STABILITY OF THE UNIQUE CONTINUATION FOR THE WAVE OPERATOR VIA TATARU INEQUALITY: THE LOCAL CASE

ROBERTA BOSI, YAROSLAV KURYLEV, MATTI LASSAS

Abstract. In 1995 Tataru proved a Carleman-type estimate for linear operators with partially analytic coefficients that is generally used to prove the unique continuation of those operators. In this paper we use this inequality to study the stability of the unique continuation in the case of the wave equation with coefficients independent of time. We prove a logarithmic estimate in a ball whose radius has an explicit dependence on the C^1-norm of the coefficients and on the other geometric properties of the operator.

Keywords: wave equation, unique continuation property, stability.
Mathematical Subject Classification 2010. Primary: 35B53, 35B60, 35L05, 93B05. Secondary: 35R30, 31B20.

1. Introduction

We consider the wave operator in $\mathbb{R}^{n+1}$,

$$P(y,D) = -D_0^2 + \sum_{j,k=1}^{n} g^{jk}(x)D_jD_k + \sum_{j=1}^{n} h^j(x)D_j + q(x), \tag{1.1}$$

where $y=(t,x)\in\mathbb{R}\times\mathbb{R}^n$ are the time-space variables, $D_0=-i\partial_t$, $D_j=-i\partial_{x_j}$. The coefficients $g^{jk}\in C^1(\mathbb{R}^n)$ are real and independent of time, and $[g^{jk}]$ is a symmetric positive-definite matrix. The coefficients $h^j, q\in L^\infty(\mathbb{R}^n)$ are complex valued and independent of time.

An operator $P(y,D)$ is said to have the unique continuation property if for any solution u to $Pu=0$ in a connected open set $\Omega\subset\mathbb{R}^{n+1}$ and vanishing on an open subset $B\subset\Omega$, it follows that u vanishes in Ω.

In the paper [20] Tataru proved for the first time the unique continuation property for (1.1) across every non-characteristic C^2-hypersurface with no limitation to the normal direction. The result is valid for a larger class of linear operators where the pseudo-convexity condition across a surface is fulfilled for the cotangent vectors with $\xi_0=0$ and it has been extended to the case of coefficients analytic in time [6, 17, 21]. The key point of these results is a Carleman-type estimate involving an exponential pseudo-differential operator.

Much is known about the consequences of this property on the uniqueness of a corresponding Cauchy problem. Actually the unique continuation property has proved to be instructive in many areas of mathematics, e.g. in studying the uniqueness for linear and nonlinear PDEs together with their blow up or traveling wave solutions [5], in studying the Anderson localization [3], in control theory to get controllability results [22], in inverse problems to obtain uniqueness and stability estimates [11].

Concerning the continuous dependence of the unique continuation property,

Date: June 16, 2014.

that is its stability, less results are available. The elliptic and the parabolic cases have been studied in several settings by using either Carleman estimates or some versions of the three ball theorem (see [1], for a review of the results). To our knowledge the hyperbolic case like (1.1) is still open for arbitrary domains and arbitrary matrix valued coefficients $g^{jk}(x)$, while there exist results for particular coefficients or domains (see [18]). This is maybe related to the difficulty of using the standard Carleman estimates for hyperbolic operators in order to prove the unique continuation close to the characteristic directions, that is the reason why Tataru's work was so important in this field.
The aim of the present work is then to prove a stability estimate for the unique continuation of the operator $P(y, D)$. We will focus on the local case, and we formulate an explicit stability estimate for the inhomogeneous operator $Pu = f$, that can be alternatively reformulated in terms of a boundary value problem. Let $\Omega \subset \mathbb{R}^{n+1}$ be a connected open set and consider a non-characteristic oriented hypersurface S written as the level set of the function $\psi : \Omega \to \mathbb{R}$, $S = \{y \in \Omega; \psi(y) = 0\}$. Assume $Pu = f$, in a ball $\Omega_1 := B(y_0, 2R)$. Moreover, let $\mathrm{supp}(u) \subset \Omega_2 := \{y \in \Omega; \psi(y) \leq 0\}$ with $\|u\|_{H^1(\Omega_1)} \leq C_1$, and let $\|f\|_{L^2(\Omega_1)} \leq \epsilon_1$ for some small $\epsilon_1 > 0$. The stable unique continuation is based upon an estimate like

$$\|u\|_{L^2(\Omega_3)} \leq \Upsilon(C_1, \epsilon_1), \tag{1.2}$$

for some ball $\Omega_3 := B(y_0, r)$ contained in Ω_1, where the right hand side goes to zero as $\epsilon_1 \to 0$. Our aim is to prove (1.2) with a function Υ that has an explicit form depending on the constants related to geometrical properties of Ω_3, Ω_1, and Ω and the norm of the coefficients g^{jk} in $C^1(\overline{\Omega})$. In this paper we consider the case where the domains Ω_1 and Ω_3 are balls centred in $y_0 \in \Omega$ and we find a logarithmic function Υ dependent on their size R and r and on the norms of g^{jk}, h^j, q and ψ. In a forthcoming paper we will use the local stability estimate to prove (1.2) for quite general domains.
Like in the elliptic case, many possible applications can be derived out of it. In particular we plan to use inequality (1.2) to obtain an explicit modulus of continuity for the inverse problem for the wave operator on manifolds. This would improve the existing inverse stability results for Riemannian manifolds, which are currently based either on compactness-type arguments, see [2, 14], or on very strong geometrical conditions for the coefficients, e.g. in [4, 12, 13]. In the unpublished manuscript [19], Tataru suggested the possibility of obtaining a stability estimate, by using Gevrey-class localizers to improve the estimates of u for low temporal frequencies.
Here we develop that idea by employing properties of subharmonic functions (see Lemma 2.7) and by performing the explicit estimate of the radii r and R and the constants. Of fundamental importance is the possibility of linking the positive lower bound for r to the geometric parameters of the domain, in order to assure that the estimate can work close to the characteristic surfaces of the operator.

We first introduce some assumptions.
Assumption A1. Let Ω be a connected open subset of $\mathbb{R} \times \mathbb{R}^n$. Let $P(y, D)$ be the wave operator (1.1), with $g^{jk}(x) \in C^1(\Omega)$, $h^j, q \in L^\infty(\Omega)$. Let $S =$

$\{y \in \Omega; \psi(y) = 0\}$ be a $C^{2,\rho}$-smooth oriented hypersurface, which is non-characteristic in Ω, for some fixed $\rho \in (0,1)$. We assume that $u \in H^1(\Omega)$ is supported in $\{y; \psi(y) \leq 0\} \cap \Omega$, and $P(y,D)u \in L^2(\Omega)$.

Assumption A2. We define $A(D_0)$ to be a pseudo-differential operator with symbol $a(\xi_0)$, $0 \leq a \leq 1$, where $a \in C_0^\infty(\mathbb{R})$ is a smooth localizer supported in $|\xi_0| \leq 2$, equal to one in $|\xi_0| \leq 1$. Furthermore let $a \in G_0^{1/\alpha}(\mathbb{R})$ for a fixed $\alpha \in (0,1)$. Here $G_0^{1/\alpha}$ is the set of Gevrey functions of class $1/\alpha$ with compact support, defined in Definition 4.1. The also define the smooth localizer $b(y)$, supported in $|y| \leq 2$, $0 \leq b \leq 1$ and equal to one in $|y| \leq 1$.

The main results of the paper are the following 2 Theorems.
The first one is a stability estimate of exponential type for the low temporal frequencies.

Theorem 1.1. *Under the Assumptions A1-A2, let $y_0 \in S$ with $\psi'(y_0) \neq 0$, and let $b \in G_0^{1/\alpha_1}(\mathbb{R}^{n+1})$ be a Gevrey functions of class $1/\alpha_1$ with compact support, such that $0 < \alpha \leq \alpha_1 < 1$.*
Then, there exist two constants R, r with $R \geq 2r > 0$ and two balls centred in y_0 of radii r and $2R$, $B(y_0,r) \subset B(y_0,2R) \subset \Omega$, such that for $\mu \geq 1$ there are constants $c_{129}, c_{131}, c_{132}$ for which,
if

$$\|u\|_{H^1(B_{2R})} = 1, \quad \|Pu\|_{L^2(B_{2R})} < 1, \quad \|A(D_0/\mu)b((y-y_0)/R)Pu\|_0 \leq e^{-\mu^\alpha},$$

then,

$$\|A(D_0/\omega)b((y-y_0)/r)u\|_{H^1} \leq c_{129}e^{-c_{132}\mu^{\alpha\cdot\alpha_1}}, \quad \forall\, \omega \leq \mu^\alpha/(3c_{131}).$$

The radii r and R are defined in Table (3.10)*, while the coefficients c_k are computed in the proof of the Theorem.*

The second result is a log-stability estimate in a ball, valid for all the temporal frequencies (see Figure 1 for the construction).

Theorem 1.2. *Under the conditions of Assumption A1 we obtain that, for each $y_0 \in S$, with $\psi'(y_0) \neq 0$, there exist two constants R, r with $R \geq 2r > 0$ and two balls centred in y_0 of radii r and $2R$, $B(y_0,r) \subset B(y_0,2R) \subset \Omega$, for which the following stability estimate holds:*

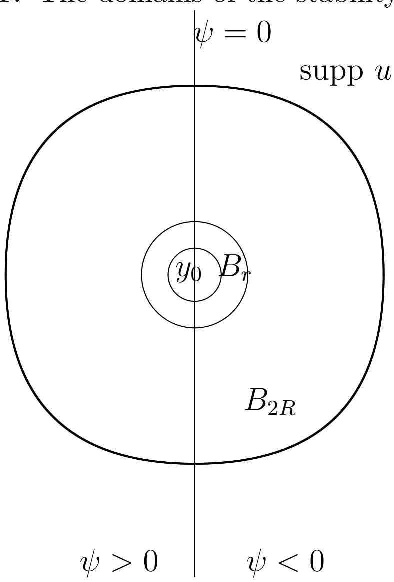
$$\|u\|_{L^2(B(y_0,r))} \leq c_{111}\frac{\|u\|_{H^1(B(y_0,2R))}}{\ln\left(1+\frac{\|u\|_{H^1(B(y_0,2R))}}{\|Pu\|_{L^2(B(y_0,2R))}}\right)}.$$

The radii r and R and the coefficient c_{111} are defined in Table (3.10)*. Moreover, for any $m \in (0,1]$ we get*

$$\|u\|_{H^{1-m}(B(y_0,r))} \leq c_{111}^m\frac{\|u\|_{H^1(B(y_0,2R))}}{\left(\ln\left(1+\frac{\|u\|_{H^1(B(y_0,2R))}}{\|Pu\|_{L^2(B(y_0,2R))}}\right)\right)^m}.$$

As a consequence, one can find in a domain $\Omega_0 \subseteq \Omega$ a uniform radius $r_0 > 0$ such that $r \geq r_0$, and where $r_0 = r_0\big(|\psi'|_{C^{1,\rho}(\Omega_0)}, |g^{jk}|_{C^1(\Omega_0)}, \min_{y\in\Omega_0}|p(y,\psi')|,$ $\min_{y\in\partial\Omega_0}|y_0 - y|\big)$.
Theorems 1.1 and 1.2 will be proved in Section 2. In Section 3 we will compute the related parameters R, r, c_k that are dependent upon the constants of the

FIGURE 1. The domains of the stability estimate



Carleman estimate of Theorem 2.1 and upon a particular geometric construction. The Appendix is devoted to the main definitions used in the article. We finally observe that even if we deal with the wave equation, the same method can be generalized to ultrahyperbolic operators of the type $-|D_a|^2 + g^{jk}(x_b)D_kD_j$, where the variable $y = (x_a, x_b)$ has a different splitting and where x_a corresponds to the conormal direction for the pseudo-convexity condition.

2. Proof of the stability estimate

Notations. We start by introducing some notations and definitions used in the rest of the article: first we consider $y = (t, x) \in \mathbb{R} \times \mathbb{R}^n$ the time-space variable and call $\xi = (\xi_0, \widetilde{\xi})$ its Fourier dual variable. We remind that the exponential pseudodifferential operator in Theorem 2.1 is defined as $e^{-\epsilon|D_0|^2/2\tau}v = \mathcal{F}^{-1}_{\xi_0\to t}e^{-\epsilon\xi_0^2/2\tau}\mathcal{F}_{t'\to\xi_0}v$, with $\mathcal{F}$ and $\mathcal{F}^{-1}$ representing respectively the Fourier transform and its inverse. Then $e^{-\epsilon|D_0|^2/2\tau}$ is an integral operator with kernel $(\frac{\tau}{2\pi\epsilon})^{1/2}e^{-\tau|t'-t|^2/2\epsilon}$. We also define $A(D_0)$ to be a pseudo-differential operator with symbol $a(\xi_0)$, $0 \le a \le 1$, where $a \in C_0^\infty(\mathbb{R})$ is a smooth localizer supported in $|\xi_0| \le 2$, equal to one in $|\xi_0| \le 1$. Hence we can write $A(\beta|D_0|/\omega)v = \mathcal{F}^{-1}_{\xi_0\to t}a(\beta|\xi_0|/\omega)\mathcal{F}_{t'\to\xi_0}v$ and the integral kernel is $(\frac{\omega}{2\pi\beta})^{1/2}\widehat{a}(\frac{\omega|t'-t|}{\beta})$. We will often work under the Assumption A2, where the symbol a is of Gevrey class. The smooth localizer $b(y)$ is supported in $|y| \le 2$ and equal to one in $|y| \le 1$.
The norm of the Sobolev space H^s_τ is defined as $\|u\|_{s,\tau} = \|(|\xi|^2+\tau^2)^{s/2}\mathcal{F}_{y\to\xi}u\|_{L^2}$, and the space H^s corresponds to the case $\tau = 1$.
According to our notations the positive coefficients denoted by c_x with $x \ge 100$ are defined just once, independently on the variables μ, τ, and they are computed explicitly in terms of the coefficients of the operator (1.1) and the geometric parameters. This is essential to finally recover the value of c_{111} and the radii R, r in Theorem 1.2.

We then introduce the Tataru inequality proved in [20] in the version presented by Hörmander [6] and adapted to the wave operator.
In the Appendix one can find the definition of conormally strongly pseudoconvex function or surface, and Gevrey function. According to Definition 4.5 and the splitting $y = (t, x)$, the conormal bundle in $\mathbb{R}^{n+1}$ with respect to the foliation x =const is defined as
$N^*F := \{(y, \xi) \in T^*\mathbb{R}^{n+1};$ with $\xi = (\xi_0, \widetilde{\xi})$ and $\xi_0 = 0\}$, and its fibre in y_0 is Γ_{y_0}.

Theorem 2.1. *Let Ω be an open subset of $\mathbb{R} \times \mathbb{R}^n$. Let $P(y, D)$ be the wave operator* (1.1)*, with $g^{jk}(x) \in C^1(\Omega)$, $h^j, q \in L^\infty(\Omega)$. Let $y_0 \in \Omega$ and $\psi \in C^{2,\rho}(\Omega)$ be real valued, for some fixed $\rho \in (0,1)$, such that $\psi'(y_0) \neq 0$ and $S = \{y; \psi(y) = 0\}$ being an oriented hypersurface non-characteristic in y_0.*
Consequently there is $\lambda > 1$ such that $\phi(y) = \exp(\lambda\psi)$ is a conormally strongly pseudoconvex function with respect to P at y_0.
Then there is a real valued quadratic polynomial f defined in (3.3) *with proper $\sigma > 0$, and a ball $B_{R_2}(y_0)$ such that $f(y) < \phi(y)$ when $y \in B_{R_2} - \{y_0\}$ and $f(y_0) = \phi(y_0)$; and f being a conormally strongly pseudoconvex function with respect to P in B_{R_2}. This implies that there exist ϵ_0, τ_0, $c_{1,T}$, $c_{2,T}, R$, such that, for each small enough $\epsilon < \epsilon_0$ and large enough $\tau > \tau_0$, we have*

$$\|e^{-\epsilon|D_0|^2/2\tau}e^{\tau f}u\|_{1,\tau} \leq c_{1,T}\,\tau^{-1/2}\|e^{-\epsilon|D_0|^2/2\tau}e^{\tau f}P(y,D)u\|_0 + c_{2,T}e^{-\tau R_2^2/4\epsilon}\|e^{\tau f}u\|_{1,\tau}.$$

Here $u \in H^1_{loc}(\Omega)$, with $P(y, D)u \in L^2(\Omega)$ and supp$(u) \subset B_R(y_0)$.

Remark 2.2. We note that the explicit estimate for the involved coefficients $\epsilon_0, \tau_0, c_{1,T}, c_{2,T}, \sigma$, R_2, R and their dependence upon the parameters of the problem have never been found. In this paper we provide proper estimates, which are summarized in Table (3.10) of Section 3.1.
Notice that this is possible under the condition that $\psi \in C^{2,\rho}(\Omega)$ instead of the usual $\psi \in C^2(\Omega)$. Furthermore we assume that S is not characteristic in y_0 and consequently in a domain $\Omega_0 \subseteq \Omega$. Actually this assumption is not required in [20, 6] where only the strongly pseudoconvexity of S in Γ_{y_0} is assumed. In Remark 3.1 we will underline such difference with an alternative condition on ψ. Anyway for the practical computations of the values in Table (3.10) we prefer to work in the stronger setting of Theorem 2.1. Our wave operator can be seen in 2 ways: (H) an hyperbolic operator with constant in time and real valued coefficients for the principal part, or (E) an operator whose principal symbol is elliptic in the set $\Gamma_\Omega \subset N^*F$. In the latter case Tataru inequality is sharper (see [20]). Here we prefer to consider just the case (H). Finally, some improvements to the assumption on the coefficients of (1.1) may be done, for example taking $\Omega_x \subset \mathbb{R}^n$ the smooth domain of definition of $q(x)$ we can assume $q \in L^n(\Omega_x)$ for $n \geq 3$, $q \in L^{2+\epsilon}(\Omega_x)$ for $n = 2$, $q \in L^\infty(\Omega_x)$ for $n = 1$.

We now proceed with the detailed proof of Theorems 1.1 and 1.2.
A first step is the following lemma, introducing a property often used in this section.

Lemma 2.3. *Let $A(D_0)$ be a pseudo-differential operator with symbol $a(\xi_0)$, where $a \in C_0^\infty(\mathbb{R})$ is a smooth localizer supported in $|\xi_0| \leq 2$ and equal one in $|\xi_0| \leq 1$. Assume that $f(y) \in C_0^\infty(\mathbb{R}^{n+1}) \cap G_0^{1/\alpha}(\mathbb{R}^1_t)$, where $0 < \alpha < 1$. Then, for every $\mu > 0$, $\beta_1 > 2$, $v \in L^2(\mathbb{R}^{n+1})$ there are two constants c_{106}, c_{107}*

independent of μ such that

$$a)\quad \|A(\beta_1 D_0/\mu)f(y)(1-A(D_0/\mu))v\|_0 \le c_{107}e^{-c_{106}\mu^\alpha}\|v\|_0 \, .$$

Moreover, if $h \in C_0^\infty(\mathbb{R}^{n+1})$ is a function such that $h \equiv 1$ on supp(f), then

$$b)\quad \|A(\beta_1 D_0/\mu)fhv\|_0 \le \|f\|_\infty \|A(D_0/\mu)h(y)v\|_0 + c_{107}e^{-c_{106}\mu^\alpha}\|hv\|_0 \, .$$

If $v \in H^m(\mathbb{R}^{n+1})$, $m \ge 1$, then the estimate above holds also in $H^m(\mathbb{R}^{n+1})$, under the additional condition $D_x^m f(y) \in G_0^{1/\alpha}(\mathbb{R}_t^1)$:

$$c)\quad \|A(\beta_1 D_0/\mu)f(1-A(D_0/\mu))v\|_m \le c_{108}e^{-c_{106}\mu^\alpha}\|v\|_m \, .$$

Proof. $a)$ On the set supp$[(1-a(\xi_0/\mu))a(\beta_1\xi_0^1/\mu)]$ one obtains $|\xi_0^1-\xi_0|^\alpha \ge (\mu - 2\mu/\beta_1)^\alpha$ and the assumption $f(t,.) \in G_0^{1/\alpha}(\mathbb{R}_t)$ implies, uniformly in x on a compact set $K \subset \mathbb{R}^n$ and for some $c_3 = c_3(\alpha, K)$, $c_{117} = c_{117}(\alpha, K)$ and $c_{106} = c_{117}(1-2/\beta_1)^\alpha/4$,

$$|\mathcal{F}_{t'\to(\xi_0^1-\xi_0)}[f(t',x)]| \le c_3 e^{-c_{117}|\xi_0^1-\xi_0|^\alpha} \le c_3 e^{-2c_{106}\mu^\alpha}e^{-c_{117}|\xi_0^1-\xi_0|^\alpha/2}.$$

We then estimate in the Fourier space the operator $A(\beta_1 D_0/\mu)f(\cdot)(1-A(D_0/\mu))$,

$$\|a(\beta_1\xi_0^1/\mu)\mathcal{F}_{t'\to\xi_0^1}\Big(f(t',x)\big(\mathcal{F}^{-1}_{\xi_0\to t'}(1-a(\xi_0/\mu))\mathcal{F}_{t\to\xi_0}[v]\big)\Big)\|_0^2$$

$$= \|a(\frac{\beta_1\xi_0^1}{\mu})\Big(\int_{\mathbb{R}}(1-a(\frac{\xi_0}{\mu}))\mathcal{F}_{t'\to(\xi_0^1-\xi_0)}[f(t',x)]\mathcal{F}_{t\to\xi_0}[v]\,d\xi_0\Big)\|_0^2$$

$$\le c_3\int_{\mathbb{R}^{n+1}}dxd\xi_0^1\Big(\int_{\mathbb{R}}(1-a(\frac{\xi_0}{\mu}))a(\frac{\beta_1\xi_0^1}{\mu})e^{-2c_{106}\mu^\alpha}e^{-c_{117}|\xi_0^1-\xi_0|^\alpha/2}|\mathcal{F}_{t\to\xi_0}[v](\xi_0,x)|\,d\xi_0\Big)^2$$

$$\le c_3\int_{\mathbb{R}^{n+1}}dxd\xi_0^1\|(1-a(\frac{\xi_0}{\mu}))a(\frac{\beta_1\xi_0^1}{\mu})e^{-2c_{106}\mu^\alpha}e^{-c_{117}|\xi_0^1-\xi_0|^\alpha/2}\|^2_{L^2(d\xi_0)}\|\mathcal{F}_{t\to\xi_0}[v](\xi_0,x)\|^2_{L^2(d\xi_0)}$$

$$\le c_3 e^{-4c_{106}\mu^\alpha}\|(1-a(\frac{\xi_0}{\mu}))a(\frac{\beta_1\xi_0^1}{\mu})e^{-c_{117}|\xi_0^1-\xi_0|^\alpha/2}\|^2_{L^2(d\xi_0d\xi_0^1)}\|\mathcal{F}_{t\to\xi_0}[v](\xi_0,x)\|^2_{L^2(d\xi_0dx)}$$

$$\le c_{107}^2 e^{-2c_{106}\mu^\alpha}\|v\|_0^2 \, ,$$

with $c_{107} = \big(c_3\frac{8}{\beta_1}\Gamma\big(\frac{1}{\alpha}\big)\frac{1}{\alpha(c_{117})^{1/\alpha}}\frac{1}{(\alpha c_{106})^{\frac{1}{\alpha-1}}})\big)^{1/2}$ and where we apply at the last step the inequalities

$$\|(1-a(\xi_0/\mu))a(\beta_1\xi_0^1/\mu)e^{-c_{117}|\xi_0^1-\xi_0|^\alpha/2}\|^2_{L^2(d\xi_0d\xi_0^1)} \le \frac{8}{\beta_1}\Gamma\Big(\frac{1}{\alpha}\Big)\frac{1}{\alpha(c_{117})^{1/\alpha}}\,\mu,$$

and $\mu\, e^{-c_{106}\mu^\alpha} \le \frac{1}{(\alpha c_{106})^{\frac{1}{\alpha-1}}}$.

$b)$ To prove the inequality we observe that

$$\begin{aligned}\|A(\beta_1 D_0/\mu)fhv\|_0 \;&\le\; \|A(\beta_1 D_0/\mu)fA(D_0/\mu)hv\|_0 + \\ &\quad\; \|A(\beta_1 D_0/\mu)f(1-A(D_0/\mu))hv\|_0\end{aligned}$$

where we can apply the estimate $a)$ to the second term at the right hand side, and where the first term is bounded by $\|f\|_\infty\|A(D_0/\mu)h(y)v\|_0$.

$c)$ The extension to H^m of $a)$ follows from

$$D^\zeta f(1-A(D_0/\mu))v = \sum_{v:v\le\zeta}\left(\begin{array}{c}\zeta\\ v\end{array}\right)\Big(D^{\zeta-v}f\Big)\Big((1-A(D_0/\mu))(D^v v)\Big).$$

By hypothesis any derivative $D^{\zeta-v}f$ belongs to $G_0^{1/\alpha}(\mathbb{R}_t)$; hence we consider all of them as a new function g, having the same Gevrey-parameters c_3, c_{117} as f. Then we apply $A(\beta_1 D_0/\mu)$ and repeat the computations of step $a)$, replacing v

with $D^v v$. The coefficient $c_{108} = c_m c_{107}$ is a proper multiple in m of c_{107}.
□

Another technical Lemma is the following result.

Lemma 2.4. *Let $\varphi(y)$ be a second order polynomial in $y = (t, x) \in \mathbb{R} \times \mathbb{R}^n$. If $\chi(s) \in G_0^{1/\alpha_1}(\mathbb{R})$, for $\alpha_1 \in (0, 1)$, then $e^{\tau\varphi(y)}\chi(\varphi(y)) \in G^{1/\alpha_1}(\mathbb{R}^{n+1})$. If supp$(\chi) = [-8\delta, \delta]$ and $b((y - y_0)/(2R)) \in G_0^{1/\alpha_1}(\mathbb{R}^{n+1})$ is a cut-off function, then there are constants c_{122}, c_{123}, such that*

$$|\mathcal{F}_{t\to\xi_0}[e^{\tau\varphi(y)}\chi(\varphi(y))b((y - y_0)/(2R))]| \leq c_{122}e^{\tau\delta - c_{123}|\xi_0|^{\alpha_1}} .$$

Proof. By assumption both $\varphi(y)$ and $e^{\tau s}$ are analytic functions (i.e. in G^1) while $\chi \in G_0^{1/\alpha_1}(\mathbb{R})$. Since $G^1 \subset G^{1/\alpha_1}$ and they are both rings by Proposition 8.4.1 of [7], we deduce that the product $e^{\tau s}\chi(s)$ is in $G^{1/\alpha_1}(\mathbb{R})$. Moreover this product has compact support, since χ is compact supported. Let us write χ as $\chi(s) = \chi_1(s/\delta)$ where χ_1 has the properties in Definition 4.3, with associated coefficient c_{1X}. By assumption we have, for $z \in \mathbb{C}$, E =supp(χ)=$[-8\delta, \delta]$, $c_{119} = \delta c_{1X}(\alpha_1)$, $B = \delta^{\alpha_1}c_{1X}(\alpha_1)$, and H_E as in Definition 4.1,

$$|(\mathcal{F}_{s\to z}\chi(s))| \leq c_{119}\exp(H_E(Im\, z) - B|Re\, z|^{\alpha_1}).$$

Consequently, for $\xi \in \mathbb{R}$,

$$\mathcal{F}_{s\to\xi}\,(e^{\tau s}\chi(s)) = \mathcal{F}_{s\to\xi+i\tau}(\chi(s)),$$

and for $\tau > 0$

$$|\mathcal{F}_{s\to\xi}\,(e^{\tau s}\chi(s))| \leq c_{119}\exp(H_E(\tau) - B|\xi|^{\alpha_1}) = c_{119}\exp(\delta\tau - B|\xi|^{\alpha_1}).$$

Hence we can estimate the derivatives:

$$\begin{aligned}
&|\partial_s^k e^{\tau s}\chi(s)| = |\int_{\mathbb{R}} e^{i\xi s}(i\xi)^k(\mathcal{F}_{s'\to\xi}\,e^{\tau s'}\chi(s'))(\xi)d\xi| \\
&\leq \int_{\mathbb{R}} c_{119}|\xi|^k\exp(\delta\tau - B|\xi|^{\alpha_1})d\xi = 2\pi c_{119}e^{\tau\delta}B^{-\frac{(k+1)}{\alpha_1}}\frac{1}{\alpha_1}\Gamma(\frac{k+1}{\alpha_1}) \\
&\leq \frac{2\pi c_{119}}{\alpha_1}e^{\tau\delta}\Gamma(2)\max_k\{B^{-\frac{(k+1)}{\alpha_1}}(\frac{k+1}{\alpha_1} - 1)^{\frac{(1-\alpha_1)}{\alpha_1}}\}\Big(\frac{k+1}{\alpha_1} - 1\Big)^{\frac{k}{\alpha_1}} \leq c_{121}^{k+1}e^{\tau\delta}k^{k/\alpha_1},
\end{aligned}$$

where $c_{120} = \big(\frac{(1-\alpha_1)}{\alpha_1\ln B}\big)^{\frac{(1-\alpha_1)}{\alpha_1}}B^{-\frac{(1-\alpha_1)}{\alpha_1\ln B}+1}$, $c_{121} = \frac{2\pi c_{119}}{\alpha_1}\Gamma(2)\big(\frac{2}{\alpha_1}\big)^{\alpha_1}c_{120}$ and

$$\Gamma\Big(\frac{k+1}{\alpha_1}\Big) = \Big(\frac{k+1}{\alpha_1} - 1\Big)\cdots\Big(\frac{k+1}{\alpha_1} - p\Big)\Gamma\Big(\frac{k+1}{\alpha_1} - p\Big) \leq \Gamma(2)\Big(\frac{k+1}{\alpha_1} - 1\Big)^{\frac{k+1}{\alpha_1}-1}$$

with $p = [\frac{k+1}{\alpha_1}]-1$. We now recall that the composition of a Gevrey function with an analytic map is still a Gevrey function, therefore we get $e^{\tau\varphi}\chi(\varphi(y)) \in G^{1/\alpha_1}$. Since $\varphi(y)$ is a second order polynomial, for $k = 0, 1, 2$ we have $|\partial_t^k\varphi(y)|_{C^0(B_R)} \leq c_{118}(R)$ and, without restriction of generality we take $c_{118} \geq 1$. Considering the composition with φ we obtain by induction, calling $m(s) = e^{\tau s}\chi(s)$, for any $k \geq 0$,

$$\partial_t^k m(\varphi(y)) = \sum_{r\in J}\frac{k!}{(2r-k)!(k-r)!}(\partial_s^r m)_{s=\varphi(y)}(\partial_t\varphi)^{2r-k}\Big(\frac{\partial_t^2\varphi}{2}\Big)^{k-r},$$

where $J = \{r \geq 0 : \ 2r \geq k \geq r\}$, and

$$|\partial_t^k e^{\tau\varphi(y)}\chi(\varphi(y))| \leq c_{121}^{k+1} e^{\tau\delta} \sum_{r\in J} \frac{k!}{2^{k-r}(2r-k)!(k-r)!} c_{118}^{r+1} r^{r/\alpha_1}$$

$$\leq c_{121}^{k+1} e^{\tau\delta} c_{118}^{k+1} \sum_{r\in J} \left(\begin{array}{c} k \\ r \end{array} \right) \frac{r!}{2^{k-r}(2r-k)!} r^{r/\alpha_1} \leq (e^{\tau\delta} c_{121}^{k+1} c_{118}^{k+1}) 2^k k^{k/\alpha_1} ,$$

with $\frac{r!}{(2r-k)!} r^{r/\alpha_1} \leq k^{k/\alpha_1}$, for r admissible. For the product we get, applying (4.1) and calling $c_{122} = \max\{4c_{118}c_{121}, c_{1X}/R\}$,

$$\partial_t^k b((y-y_0)/(2R)) \leq R^{-k} c_{1X}^{1+k} k^{k/\alpha_1},$$
$$\partial_t^k [m(\varphi(y)) b((y-y_0)/(2R))] \leq e^{\tau\delta} c_{122}^{2+k} k^{k/\alpha_1}.$$

Consider the partial Fourier transform $\mathcal{F}_{t\to\xi_0}$ in time of $\partial_t^k(e^{\tau\varphi}\chi(\varphi)b((y-y_0)/(2R))$; from the estimate above it follows that

$$|\xi_0|^k |\mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)b((y-y_0)/(2R)))| \leq e^{\tau\delta} c_{122}^{k+1} \, k^{k/\alpha_1} .$$

This implies that

$$|\mathcal{F}_{t\to\xi_0}[e^{\tau\varphi}\chi(\varphi)]| \leq e^{\tau\delta} c_{122}^{k+1} \frac{k^{k/\alpha_1}}{|\xi_0|^k} \leq c_{122} e^{\tau\delta - k} \leq c_{122} e^{\tau\delta - c_{123}|\xi_0|^{\alpha_1}+1}$$

where for each ξ_0 we choose k as the largest integer such that $c_{122}|\xi_0|^{-1}k^{1/\alpha_1} < e^{-1}$. Since $k > [e^{-1}c_{122}^{-1}|\xi_0|]^{\alpha_1} - 1$, we get the result once we choose $c_{123} = (ec_{122})^{-\alpha_1}$. □

In Theorem 2.1 we referred to the radius R, that is defined in Table (3.10) as $R := qR_2$ with $q = \frac{1}{4}\left(16 + \frac{1}{16}\right)^{-1/2}$, and where R_2 in the same table is computed in terms of the pseudoconvexity constants introduced in subsection 3.1. By using those quantities one can introduce the geometric construction of figure 2. Let $f(y)$ be the second order polynomial defined in Theorem 2.1, with $\phi = e^{\lambda\psi}$ and $y_0 \in \Omega$. Recall that $f(y_0) = \phi(y_0)$.

Proposition 2.5. *Let δ be a positive constant such that*

$$0 < \delta \leq n|\phi''|_{C^{0,\rho}} q^2 R_2^{2+\rho}/8 \,, \tag{2.1}$$

and

$$\varphi(y) := f(y) - f(y_0) = \sum_{0<|v|\leq 2} (\partial^v \phi)(y_0)\,(y-y_0)^v/v! - \sigma|y-y_0|^2 \,. \tag{2.2}$$

Then, $\{y \in B(y_0, R_2);\ \phi \leq \phi(y_0)\} \cap \{y \in B(y_0, R_2);\ \varphi(y) \geq -8\delta\} \subset B(y_0, R)$. In addition, let r be a positive constant such that

$$r \leq \frac{n|\phi''|_{C^{0,\rho}(B_{R_2})} q^2 R_2^{2+\rho}}{2|\phi'|_{C^0(B_{R_2})} + 10n|\phi''|_{C^{0,\rho}(B_{R_2})} R_2^{1+\rho}}. \tag{2.3}$$

Then the ball $B(y_0, 2r) \subset \{y;\ |\varphi(y)| \leq \delta\}$.

We postpone the proof of the proposition till the end of section 3.

In the following Lemma we show how an exponential decay for the L^2-norm of a proper localization of Pu is transmitted to the right hand side of Tataru inequality.

Lemma 2.6. *Under the Assumptions A1, let $y_0 \in \Omega$ and φ be the quadratic polynomial* (2.2)*. Let $0 < \alpha,\, \alpha_1 < 1$ and $\chi(s) \in G_0^{1/\alpha_1}(\mathbb{R})$ be a localizer supported in $[-8\delta, \delta]$ and equal 1 in $[-7\delta, \delta/2]$. Let $\mu > 0$, $\delta > 0$, be given constants, $b \in C_0^\infty(\mathbb{R}^{n+1})$ and $a \in C_0^\infty(\mathbb{R})$. Let $A(D_0)$ be a pseudodifferential operator with symbol a and $\mu_* = \min\{\mu^\alpha, \mu^{\alpha_1}\}$. If*

$$\|u\|_{H^1(B_{2R})} = 1, \quad \|Pu\|_{L^2(B_{2R})} < 1, \quad \|A(D_0/\mu)b((y-y_0)/R)Pu\|_0 \le e^{-\mu^\alpha},$$

then for each $\tau \ge 0$, there are constants c_{110}, c_{109} such that

$$\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi}\chi(\varphi)P(y,D)u\|_0 \le c_{110} e^{2\tau\delta - c_{109}\mu_*}.$$

Proof. Define $a_{\mu/3}(s) := a(3s/\mu)$, hence $\operatorname{supp}(1 - a_{\mu/3}(\xi_0)) \subset \{|\xi_0| \ge \mu/3\}$. Then, we get:

$$\begin{aligned}
&\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi}\chi(\varphi)P(y,D)u\|_0 = \|e^{-\epsilon\xi_0^2/(2\tau)}\mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)P(y,D)u)\|_0 \\
\le\ & \|(1 - a_{\mu/3}(\xi_0))e^{-\epsilon\xi_0^2/(2\tau)}\mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)P(y,D)u)\|_0 \\
&+ \|a_{\mu/3}(\xi_0)e^{-\epsilon\xi_0^2/(2\tau)}\mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)P(y,D)u)\|_0 \ =: I_1 + I_2\,.
\end{aligned}$$

By our construction we have that $b((y-y_0)/R) = 1$ on $\operatorname{supp}(\chi(\varphi)Pu)$, hence we can write $\chi(\varphi)P(y,D)u = \chi(\varphi)b((y-y_0)/R)P(y,D)u$.
The first integral can be estimated for $\tau < c_{127}\mu$ as follows, where $c_{127} = \sqrt{\epsilon/(36\,\delta)}$,

$$\begin{aligned}
I_1 \ \le\ & e^{-\epsilon\mu^2/(18\tau)}\|\mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)P(y,D)u)\|_0 \le e^{-\epsilon\mu^2/(18\tau)}\|e^{\tau\varphi}\chi(\varphi)P(y,D)u\|_0 \\
\le\ & e^{-\epsilon\mu^2/(18\tau)+\tau\delta}\|\chi(\varphi)P(y,D)u\|_0 \le e^{-c_{127}\delta\mu}\|b((y-y_0)/R)P(y,D)u\|_0,
\end{aligned}$$

where we have $-\epsilon\mu^2/(18\tau) + \tau\delta \le -c_{127}\delta\mu$. Notice that the estimate for I_1 holds only for $\tau < c_{127}\mu$. If instead $\tau \ge c_{127}\mu$ then

$$\begin{aligned}
\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi}\chi(\varphi)P(y,D)u\|_0 \ &\le\ e^{\delta\tau}\|\chi(\varphi)P(y,D)u\|_0 \\
&\le\ e^{2\delta\tau - c_{127}\delta\mu}\|b((y-y_0)/R)P(y,D)u\|_0
\end{aligned}$$

since $e^{\delta\tau} = e^{2\delta\tau - \delta\tau} \le e^{2\delta\tau - c_{127}\delta\mu}$. For the second integral we get

$$\begin{aligned}
I_2 =\ & \|e^{-\epsilon\xi_0^2/(2\tau)}a_{\mu/3}(\xi_0)\mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)b((y-y_0)/R)P(y,D)u)\|_0 \\
\le\ & \|A_{\mu/3}(D_0)e^{\tau\varphi}\chi(\varphi)b((y-y_0)/R)P(y,D)u\|_0 \\
\le\ & \|A_{\mu/3}(D_0)e^{\tau\varphi}\chi(\varphi)b((y-y_0)/(2R))A_\mu(D_0)b((y-y_0)/R)P(y,D)u\|_0 + \\
& \|A_{\mu/3}(D_0)e^{\tau\varphi}\chi(\varphi)b((y-y_0)/(2R))(1 - A_\mu(D_0))b((y-y_0)/R)P(y,D)u\|_0 \\
=:\ & I_3 + I_4\,.
\end{aligned}$$

To estimate I_3 we apply the assumption and we obtain:

$$\begin{aligned}
&\|A(3D_0/\mu)e^{\tau\varphi}\chi(\varphi)b((y-y_0)/(2R))A(D_0/\mu)b((y-y_0)/R)Pu\|_0 \\
&\le e^{\tau\delta}\|A(D_0/\mu)b((y-y_0)/R)Pu\|_0 \le e^{\tau\delta - \mu^\alpha}.
\end{aligned}$$

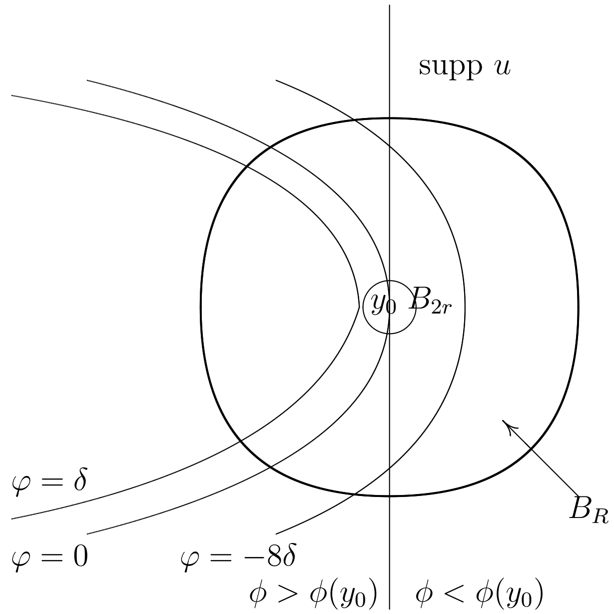
To estimate I_4 we apply Lemma 2.3 and Lemma 2.4. By the estimates for $f(y)$ at its derivatives in Step 3 of section 3.1 we deduce for $k = 0, 1, 2$

$$|\partial_t^k\varphi(y)| \le c_{118}(\phi) := 1 + |\phi'|_0(1+R_2) + 5n|\phi''|_{0,\rho}R_2^{\rho+1} + |\phi''|_0(1+R_2^2) + \sigma(2+R_2^2).$$

Lemma 2.4 and the properties of $e^{\tau\varphi}\chi(\varphi)$ imply that

$$|\mathcal{F}_{t'\to(\xi_0^1-\xi_0)}[e^{\tau\varphi}\chi(\varphi)(t',x)b((y'-y_0)/(2R))]| \le c_{122}e^{\tau\delta}e^{-c_{123}\mu^{\alpha_1}/(2\,3^{\alpha_1})}e^{-c_{123}|\xi_0^1-\xi_0|^{\alpha_1}/2},$$

FIGURE 2. Geometric construction around y_0



since $|\xi_0^1 - \xi_0| \geq \mu - 2\mu/3 = \mu/3$ on $\mathrm{supp}[(1 - a(\xi_0/\mu))a(3\xi_0^1/\mu)]$.
To estimate I_4 we then apply Lemma 2.3.a) and using the fact that $f = e^{\tau\varphi}\chi(\varphi)b((y - y_0)/(2R))$ and recomputing the constants, we get

$$\begin{aligned}\|A(3D_0/\mu)e^{\tau\varphi}\chi(\varphi)b((y-y_0)/(2R))(1 - A(D_0/\mu))b((y-y_0)/R)Pu\|_0^2 \\ \leq c_{110}^2 e^{2\tau\delta - c_{128}\mu^{\alpha_1}}\|b((y-y_0)/R)Pu\|_0^2\,,\end{aligned}$$

with $c_{128} = \frac{1}{3^{\alpha_1}2}c_{123}$ and $c_{110} = \left(c_{122}(8/3)\Gamma(1/\alpha_1)/[\alpha_1 c_{123}^{1/\alpha_1}(\alpha_1 c_{128})^{1/(\alpha_1-1)}]\right)^{1/2}$.
Calling $c_{109} = \min(\sqrt{\epsilon\,\delta/36}, c_{128}/2, 1)$ we finally get the result. $\square$

We now prove Theorem 1.1, stating an estimate of inverse exponential type for the temporal frequencies $|\xi_0| \leq 2\omega$.

Proof of Theorem 1.1.
If $y_0 \in S$ is as in the Assumption A1, then by Theorem 2.1 there exists $\lambda > 1$ such that $\phi(y) = \exp(\lambda\psi)$ is a conormally strongly pseudoconvex function with respect to P in Ω. Then we introduce the function φ defined in (2.2) as the second order polynomial approximation of the conormally pseudoconvex function $\phi - \phi(y_0)$ around y_0, translated by $-\sigma|y - y_0|^2$.
In Table (3.10) we found σ independent of y_0 so that φ also satisfies the conormally pseudoconvexity condition w.r.t. P in the ball $B(y_0, R_2)$. In Proposition 2.5 we also computed a δ independent of y_0 so that

$$\{y;\, \phi \leq \phi(y_0)\} \cap \{y;\, \varphi > -8\delta\} \subset B(y_0, R).$$

Given δ, we found $r > 0$ so that $B(y_0, 2r) \subset \{y;\, |\varphi(y)| \leq \delta\}$.
Let $\chi_1 \in G_0^{1/\alpha_1}(\mathbb{R})$ be a smooth cutoff function which is 0 on $(-\infty, -8] \cup [1, \infty)$, 1 in $[-7, 1/2]$ and $0 \leq \chi_1 \leq 1$. Define its scaled version $\chi(s) := \chi_1(s/\delta)$. Then,

$$P\chi(\varphi)u = \chi(\varphi)Pu + [P, \chi(\varphi)]u$$

where, since u is supported in $\{y;\, \phi \le \phi(y_0)\}$, it follows from Proposition 2.5 that

$$\operatorname{supp}(\chi(\varphi)u) \subset \{y;\, \phi(y) \le \phi(y_0)\} \cap \{y;\, -8\delta < \varphi(y) < \delta\} \cap \{y;\, |y - y_0| \le R\},$$

and $[P, \chi(\varphi)]$ is a partial differential operator of order one and satisfies $\operatorname{supp}([P, \chi(\varphi)]u(y)) \subset \{y;\, -8\delta < \varphi(y) < -7\delta\}$.
We now apply the estimate of Theorem 2.1 to χu to obtain, for all $\tau > \tau_0$,

$$\tau \|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi}\chi(\varphi)u\|^2_{1,\tau} \le c^2_{1,T}\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi}\chi(\varphi)Pu\|^2_0$$
$$+ c^2_{1,T}\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi}[P, \chi(\varphi)]u\|^2_0 + c^2_{2,T}\tau\|e^{\tau(\varphi-d)}\chi(\varphi)u\|^2_{1,\tau},$$

where $d = R_2^2/(4\epsilon)$. We refer to Table (3.10) for all the involved parameters. According to our construction, δ is chosen such that $d > 8\delta$. To estimate the first term at the right hand side we apply Lemma 2.6.
The second term can be bounded by

$$c^2_{1,T}\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi}[P, \chi(\varphi)]u\|^2_0 \le c_{114}e^{-14\tau\delta}\|u\|^2_{H^1(B_{2R})},$$

with $c_{114} = c^2_{1,T}|g|^2_{C^1}|\chi_1|^2_{C^2}(1+|\varphi'|^4_{C^0}/\delta^4+|\varphi''|^2_{C^0}/\delta^2)$, since $\chi'(\varphi) = \chi_1'(\varphi/\delta)(\varphi'/\delta)$ and $\chi''(\varphi) = \chi_1''(\varphi/\delta)(\varphi' \cdot \varphi')/\delta^2 + \chi_1'(\varphi/\delta)(\varphi''/\delta)$.
Applying $\|\chi(\varphi)u\|^2_1 \le (1 + |\chi_1'|^2_{C^0}/\delta^2)\|u\|^2_{H^1(B_{2R})}$, the third term is such that

$$c^2_{2,T}\tau\|e^{\tau(\varphi-d)}\chi(\varphi)u\|^2_{1,\tau} \le c^2_{2,T}(|\varphi'|^2_{C^0} + 1)\tau^3 e^{-14\tau\delta}\|\chi(\varphi)u\|^2_1 \le c_{115}\, e^{-13\tau\delta},$$

with $c_{115} = c^2_{2,T}(|\varphi'|^2_{C^0} + 1)(3^3e^{-3}/\delta^3)(1 + |\chi_1'|^2_{C^0}/\delta^2)$. Since $\tau_0 \ge 1$ so that $(1 + \tau_0)/2 \le \tau$, we get

$$\text{(2.4)}\quad \frac{(1+\tau_0)}{2}\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi}\chi(\varphi)u\|^2_{1,\tau} \le c_{116}e^{4\delta\tau}(e^{-2c_{109}\mu^\alpha} + e^{-16\delta\tau}),\ \forall\tau > \tau_0,$$

where $c_{116} := 3\max(c^2_{1,T}c^2_{110}, c_{114}, c_{115})$. We want to extend the previous estimate to the complex upper half-space. Define, for $\tau \ge 0$

$$N(\tau) := \frac{1}{2}(1 + \tau_0)\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi}\chi(\varphi)u\|^2_{1,\tau},$$

and we get

$$N(\tau) = \frac{1}{2}(1 + \tau_0)\|\sqrt{|\xi|^2 + \tau^2}\mathcal{F}_{t\to\xi_0}\mathcal{F}_{x\to\tilde\xi}[\mathcal{F}^{-1}_{\xi_0\to t}e^{-\epsilon\xi_0^2/(2\tau)}\mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)u)]\|^2_0$$
$$= \frac{1}{2}(1 + \tau_0)\|\sqrt{|\xi|^2 + \tau^2}e^{-\epsilon\xi_0^2/(2\tau)}\mathcal{F}_{t\to\xi_0}\mathcal{F}_{x\to\tilde\xi}(e^{\tau\varphi}\chi(\varphi)u)\|^2_0$$
$$= \frac{(1+\tau_0)}{2}\int_{\mathbb{R}^{n+1}} d\tilde\xi d\xi_0(|\xi|^2 + \tau^2)e^{-\epsilon\xi_0^2/(2\tau)}\mathcal{F}_{y\to\xi}(e^{\tau\varphi}\chi(\varphi)u)\overline{e^{-\epsilon\xi_0^2/(2\tau)}\mathcal{F}_{y\to\xi}(e^{\tau\varphi}\chi(\varphi)u)}.$$

We first extend the estimate (2.4) to the case $0 \le \tau \le \tau_0$. Define first $c_{112} = (1 + |\varphi'|^2_{C^0})$ and

$$\text{(2.5)}\qquad c_{113} = \max\{c_{116}, c_{112}(1 + \tau_0^3)(1 + |\chi_1'|^2_{C^0}/\delta^2)e^{12\delta\tau_0}\}.$$

Using that $\varphi \le 0$ on $\operatorname{supp}(\chi u)$, we have

$$\text{(2.6)}\quad N(\tau) \le \frac{1}{2}(1 + \tau_0)\int_{\mathbb{R}^{n+1}} d\tilde\xi d\xi_0\Big(|\xi\mathcal{F}_{y\to\xi}(e^{\tau\varphi}\chi(\varphi)u)|^2 + \tau^2|\mathcal{F}_{y\to\xi}(e^{\tau\varphi}\chi(\varphi)u)|^2\Big)$$
$$\le \frac{1}{2}(1 + \tau_0)\int_{\mathbb{R}^{n+1}} dxdt\Big(|\nabla_y(e^{\tau\varphi(y)}\chi(\varphi)u)|^2 + \tau_0^2|e^{\tau\varphi}\chi(\varphi)u|^2\Big)$$
$$\le c_{112}(1 + \tau_0^3)\|\chi(\varphi)u\|^2_1 \le c_{113}e^{-12\delta\tau_0} \le c_{113}e^{4\delta\tau}(e^{-2c_{109}\mu^\alpha} + e^{-16\delta\tau})\,.$$

We now consider $z \in \mathbb{C}$ with $Im(z) > 0$ and rewrite the previous expression in the complex half-space by replacing τ with $-iz$:

$$
\begin{aligned}
(2.7)\qquad & N(-iz) := \\
& \frac{1+\tau_0}{2}\int_{\mathbb{R}^{n+1}} d\xi(|\xi|^2+|z|^2)e^{\frac{\epsilon\xi_0^2}{i2z}}\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)\overline{e^{\frac{\epsilon\xi_0^2}{i2z}}\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)} \\
&= \frac{1}{2}(1+\tau_0)\int_{\mathbb{R}^{n+1}} d\xi e^{-\frac{\epsilon\xi_0^2 Im\, z}{|z|^2}}(|\xi\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)|^2+|z\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)|^2) \\
&\le \frac{1}{2}(1+\tau_0)\int_{\mathbb{R}^{n+1}} d\xi(|\xi\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)|^2+|z\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)|^2) \\
&= \frac{1}{2}(1+\tau_0)\int_{\mathbb{R}^{n+1}} dy\Big(|\nabla_y(e^{-iz\varphi}\chi(\varphi)u)|^2+|z(e^{-iz\varphi}\chi(\varphi)u)|^2\Big) \\
&= \frac{1}{2}(1+\tau_0)\int_{\mathbb{R}^{n+1}} dy\Big(|e^{-iz\varphi}[\nabla_y(-iz\varphi)\,(\chi(\varphi)u)+\nabla_y(\chi(\varphi)u))]|^2+ \\
&\qquad\qquad +|z|^2|e^{-iz\varphi}\chi(\varphi)u)|^2\Big) \\
&\le \frac{1}{2}(1+\tau_0)c_{112}\Big((1+|z|^2)\|\chi(\varphi)u\|_0^2+\|\nabla_y\chi(\varphi)u\|_0^2\Big) \\
&\le \frac{1}{2}(1+\tau_0)c_{112}\,(1+|z|^2)\|\chi(\varphi)u\|_1^2 \le c_{113}(1+|z|^2)\,.
\end{aligned}
$$

In the following we want to apply the properties of subharmonic functions. We notice that the function $U(y,z) := e^{-\epsilon|D_0|^2/(-2iz)}e^{-iz\varphi}\chi(\varphi)u(y)$ is analytic in z such that $Im\, z > 0$, and that $N(-iz)$ is subharmonic in z as integral in one parameter of the sum of two squares of the absolute values of analytic functions. Our aim is now to estimate the H^1 norm of $A(D_0/\omega)b((y-y_0)/R)u)(y)$ where $\omega = \mu^\alpha/\beta$, for some $\beta > 0$ to be determined.
Let $\eta(s) := \eta_1(s/\delta)$, for η_1 of Gevrey class $1/\alpha_1$ with support in $[-4,1]$ and equal to one in $[-3,1/2]$. Call $\widetilde{\mu} = \mu^\alpha$ and $\widehat{\eta} = \mathcal{F}_{s\to z}\eta$ to shorten the notation. First define F as

$$F(y) := A(\beta D_0/\widetilde{\mu})(\eta(\varphi)u)(y).$$

Due to the regularity of η we can write the following foliation with respect to the level sets of φ:

$$\eta(\varphi)(y') = \int_{\mathbb{R}} \eta(s)\,\delta(s-\varphi(y'))\,ds = \int_{\mathbb{R}} \overline{\widehat{\eta}}(z)e^{-iz\varphi(y')}\,dz\,.$$

We remind that, according to our construction, $\chi(\varphi) = 1$ on $\mathrm{supp}(\eta(\varphi)u)$, and then $\eta(\varphi)u = \eta(\varphi)\chi(\varphi)u$.
Consequently we rewrite F as:

$$F(y) = A(\beta D_0/\widetilde{\mu})(\eta(\varphi)\chi(\varphi)u)(y) = \int_{\mathbb{R}} \overline{\widehat{\eta}}(\bar{z})(A(\beta D_0/\widetilde{\mu})e^{-iz\varphi}\chi(\varphi)u)(y)\,dz\,.$$

We remind that $A(\beta D_0/\widetilde{\mu})$ is an integral operator with kernel

$$k(t,t') = \frac{\widetilde{\mu}}{\beta}\widehat{a}\Big(\frac{\widetilde{\mu}}{\beta}(t'-t)\Big).$$

Hence the previous equality is justified by Fubini's theorem, because for $y' = (t',x)$ the integrand $|\overline{\widehat{\eta}}(\bar{z})k(t,t')e^{-iz\varphi(y')}\chi(\varphi(y'))u(y')|$ is bounded by the function $ce^{-|z|^{\alpha_1}}e^{-|t-t'|^\alpha}u(t',x) \in L^1(\mathbb{R}_z \times \mathbb{R}_{t'})$.

Since $\eta \in C_0^\infty$, then the Fourier-Laplace transform $\widehat{\eta}(z)$ is holomorphic for $z \in \mathbb{C}$, and hence $\overline{\widehat{\eta}}(\bar{z})$ is also holomorphic. We then need a good estimate for both $\overline{\widehat{\eta}}(\bar{z})$ and $A(\beta D_0/\widetilde{\mu})\,(e^{-iz\varphi}\chi(\varphi)u(y))$ in the upper half plane.
From the Gevrey class condition, we compute:

$$|\widehat{\eta}(z)| = |\delta\widehat{\eta}_1(\delta z)| \leq \delta c_{101}\exp(\delta \sup_{w\in supp(\eta_1)} \langle w, Im\, z\rangle - c_{102}\delta^{\alpha_1}|Re\, z|^{\alpha_1})\,.$$

By considering the domain $Im\bar{z} = -Im\, z < 0$, we have

$$\begin{aligned} |\overline{\widehat{\eta}}(\bar{z})| &\leq \delta c_{101}\exp(\delta \sup_{w\in[-4,1]} \langle w, Im\bar{z}\rangle - c_{102}\delta^{\alpha_1}|Re\bar{z}|^{\alpha_1}) \\ &\leq \delta c_{101}\exp(-4\delta Im\bar{z} - c_{102}\delta^{\alpha_1}|Re\, z|^{\alpha_1})\,, \end{aligned}$$

where $c_{101} = c_{101}(\alpha_1)$ is a given constant, $c_{102} = c_{102}(\alpha_1, c_{101})$.
We now change path of integration in the upper half plane $Im\, z > 0$:

$$F(y) = \int_{\Gamma_1\cup\Gamma_2} \overline{\widehat{\eta}}(\bar{z})A(\beta D_0/\widetilde{\mu})\,(e^{-iz\varphi}\chi(\varphi)u(y))\,dz$$

with $\Gamma_1 = \{z \in \mathbb{R} : |z| \geq \frac{1}{\sqrt{2}}c_{130}\widetilde{\mu}\}$ and Γ_2 the open rectangle inside the ball $|z| \leq c_{130}\widetilde{\mu}$ defined as $\Gamma_2 = \{z \in \mathbb{C} : Re\, z = -\frac{1}{\sqrt{2}}c_{130}\widetilde{\mu},\ 0 \leq Im\, z \leq \frac{1}{\sqrt{2}}c_{130}\widetilde{\mu}\} \cup \{z \in \mathbb{C} : |Re\, z| \leq \frac{1}{\sqrt{2}}c_{130}\widetilde{\mu},\ Im\, z = \frac{1}{\sqrt{2}}c_{130}\widetilde{\mu}\} \cup \{z \in \mathbb{C} : Re\, z = \frac{1}{\sqrt{2}}c_{130}\widetilde{\mu},\ 0 \leq Im\, z \leq \frac{1}{\sqrt{2}}c_{130}\widetilde{\mu}\}$.
Hence,

$$\begin{aligned} \|F\|_{H^1} \leq\ & \int_{\Gamma_1} |\overline{\widehat{\eta}}(\bar{z})|\|A(\beta D_0/\widetilde{\mu})\,(e^{-iz\varphi}\chi(\varphi)u(y))\|_{H^1}\,|dz| \\ &+ \int_{\Gamma_2} |\overline{\widehat{\eta}}(\bar{z})|\|A(\beta D_0/\widetilde{\mu})\,(e^{-iz\varphi}\chi(\varphi)u(y))\|_{H^1}\,|dz| := I_{\Gamma_1} + I_{\Gamma_2}. \end{aligned}$$

Along Γ_1, with $z = Re\, z$, we have both $|\widehat{\eta}(\bar{z})| \leq \delta c_{101}\exp(-c_{102}\delta^{\alpha_1}|z|^{\alpha_1})$, and

$$\begin{aligned} \|A(\beta D_0/\widetilde{\mu})e^{-iz\varphi}\chi(\varphi)u(y)\|_{H^1}^2 &\leq \|e^{-iz\varphi}\chi(\varphi)u(y)\|_{H^1}^2 \\ &\leq 2(|z|^2|\varphi'|_{C^0}^2 + 1)\|\chi(\varphi)u\|_{H^1}^2 \\ &\leq (|z|^2+1)c_{113}\,. \end{aligned}$$

The final estimate for I_{Γ_1} is

$$\begin{aligned} I_{\Gamma_1} &\leq 2\delta c_{101}\sqrt{c_{113}}\int_{\frac{1}{\sqrt{2}}c_{130}\widetilde{\mu}}^{+\infty} \sqrt{s^2+1}e^{-c_{102}\delta^{\alpha_1}s^{\alpha_1}}ds \\ &\leq 2\delta c_{101}\sqrt{c_{113}}e^{-c_{102}\delta^{\alpha_1}(\frac{1}{2\sqrt{2}}c_{130})^{\alpha_1}\widetilde{\mu}^{\alpha_1}} \int_{\mathbb{R}} \sqrt{s^2+1}\,e^{-c_{102}\delta^{\alpha_1}s^{\alpha_1}/2}ds \\ &\leq 2c_{101}\sqrt{c_{113}}e^{-c_{102}\delta^{\alpha_1}(\frac{1}{2\sqrt{2}}c_{130})^{\alpha_1}\widetilde{\mu}^{\alpha_1}} \int_{\mathbb{R}} \sqrt{(s/\delta)^2+1}\,e^{-c_{102}s^{\alpha_1}/2}ds\,. \end{aligned}$$

In I_{Γ_2} we multiply and divide by the invertible operator $e^{\epsilon|D_0|^2/(2iz)}$

$$\begin{aligned} I_{\Gamma_2} &= \int_{\Gamma_2} |\overline{\widehat{\eta}}(\bar{z})|\|A(\beta D_0/\widetilde{\mu})e^{-\epsilon|D_0|^2/(2iz)}e^{\epsilon|D_0|^2/(2iz)}e^{-iz\varphi}u(y)\|_{H^1}|dz| \\ &\leq \delta c_{101}\int_{\Gamma_2} e^{4\delta Im\, z - c_{102}\delta^{\alpha_1}|Re\, z|^{\alpha_1}}\|A(\beta D_0/\widetilde{\mu})e^{-\epsilon|D_0|^2/(2iz)}\|_{\mathcal{B}(H^1)}\cdot \\ &\qquad \cdot\|e^{-\epsilon|D_0|^2/(-2iz)}e^{-iz\varphi}\chi(\varphi)u(y)\|_{H^1}|dz|. \end{aligned}$$

In the region $\Gamma_2 \subset \{z : c_{130}\widetilde{\mu}/\sqrt{2} \le |z| \le c_{130}\widetilde{\mu}\}$ the norm in $\mathcal{B}(H^1)$ can be estimated, independently of $\widetilde{\mu}$, via the Fourier symbol of the product

$$|a(\beta\xi_0/\widetilde{\mu})e^{-\epsilon\xi_0^2/(2iz)}| = |a(\beta\xi_0/\widetilde{\mu})e^{\frac{\epsilon\xi_0^2 Im\, z}{2|z|^2}}| \le \exp\Big(\frac{\epsilon(2\widetilde{\mu})^2 Im\, z}{2\beta^2|c_{130}\widetilde{\mu}/\sqrt{2}|^2}\Big) = \exp\Big(\frac{4\epsilon Im\, z}{\beta^2 c_{130}^2}\Big);$$

while the latter H^1-norm is related to the estimate (2.11) for $N(-iz)$

$$\|e^{-\epsilon|D_0|^2/(-2iz)}e^{-iz\varphi}\chi(\varphi)u(y)\|_{H^1}^2 \le \frac{N(-iz)}{\min\{1,\, \widetilde{\mu}^2c_{130}^2/2\}} \le \frac{2c_{113}(1+|z|^2)}{\min\{1,\, c_{130}^2/2\}}e^{-10\delta Im\, z}$$

where we use $(|\xi|^2+1) \le \frac{1}{\min\{1,\, \widetilde{\mu}^2c_{130}^2/2\}}(|\xi|^2+|z|^2)$ in the first inequality, and and $\widetilde{\mu}\ge 1$ in the second. Hence,

$$\begin{aligned} I_{\Gamma_2} &\le \delta c_{101}\Big(\frac{2c_{113}(1+\widetilde{\mu}^2c_{130}^2)}{\min\{1,\, c_{130}^2/2\}}\Big)^{1/2}\int_{\Gamma_2} e^{4\delta Im\, z - c_{102}\delta^{\alpha_1}|Re\, z|^{\alpha_1}}e^{\frac{4\epsilon Im\, z}{\beta^2c_{130}^2}}e^{-5\delta Im\, z}|dz| \\ &\le \delta c_{101}\Big(\frac{2c_{113}(1+\widetilde{\mu}^2c_{130}^2)}{\min\{1,\, c_{130}^2/2\}}\Big)^{1/2}\int_{\Gamma_2} e^{-c_{102}\delta^{\alpha_1}|Re\, z|^{\alpha_1}}e^{-\delta Im\, z/2}|dz| \end{aligned}$$

where we choose ϵ and β such that $\epsilon \le \delta\beta^2c_{130}^2/8$. Actually by our choice of c_{130} the inequality can be written as

$$\epsilon \le \frac{9\beta^2}{2^{47}\delta}\min\Big(\frac{\epsilon\delta}{36}, \frac{c_{123}^2}{4(3)^{2\alpha^1}}, 1\Big).$$

The latter relation is satisfied for any $\epsilon \le \epsilon_0$ and $\beta \ge c_{131}$, where $c_{131} = \max\{\sqrt{2}(16)^6,\, (\sqrt{2}(16)^6 3^{(\alpha_1-1)}\sqrt{\epsilon_0\delta})/c_{123},\, ((16)^6\sqrt{\epsilon_0\delta})/(3\sqrt{2})\}$, with ϵ_0 computed in Table 3.10.
Denoting $z = x'+iy'$ we conclude the estimate

$$\begin{aligned} &\delta\int_{\Gamma_2} e^{-c_{102}\delta^{\alpha_1}|Re\, z|^{\alpha_1}-\delta Im\, z/2}|dz| \\ &\le 2\delta\int_0^{\frac{c_{130}\widetilde{\mu}}{\sqrt{2}}} e^{-c_{102}\delta^{\alpha_1}\frac{(c_{130}\widetilde{\mu})^{\alpha_1}}{\sqrt{2}^{\alpha_1}}}e^{-\delta y'/2}dy' + \delta\int_{-\frac{c_{130}\widetilde{\mu}}{\sqrt{2}}}^{\frac{c_{130}\widetilde{\mu}}{\sqrt{2}}} e^{-c_{102}\delta^{\alpha_1}|x'|^{\alpha_1}}e^{-\delta\frac{c_{130}\widetilde{\mu}}{2\sqrt{2}}}dx' \\ &\le 2\delta e^{-c_{102}\delta^{\alpha_1}\frac{(c_{130})^{\alpha_1}}{(\sqrt{2})^{\alpha_1}}\widetilde{\mu}^{\alpha_1}}\int_0^{+\infty} e^{-\delta y'/2}dy' + \delta e^{-\delta\frac{c_{130}}{2\sqrt{2}}\widetilde{\mu}}\int_{\mathbb{R}} e^{-c_{102}\delta^{\alpha_1}|x'|^{\alpha_1}}dx' \\ &\le 2e^{-c_{102}\delta^{\alpha_1}\frac{(c_{130})^{\alpha_1}}{(\sqrt{2})^{\alpha_1}}\widetilde{\mu}^{\alpha_1}}\int_0^{+\infty} e^{-y'/2}dy' + e^{-\delta\frac{c_{130}}{2\sqrt{2}}\widetilde{\mu}}\int_{\mathbb{R}} e^{-c_{102}|x'|^{\alpha_1}}dx'. \end{aligned}$$

Comparing the estimates for I_{Γ_1} and I_{Γ_2}, recalling that $e^{-c\widetilde{\mu}} \le e^{-c\widetilde{\mu}^{\alpha_1}}$ and choosing the largest constants, we obtain the final estimate for $F(y)$

$$\|A(\beta D_0/\widetilde{\mu})(\eta(\varphi)\chi(\varphi)u)(y)\|_{H^1} \le c_{136}e^{-c_{137}\widetilde{\mu}^{\alpha_1}}, \tag{2.8}$$

with $c_{137} = \frac{1}{2}\big(c_{102}\delta^{\alpha_1}\frac{(c_{130})^{\alpha_1}}{(\sqrt{2})^{\alpha_1}} + \delta\frac{c_{130}}{2\sqrt{2}}\big) + \frac{1}{2}c_{102}\delta^{\alpha_1}(\frac{1}{2\sqrt{2}}c_{130})^{\alpha_1}$ and
$c_{136} = 2c_{101}\sqrt{c_{113}}\int_{\mathbb{R}}\sqrt{(s/\delta)^2+1}\, e^{-c_{102}s^{\alpha_1}/2}ds + c_{101}\Big(\frac{2c_{113}(1+c_{130}^2)}{\min\{1,\, c_{130}^2/2\}}\Big)^{\frac{1}{2}}\Big(2\int_0^{+\infty}e^{-y'/2}dy' + \int_{\mathbb{R}}e^{-c_{102}|x'|^{\alpha_1}}dx'\Big)$.
One can prove a similar estimate with $\eta(\varphi)$ replaced by $b((y-y_0)/r)$. By construction we have chosen r so that $\mathrm{supp}(b((y-y_0)/r))\cap\mathrm{supp}\, u \subset \{y;\, \eta(\varphi(y)) =$

$1)\} \cap \operatorname{supp} u$ and we write

$$A(\frac{3\beta D_0}{\widetilde{\mu}})b(\frac{y-y_0}{r})u(y) = A(\frac{3\beta D_0}{\widetilde{\mu}})b(\frac{y-y_0}{r})A(\frac{\beta D_0}{\widetilde{\mu}})\eta(\varphi)u(y)$$
$$+A(\frac{3\beta D_0}{\widetilde{\mu}})b(\frac{y-y_0}{r})[1-A(\frac{\beta D_0}{\widetilde{\mu}})]\eta(\varphi)u(y) := J_1 + J_2.$$

From (2.8), J_1 has the desired estimate

$$\|J_1\|_1 \leq c_{136}\Big(1+\frac{|b'|_{C^0}}{r}\Big)e^{-c_{137}\widetilde{\mu}^{\alpha_1}},$$

due to the fact that $A(\frac{3\beta D_0}{\widetilde{\mu}})b(\frac{y-y_0}{r})$ is a bounded operator. To estimate J_2 we apply Lemma 2.3.c) using the fact that $b \in G^{1/\alpha_1}(\mathbb{R}^{n+1})$:

$$\|J_2\|_1 \leq c_{134}e^{-c_{135}\widetilde{\mu}^{\alpha_1}},$$

where $c_{134} = \Big((rc_{1X})\frac{8}{3}\Gamma\Big(\frac{1}{\alpha_1}\Big)\frac{1}{\alpha_1(r^{\alpha_1}c_{2X})^{1/\alpha_1}(\alpha_1 c_{135})^{1/(\alpha_1-1)}}\Big)^{1/2}$, $c_{135} = r^{\alpha_1}c_{2X}\frac{1}{23^{\alpha^1}}$. This concludes the proof of Theorem 1.1 by choosing $c_{129} = \max\{c_{134}, c_{136}\}$, $c_{132} = \min\{c_{135}, c_{137}\}$.

□

Here we show in details the estimate for the function N applied in the proof of the previous Lemma.

Lemma 2.7. *Let us define* $N_1(-iz) := N(-iz)/|1-iz|^2$*, where* $N(-iz)$ *is defined in* (2.7). *For* $z \in \mathbb{C} \cap \{Im\, z \geq 0\}$, $N_1(-iz)$ *satisfies the inequalities*

$$(2.9)\quad N_1(-iz) \leq c_{113}e^{4\delta Im\, z}(e^{-2c_{109}\widetilde{\mu}} + e^{-16\delta Im\, z}),\ z \in \mathbb{R}\cup\{Re\, z = 0,\ Im\, z \geq 0\},$$
$$N_1(-iz) \leq c_{113}, \quad Im\, z > 0.$$

where c_{113} *is given in* (2.5) *and* c_{109} *is defined in Lemma 2.6. Therefore, there exists some constant* c_{130} *independent of* $\widetilde{\mu}$*, so that*

$$(2.10)\qquad N_1(-iz) \leq 2c_{113}e^{-10\delta Im\, z}, \quad |z| \leq c_{130}\widetilde{\mu},\ Im\, z \geq 0,$$

with $c_{130} = \dfrac{3c_{109}}{4\delta}\Big(\dfrac{1}{16}\Big)^5$.
Consequently, in the region $|z| \leq c_{130}\widetilde{\mu}$ *with* $Im\, z \geq 0$,

$$(2.11)\qquad N(-iz) \leq 2c_{113}(1+|z|^2)e^{-10\delta Im\, z}.$$

Proof. Since $N_1(-iz) \leq N(-iz)$, the estimates (2.9) for N_1 follow from the equivalent estimates for N proved in (2.4), (2.6) and (2.7). To show (2.10) we first consider $z = x' + iy'$ in the region $x' > 0$, $y' > 0$. Here we define the analytic function

$$h(z) = e^{2i\delta z}e^{-8\delta i(z - C_1\widetilde{\mu}^{(1-\kappa)}z^\kappa)},$$

where $z = |z|e^{i\theta}$, $z^\kappa = \exp(\kappa \ln z)$, with $\ln z = \ln|z| + i\theta$, $\theta \in [0, \pi/2]$, and C_1 is a constant to be determined. Taking $\kappa = 6/5$, so that $1 < \kappa < 2$ and close to 1, we write $h(z)$ as

$$\begin{aligned} h(z) &= \exp(2\delta(ix' - y'))\exp(-8\delta[-y' + C_1\widetilde{\mu}^{1-\kappa}|z|^\kappa \sin(\kappa\theta)])\cdot \\ &\quad \cdot \exp(-8\delta i[x' - C_1\widetilde{\mu}^{1-\kappa}|z|^\kappa\cos(\kappa\theta)]), \end{aligned}$$

and use h and its inverse to estimate N_1. Consider

$$N_1(-iz) = N_2(-iz)|h^{-1}(z)|^2,$$

where $N_2(-iz)$ is the subharmonic function in the first quadrant given by:

$$N_2(-iz):=N_1(-iz)|h(z)|^2=\int_{\mathbb{R}^{n+1}}(|\xi|^2+|z|^2)\frac{|h(z)|^2}{|1-iz|^2}|e^{\frac{-\epsilon\xi_0^2}{-i2z}}\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)|^2d\widetilde{\xi}d\xi_0.$$

We observe that:

a. On the real axis $y'=0$ we have $|h(x')|=1$, therefore

$$N_2(-iz)\le N_1(-iz)\le 2c_{113}\,.$$

b. On the positive imaginary axis $y'>0$, $x'=0$,

$$|h(iy')|=\exp(-2\delta y')\exp(a(y')),\quad \text{with}\quad a(y'):=8\delta(y'-C_1\widetilde{\mu}^{1-\kappa}(y')^\kappa s_\kappa)\,.$$

where we have $s_\kappa=\sin(\kappa\pi/2)>1/2$.

Then $a(y')$ achieves its maximum at $y'_M=\widetilde{\mu}(C_1\kappa s_\kappa)^{1/(1-\kappa)}$ with the value $a(y'_M)=\dfrac{8\delta\widetilde{\mu}(\kappa-1)}{\kappa(C_1\kappa s_\kappa)^{1/(\kappa-1)}}$. We choose $C_1\ge\dfrac{1}{\kappa s_\kappa}\Big(\dfrac{8(\kappa-1)}{\kappa c_{109}}\Big)^{\kappa-1}\delta^{\kappa-1}$, so that we have $-c_{109}\widetilde{\mu}+a(y'_M)\le 0$, and consequently, by the estimates of N_1 and $|h|^2$,

$$\begin{aligned}
N_2(-iz) &\le \Big(c_{113}e^{4\delta y'}(e^{-2c_{109}\widetilde{\mu}}+e^{-16\delta y'})\Big)e^{-4\delta y'}e^{2a(y')}\\
&= c_{113}(e^{2(-c_{109}\widetilde{\mu}+a(y))}+e^{-16\delta y'+2a(y')})\\
&\le c_{113}(e^{2(-c_{109}\widetilde{\mu}+a(y'_M))}+e^{-16\delta C_1\widetilde{\mu}^{1-\kappa}|z|^\kappa s_\kappa})\le 2c_{113}\,.
\end{aligned}$$

c. In the region $y'>0$, $x'>0$, we get

$$|h(z)|=\exp(-2\delta y')\,\exp[8\delta|z|(\sin\theta-C_1\widetilde{\mu}^{1-\kappa}|z|^{\kappa-1}\sin(\kappa\theta))]\le e^{-2\delta y'}e^{c(\widetilde{\mu})}.$$

Indeed for any $\theta\neq 0$ fixed we can compute the maximum in $|z|$ of that expression and apply $1/2<\sin(\kappa\theta)/\sin\theta\le\kappa$ to obtain

$$\max_{r\ge 0}[8\delta r(\sin\theta-C_1\widetilde{\mu}^{1-\kappa}r^{\kappa-1}\sin(\kappa\theta))]\le c(\widetilde{\mu}):=\frac{8\delta(\kappa-1)}{\kappa(\kappa C_1/2)^{1/(\kappa-1)}}\widetilde{\mu}$$

that implies

$$N_2(-iz)\le c_{113}e^{-4\delta y'}e^{2c(\widetilde{\mu})}\,.$$

In order to get rid of the $\widetilde{\mu}$ dependency in this estimate, we apply the Phragmen-Lindelöf Theorem 4.4 for subharmonic functions in the sector $x'\ge 0, y'\ge 0$ to obtain

$$N_2(-iz)\le 2c_{113}$$

and we note that c_{113} is independent of $\widetilde{\mu}$. To prove (2.10) observe that, for $\kappa=6/5$, we have $\sin(\kappa\pi/2)>1/2$ and the following inequality is valid in the region $|z|=c_{130}\widetilde{\mu}$ with $Re\,z\ge 0$, $Im\,z\ge 0$,

$$|h^{-1}(z)|=\exp(2\delta Im\,z)\,\exp(-8\delta Im\,z+8\delta C_1\widetilde{\mu}^{1-\kappa}|z|^\kappa\sin(\kappa\theta))\le\exp(-5\delta Im\,z)\,,$$

where

$$c_{130}:=\frac{3c_{109}}{4\delta}\Big(\frac{1}{16}\Big)^5\le\min_{\theta\in[0,\pi/2]}\Big(\frac{\sin\theta}{8C_1\sin(\kappa\theta)}\Big)^{\frac{1}{\kappa-1}}.$$

Indeed we see that $8C_1\widetilde{\mu}^{1-\kappa}(c_{130}\widetilde{\mu})^{\kappa-1}\sin(\kappa\theta)\le\sin\theta$, proving the derived estimate, and also (2.11) follows consequently.

Next, we observe that the same estimate (2.10) can be obtained in the sector $Re\,z\le 0$, $Im\,z\ge 0$ by applying the following arguments:

In the region $x'<0$, $y'>0$, with $z=x'+iy'$, we define $w=-\bar{z}=-x'+iy'$, belonging to the first quadrant, and $N_3(-iw):=N_1(-i(-\bar{z}))$ and $N_4(-iw):=N_1(-i(-\bar{z}))|h(-\bar{z})|^2$. Notice that $h(w)$ is an antiholomorphic function in w, and therefore $|h(w)|$ is subharmonic. Also $N_3(-iw), N_4(-iw)$ are subharmonic and they satisfy the same estimates as $N_1(-iz), N_2(-iz)$. We then apply the same procedure as in the first step with N_1, N_2 replaced by N_3, N_4. $\square$

We now can complete the proof of the logarithmic stability estimate in Theorem 1.2.

Proof of Theorem 1.2. We consider two cases:
Case A. Assume $\|Pu\|_{L^2(B_{2R})} \geq \|u\|_{H^1(B_{2R})}/e$. Then the estimate is trivial as

$$\|u\|_{L^2(B_r)} \leq \|u\|_{H^1(B_{2R})} \leq \ln(1+e)\frac{\|u\|_{H^1(B_{2R})}}{\ln\left(1+\frac{\|u\|_{H^1(B_{2R})}}{\|Pu\|_{L^2(B_{2R})}}\right)}.$$

Case B. Assume now $\|Pu\|_{L^2(B_{2R})} < \|u\|_{H^1(B_{2R})}/e$ and without restriction of generality take $\|u\|_{H^1(B_{2R})} = 1$. Our aim is to consider separetely estimates for low and high temporal frequencies. Let $A(D_0)$ be a pseudo-differential operator with symbol $a(\xi_0)$, where $a \in G_0^{1/\alpha}(\mathbb{R})$ with $\alpha \in (0,1)$ is a smooth Gevrey class localizer that is supported in $|\xi_0| \leq 2$, equal to one in $|\xi_0| \leq 1$ and $0 \leq a \leq 1$. Then $a(\beta\xi_0/\widetilde{\mu})$ is a scaled version of it, where $\widetilde{\mu} > 1$ is the parameter to be optimized, and $\beta > 0$ an adjusting constant. Let $b \in G_0^{1/\alpha_1}(\mathbb{R}^{n+1})$ with $0 < \alpha \leq \alpha_1$ be another localizer supported in B_2, equal to one in B_1 and $0 \leq b \leq 1$.
Observe that according to our geometric construction we have (see Proposition 2.5):

$$B_r \subset \operatorname{supp} b((y-y_0)/r) \subseteq B_{2r} \subset B_R \subset \operatorname{supp} b((y-y_0)/R) \subseteq B_{2R}\,,$$

and hence $\|u\|_{L^2(B_r)} \leq \|b((y-y_0)/r)u\|_{L^2}$.
Then we perform the splitting:

$$b((y-y_0)/r)u = A(\beta D_0/\widetilde{\mu})b((y-y_0)/r)u + (1-A(\beta D_0/\widetilde{\mu}))b((y-y_0)/r)u.$$

For high temporal frequencies $|\xi_0| \geq \widetilde{\mu}/\beta$ we estimate as follows:

$$\begin{aligned}\|(1-A(\beta D_0/\widetilde{\mu}))b((y-y_0)/r)u(y)\|^2_{L^2} &= \|(1-a(\frac{\beta\xi_0}{\widetilde{\mu}}))\,\mathcal{F}_{t\to\xi_0}(b((y-y_0)/r)u(y))\|^2_{L^2} \\ &\leq \frac{\beta^2}{\widetilde{\mu}^2}\int_{|\xi_0|>\widetilde{\mu}/\beta}\int_{\mathbb{R}^n}|\xi_0\mathcal{F}_{t\to\xi_0}(b((y-y_0)/r)u(t,x))|^2 dx d\xi_0 \\ &\leq \frac{\beta^2}{\widetilde{\mu}^2}\|b((y-y_0)/r)u(y)\|^2_{H^1} \leq \frac{\beta^2}{\widetilde{\mu}^2}\Big(1+\frac{|b'|^2_{C^0}}{r^2}\Big)\|u(y)\|^2_{H^1(B_R)}\,.\end{aligned}$$

For low temporal frequencies we first choose $\widetilde{\mu}$ such that $\|Pu\|_{L^2(B_{2R})} = e^{-\widetilde{\mu}} < e^{-1}$. Then we take $\mu > 1$, such that $\widetilde{\mu} = \mu^\alpha$. Hence for A and b as above we get for all $\zeta > 0$:

$$\|A(\zeta D_0/\mu)b((y-y_0)/R)Pu\|_0 \leq \|Pu\|_{L^2(B_{2R})} = e^{-\widetilde{\mu}}. \tag{2.12}$$

For $\zeta = 1$ in (2.12) we can apply Theorem 1.1 to obtain

$$\|A(\beta D_0/\widetilde{\mu})b((y-y_0)/r)u\|_{L^2} \leq c_{129}e^{-c_{132}\widetilde{\mu}^{\alpha_1}},\ \beta \geq 3c_{131}. \tag{2.13}$$

By collecting the previous estimates for low and high temporal frequencies we conclude that, as $\widetilde{\mu} \geq 1$,

$$\begin{aligned}\|u\|_{L^2(B_r)} &\leq \frac{\beta}{\widetilde{\mu}}\Big(1+\frac{|b'|^2_{C^0}}{r^2}\Big)^{1/2} + c_{129}e^{-c_{132}\widetilde{\mu}^{\alpha_1}} \leq \frac{c_{105}}{\widetilde{\mu}} = \frac{c_{105}}{-\ln(\|Pu\|_0)} \\ &\leq 2c_{105}\frac{\|u\|_{H^1(B_{2R})}}{\ln\Big(1+\frac{\|u\|_{H^1(B_{2R})}}{\|Pu\|_{L^2(B_{2R})}}\Big)},\end{aligned}$$

where $c_{105} = \beta\Big(1+\frac{|b'|^2_{C^0}}{r^2}\Big)^{1/2} + c_{129}$ and in the last step we apply $\ln(y) \geq \ln(1+y)/2$ for $y = \|u\|_{H^1(B_{2R})}/\|Pu\|_{L^2(B_{2R})} > e$, and then we return to the original notation. Defining $c_{111} = \ln(1+e) + 2c_{105}$ we obtain the result. □

3. Geometric constants

3.1. **Pseudoconvexity constants.** In the following we work under the following assumptions, derived from the ones in Theorem 1.2:

A3. We consider the case of the wave operator (1.1) with principal symbol $p(y,\xi) = -\xi_0^2 + \sum_{jk=1}^n g^{jk}(x)\xi_j\xi_k$, with $0 < a_1\,\delta^{jk} \leq g^{jk}(x) \leq b_1\,\delta^{jk}$, $a_1,\, b_1 > 0$.
Call $\xi = (\xi_0, \widetilde{\xi}) \in \mathbb{R}\times\mathbb{R}^n$, where $|\widetilde{\xi}|^2 = \sum_{j=1}^n \xi_j^2$.

A4. We fix a function $\psi \in C^{2,\rho}(\mathbb{R}^{n+1})$, for some $\rho \in (0,1)$, such that $p(y,\psi'(y)) \neq 0$ and $\psi'(y) \neq 0$ in a domain $\Omega_0 \subseteq \Omega$, containing the point y_0 lying on the level set $S = \{y;\, \psi(y) = 0\}$. In particular we assume that $|\psi'(y)| \geq C_l$ in Ω_0 for $C_l > 0$.

Moreover we use Einstein's convention for the repeated indexes.
To get Tataru inequality we proceed in three Steps. In Table (3.10) are listed the computed constants.

Step 1 . Given a function $\psi \in C^{2,\rho}(\mathbb{R}^{n+1})$ fulfilling the assumptions above in a domain Ω_0, we find positive constants M_2, M_1, M_P such that the following inequality holds true

$$\begin{aligned}M_2\xi_0^2 + M_1\Big(\frac{|p(y,\xi+i\tau\psi'(y))|^2}{\tau^2+|\xi|^2} + |\langle p'_\xi(y,\xi+i\tau\psi'(y)),\psi'(y)\rangle|^2\Big) \\ +\frac{\{\overline{p(y,\xi+i\tau\psi'(y))}, p(y,\xi+i\tau\psi'(y))\}}{2i\tau} \geq M_P(\tau^2+|\xi|^2)\end{aligned} \tag{3.1}$$

for every $\xi \in \mathbb{R}\times\mathbb{R}^n$, $\xi \neq 0$, $\tau \in \mathbb{R}$. The previous inequality proves that the hypersurface $S = \{y;\, \psi(y) = 0\}$ is conormally strongly pseudoconvex w.r.t. P in Ω_0.

Step 2 . For $\phi = e^{\lambda\psi}$, with y_0 on the level set $\phi(y) = 1$, we find $\lambda > 0$ such that the following inequality holds true

$$\begin{aligned}M_2\xi_0^2 + \frac{M_1}{\min\{1,\lambda^2\phi^2(y)\}}\frac{|p(y,\xi+i\tau\phi'(y))|^2}{\tau^2+|\xi|^2} \\ +\frac{1}{\lambda\phi(y)}\frac{\{\overline{p(y,\xi+i\tau\phi'(y))}, p(y,\xi+i\tau\phi'(y))\}}{2i\tau} \geq M_P\min\{1,\lambda^2\phi^2(y)\}(\tau^2+|\xi|^2)\end{aligned} \tag{3.2}$$

for every $\xi \in \mathbb{R}\times\mathbb{R}^n$, $\xi \neq 0$, $\tau \in \mathbb{R}$. The previous inequality proves that the function ϕ is conormally strongly pseudoconvex w.r.t. P in Ω_0.

Step 3 . We consider a perturbation of ϕ by the shifted 2nd order polynomial centred in the point y_0,

$$f(y) = \sum_{|v|\le 2} (\partial^v \phi)(y_0)\,(y-y_0)^v/v! - \sigma|y-y_0|^2. \tag{3.3}$$

In a ball $B(y_0,R_1)\subset\Omega_0$ where $f'\neq 0$ we define

$$\phi_0 = \min_{y\in B(y_0,R_1)} \phi(y), \qquad \phi_M = \max_{y\in B(y_0,R_1)} \phi(y).$$

We find σ and $R_2>0$ small enough such that in the ball $B(y_0,R_2)$ the following inequalities hold true: $f(y)<\phi(y)$ in $B(y_0,\,R_2)\backslash\{y_0\}$, and

$$M_2\xi_0^2 + 2M_1\frac{|p(y,\xi+i\tau f'(y))|^2}{\tau^2+|\xi|^2} + \frac{\{\overline{p(y,\xi+i\tau f'(y))}, p(y,\xi+i\tau f'(y))\}}{(\lambda\phi_0)2i\tau} \tag{3.4}$$
$$\ge \frac{1}{2}(\tau^2+|\xi|^2)\,.$$

The previous inequality proves that the function f is conormally strongly pseudoconvex w.r.t. P in $B(y_0,R_2)$.

Proof of STEP 1: We recall that

$$\begin{aligned}
p(y,\xi+i\tau\psi'(y)) &= p(y,\xi)-\tau^2 p(y,\psi')+i\tau\{p,\psi\}\\
|p(y,\xi+i\tau\psi'(y))|^2 &= |p(y,\xi)-\tau^2p(y,\psi')|^2+\tau^2|\{p,\psi\}|^2\\
&= |p(y,\xi)|^2+\tau^4|p(y,\psi')|^2-2\tau^2p(y,\xi)p(y,\psi')+\tau^2|\{p,\psi\}|^2\\
\langle p'_\xi(y,\xi+i\tau\psi'(y),\psi'(y)\rangle &= \{p,\psi\}(y,\xi)+i2\tau p(y,\psi')\\
|\langle p'_\xi(y,\xi+i\tau\psi'(y),\psi'(y)\rangle|^2 &= |\{p,\psi\}(y,\xi)|^2+4\tau^2|p(y,\psi')|^2
\end{aligned}$$

We have to estimate the quantities

$$\begin{aligned}
I_{1,\psi} &:= \frac{|p(y,\xi+i\tau\psi'(y))|^2}{\tau^2+|\xi|^2}+|\langle p'_\xi(y,\xi+i\tau\psi'(y),\psi'(y)\rangle|^2,\\
I_{2,\psi} &:= \frac{\{\overline{p(y,\xi+i\tau\psi'(y))},p(y,\xi+i\tau\psi'(y))\}}{2i\tau}\\
&= \{p,\{p,\psi\}\}(y,\xi)+\tau^2\{p,\{p,\psi\}\}(y,\psi'(y)),
\end{aligned}$$

where the last equality holds for our second order wave operator.
For the second term we get, by setting $a^{00}=-1$, $a^{j0}=0$, $a^{jk}=g^{jk}$, $j,k=1...n$,

$$\begin{aligned}
I_{2,\psi} = &\sum_{l,m=0}^n \xi_l\xi_m\Big(4\sum_{j,k=0}^n a^{jl}\psi''_{jk}a^{km}+4\sum_{j,k=0}^n a^{jl}\partial_{x_j}a^{km}\psi'_k-2\sum_{j,k=0}^n \partial_{x_j}a^{lm}a^{kj}\psi'_k\Big)\\
&+\tau^2\sum_{l,m=0}^n \psi'_l\psi'_m\Big(4\sum_{j,k=0}^n a^{jl}\psi''_{jk}a^{km}+2\sum_{j,k=0}^n a^{jl}\partial_{x_j}a^{km}\psi'_k\Big)\ge -C_3(|\xi|^2+\tau^2)
\end{aligned}$$

where C_3 is defined as follows

$$\begin{aligned}
&\max_{y\in\Omega_0}\Big(4\sum_{j,k} a^{jl}\psi''_{jk}a^{km}+4\sum_{j,k}a^{jl}\partial_{x_j}a^{km}\psi'_k-2\sum_{j,k}\partial_{x_j}a^{lm}a^{kj}\psi'_k\Big)(2+\psi'_l\psi'_m)\\
&\qquad\le 20(1+n^2|g^{jl}|^2_{C^1})|\psi'|_{C^1}(1+|\psi'|^2_{C^0})\ :=C_3.
\end{aligned}$$

For the first term we get

$$
\begin{aligned}
I_{1,\psi} = \; & \frac{|p(y,\xi)|^2}{\tau^2+|\xi|^2} + \tau^2|p(y,\psi')|^2\Big(4+\frac{\tau^2}{\tau^2+|\xi|^2}\Big) - 2\frac{\tau^2}{\tau^2+|\xi|^2}p(y,\xi)p(y,\psi') \\
& +|\{p,\psi\}|^2\Big(1+\frac{\tau^2}{\tau^2+|\xi|^2}\Big) \\
\ge \; & \frac{|p(y,\xi)|^2}{\tau^2+|\xi|^2}(1-\omega) + \tau^2|p(y,\psi')|^2\Big(4|\xi|^2+\big(5-\frac{1}{\omega}\big)\tau^2\Big)\frac{1}{\tau^2+|\xi|^2} \\
& +|\{p,\psi\}|^2\Big(1+\frac{\tau^2}{\tau^2+|\xi|^2}\Big) \\
\ge \; & \frac{1}{\tau^2+|\xi|^2}\Big(|p(y,\xi)|^2(1-\omega)+4|p(y,\psi')|^2|\xi|^2\tau^2+|p(y,\psi')|^2\big(5-\frac{1}{\omega}\big)\tau^4 \\
& +|\{p,\psi\}|^2(2\tau^2+|\xi|^2)\Big)
\end{aligned}
$$

where by Young's inequality, $2p(y,\xi)\tau^2 p(y,\psi') \le \omega|p(y,\xi)|^2 + \frac{1}{\omega}\tau^4|p(y,\psi')|^2$, where we choose $\omega \in (0,1)$ such that $(4 \ge)\ 5-\frac{1}{\omega} > 0$.
We now split the estimate into two parts:
<u>Case 1</u>: If $p(y,\xi) > 0$, then

$$
\begin{aligned}
& |p(y,\xi)| = p(y,\xi) = -\xi_0^2 + \sum_{kj} g^{kj}\xi_k\xi_j \ge a_1|\widetilde{\xi}|^2 - \xi_0^2\,, \\
& |p(y,\xi)|^2 = (-\xi_0^2 + \sum_{kj} g^{kj}\xi_k\xi_j)^2 \ge (a_1|\widetilde{\xi}|^2-\xi_0^2)(-\xi_0^2+\sum_{kj} g^{kj}\xi_k\xi_j) \\
& = \xi_0^4 + a_1|\widetilde{\xi}|^2(\sum_{kj} g^{kj}\xi_k\xi_j) - \xi_0^2[\sum_{kj} g^{kj}\xi_k\xi_j + a_1|\widetilde{\xi}|^2] \ge \xi_0^4 + a_1^2|\widetilde{\xi}|^4 - (b_1+a_1)|\widetilde{\xi}|^2\xi_0^2\,.
\end{aligned}
$$

Our aim is to find M_2, M_1, M_P such that $M_2\xi_0^2 + M_1 I_{1,\psi} + I_{2,\psi} \ge M_P(\tau^2+|\xi|^2)$.
Hence,

$$
\begin{aligned}
& M_2\xi_0^2 + M_1 I_{1,\psi} + I_{2,\psi} \ge M_2\xi_0^2 - C_3(\tau^2+|\xi|^2) + \\
& M_1\Big[\frac{|p(y,\xi)|^2}{\tau^2+|\xi|^2}(1-\omega) + \frac{\tau^2|p(y,\psi')|^2}{\tau^2+|\xi|^2}\Big(4|\xi|^2+\big(5-\frac{1}{\omega}\big)\tau^2\Big) + |\{p,\psi\}|^2\Big(1+\frac{\tau^2}{\tau^2+|\xi|^2}\Big)\Big] \\
& \ge \frac{1}{\tau^2+|\xi|^2}\Big(M_2(\tau^2\xi_0^2+|\xi|^2\xi_0^2) + M_1(1-\omega)[a_1^2|\widetilde{\xi}|^4+\xi_0^4-(a_1+b_1)|\widetilde{\xi}|^2\xi_0^2] + \\
& +M_1|p(y,\psi')|^2\big[4\tau^2|\xi|^2+\big(5-\frac{1}{\omega}\big)\tau^4\big] - C_3(\tau^4+|\xi|^4+2\tau^2|\xi|^2)\Big) \\
& \ge M_P(\tau^2+|\xi|^2)\,.
\end{aligned}
$$

To fulfil the last inequality we have to solve the system of 7 inequalities:

$$\begin{array}{rcl} \Big(M_1(1-\omega)a_1^2 - C_3\Big)\,|\widetilde{\xi}|^4 & \geq & M_P|\widetilde{\xi}|^4 \\ \Big(M_1(1-\omega) + M_2 - C_3\Big)\,\xi_0^4 & \geq & M_P\xi_0^4 \\ \Big(M_2 - (b_1+a_1)M_1(1-\omega) - 2C_3\Big)\,|\widetilde{\xi}|^2\xi_0^2 & \geq & 2M_P|\widetilde{\xi}|^2\xi_0^2 \\ \Big(4M_1|p(y,\psi')|^2 - 2C_3\Big)\,\tau^2|\widetilde{\xi}|^2 & \geq & 2M_P\tau^2|\widetilde{\xi}|^2 \\ \Big(4M_1|p(y,\psi')|^2 - 2C_3 + M_2\Big)\,\tau^2\xi_0^2 & \geq & 2M_P\tau^2\xi_0^2 \\ \Big(M_1|p(y,\psi')|^2\big(5-\dfrac{1}{\omega}\big) - C_3\Big)\,\tau^4 & \geq & M_P\tau^4\,. \end{array}$$

Case 2: If $p(y,\xi) \leq 0$, then

$$|p(y,\xi)| = -p(y,\xi) = \xi_0^2 - \sum_{kj} g^{kj}\xi_k\xi_j \geq 0 \;\Rightarrow\; \xi_0^2 \geq \sum_{kj} g^{kj}\xi_k\xi_j \geq a_1|\widetilde{\xi}|^2.$$

Once again we look for M_2, M_1, M_P such that $M_2\xi_0^2 + M_1 I_{1,\psi} + I_{2,\psi} \geq M_P(\tau^2 + |\xi|^2)$:

$$\begin{aligned} M_2\xi_0^2 + M_1 I_{1,\psi} + I_{2,\psi} &\geq M_1\Big[\frac{\tau^2|p(y,\psi')|^2}{\tau^2+|\xi|^2}\Big(4|\xi|^2 + \big(5-\frac{1}{\omega}\big)\tau^2\Big) \\ &\qquad +|\{p,\psi\}|^2\Big(1+\frac{\tau^2}{\tau^2+|\xi|^2}\Big)\Big] + M_2\big(\frac{\xi_0^2}{2}+\frac{\xi_0^2}{2}\big) - C_3(\tau^2+|\xi|^2) \\ &\geq \frac{1}{\tau^2+|\xi|^2}\Big(M_1|p(y,\psi')|^2\big[4\tau^2|\xi|^2 + \big(5-\frac{1}{\omega}\big)\tau^4\big] - C_3(\tau^4+|\xi|^4+2\tau^2|\xi|^2) \\ &\qquad +M_2\big(\frac{\xi_0^2}{2}+\frac{a_1|\widetilde{\xi}|^2}{2}\big)(\tau^2+|\xi|^2)\Big) \\ &\geq M_P(\tau^2+|\xi|^2)\,. \end{aligned}$$

To get the last inequality we have to solve the system of 3 inequalities:

$$\begin{array}{rcl} \Big(4M_1|p(y,\psi')|^2 - 2C_3 + \dfrac{M_2}{2}\min\{a_1,1\}\Big)\,\tau^2|\xi|^2 & \geq & 2M_P\tau^2|\xi|^2 \\ \Big(M_1|p(y,\psi')|^2\big(5-\dfrac{1}{\omega}\big) - C_3\Big)\,\tau^4 & \geq & M_P\tau^4 \\ \Big(\dfrac{M_2}{2}\min\{a_1,1\} - C_3\Big)\,|\xi|^4 & \geq & M_P|\xi|^4\,. \end{array}$$

From Case 1 and 2 we obtain two systems of inequalities for the coefficients; by choosing $\omega = 1/2$ and solving them, the pseudoconvexity estimate (3.1) holds with M_1, M_2 as in Table (3.10) and with M_P a free parameter to be set in the following.

Remark 3.1. 1. Notice that the estimate is valid also in the limit $\tau \to 0$. Indeed, for $\xi \neq 0$

$$\begin{array}{rcl} M_2\xi_0^2 + M_1 I_{1,\psi} + I_{2,\psi} & = & M_2\xi_0^2 + M_1\big(\dfrac{|p(y,\xi)|^2}{|\xi|^2} + |\{p,\psi\}|^2\big) + \{p,\{p,\psi\}\} \\ & \geq & M_2\xi_0^2 - C_3|\xi|^2 + M_1\dfrac{|p(y,\xi)|^2}{|\xi|^2} \geq M_P|\xi|^2. \end{array}$$

2. From the constraint on M_1 one can understand the reason for the assumption $p(y,\psi')\neq 0$. Actually, as observed in [6] and by other authors, in the case $p(y_0,\psi'(y_0))=0$ the estimate (3.1) is still possible if $\{p,\{p,\psi\}\}(y_0,\psi'(y_0))>0$. Indeed, in that case there are positive constants C_4, C_5 such that $I_{2,\psi}\geq C_5\tau^2-C_4|\xi|^2$, and one can proceed as above to get (3.1) with different coefficients.

Proof of STEP 2: Let $\phi(y)=e^{\lambda\psi(y)}$, $\tau_1=\tau\lambda\phi(y)$, and recall that

$$\tau\phi'(y)\;=\;\tau\lambda\phi(y)\psi'(y)\;=\;\tau_1\psi'(y),\quad \phi''(y)\;=\;\lambda\phi(y)(\psi''(y)+\lambda\psi'(y)\otimes\psi'(y)),$$

where $\phi'(y)\neq 0$ in Ω_0. Then for $\tau\neq 0$ (see [6], Lemma 4.2)

$$\frac{\{\overline{p(y,\xi+i\tau\phi'(y))},p(y,\xi+i\tau\phi'(y))\}}{\lambda\phi(y)(2i\tau)}=\frac{1}{2i\tau_1}\{\overline{p(y,\xi+i\tau_1\psi'(y))},p(y,\xi+i\tau_1\psi'(y))\}$$
$$+\lambda|\langle p'_\xi(y,\xi+i\tau_1\psi'(y),\psi'(y)\rangle|^2\,,$$

where at the right hand side one has first to perform the derivatives and next to substitute τ_1 (which consequently must not be seen as a function of y and τ in the bracket). In the case $\tau=0$,

$$\frac{\{p,\{p,\phi\}\}(y,\xi)}{\lambda\phi(y)}=\{\{p,\{p,\psi\}\}(y,\xi)+\lambda|\langle p'_\xi(y,\xi),\psi'(y)\rangle|^2.$$

Hence by substituting in (3.1) the variables τ_1,ξ and for

$$\lambda\geq M_1,$$

we obtain $\tau_1^2+|\xi|^2\geq\min\big(1,\,\lambda^2\phi^2(y)\big)\,(\tau^2+|\xi|^2)$, and finally (3.2).

Proof of STEP 3: For simplicity we now consider λ and a domain $B(y_0,R_1)$ where $\phi_0=e^{-1}\leq\phi(y)\leq e=\phi_M$ and $\min\big(1,\,\lambda^2\phi^2(y)\big)=1$. Since $|\psi(y)-\psi(y_0)|\leq|\psi'|_{C^0(\Omega_0)}R_1$, then we choose

$$R_1\leq\min\{1,\min_{\Omega_0}|y_0-y|,\frac{1}{\lambda|\psi'|_{C^0(\Omega_0)}}\},\quad \lambda\geq e\,. \tag{3.5}$$

We then rewrite f as

$$\begin{aligned}f(y)&=\phi(y_0)+\sum_{j=1}^n\partial_j\phi(y_0)(x_j-x_{0,j})+\partial_t\phi(y_0)(t-t_0)\\&+\frac12\sum_{j,k=1}^n\partial^2_{j,k}\phi(y_0)(x_j-x_{0,j})(x_k-x_{0,k})+\sum_{j=1}^n\partial^2_{j,t}\phi(y_0)(x_j-x_{0,j})(t-t_0)\\&+\frac12\partial^2_t\phi(y_0)(t-t_0)^2-\sigma|x-x_0|^2-\sigma|t-t_0|^2\end{aligned}$$

and its derivatives, by identifying ∂_t with ∂_0, and calling δ_{ab} the Kroenecker symbol,

$$\begin{aligned}f'_j(y)\;&=\;\phi'_j(y_0)+\sum_{h=1}^n\phi''_{jh}(y_0)(x_h-x_{0h})+\\&\quad+\phi''_{tj}(y_0)(t-t_0)-2\sigma\big((x_j-x_{0j})(1-\delta_{0j})+(t-t_0)\delta_{0j}\big)\\f''_{jm}(y)\;&=\;\phi''_{jm}(y_0)-2\sigma\,\delta_{jm},\quad j,m\in\{0,1,\dots,n\}.\end{aligned}$$

First of all we ask for $f' \neq 0$ in the ball $|y-y_0| \leq R_2$

$$\begin{aligned} |f'(y)| &\geq |\phi'(y_0)| - |\phi''(y_0)||y-y_0| - 2\sigma|y-y_0| \\ &\geq |\phi'(y_0)| - |\phi''(y_0)|R_2 - 2\sigma R_2 \\ &\geq |\phi'(y_0)|/2 \end{aligned}$$

which implies the following constraint on R_2,

$$|\phi''(y_0)|R_2 + 2\sigma R_2 \leq |\phi'(y_0)|/2 \tag{3.6}$$

In order to pass from (3.2) to (3.4) we compute

$$\begin{aligned} &|p(y,\xi+i\tau\phi'(y))|^2 = \\ &|p(y,\xi) - \tau^2 p(y,f') + \tau^2(p(y,f') - p(y,\phi'))|^2 + \tau^2|\{p,f\} + (\{p,\phi-f\})|^2 \\ &\leq 2|p(y,\xi) - \tau^2 p(y,f')|^2 + 2\tau^4|p(y,\phi') - p(y,f')|^2 \\ &\quad +2\tau^2|\{p,f\}|^2 + 2\tau^2|(\{p,\phi-f\})|^2 \\ &\leq 2|p(y,\xi+i\tau f'(y))|^2 + 2\tau^4|p(y,\phi') - p(y,f')|^2 + 2\tau^2|(\{p,\phi-f\})|^2 \\ &\leq 2|p(y,\xi+i\tau f'(y))|^2 + 2\tau^4\eta_1 + 2\tau^2|\xi|^2\eta_2 \\ &\leq 2|p(y,\xi+i\tau f'(y))|^2 + \eta_2(1+|\phi'|^2_{C^0} + |f'|^2_{C^0})\,(\tau^2+|\xi|^2)^2 \end{aligned}$$

where η_1, η_2 are

$$\begin{aligned} &|p(y,\phi') - p(y,f')|^2 = \left| -(\phi'_t)^2 + \sum_{jk=1}^n g^{jk}\phi'_j\phi'_k + (f'_t)^2 - \sum_{jk=1}^n g^{jk} f'_j f'_k \right|^2 \\ &\leq 2|f'_t - \phi'_t|^2(|\phi'_t| + |f'_t|)^2 + 2|\sum_{jk=1}^n g^{jk}((\phi'_j - f'_j)\phi'_k + f'_j(-f'_k+\phi'_k))|^2 \\ &\leq 4(1+n^4|g^{jk}|^2_{C^0})(|\phi'|^2_{C^0} + |f'|^2_{C^0})|f'-\phi'|^2_{C^0} \;:=\; \eta_1 \end{aligned}$$

and

$$\begin{aligned} |\{p,\phi-f\}|^2 &= |2\xi_0(f'_t - \phi'_t) + 2\sum g^{jk}\xi_j(\phi'_k - f'_k)|^2 \\ &\leq 8(1+n^4|g^{jk}|^2_{C^0})|f'-\phi'|^2_{C^0}|\xi|^2 \;:=\; \eta_2|\xi|^2\,. \end{aligned}$$

Next

$$\begin{aligned} &\frac{\{\overline{p(y,\xi+i\tau\phi'(y))}, p(y,\xi+i\tau\phi'(y))\}}{2i\tau} = \{p,\{p,\phi\}\}(y,\xi) + \tau^2\{p,\{p,\phi\}\}(y,\phi'(y)) \\ &\leq \{p,\{p,f\}\}(y,\xi) + \tau^2\{p,\{p,f\}\}(y,f'(y)) + |\{p,\{p,\phi-f\}\}(y,\xi)| \\ &+\tau^2|\{p,\{p,\phi-f\}\}(y,\phi'(y))| + \tau^2|\{p,\{p,f\}\}(y,\phi'(y)) - \{p,\{p,f\}\}(y,f'(y))| \\ &\leq \{p,\{p,f\}\}(y,\xi) + \tau^2\{p,\{p,f\}\}(y,f'(y)) + \eta_3|\xi|^2 + \eta_4\tau^2 + \eta_5\tau^2\,. \end{aligned}$$

Where η_3, η_4, η_5, are defined as follows

$$\begin{aligned} &\{p,\{p,\phi-f\}\}(y,\xi) = 4(\phi''_{tt} - f''_{tt})\xi_0^2 + \sum_{l,m=1}^n \xi_l\xi_m\Big(4\sum_{j,k=1}^n g^{jl}(\phi''_{jk} - f''_{jk})g^{km} \\ &\qquad +4\sum_{j,k=1}^n g^{jl}\partial_{x_j}g^{km}(\phi'_k - f'_k) - 2\sum_{j,k=1}^n \partial_{x_j}g^{lm}g^{kj}(\phi'_k - f'_k)\Big) \\ &\leq 4|\phi''-f''|_{C^0}\xi_0^2 + \Big(4|\phi''-f''|_{C^0}|g^{jl}|^2_{C^0}n^4 + 6|g^{jl}\partial_{x_j}g^{km}|_{C^0}n^4|\phi'-f'|_{C^0}\Big)|\widetilde{\xi}|^2 \\ &\leq 10(1+n^4|g^{jl}|^2_{C^1})\big(|\phi''-f''|_{C^0} + |\phi'-f'|_{C^0}\big)|\xi|^2 \;:=\; \eta_3|\xi|^2\,. \end{aligned}$$

Analogously, by setting $\xi = \phi'(y)$,

$$\{p,\{p,\phi-f\}\}(y,\phi') \le \eta_3|\phi'|^2_{C^0} \;:= \eta_4\,.$$

Then, substituting $\phi - f$ with f and ξ with ϕ' or f' in the computations for η_3,

$$\begin{aligned}
&|\{p,\{p,f\}\}(y,\phi'(y)) - \{p,\{p,f\}\}(y,f'(y))| \le 4|f''|_{C^0}(|\phi'|_{C^0}+|f'|_{C^0})|\phi'-f'|_{C^0}\\
&\quad+\Big(4|f''|_{C^0}|g^{jl}|^2_{C^0}n^4 + 6|g^{jl}\partial_{x_j}g^{km}|_{C^0}n^4|f'|_{C^0}\Big)(|\phi'|_{C^0}+|f'|_{C^0})|\phi'-f'|_{C^0}\\
&\le 10|f'|_{C^1}(1+n^4|g^{jl}|^2_{C^1})(|\phi'|_{C^0}+|f'|_{C^0})|\phi'-f'|_{C^0} := \eta_5\,.
\end{aligned}$$

Summing up:

$$\begin{aligned}
&M_2\xi_0^2 + \frac{M_1}{\min\{1,\lambda^2\phi^2\}}\left(\frac{2|p(y,\xi+i\tau f'(y))|^2 + 2\eta_1\tau^4 + 2\eta_2\tau^2|\xi|^2}{\tau^2+|\xi|^2}\right)\\
&+\frac{\{\overline{p(y,\xi+i\tau f'(y))}, p(y,\xi+i\tau f'(y))\}}{(\lambda\phi)2i\tau} + \frac{1}{\lambda\phi}(\eta_3|\xi|^2 + (\eta_4+\eta_5)\tau^2)\\
&\qquad\qquad \ge M_P\min\{1,\lambda^2\phi^2\}(\tau^2+|\xi|^2)\,.
\end{aligned}$$

Without restrictions of generality we can take $M_P = 1$, while on the ball $B_{R_2}(y_0) \subset B(y_0,R_1)$ we have also that $\min\{1,\lambda^2\phi_0^2\} = 1$. Then

$$\begin{aligned}
&M_2\xi_0^2 + 2M_1\frac{|p(y,\xi+i\tau f'(y))|^2}{\tau^2+|\xi|^2} + \frac{\{\overline{p(y,\xi+i\tau f'(y))}, p(y,\xi+i\tau f'(y))\}}{(\lambda\phi)2i\tau} +\\
&\ge (\tau^2+|\xi|^2) - \eta_2 M_1(1+|\phi'|^2_{C^0}+|f'|^2_{C^0})(\tau^2+|\xi|^2) - \frac{\eta_3}{\lambda\phi}|\xi|^2 - \frac{1}{\lambda\phi}(\eta_3|\phi'|^2_{C^0}+\eta_5)\tau^2\\
&\ge \Big(1-\eta_2M_1(1+|\phi'|^2_{C^0}+|f'|^2_{C^0}) - \frac{\eta_3}{\lambda\phi}(1+|\phi'|^2_{C^0}+|f'|_{C^1}(|\phi'|_{C^0}+|f'|_{C^0}))\Big)(\tau^2+|\xi|^2)\\
&:= M_R\,(\tau^2+|\xi|^2)
\end{aligned}$$

where we used $\eta_5 \le \eta_3|f'|_{C^1}(|\phi'|_{C^0}+|f'|_{C^0})$. Furthermore on σ we must set the constraint $f < \phi$ for $y \ne y_0$. Define $v(s) = \phi(p(s))$, $p(s) = y_0 + s(y-y_0)$, then there is a $q \in (0,1)$ such that $v(1) = v(0) + v'(0) + \frac{1}{2}v''(q)$. Hence,

$$\begin{aligned}
&|v(1)-v(0)-v'(0)-\frac{1}{2}v''(0)| = \frac{1}{2}|v''(q)-v''(0)|\\
&= |\sum_{|\zeta|=2}\frac{1}{\zeta!}(\partial^\zeta\phi(p(q)) - \partial^\zeta\phi(y_0))\,(y-y_0)^\zeta| \le c_T|y-y_0|^{\rho+2},\\
&|\phi(y) - \sum_{|\zeta|\le 2}\frac{1}{\zeta!}(\partial^\zeta\phi)(y_0)\,(y-y_0)^\zeta| \le c_T|y-y_0|^{\rho+2},\ \text{for } c_T = n\max_{|\zeta|=2}|\partial^\zeta\phi|_{C^{0,\rho}}\,.
\end{aligned}$$

On the set $|y-y_0| \le R_2$, $y \ne y_0$, we now consider the inequality

$$f(y)-\phi(y) \le -\sigma|y-y_0|^2 + c_T|y-y_0|^{\rho+2} \le -(\sigma - c_TR_2^\rho)|y-y_0|^2 < 0\,.$$

This is satisfied by taking

$$\sigma := 2c_TR_2^\rho = 2n|\phi''|_{C^{0,\rho}(B_{R_2}(y_0))}R_2^\rho. \tag{3.7}$$

With this choice the constraint (3.6) becomes, since $R_2^{1+\rho} \le R_2$ and C_l as in A2,

$$(|\phi''|_0 + 4n|\phi''|_{0,\rho})R_2 \le \lambda C_l/2\,. \tag{3.8}$$

Hence, the main quantities can be estimated as follows. If not else specified, the C^0, C^1,C^2, $C^{0,\rho}$ norms of ψ and g^{jk} are referred to the given domain $B(y_0, R_1)$, while the ones for ϕ and f are referred to the smaller ball $|y - y_0| \leq R_2$, with radius $R_2 \leq R_1$ to be determined:

$$
\begin{array}{rcl}
|\phi'|_{C^0(B_{R_2})} & \leq & \lambda\phi_M|\psi'|_{C^0} \\
|\phi''|_{C^0(B_{R_2})} & \leq & \lambda\phi_M(|\psi''|_{C^0} + \lambda|\psi'|^2_{C^0}) \\
|\phi|_{C^{0,\rho}(B_{R_2})} & \leq & |\phi'|_{C^0}|\psi|_{C^{0,\rho}} \leq \lambda\phi_M|\psi|_{C^{0,1}}R_2^{1-\rho} \\
|\phi''|_{C^{0,\rho}(B_{R_2})} & \leq & \lambda|\phi\psi''|_{0,\rho} + \lambda^2|\phi\psi' \circ \psi'|_{0,\rho} \leq \lambda\phi_M|\psi''|_{0,\rho} + \lambda^2\phi_M|\psi|_{0,1}|\psi''|_0 R_2^{1-\rho} \\
& & +2\lambda^2\phi_M|\psi'|_0|\psi'|_{0,1}R_2^{1-\rho} + \lambda^3\phi_M|\psi|_{0,1}|\psi'|_0^2 R_2^{1-\rho} \\
|\phi' - f'|_{C^0(B_{R_2})} & \leq & \sup\limits_j |\sum\limits_{k=0}^{n}(\partial^k\phi'_j(p(\widetilde{q})) - \partial^k\phi'_j(y_0))\,(y_k - y_{0,k})| + 2\sigma|y - y_0| \\
& \leq & n|\phi''|_{0,\rho}|y - y_0|^{1+\rho} + 2\sigma|y - y_0| \leq 5n|\phi''|_{0,\rho}R_2^{1+\rho} \\
|\phi'' - f''|_{C^0(B_{R_2})} & \leq & |\phi''|_{C^{0,\rho}}|y - y_0|^\rho + 2\sigma \leq (2n+1)|\phi''|_{C^{0,\rho}}R_2^\rho \\
|f'|_{C^0(B_{R_2})} & \leq & |\phi'|_{C^0} + |\phi' - f'|_{C^0} \leq |\phi'|_{C^0} + 5n|\phi''|_{0,\rho}R_2^{1+\rho} \\
|f''|_{C^0(B_{R_2})} & \leq & |\phi''|_{C^0} + 2\sigma = |\phi''|_{C^0} + 4n|\phi''|_{C^{0,\rho}}R_2^\rho.
\end{array}
$$

We can now end up the estimate above

$$
\begin{array}{rcl}
\eta_2 & \leq & c(|g^{jk}|_{C^0})\,|\phi' - f'|^2_{C^0} \\
\eta_3 & \leq & c(|g^{jk}|_{C^1})\,(|\phi' - f'|_{C^0} + |\phi'' - f''|_{C^0}) \\
\eta_5 & \leq & \eta_3|f'|_{C^1}(|\phi'|_{C^0} + |f'|_{C^0}).
\end{array}
$$

Call $c_{100}(g) = 10(1 + n^4|g^{jk}|^2_{C^1(B(y_0,R_1))})$ the biggest constant entering in the estimates for η_j. Then, for $R_2 < 1$

$$
\begin{array}{l}
M_R := 1 - c_{100}(g)\Big[|\phi' - f'|^2_{C^0}M_1(1 + |\phi'|^2_{C^0} + |f'|^2_{C^0}) \\
+(|\phi' - f'|_{C^0} + |\phi'' - f''|_{C^0})\dfrac{1}{\lambda\phi_0}\big(1 + |\phi'|^2_{C^0} + |f'|_{C^1}(|\phi'|_{C^0} + |f'|_{C^0})\big)\Big] \\
\geq 1 - c_{100}(g)\Big[((5n)^2R_2^{2(1+\rho)}|\phi''|^2_{C^{0,\rho}})M_1(1 + 5|\phi'|^2_{C^0}) \\
+(10nR_2^\rho|\phi''|_{C^{0,\rho}})\dfrac{1}{\lambda\phi_0}\big(1 + |\phi'|^2_{C^0} + (2|\phi'|_{C^0} + |\phi''|_{C^0} + 4n|\phi''|^2_{0,\rho}R_2^\rho)(3|\phi'|_{C^0})\big)\Big]
\end{array}
$$

In the last step we used the following constraint on B_{R_2}: $|f'|_{C^0} \leq 2|\phi'|_{C^0}$, that is a consequence of (3.8).
Defining the term $|\lambda\psi|_{max}$ as

$$
|\phi''|_{C^{0,\rho}} \leq \phi_M \max(\lambda|\psi''|_{0,\rho},\, \lambda^2|\psi|_{0,1}|\psi''|_0,\, \lambda^3|\psi|_{0,1}|\psi'|_0^2) := |\lambda\psi|_{max}\,,
$$

we can refine condition (3.8) and add an extra conditions on R_2^ρ (that is qualitatively equivalent to $|f''|_{C^0} \leq 2|\phi''|_{C^0}$)

$$
\begin{array}{l}
(\lambda\phi_M(|\psi''|_{C^0} + \lambda|\psi'|^2_{C^0}) + 4n|\lambda\psi|_{max})R_2 \leq \lambda C_l/2, \\
4n|\phi''|_{0,\rho}R_2^\rho \leq 4n|\lambda\psi|_{max}R_2^\rho \leq \lambda\phi_M(|\psi''|_{C^0} + \lambda|\psi'|^2_{C^0}),
\end{array}
\tag{3.9}
$$

where we apply the previous estimates to the norms of ϕ', ϕ''. By including the numeric constants into c_{100}, we can then write

$$\begin{aligned} M_R &\geq 1 - c_{100}(g)\Big[|\lambda\psi|^2_{max} R_2^{2(1+\rho)} M_1 (1 + \lambda^2\phi_M^2|\psi'|_0^2) \\ &\quad + |\lambda\psi|_{max} R_2^{\rho} \frac{1}{\lambda\phi_0}\big(1 + \lambda^2\phi_M^2|\psi'|_0^2 + \lambda^2\phi_M^2(|\psi'|_0|\psi''|_0 + \lambda|\psi'|_0^3)\big)\Big]. \end{aligned}$$

We first require that R_2 is such that:

$$\begin{aligned} c_{100}(g)|\lambda\psi|^2_{max} R_2^{2(1+\rho)} M_1 (1 + \lambda^2\phi_M^2|\psi'|_0^2 &\leq 1/4, \\ c_{100}(g)|\lambda\psi|_{max} R_2^{\rho} \frac{1}{\lambda\phi_0}\big(1 + \lambda^2\phi_M^2|\psi'|_0^2 + \lambda^2\phi_M^2(|\psi'|_0|\psi''|_0 + \lambda|\psi'|_0^3)\big) &\leq 1/4. \end{aligned}$$

Then we add the previous two constraints (3.9). The resulting upper bound for R_2 is in Table (3.10).

We now collect in the following table all the constants computed in Step 1, 2, 3 and in the following sections. If not else specified, the C^0, C^1,C^2, $C^{0,\rho}$ norms of ψ and g^{jk} are referred to the domain $B(y_0, R_1)$, while the ones for ϕ and f are referred to the smaller ball $B_{R_2} : |y - y_0| \leq R_2$. In case of special geometries where ψ is given explicitly, the constraints in the table can be improved.

(3.10) Table for the constants computed under the assumptions $A3$, $A4$ and the notations of Step 1, 2, 3 at the beginning of the section.

Name		Limit Value
C_3	$\geq$	$20(1+n^2\vert g^{jk}\vert^2_{C^1(\Omega_0)})\vert\psi'\vert_{C^1(\Omega_0)}(1+\vert\psi'\vert^2_{C^0(\Omega_0)})$
M_1	$\geq$	$(M_P+C_3)\max_{y\in\Omega_0}\left\{\frac{2}{a_1^2},\ \frac{1}{2\vert p(y,\psi')\vert^2}\right\}$
M_2	$\geq$	$\frac{2}{\min\{1,a_1\}}(M_P+C_3)+\frac{(b_1+a_1)}{2}M_1$
M_P	$\leq$	1
λ	$\geq$	$\max\{M_1, e, \frac{2\vert\psi''\vert_{C^0(\Omega_0)}}{C_l^2}\}$
ϕ_0	$\geq$	e^{-1}
ϕ_M	$\leq$	e
R_1	$\leq$	$\min\{1, \min_{y\in\partial\Omega_0}\vert y_0-y\vert, \frac{1}{\lambda\vert\psi'\vert_{C^0(\Omega_0)}}\}$
R_2	$\leq$	$\min\Big\{R_1,\ \Big(\frac{C_l}{2\phi_M(\vert\psi''\vert_{C^0(B(R_1))}+\lambda\vert\psi'\vert^2_{C^0(B(R_1))})}\Big),\ \Big(\frac{\lambda\phi_M(\vert\psi''\vert_{C^0(B(R_1))}+\lambda\vert\psi'\vert^2_{C^0(B(R_1))})}{4n\vert\lambda\psi\vert_{max}}\Big)^{\frac{1}{\rho}},$ $\Big(\frac{1}{4c_{100}(g)\vert\lambda\psi\vert^2_{max}M_1(1+\lambda^2\phi_M^2\vert\psi'\vert^2_{C^0(B(R_1))}}\Big)^{\frac{1}{2+2\rho}},$ $\Big(\frac{\lambda\phi_0}{4c_{100}(g)\vert\lambda\psi\vert_{max}\big(1+\lambda^2\phi_M^2\vert\psi'\vert^2_{C^0(B(R_1))}+\lambda^2\phi_M^2(\vert\psi'\vert_{C^0(B(R_1))}\vert\psi''\vert_{C^0(B(R_1))}+\lambda\vert\psi'\vert^3_{C^0(B(R_1))}\big)}\Big)^{\frac{1}{\rho}}\Big\}$
σ	$\geq$	$2n\vert\phi''\vert_{C^{0,\rho}(B_{R_2})}R_2^\rho$
ϵ_0	$\leq$	$\frac{1}{2n\vert f''\vert_{C^0(B_{R_2})}}$
τ_0	$\geq$	$\max\{1,\ 64\Big(4M_1+\frac{1}{4\lambda\phi_0}\Big)\Big(\vert f''\vert^2_{C^0}(1+n^2\vert g^{jk}\vert_{C^0})^2+n\vert h\vert^2_{L^\infty}(2+2\vert f'\vert^2_{C^0})+$ $2\vert q\vert^2_{L^\infty}\Big)\}$
R	$\leq$	$\frac{1}{4}\Big(16+\frac{1}{16}\Big)^{-1/2}R_2$
δ	$\leq$	$n\frac{1}{32}\Big(16+\frac{1}{16}\Big)^{-1}\vert\phi''\vert_{C^{0,\rho}(B_{R_2})}R_2^{2+\rho}$
r	$\leq$	$\frac{n\vert\phi''\vert_{C^{0,\rho}(B_{R_2})}\frac{1}{4}\left(16+\frac{1}{16}\right)^{-1}R_2^{2+\rho}}{\vert\phi'\vert_{C^0(B_{R_2})}+5n\vert\phi''\vert_{C^{0,\rho}(B_{R_2})}R_2^{1+\rho}}$
r_0	$\leq$	$\frac{n\lambda^2C_l^2\frac{1}{4}\left(16+\frac{1}{16}\right)^{-1}R_2^{2+\rho}}{2e\left(\vert\phi'\vert_{C^0(B_{R_2})}+5n\vert\phi''\vert_{C^{0,\rho}(B_{R_2})}R_2^{1+\rho}\right)}$
$c_{1,T}$	$\geq$	$\sqrt{4\Big(\frac{4M_1}{\tau_0}+\frac{1}{4(\lambda\phi_0)}\Big)}$
$c_{2,T}$	$\geq$	$(\frac{1}{2}+\sqrt{2M_2})(1+\frac{2\vert\chi_1'\vert_{C^0}}{\tau_0\kappa})+\frac{c_{1,T}}{\sqrt{\tau_0}}c_{133}$
c_{111}	$\geq$	$\ln(1+e)+6c_{131}\Big(1+\frac{\vert b'\vert^2_{C^0}}{r^2}\Big)^{1/2}+2c_{129}$

Here the coefficients $c_{129}, c_{131}, c_{133}$ are defined and derived: c_{133} in subsection 3.2; c_{129} and c_{131} in the proof of Theorem 1.1.

3.2. **Tataru inequality for the wave operator.** We now go quickly trough [6] to compute the coefficients of the inequality in Theorem 2.1. We decompose the wave operator (1.1) into the sum of its principal part P_2 and the lower order part P_1

$$\begin{aligned} P_2(y,D) &= -D_0^2+g^{jk}(x)D_jD_k \\ P_1(y,D) &= h^j(x)D_j+q(x) \end{aligned}$$

We then consider the conjugate operator $P(y, D + i\tau f')$ and split it into its principal part P_3 and the lower order part P_4

$$\begin{aligned}
P(y, D + i\tau f'(y)) &= e^{\tau f(y)} P(y,D) e^{-\tau f(y)} = P_3(y, D, \tau) + P_4(y, D, \tau)\\
P_3(y, D, \tau) &= P_2(y, D) + \tau^2((f_0')^2 - g^{jk} f_j' f_k') + 2i\tau(-f_0' D_0 + g^{jk} f_j' D_k)\\
P_4(y, D, \tau) &= -\tau(f_0'' - g^{jk} f_{jk}'') + P_1(y, D + i\tau f')
\end{aligned}$$

The principal symbol of $P(y, D)$ and $P(y, D + i\tau f')$ are respectively

$$\begin{aligned}
p(y, \xi) &= -\xi_0^2 + g^{jk}(x)\xi_j\xi_k\\
p(y, \xi + i\tau f') &= p(y, \xi) - \tau^2 p(y, f') + i\tau\{p, f\}
\end{aligned}$$

Since f is a quadratic function and the coefficients g^{jk} are time independent we can write the following expression

$$\begin{aligned}
e^{-\epsilon|D_0|^2/(2\tau)} e^{\tau f} P(y, D) u &= e^{-\epsilon|D_0|^2/(2\tau)} P(y, D + i\tau f') e^{\tau f} u\\
&= P(y, D - \epsilon f'' \cdot (D_0, 0) + i\tau f') e^{-\epsilon|D_0|^2/(2\tau)} e^{\tau f} u
\end{aligned}$$

Call $\vec{D} = D - \epsilon f'' \cdot (D_0, 0)$ and $\vec{\xi_j} = \xi_j - \epsilon f_{j0}''\xi_0$, $j = 0, 1, \ldots, n$.
If ϵ is such that $2n\epsilon|f''|_{C^0} \leq 1$, then we get $|\vec{\xi_j}|^2 \leq 2|\xi_j|^2 + 2\epsilon^2|f''|_{C^0}^2\xi_0^2$ and $\frac{1}{2}|\xi|^2 \leq |\vec{\xi}|^2 \leq 2|\xi|^2$.
Since $p(y, \xi + i\tau f')$ is the symbol of $P_3(y, D, \tau)$, then $p(y, \widehat{\xi} + i\tau f')$ is the symbol of $P_3(y, \vec{D}, \tau)$. Now set $\vec{\xi}$ in place of ξ into the inequality (3.4), which becomes, for $V \in C_0^\infty(B(y_0, R_2))$,

$$2M_2\||D_0|V\|^2 + 4M_1\|P_3(y, \vec{D}, \tau)V\|_{-1,\tau}^2 + \frac{\mathrm{Im}\langle \mathrm{Re}(P_3(y, \vec{D}, \tau))V,\, \mathrm{Im}(P_3(y, \vec{D}, \tau))V\rangle}{(\lambda\phi_0)2\tau} \geq \frac{1}{4}\|V\|_{1,\tau}^2\,.$$

Observing that $\|W\|_0^2 \geq \tau^2\|W\|_{-1,\tau}^2$, $\|P_3W\|_0^2 \geq 2\mathrm{Im}\langle(\mathrm{Re}P_3)W, (\mathrm{Im}P_3)W\rangle$ and for $\tau \geq 1$ we get

$$2M_2\||D_0|V\|^2 + \Big(\frac{4M_1}{\tau} + \frac{1}{4(\lambda\phi_0)}\Big)\frac{\|P_3(y, \vec{D}, \tau)V\|_0^2}{\tau} \geq \frac{1}{4}\|V\|_{1,\tau}^2\,. \tag{3.11}$$

We now estimate the error term E_1:

$$\begin{aligned}
E_1 &:= \|P(y, \vec{D} + i\tau f')V - P_3(y, \vec{D}, \tau)V\|_0^2\\
&= \| - \tau(f_0'' - g^{jk} f_{jk}'')V + P_1(y, \vec{D} + i\tau f')V\|_0^2\\
&\leq 2\tau^2\||f''|_{C^0}(1 + n^2|g^{jk}|_{C^0})V\|_0^2 + 2\|h^j D_j V - \epsilon h^j f_{0j}'' D_0 V\|_0^2\\
&\quad +2\tau^2\|(n|h|_{L^\infty}|f'|_{C^0} + |q|_{L^\infty})V\|_0^2\\
&\leq 4\Big(|f''|_{C^0}^2(1 + n^2|g^{jk}|_{C^0})^2 + n|h|_{L^\infty}^2(2 + 2|f'|_{C^0}^2) + 2|q|_{L^\infty}^2\Big)\|V\|_{1,\tau}^2.
\end{aligned}$$

Now choose $\tau_0 > 1$ such that $\frac{2}{\tau_0}\Big(4M_1 + \frac{1}{4\lambda\phi_0}\Big)E_1 \leq \frac{1}{8}\|V\|_{1,\tau}^2$ and call $c_{1,T} := \sqrt{4\Big(\frac{4M_1}{\tau_0} + \frac{1}{4(\lambda\phi_0)}\Big)}$. From (3.11) and $\|P_3(\vec{D})v\|^2 \leq 2E_1 + 2\|P(\vec{D})v\|^2$, we have after multiplying by 2 and squaring, for $\tau \geq \tau_0$,

$$\sqrt{2M_2}\||D_0|V\|_0 + c_{1,T}\frac{\|P(y, \vec{D} + i\tau f')V\|_0}{\sqrt{\tau}} \geq \frac{1}{2}\|V\|_{1,\tau}\,. \tag{3.12}$$

Now consider $u \in H^1(B_{\kappa/4})$ and define $v := e^{-\epsilon|D_0|^2/(2\tau)}e^{\tau f}u$, $V := \chi_1(t/(2\kappa))v$, with χ_1 as in (4.3) with $N = 1$, $B_1 = [-1, 1]$ $B_2 = [-2, 2]$. Hence, $\mathrm{supp}(V) \subset \{y;\ |t| \le 4\kappa, |x| \le \kappa/4\} \subset \{y;\ |y - y_0| \le R_2\}$ with $\kappa = \left(16 + \frac{1}{16}\right)^{-1/2} R_2$. Due to Lemma 3.4 in [6] (see also Lemma 2.79 in [10]) the following inequalities hold:

$$\begin{aligned}
&\|P(y, \vec{D} + i\tau f')V - \chi_1(t/(2\kappa))e^{-\epsilon|D_0|^2/(2\tau)}e^{\tau f}P(y, D)u\|_0 \\
&= \|[P(y, \vec{D} + i\tau f'), \chi_1(t/(2\kappa))]v\|_0 \\
&\le c_{133}\|(1 - \chi_1(t/\kappa))(\nabla + \tau)v\|_0 \le c_{133}e^{-\tau\kappa^2/(4\epsilon)}\|e^{\tau f}u\|_{1,\tau},
\end{aligned}$$

and

$$\begin{aligned}
\||D_0|V\|_0 &\le \||D_0|v\|_0 + \frac{2|\chi_1'|_{C^0}}{\kappa}\|(1 - \chi_1(t/\kappa))v\| \\
&\le \frac{2\kappa\tau}{\epsilon}\|v\|_0 + \left(1 + \frac{2|\chi_1'|_{C^0}}{\tau_0\kappa}\right)e^{-\tau\kappa^2/(4\epsilon)}\|e^{\tau f}u\|_{1,\tau},
\end{aligned}$$

and

$$\|v\|_{1,\tau} \le \|V\|_{1,\tau} + \left(1 + \frac{2|\chi_1'|_{C^0}}{\tau_0\kappa}\right)e^{-\tau\kappa^2/(4\epsilon)}\|e^{\tau f}u\|_{1,\tau},$$

for $\tau \ge \tau_0$ and $c_{133} = 2(1+n^2|g^{jk}|_{C^0})\left(\frac{|\chi_1''|_{C^0}}{\tau_0\kappa^2} + \frac{|\chi_1'|_{C^0}}{\kappa}(1+|f'|_{C^0} + \frac{|h|_{L^\infty}}{\tau_0})\right)$. As last step we use the above relations to estimate the terms of (3.12) and we notice that $\sqrt{2M_2}\frac{2\kappa}{\epsilon_0} < \frac{1}{4}$ according to our choice of the parameters. Therefore, for $\tau > \tau_0$, we obtain the Tataru inequality of Theorem 2.1 with coefficients as in Table 3.10.

Remark 3.2. Observe that, according to the computations above, ϵ cannot be arbitrarily smaller that ϵ_0, since this affects the size of R_2 and τ.

3.3. **Proof of Proposition 2.5.** In the previous subsection we considered $u \in H^1(B_R)$ where the radius R is defined as $R := qR_2$ with $q = \frac{1}{4}\left(16 + \frac{1}{16}\right)^{-1/2}$, with R_2 defined in Table (3.10).
Let us compute δ such that the region $I_B := \{y \in B(y_0, R_2);\ f(y) - \phi(y_0) \ge -8\delta\} \cap \{y \in B(y_0, R_2); \phi \le \phi(y_0)\}$ is inside the ball $B(y_0, R)$. By assumption, in $B(y_0, R_2) - \{y_0\}$ we have

$$f(y) - \phi(y) < -c_T R_2^\rho |y - y_0|^2.$$

Moreover, in I_B we have $f(y) - \phi(y_0) \le f(y) - \phi(y)$.
Hence, the limit case is reached along the boundary $\{y;\ |y - y_0| = R\}$, where

$$f(y) - \phi(y_0) < -c_T q^2 R_2^{2+\rho}.$$

Define δ such that $-c_T q^2 R_2^{2+\rho} \le -8\delta$, i.e.

$$\delta = c_T q^2 R_2^{2+\rho}/8 = n|\phi''|_{C^{0,\rho}} q^2 R_2^{2+\rho}/8\,.$$

Under this condition, the set I_B is inside $B(y_0, R)$.
In order to compute the smaller radius r we apply a rougher estimate, using the definition of f. Consider $\{y;\ |f - \phi(y_0)| \le \delta\} \cap \{y;\ |y - y_0| \le 2r\}$, then

$$|f(y) - \phi(y_0)| \le |f'|_{C^0(B_{R_2})}|y - y_0| \le |f'|_{C^0}2r \le \delta.$$

Hence the solution is $r \le \frac{\delta}{2|f'|_{C^0(B_{R_2})}}$, that is guaranteed by

$$r \le \frac{n|\phi''|_{C^{0,\rho}(B_{R_2})} q^2 R_2^{2+\rho}}{2|\phi'|_{C^0(B_{R_2})} + 10n|\phi''|_{C^{0,\rho}(B_{R_2})} R_2^{1+\rho}} \quad (\le R_2/10).$$

By hypothesis $\phi'(y_0) \neq 0$, hence the denominator does not vanish.
If we choose $\lambda > 2|\psi''|_{C^0(\Omega_0)}/C_l^2$ and apply $\psi'(y) > C_l$ we obtain in B_{R_1}

$$\phi''(y) = \phi\lambda(\psi'' + \lambda\psi' \times \psi') \ge e^{-1}\lambda^2 C_l^2/2.$$

Consequently $\phi''(y) \neq 0$ and $|\phi''|_{C^{0,\rho}(B_{R_2})} > C_\rho$, with $C_\rho := e^{-1}\lambda^2 C_l^2/2 > 0$ we get an uniform lower bound for r in B_{R_1}

$$r_0 \le \frac{nC_\rho q^2 R_2^{2+\rho}}{2|\phi'|_{C^0(B_{R_2})} + 10n|\phi''|_{C^{0,\rho}(B_{R_2})} R_2^{1+\rho}}.$$

Acknowledgment RB was supported by the postdoc programme of the Hausdorff Center for Mathematics in Bonn and by the AXA Mittag-Leffler Fellowship Project, funded by the AXA Research Fund. YK and RB were partly supported by EPSRC. ML and RB were partly supported by the Academy of Finland, CoE-project 250215 and project 273979. This work was partly done at the Isaac Newton Institute for Mathematical Sciences and the Mittag-Leffler Institute. The authors would like to express their gratitude to these institutions for their hospitality.

4. Appendix

We recall the results on Gevrey class functions that are used in the article. The reference is [7, 18].

Definition 4.1. Let L_s be an increasing sequence of positive numbers such that

$$L_0 = 1, \quad s \le L_s, \quad L_{s+1} \le CL_s,$$

for some constant $C > 1$. We denote by C^L the set of all $u \in C^\infty(X)$ (with $X \subset \mathbb{R}^N$ open subset) for which to every compact set $K \subset X$ there is a constant C_K such that

$$|D^\zeta u(x)| \le C_K (C_K L_{|\zeta|})^{|\zeta|}, \quad x \in K,$$

for all multi-indices ζ. By Stirling's formula we could replace $|\zeta|^{|\zeta|}$ by $|\zeta|!$. $C^L(X)$ is a ring which is closed under differentiation. If $f : Y \to X$ is and analytic map from the open set $Y \subset \mathbb{R}^N$ to the open set $X \subset \mathbb{R}^N$, then the composition with f defines the map $f^* : C^L(Y) \to C^L(X)$, $f^*u = u \circ f$.
The class $C^L(X)$ with $L_s = (s+1)^m$ and $m > 1$ is called the *Gevrey class of order* m and denoted by $G^m(X)$. If $m = 1$, then $G^1(X)$ is the set of real analytic functions in X.
We denote by $G_0^m(\mathbb{R}^N)$ the set $G_0^m(\mathbb{R}^N) = G^m(\mathbb{R}^N) \cap C_0^\infty(\mathbb{R}^N)$. For $m > 1$ one has $\sum 1/k^m < \infty$, and it follows from Th.1.4.2 in [7] that G_0^m is so large that one can find cutoff functions there; it is of course an algebra.
In particular, let $f, g \in G^m(\mathbb{R}^N)$ and let $K \subset R^N$ be a compact set, then by calling $c_{1,f}$ and $c_{1,g}$ the constants C_K for f and g, we get $fg \in G^m(\mathbb{R}^N)$ such that for $c_P = \max\{c_{1,f}, c_{1,g}\}$

$$|D^\kappa(f(x)g(x))| \le 2^{|\kappa|} c_P^{|\kappa|+2} |\kappa|^{m|\kappa|}, \quad x \in K. \tag{4.1}$$

Moreover, if E is a compact set in $\mathbb{R}^N$, then we define the *'the supporting function of E'* H_E as ([7], (4.3.1), p. 105)

$$H_E(\xi) = \sup_{x\in E}\langle x,\xi\rangle, \quad \xi \in \mathbb{R}^N.$$

In the present paper we widely use the Paley-Wiener-Schwartz Theorem for Gevrey class functions. As reference we give the following statement proved in [8] for a proper subset $\gamma_0^m(\mathbb{R}^N)$ of $G_0^m(\mathbb{R}^N)$. The Theorem can also be reformulated for $\phi \in G_0^m$ once we substitute the sentence "to every $B > 0$ there exists a constant C_B such that" with "there exist positive constants B and C such that". The proof is the same.

Theorem 4.2. *([8], Th.12.7.4, p.137) An entire function $\Phi(\zeta), \zeta \in \mathbb{C}^N$, is the Fourier-Laplace transform of a function $\phi \in \gamma_0^m(\mathbb{R}^N)$ with support in the compact convex set K with supporting function H_K if and only if to every $B > 0$ there exists a constant C_B such that*

$$|\Phi(\zeta)| \le C_B \exp(H_K(Im\zeta) - B|Re\zeta|^{1/m}), \quad \zeta \in \mathbb{C}^N.$$

In particular we can introduce the main properties of the Gevrey class localizers used in the paper.

Definition 4.3. Define $\chi_1 \in G_0^m(\mathbb{R}^N)$ and $\chi_\delta(v) := \chi_1(v/\delta)$ such that $\chi_1 = 1$ in a ball $B_1 \subset \mathbb{R}^N$, $\chi_1 = 0$ outside a larger ball B_2, and $0 \le \chi_1 \le 1$. Hence, $\mathcal{F}_{v\to\zeta}\chi_\delta(v) = \delta^N \mathcal{F}_{v\to\delta\zeta}\chi_1(v)$ and

$$\begin{aligned}
(4.2)\quad |D^\kappa \chi_1(v)| &\le c_{1X}^{|\kappa|+1}|\kappa|^{m|\kappa|}, \quad v \in B_2,\\
(4.3)\,|\mathcal{F}_{v\to\zeta}\chi_1(v)| &\le c_{1X}\exp(H_{B_2}(\mathrm{Im}\zeta) - c_{2X}|\mathrm{Re}\zeta|^{1/m}), \quad \zeta \in \mathbb{C},\\
(4.4)\,|\mathcal{F}_{v\to\zeta}\chi_\delta(v)| &\le \delta^N c_{1X}\exp(\delta H_{B_2}(\mathrm{Im}\zeta) - c_{2X}\delta^{1/m}|\mathrm{Re}\zeta|^{1/m}), \quad \zeta \in \mathbb{C},
\end{aligned}$$

with $c_{1X} = c_{1X}(m)$ a given number, and $c_{2X} = 1/(eNc_{1X})^{1/m}$.

In the following we present the Phragmen-Lindelöf Theorem for subharmonic functions used in Lemma 2.7.

Theorem 4.4. *([15], Ch. 7.3.) Let D be an angle of opening π/λ, and let $u(z)$ be a function subharmonic in this angle, satisfying an asymptotic estimate*

$$u(z) < |z|^\rho, \quad a.e., \quad \rho < \lambda,$$

and bounded by a constant M on the boundary of the angle. Then $u(z) \le M$ inside the full angle D.

We now recall the concept of conormal pseudoconvexity for operators as given in [20, 21].
If S is a C^2-oriented hypersurface, we can represent it as level set surface of a C^2-function:

$$S := \{y;\, \psi(y) = 0\}$$

where $\psi' \ne 0$ on S.

Definition 4.5. Decompose the coordinates of $\mathbb{R}^N$ into $y = (y', y'')$.
The conormal bundle of the foliation F of $\mathbb{R}^N$ with the surfaces y'' =const is the set

$$N^*F := \{(y,\xi) \in T^*\mathbb{R}^N;\ \text{with } \xi = (\xi',\xi'') \text{ and } \xi' = 0\}.$$

Its reduction to a subset $K \subset \mathbb{R}^N$ is

$$\Gamma_K := \{(y,\xi) \in T^*K,\, \xi' = 0\},$$

while its fibre in y_0 is

$$\Gamma_{y_0} := \{(y_0,\xi) \in N^*F\}.$$

Let $P(y,D)$ be a partial differential operator of order m with smooth coefficients. Denote by $p(y,\xi)$ its principal symbol.

Definition 4.6. Let S be a smooth oriented hypersurface which is a level surface of a C^2 function ψ, and $y_0 \in S$, $\psi'(y_0) \neq 0$. We say that S is *conormally strongly pseudoconvex* with respect P at y_0 if

$$\begin{aligned}
&(4.5) \qquad Re\{\overline{p},\{p,\psi\}\}(y_0,\xi) > 0 \\
&\qquad\qquad \text{on} \quad p(y_0,\xi) = \{p,\psi\}(y_0,\xi) = 0, \quad 0 \neq \xi \in \Gamma_{y_0}; \\
&(4.6) \qquad \{\overline{p(y,\xi+i\tau\psi'(y))}, p(y,\xi+i\tau\psi'(y))\}/(2i\tau) > 0 \\
&\qquad\qquad \text{on}\, y = y_0, \text{ such that } \quad 0 \neq \xi \in \Gamma_{y_0},\, \tau > 0, \\
&\qquad\qquad \text{and } p(y_0,\xi+i\tau\psi'(y_0)) = \{p(y,\xi+i\tau\psi'(y)),\psi(y)\}(y=y_0) = 0.
\end{aligned}$$

Definition 4.7. A C^2 real valued function ψ is *conormally strongly pseudoconvex* with respect to P at y_0 if

$$\begin{aligned}
&(4.7) \qquad Re\{\overline{p},\{p,\psi\}\}(y_0,\xi) > 0 \\
&\qquad\qquad \text{on} \quad p(y_0,\xi) = 0, \quad 0 \neq \xi \in \Gamma_{y_0}; \\
&(4.8) \qquad \{\overline{p(y,\xi+i\tau\psi'(y))}, p(y,\xi+i\tau\psi'(y))\}/(2i\tau) > 0 \\
&\qquad\qquad \text{on } y = y_0, \text{ such that } p(y_0,\xi+i\tau\psi'(y_0)) = 0, \quad 0 \neq \xi \in \Gamma_{y_0},\, \tau > 0.
\end{aligned}$$

Hence, the term 'conormally strongly pseudoconvex' means 'strongly pseudoconvex in N^*F or in a subset Γ_K'. Definition 4.6 implies that if Ω_0 is a sufficiently small neighborhood of y_0, then there are constants such that an inequality like (3.1) holds, while Definition 4.7 implies that the inequality (3.2) holds for the function $\phi = e^{\lambda\psi}$.

For second order differential operators the definitions above are simpler. In particular, for the wave operator (1.1) the conditions are void for noncharacteristic surfaces ψ, as shown in section 3.1, see also Remark 3.1.

References

[1] G. Alessandrini, L. Rondi, E. Rosset, S. Vessella, *The stability for the Cauchy problem for elliptic equations*, Inverse Problems 25 (12), (2009).

[2] M. Anderson, A. Katsuda, Y. Kurylev, M. Lassas, M. Taylor, *Boundary regularity for the Ricci equation, Geometric Convergence, and Gel'fand's Inverse Boundary Problem*, Inventiones Mathematicae, 158, 261-321, (2004).

[3] J. Bourgain, E. C. Kenig, *On localization in the continuous Anderson-Bernoulli model in higher dimension*, Invent. Math. 161, no. 2, 389426, (2005).

[4] M. Eller, V. Isakov, G. Nakamura, D. Tataru, *Uniqueness and stability in the Cauchy problem for Maxwell and elasticity systems*, Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XIV, 329349, Stud. Math. Appl., 31, North-Holland, Amsterdam, (2002).

[5] L. Escauriaza, C.E. Kenig, G. Ponce, L. Vega, *Unique continuation for Schrödinger evolutions, with applications to profiles of concentration and traveling waves.* Comm. Math. Phys. 305, no. 2, 487512, (2011).

[6] L. Hoermander, *On the uniqueness of the Cauchy problem under partial analiticity assumptions*, in "Geometrical optics and related topics", Birkaeuser, (1997).
[7] L. Hoermander, The analysis of linear partial differential equations I, Springer, (1990).
[8] L. Hoermander, The analysis of linear partial differential equations II, Springer, (1990).
[9] L. Hoermander, The analysis of linear partial differential equations IV, Springer, (1990).
[10] A. Katchalov, Y. Kurylev, M. Lassas, Inverse boundary spectral problems, Chapman & Hall/CRC, (2001).
[11] V. Isakov, Inverse Problems for Partial Differential Equations, Springer, (2005).
[12] O. Imanuvilov, V. Isakov, M. Yamamoto, *An inverse problem for the dynamical Lame system with two sets of boundary data*, Comm. Pure Appl. Math., 56, 1366-1382, (2003).
[13] O. Imanuvilov, V. Isakov, M. Yamamoto, *New realization of the pseudoconvexity and its application to an inverse problem*, Appl. Anal. 88, no. 5, 637652, (2009).
[14] Y. Kurylev, M. Lassas, T. Yamaguchi, *Uniqueness and Stability in Inverse Spectral Problems for Collapsing Manifolds*, preprint.
[15] B. Ya. Levin, Lectures on entire functions, American Mathematical Society, (1996).
[16] L. Robbiano, *Fonction de coût et contrôle des solutions des équations hyperboliques*, Asym. Anal. 10, 95-115, (1995).
[17] L. Robbiano, C. Zuily, *Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients,* Invent. Math. 131, no. 3, (1998).
[18] L. Rodino, *Linear Partial Differential Operators in Gevrey Spaces*, World Scientific, (1993).
[19] D. Tataru, *Carleman estimates, unique continuation and applications*, unpublished manuscript, see $http://math.berkeley.edu/\sim tataru/ucp.html$.
[20] D. Tataru, *Unique continuation for solutions to PDE's; Between Hörmander's theorems and Holmgren's theorem*, Comm. PDE, 20 (5& 6), 855-884 (1995).
[21] D. Tataru, *Unique continuation operators with partially analytic coefficients*, J. Math. Pures Appl., 78, 505-521 (1999).
[22] D. Tataru, *Carleman estimates, unique continuation and controllability for anizotropic PDEs*. Optimization methods in partial differential equations, 267279, Contemp. Math., 209, Amer. Math. Soc., Providence, RI, (1997).

Department of Mathematics and Statistics, University of Helsinki
P.O. Box 68, FI-00014 Helsinki.
E-mail address: roberta.bosi@helsinki.fi

Department of Mathematics, University College London
Gower Street, LONDON WC1E 6BT
E-mail address: y.kurylev@ucl.ac.uk

Department of Mathematics and Statistics, University of Helsinki
P.O. Box 68, FI-00014 Helsinki.
E-mail address: matti.lassas@helsinki.fi