$$

Some early results for the inverse conductivity problem

- Calderón 1980: Linearized problem.
- Sylvester-Uhlmann 1987, Nachman 1988: Smooth conductivities in 3D.
- Nachman 1996: Smooth conductivities in 2D.
- Isaacson-Mueller-Newell-Siltanen 2004: Numerical reconstruction algorithm.
- Astala-Päivärinta 2006: Bounded conductivities in 2D.
- Sylvester 1990, Astala-L.-Päivärinta 2005: Anisotropic conductivity in 2D.

All these results need assumptions like

$$c_1 I \leq \sigma(x) \leq c_2 I, \quad c_1, c_2 > 0.$$

For degenerate conductivities we define

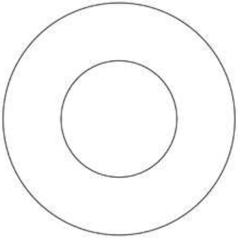
$$\langle \Lambda_\sigma f, f \rangle_{L^2(\partial\Omega)} = \inf \left\{ \int_\Omega \sigma \nabla u \cdot \nabla u \, dx \;\middle|\; u \in W^{1,1}(\Omega),\; u|_{\partial\Omega} = f \right\}.$$

Theorem

(Astala-L.-Päivärinta 2011) Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and σ_1, σ_2 be measurable, real-valued functions such that

$$\int_\Omega \exp\left(\exp\left(\sigma_j(x) + \frac{1}{\sigma_j(x)}\right)\right) dx < \infty, \quad j = 1, 2.$$

Assume that $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$. Then $\sigma_1 = \sigma_2$.



A "tunneling effect" in classical electrostatics:

Conductivity is continuous and zero on a circle Σ. However, we can determine the conductivity inside Σ.

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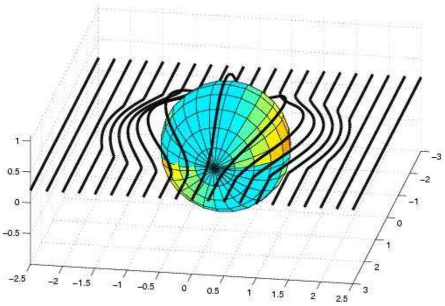


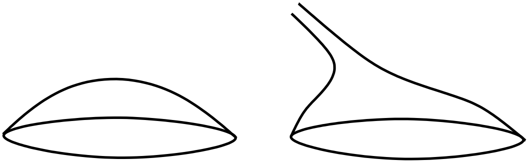
Non-uniqueness results (L.-Taylor-Uhlmann 2003) Let (M, g) be a compact 2-dimensional manifold. Let $x_0 \in M$, and consider manifold

$$\widetilde{M} = M \setminus \{x_0\}$$

with the metric

$$\widetilde{g}_{ij}(x) = \frac{1}{d_M(x, x_0)^2} g_{ij}(x).$$

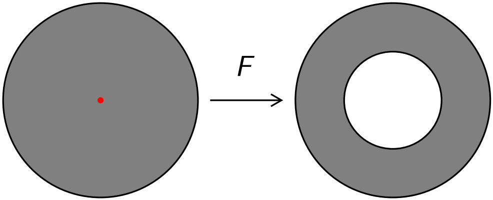
Using Brownian motion, one can see that $\Lambda_{M,g} = \Lambda_{\widetilde{M},\widetilde{g}}$.



A ball with a cloaked pocket: A counter example for inverse conductivity problem (Greenleaf-L.-Uhlmann 2003)

Let $B(2) \subset \mathbb{R}^3$ be a ball of radius 2 and $B(1) \subset \mathbb{R}^3$ a ball of radius 1. Consider the blow-up map

$$F : B(2) \setminus \{0\} \to B(2) \setminus \overline{B}(1), \quad F(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}.$$



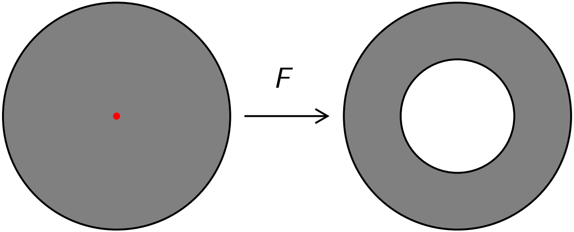
We denote

$$\widetilde{g} = F_* g, \quad \text{that is,} \quad \widetilde{g}_{jk}(y) = \sum_{p,q=1}^{n} \frac{\partial x^p}{\partial y^j} \frac{\partial x^q}{\partial y^k} \, g_{pq}(x).$$

Let $g_{jk} = \delta_{jk}$ be the Euclidian metric in $B(2)$ and $\gamma = 1$ be the corresponding conductivity. Using $\widetilde{g} = F_* g$ we define

$$\widetilde{\sigma}^{jk} = \begin{cases} |\widetilde{g}|^{1/2}\widetilde{g}^{jk} & \text{for } x \in B(2) \setminus \overline{B}(1), \\ a^{jk} & \text{for } x \in B(1). \end{cases}$$

One eigenvalue of the conductivity matrix $\widetilde{\sigma}(x)$ tends to zero when $|x|$ goes to 1.

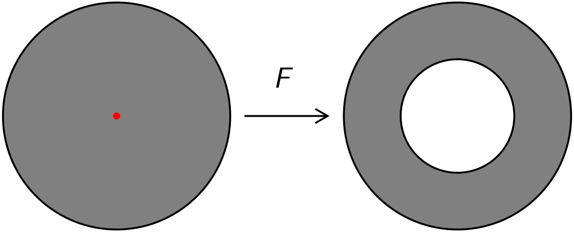


In the spherical coordinates

$$(r, \phi, \theta) \mapsto (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$$

we have

$$\widetilde{\sigma} = \begin{pmatrix} 2(r-1)^2 \sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}, \quad 1 < |x| \leq 2.$$

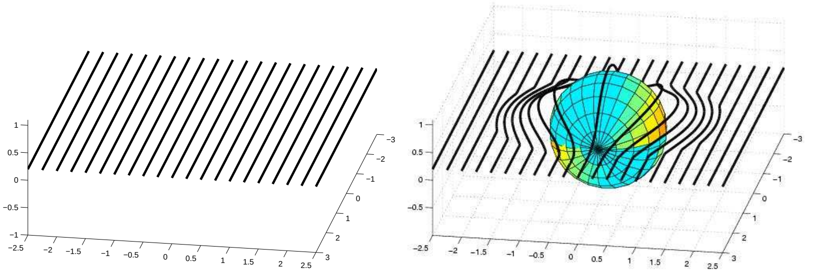


Let $\Omega = B(2)$. The Cauchy data for $H^1 \cap L^\infty$-solutions is

$$C_1(\widetilde{\sigma}) = \{(u|_{\partial\Omega}, \nu \cdot \widetilde{\sigma} \nabla u|_{\partial\Omega}) \ : \ u \in H^1(\Omega) \cap L^\infty(\Omega), \ \nabla \cdot \widetilde{\sigma} \nabla u = 0\}.$$

Theorem

(Greenleaf-L.-Uhlmann 2003) The boundary measurements for $\widetilde{\sigma}$ and $\gamma = 1$ coincide, $C_1(\widetilde{\sigma}) = C_1(\gamma)$.



Cloaking for the 2-dimensional conductivity equation is analyzed in Kohn-Shen-Vogelius-Weinstein 2007.

Idea of the proof. When $\widetilde{u} \in H^1(B(2))$, we have $\widetilde{\sigma}\nabla\widetilde{u}$ is a integrable function. Assume next that

$$\nabla\cdot\widetilde{\sigma}(x)\nabla\widetilde{u}(x) = 0.$$

Let $F^{-1} : B(2) \setminus \overline{B}(1) \to B(2) \setminus \{0\}$ and $u = \widetilde{u} \circ F$. Then

$$\Delta u(x) = 0 \quad \text{on } B(2) \setminus \{0\}.$$

The capacitance of $\{0\} \subset \mathbb{R}^3$ is zero and u is bounded. Thus u is harmonic in the whole ball $B(2)$,

$$\Delta u(x) = 0 \quad \text{on } B(2).$$

Moreover, the Cauchy data of u and $\widetilde{u}$ are the same,

$$(u|_{\partial B(2)}\,,\, \nu\cdot\nabla u|_{\partial B(2)}) = (\widetilde{u}|_{\partial B(2)}\,,\, \nu\cdot\widetilde{\sigma}\nabla\widetilde{u}|_{\partial B(2)}).$$

This shows that $C_1(\widetilde{\sigma}) \subset C_1(\gamma)$, where $\gamma = 1$.

All boundary measurements for the homogeneous conductivity $\gamma = 1$ and the degenerated conductivity $\widetilde{\sigma}$ are the same.

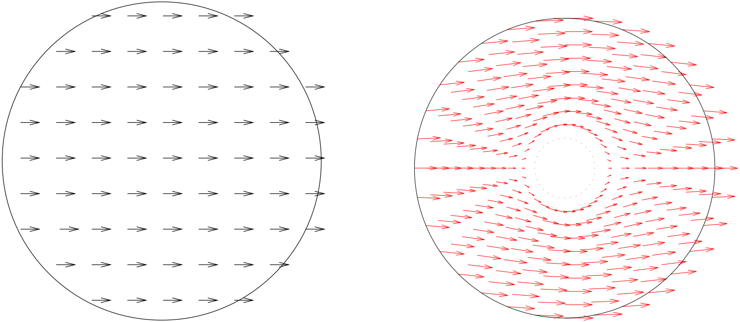
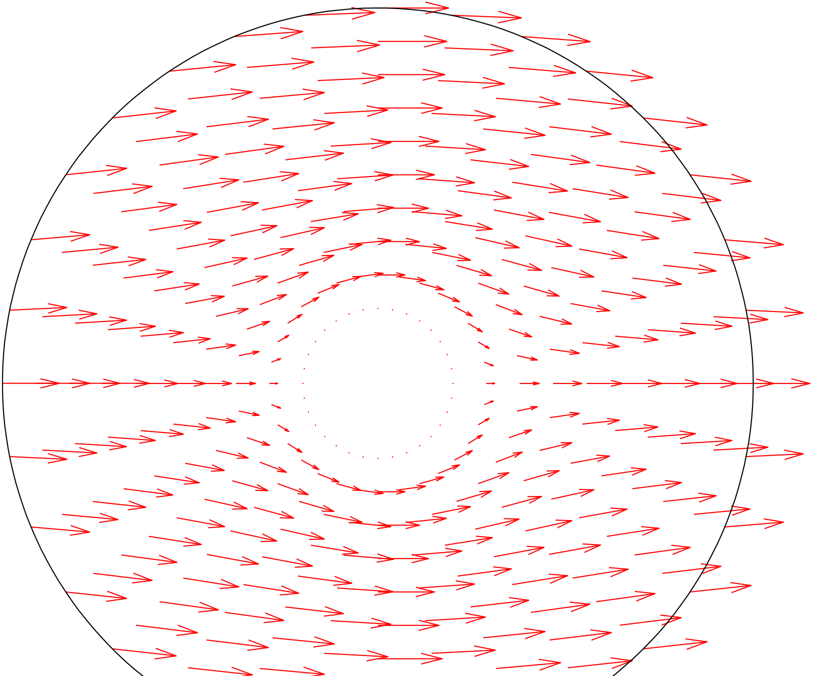


Figure: Analytic solutions for the currents.

An analytic solution for current with $\widetilde{\sigma}$

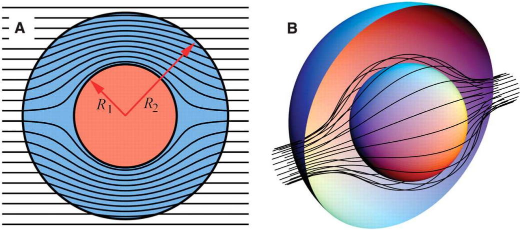


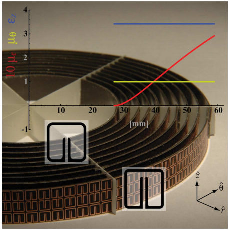
In 2006, metamaterial cloaks were proposed in

1. *Optical Conformal Mapping* by Leonhardt, Science 2006,
2. *Controlling electromagnetic fields* by Pendry, Schurig and Smith, Science 2006.

Proposal 2 is based on the similar blow-up map of a point as the above map F. It was experimentally tested in

3. Metamaterial Electromagnetic Cloak at Microwave Frequencies by Schurig et al, Science 2006.



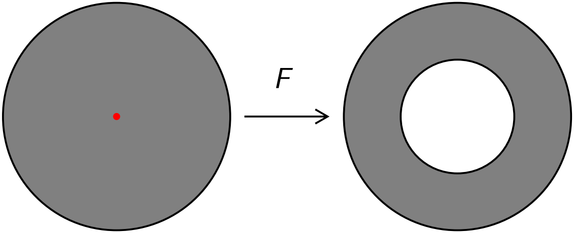


Figures: Pendry et al 2006 and Schurig et al 2006.

The principle of cloaking for Maxwell's equations

As before, let $g_{ij} = \delta_{ij}$ and $\widetilde{g} = F_* g$ with

$$F : B(2) \setminus \{0\} \to B(2) \setminus \overline{B}(1), \quad F(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}.$$

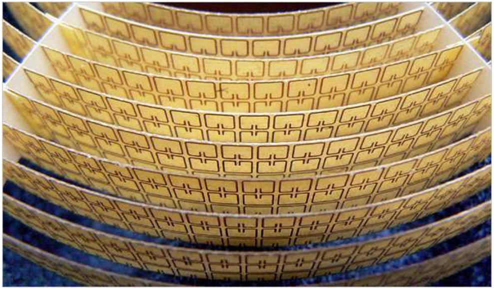


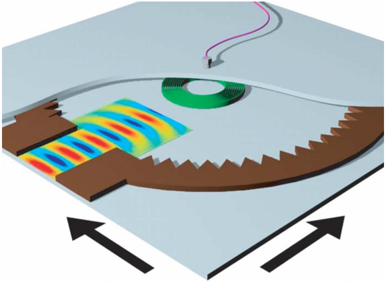
Define the permittivity $\widetilde{\varepsilon}$ and the permeability $\widetilde{\mu}$ by

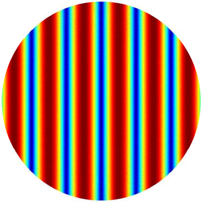
$$\widetilde{\varepsilon} = \widetilde{\mu} = \begin{cases} |\widetilde{g}|^{1/2} \widetilde{g}^{jk} & \text{for } x \in B(2) \setminus \overline{B}(1), \\ a^{jk} & \text{for } x \in B(1). \end{cases}$$

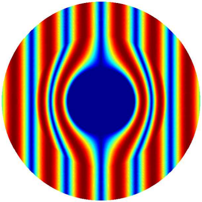
Note that $\widetilde{\varepsilon} = \widetilde{\mu}$ coincide with the above matrix $\widetilde{\sigma}$.

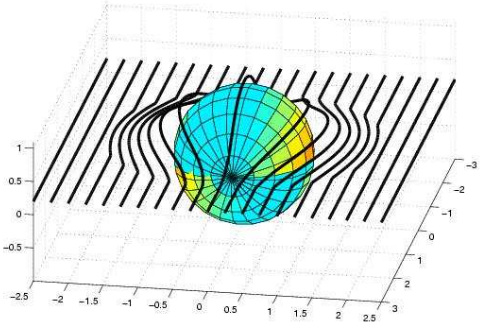
Figure: Invisibility cloak for 4 cm waves build using metamaterials, Schurig et al, Science 2006.







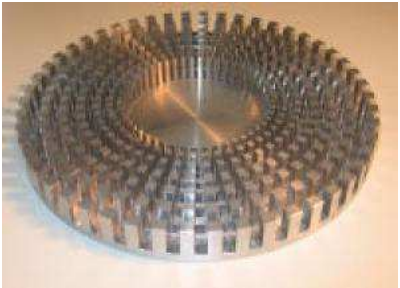


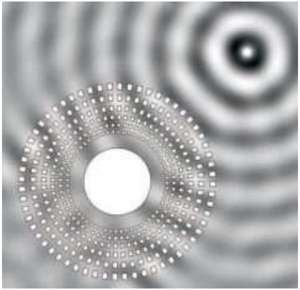


The presently built materials work with a single frequency. Because of causality, the present invisibility cloaks can not work in vacuum with all frequencies.

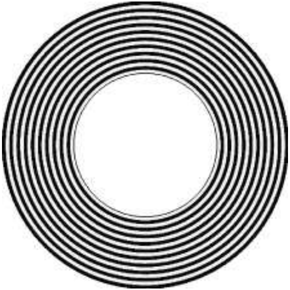
Tsunami cloak

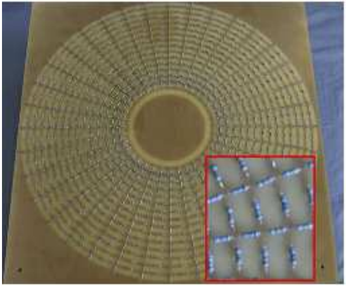
A cloak for surface water waves (M. Farhat et al 2008)





Experiments of cloaking for the conductivity equation

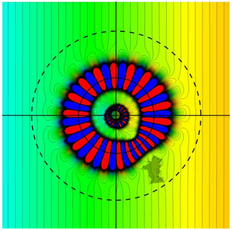


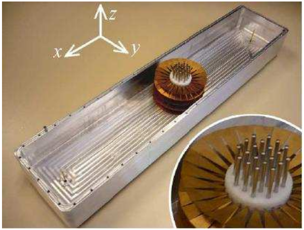


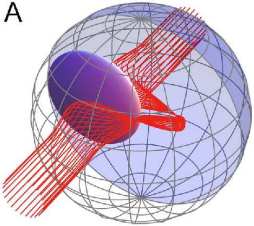
In theory, the cloaking conductivity can be obtained as a homogenization limit of spherical layers of isotropic materials (Greenleaf-L.-Uhlmann 2003 and Greenleaf-Kurylev-L.-Uhlmann 2011).
Implementation of a dc-cloak in F. Yang, Z.L. Mei, T.Y. Jin, T.J,Cui: PRL 2012.

Other proposals for cloaking:

- Milton and Nicorovici, Proc. Royal Soc. 2006: Anomalous resonance.
- Tretyakov et al, Phys. Rev. Lett. 2009: Transmission line cloaking.
- Leonhardt and Tyc, Science 2009: Non-Euclidean Cloaking.







Milton et al (Left), Tretyakov et al (Middle), Leonhardt-Tyc (Right)

Mathematics of cloaking for Helmholtz equation

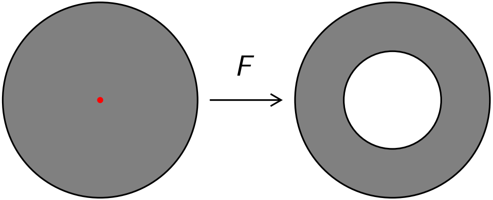
Let $g_{jk} = \delta_{jk}$ and define

$$\widetilde{g}(x) = \begin{cases} F_* g & \text{for } x \in B(2) \setminus \overline{B}(1), \\ \delta_{jk} & \text{for } x \in B(1). \end{cases}$$

This metric is singular on $\Sigma = \partial B(1)$. We consider $k \geq 0$ and

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = 0, \quad \Delta_{\widetilde{g}} \widetilde{u} = \sum_{j,k=1}^{3} |\widetilde{g}|^{-\frac{1}{2}} \frac{\partial}{\partial x^j} (|\widetilde{g}|^{\frac{1}{2}} \widetilde{g}^{jk} \frac{\partial \widetilde{u}}{\partial x^k}).$$

This equation models a time-harmonic acoustic wave.



Definition

1. We denote $\widetilde{u} \in H^1(N, |\widetilde{g}|^{1/2}dx)$ if

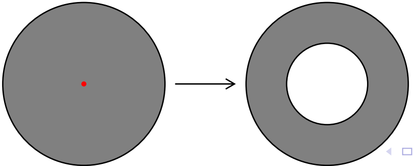
$$\widetilde{u} \in L^2(N,\ |\widetilde{g}|^{1/2}dx) \quad \text{and} \quad \int_{N\setminus\Sigma} |\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u}\,\partial_j\widetilde{u}\,dx < \infty.$$

2. A measurable function $\widetilde{u}$ on $N = B(2)$ is a **finite energy solution** of the Helmholtz equation,

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = \widetilde{f} \quad \text{on } N,$$

if $\widetilde{u} \in H^1(N, |\widetilde{g}|^{1/2}dx)$ and for all $\widetilde{\psi} \in C_0^\infty(N)$,

$$\int_N [-(|\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i\widetilde{u})\partial_j\widetilde{\psi} + k^2\widetilde{u}\widetilde{\psi}|\widetilde{g}|^{1/2}]dx = \int_N \widetilde{f}\,\widetilde{\psi}\,|\widetilde{g}|^{1/2}dx.$$



Theorem

(Greenleaf-Kurylev-L.-Uhlmann 2007)
A function $\widetilde{u}$ is a weak finite energy solution to

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u}(x) = H(x) \quad \text{on } B(2)$$

with a source H supported in the ball $B(1)$ if and only if $u = \widetilde{u} \circ F$ extends to a solution of the Euclidean Helmholtz equation

$$(\Delta + k^2)u(x) = 0 \quad \text{on } B(2)$$

and the restriction of $\widetilde{u}$ inside the cloak satisfies

$$\begin{aligned} (\Delta + k^2)\widetilde{u}(x) &= H(x) \quad \text{on } B(1), \\ \partial_\nu \widetilde{u}|_{\partial B(1)} &= 0. \end{aligned}$$

Thus if k^2 is a Neumann eigenvalue inside the cloak, solutions do not exists for some sources H.

Idea of the proof. Assume that $\widetilde{u}$ is a finite energy solution to

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = 0 \quad \text{on } B(2).$$

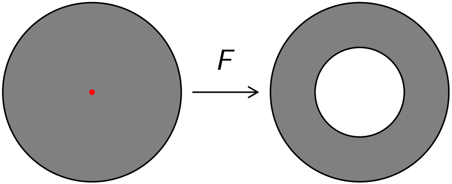
Let $\widetilde{u}^+ = \widetilde{u}|_{B(2)\setminus\overline{B}(1)}$ and $u = \widetilde{u}^+ \circ F$.
As $\widetilde{u} \in H^1(N, |\widetilde{g}|^{1/2}dx)$, we have $u \in H^1(B(2) \setminus \{0\})$ and

$$(\Delta + k^2)u = 0 \quad \text{on } B(2) \setminus \{0\}. \tag{1}$$

As the Hausdorff dimension $\{0\} \subset \mathbb{R}^3$ is $0 \leq 3 - 2$, we have $H^1(B(2) \setminus \{0\}) \equiv H^1(B(2))$, that is,

$$u = u^{ext}|_{B(2)\setminus\{0\}}, \quad \text{where } u^{ext} \in H^1(B(2), dx).$$

Moreover, u^{ext} satisfies the Helmholtz equation (1) on $B(2)$.



Boundary condition inside the cloak. We consider the finite energy solution $\widetilde{u}$ of equation

$$(\Delta_{\widetilde{g}} + k^2)\widetilde{u} = H \quad \text{on } B(2).$$

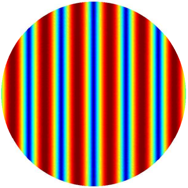
For all $\widetilde{\psi} \in C_0^\infty(B(2))$, $u = \widetilde{u} \circ F$, and $\psi = \widetilde{\psi} \circ F$,

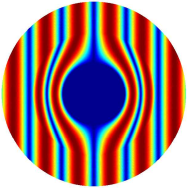
$$\begin{aligned}
& \int_{B(1)} H\widetilde{\psi}\, dx \\
= \; & \int_{(B(2)\setminus \overline{B}(1))\cup B(1)} [-(|\widetilde{g}|^{1/2}\widetilde{g}^{ij}\partial_i \widetilde{u})\partial_j \widetilde{\psi} + k^2 \widetilde{u}\widetilde{\psi}|\widetilde{g}|^{1/2}]dx \\
= \; & \int_{B(2)\setminus\{0\}} [-\nabla u \cdot \nabla \psi + k^2 u\psi]dx + \int_{B(1)} [-\nabla \widetilde{u} \cdot \nabla \widetilde{\psi} + k^2 \widetilde{u}\widetilde{\psi}]dx \\
= \; & 0 + \int_{B(1)} H\widetilde{\psi}\, dx - \int_{\partial B(1)} \partial_\nu \widetilde{u} \cdot \widetilde{\psi}\, dS.
\end{aligned}$$

This means that we have the Neumann boundary condition $\partial_\nu \widetilde{u}|_{\partial B(1)-} = 0$ on the boundary of $B(1)$.

Outline:

- Inverse conductivity problem and transformation optics
- Metamaterials
- Inverse conductivity problem
- Perfect invisibility cloaks
- **Approximate invisibility cloaks**
- Topology and cloaking: Wormholes





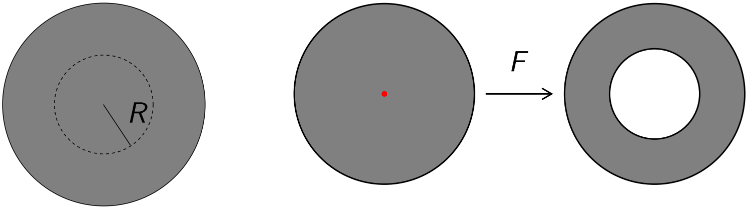
Approximate cloaks.

Ruan et al 2007, Greenleaf-Kurylev-L.-Uhlmann 2008, Kohn-Onofrei-Weinstein-Vogelius 2009, Liu-Zhou 2011.

In the layer $1 < |x| < R$, we replace $\widetilde{g}$ by identity matrix: Let $R > 1$, $L > 2$. The approximative cloaking metric is

$$(g_R)_{jk} = \begin{cases} \delta_{jk} & \text{for } x \in B(L) \setminus B(2), \\ \widetilde{g}_{jk} & \text{for } x \in B(2) \setminus \overline{B}(R), \\ \delta_{jk} & \text{for } x \in B(R), \end{cases}$$

where $\widetilde{g} = F_* g$, $g_{jk} = \delta_{jk}$.



Let us consider cylindrical cloak, that is, cloaking in $\mathbb{R}^2$.

Theorem

(L.-Zhou, MRL 2011) Assume that u_R is the solution in an approximative cloak,

$$\nabla \cdot \sigma_R \nabla u + \omega^2 \kappa_R u_R = H \quad \text{in } B(L) \subset \mathbb{R}^2.$$

When $\omega > 0$ is not an eigenfrequency and $R \to 1$, u_R converges to the limit u_1 satisfying

$$\begin{aligned} &(\nabla^2 + \omega^2) u_1 = H \quad \text{in} \;\; B(1), \\ &\partial_r u_1 + (-\partial_\theta^2)^{1/2} u_1 \big|_{\partial B(1)} = 0. \end{aligned}$$

Observe that the above boundary condition is non-local.

Choosing $k^2 = \kappa(R)$ to be close to an eigenvalue, we can obtain 'cloaked sensors' and the 'Schrödinger's hat' devices.

Outline:

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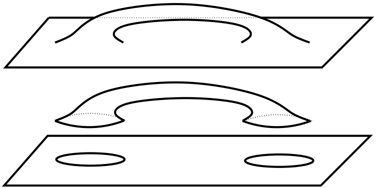
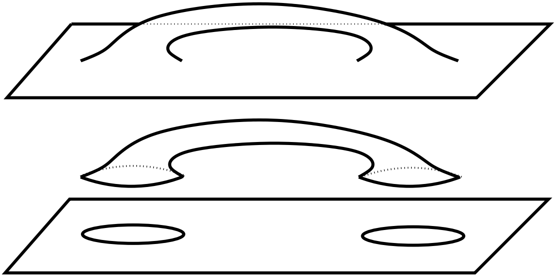
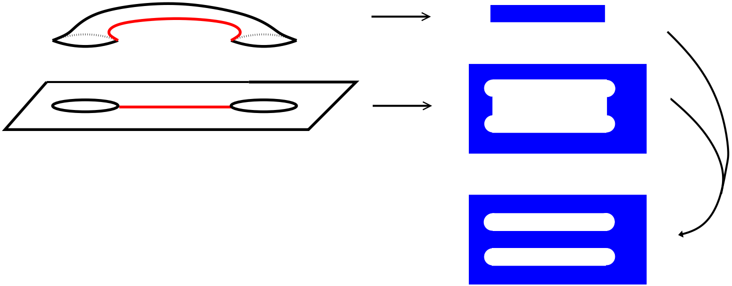


Electromagnetic wormhole device
(Greenleaf-Kurylev-L.-Uhlmann 2007)
Wormhole makes a shortcut for two distant places in the space.



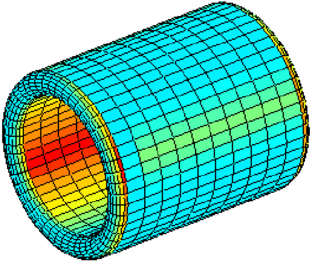


Let g be a smooth metric on the wormhole manifold M. Using a closed path $\gamma \subset M$ and a diffeomorphism

$$F : M \setminus \gamma \to \mathbb{R}^3 \setminus K$$

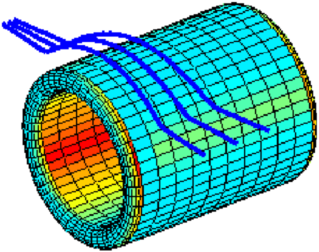
where K is a smooth tube. Define the metric $\widetilde{g} = F_* g$ in $\mathbb{R}^3 \setminus K$. In the space $\mathbb{R}^3 \setminus K$, with the singular metric $\widetilde{g}$, all observations far away from K are the same as on the wormhole manifold.

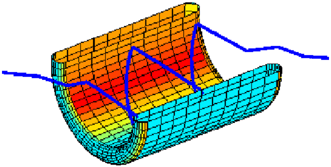
Building a wormhole for electromagnetic waves.



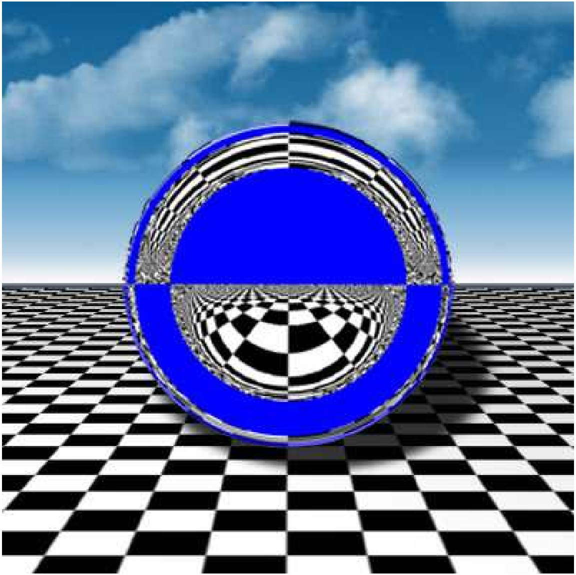
We take an obstacle and the soft-and-hard material on the boundary and cover it with "invisibility cloaking material".

Ray tracing simulations:



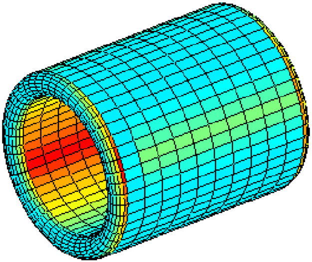


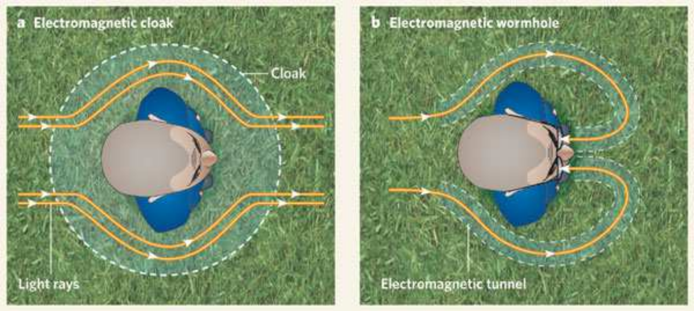
Ray tracing simulations:



Possible applications in future:

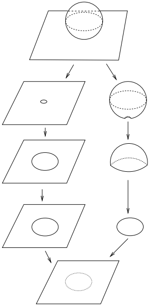
- Invisible optical cables.
- 3D video displays: ends of invisible tunnels work as light source in 3D voxels.
- Virtual magnetic monopoles.
- Scopes for Magnetic Resonance Imaging devices.

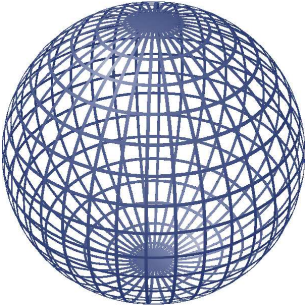




Invisibility cloak and looking behind with wormholes
(Figure: L. Tsakmakidis and O. Hess, Nature 2008)

Harry Potter's cloak inside out: An invisibility sack





Take a sphere $\mathbb{S}^3$ and the Euclidean space $\mathbb{R}^3$. Remove the South pole SP and the origin O. The sets $\mathbb{S}^3 \setminus \{SP\}$ and $\mathbb{R}^3 \setminus \{O\}$ are blow up and glued together. We obtain the space $\mathbb{R}^3$ with a singular metric. This gives us a two-sided invisibility cloak. Observations inside the cloaked region are the same as those in $\mathbb{S}^3$.

An early announcement: Balehowsky-L.-Pankka-Sirviö:
Waves on any compact orientable 3-manifold can be simulated in a metamaterial device.

Thank you for your attention!