Semiglobal boundary rigidity for Riemannian metrics

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1 Introduction and statement of the result

Let (M, g) be a Riemannian manifold with boundary. In this paper we consider the problem of determining the metric g from its associated boundary distance function

$$d_g(x, y) = \text{dist}\,(x, y), \quad x, y \in \partial M,$$

that is the geodesic distance between boundary points. This problem arose in rigidity questions in Riemannian geometry [12], [6], [8]. For the case in which M is a bounded domain of Euclidean space and the metric is conformal to the Euclidean one, this problem is called the inverse kinematic problem which arose in Geophysics and has a long history starting at least in the early part of the 20th century with Herglotz [10]. He considered the case where M is a ball $\{x \in \mathbf{R}^3 \mid r = |x| \leq R\}$ equipped with a spherically symmetric metric $ds^2 = dx^2/c^2(r)$ where $c(r)$ is a positive function depending only on the radius $r = |x|$. Herglotz found a formula to determine $c(r)$ from the boundary distance function. Physically this corresponds to the case of a spherically symmetric Earth model with an index of refraction depending only on the radius. The boundary distance function corresponds to the travel times of e.g. acoustic waves going through the Earth and measured at the surface. The general problem for the case that the sound speed depends on all variables has been extensively studied (see for instance [17] and the references given there). Also, this problem has a close connection for other inverse problems related to determining the sound speed from boundary measurements, see [20].

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The problem of determining the metric tensor from the boundary distance function is not in general uniquely solvable. Indeed, let $\psi : M \to M$ be a diffeomorphism that fixes the boundary, $\psi|_{\partial M} = \mathrm{Id}$. Since ψ is the isometry of the Riemannian manifold $(M, \psi^* g)$ onto (M, g), it is easy to see that the boundary distance functions of the metrics g and $\psi^* g$ coincide. A Riemannian manifold (M, g) is said to be boundary rigid if this is the only obstruction to unique identifiability of the metric. More precisely, (M, g) is boundary rigid if, for any other Riemannian metric g' on M, the equality $d_g = d_{g'}$ implies existence of a diffeomorphism $\psi : M \to M$ which is the identity on the boundary and such that $\psi^* g = g'$.

There are evident examples of manifolds that are not boundary rigid. Indeed, one can construct a metric g with a point $x_0 \in M$ such that $d_g(x_0, \partial M) > \sup_{x,y\in\partial M} d_g(x, y)$. For such a metric, d_g is independent of a change of g in a neighborhood of x_0. Therefore it is necessary to impose some a-priori restrictions on the metric. One such restriction is to assume that the Riemannian manifold is simple, i.e., given two points there is a unique geodesic joining the points and ∂M is strictly convex. ∂M is strictly convex if the second fundamental form of the boundary is positive definite in every boundary point.

Although the boundary rigidity problem has been extensively studied last two decades, there are very few global results for this problem. It is proved that a simple metric is uniquely determined in a prescribed conformal class by the boundary distance function [14], [2], [6]. In the two-dimensional case, boundary rigidity is proved for metrics of constant Gaussian curvature [12] and of nonpositive curvature [5], [15]. Boundary rigidity of flat metrics is proved in the multidimensional case [8]. Only recently some local results were obtained in [7] and [19] in which one assumes that the metric is a-priori close to a given metric. In the latter articles, the linearized version of the boundary rigidity problem is used which turns out to be the integral geometry problem for tensor fields (see [18] and the references given there).

In this paper we prove a semiglobal result, that is on a bounded domain of Euclidean space with smooth boundary we prove that two metrics with the same boundary distance function are isometric via an isometry which is the identity at the boundary provided that one metric is sufficient close to the Euclidean metric and the other satisfies an a-priori bound on the curvature (see Theorem 1.1). To obtain this result we show also that the boundary distance function determines all boundary values of derivatives of the metric tensor, that is, the C^∞-jet of the metric at the boundary (see Theorem 2.1). By combining Theorem 2.1 with a result of [11] we can prove that the distance function d_g determines a class of real-analytic manifolds (see Theorem 2.2 for a precise statement).

Before we state the result we introduce some notation and definitions.

Let (M, g) be a compact Riemannian manifold. If T is a real covariant tensor field of rank m on M, then its modulus, which is defined in local coordinates by

$$|T|_g^2 = g^{i_1 j_1} \dots g^{i_m j_m} T_{i_1 \dots i_m} T_{j_1 \dots j_m},$$

is independent of the choice of coordinates. By

$$\|T\|_{C^k(M,g)} = \sum_{l=0}^{k} \sup_{x\in M} |\underbrace{\nabla \dots \nabla}_{l} T(x)|_g$$

we denote the C^k-norm of the tensor field T. Here ∇ denotes the covariant derivative. We remark that $\|T\|_{C^k(M,g)}$ is also invariantly defined, i.e., is independent of the choice of coordinates. The same holds for the H^k-norm of T which is defined by

$$\|T\|^2_{H^k(M,g)} = \sum_{l=0}^{k} \int_M |\underbrace{\nabla \dots \nabla}_{l} T(x)|^2_g \, dV_g(x),$$

where dV_g is the Riemannian volume form.

For $x \in M$ and a two-dimensional subspace σ of the tangent space T_xM, we denote by $\mathrm{Sec}\,(x,\sigma)$ the sectional curvature at the point x in the direction σ. For $0 \neq \xi \in T_xM$, we denote

$$\mathrm{Sec}\,(x,\xi) = \sup_{\sigma \ni \xi} K(x,\sigma), \qquad \mathrm{Sec}^+(x,\xi) = \max\{\mathrm{Sec}\,(x,\xi), 0\}.$$

We also define

$$\kappa^+(M,g) = \sup \int_0^l t\, \mathrm{Sec}^+(\gamma(t), \dot\gamma(t))\, dt,$$

where the supremum is taken over all unit speed geodesics $\gamma : [0,l] \to M$. Finally we denote by R_g the curvature tensor of the metric g and e the Euclidean metric in $\mathbf{R}^n$. Our main result is as follows:

Theorem 1.1 *Let $D \subset \mathbf{R}^n$ be a closed bounded domain with a smooth strictly convex (with respect to the Euclidean metric) boundary ∂D. Let $K > 0$ and g be a Riemannian metric on D satisfying the conditions*

$$\|R_g\|_{C^k(D,g)} \le K, \tag{1.1}$$

$$\kappa^+(D,g) < 1/4, \tag{1.2}$$

where $k = [n/2] + 18$ and $[n/2]$ denotes the integer part of $n/2$. Let g' be another Riemannian metric on D satisfying

$$d_g = d_{g'}.$$

Then there exists $\varepsilon = \varepsilon(K, D, n) > 0$ such that if

$$\|g'_{ij} - \delta_{ij}\|_{C^l(D,e)} < \varepsilon \tag{1.3}$$

with $l = [n/2] + 20$, then the metrics g and g' are isometric via an isometry which is the identity on the boundary.

We remark that the hypothesis (1.2) guarantees invertibility, modulo the natural obstruction, of the ray transform of the metric g [18].

In Section 2 we prove that we can determine from the boundary distance function, up to the natural obstruction, all the derivatives of the metrics at the boundary. This will allow us to extend the two metrics g, g' as in Theorem 1.1 to be the same outside D with the same boundary distance function on any bounded set containing D. Therefore we can reduce the proof of Theorem 1.1 to the case that D is a ball of sufficiently large radius (see Lemma 2.3).

In Section 3 we use special coordinates, called semigeodesic coordinates (or boundary normal coordinates) to eliminate the nonuniqueness caused by an isometry. The proof of Theorem 1.1 is reduced to show that metrics g and g' coincide in these coordinates. This is accomplished by showing that the hypothesis that the boundary distance function is the same on the ball and the conditions of Theorem 1.1 imply that in fact the metric g is also close to the Euclidean metric on an appropriate cube. This is stated in Lemma 3.1. The latter lemma implies Theorem 1.1 with the help of the main result of [19].

In Section 4 we estimate the components of a metric tensor in semigeodesic coordinates through the curvature tensor. In fact this estimation is a weak version of Cheeger's method for proving precompactness of families of metrics under some curvature conditions [3].

Lemma 3.1 is proved in Section 5. The main ingredients here are the interpolation inequality and the stability estimate for the inverse of the ray transform. In order to use the ray transform we need the boundary distance function of a strictly convex smooth domain rather than of a cube.

2 The boundary C^∞-jet of a metric is determined by the boundary distance function

Given a connected Riemannian manifold (M, g) with boundary, we denote by $d_g : \partial M \times \partial M \to \mathbf{R}$ the boundary distance function. We recall that, for a diffeomorphism $\varphi : M \to M$ which is the identity at the boundary, i.e. $\varphi|_{\partial M} = \mathrm{Id}$, the metrics g and $g' = \varphi^* g$ have the same boundary distance functions, that is, $d_g = d_{g'}$. We say that the boundary ∂M is convex if the following holds: for every two points $p_0, p_1 \in \partial M,\ p_0 \neq p_1$, there exists a geodesic $\gamma : [0,1] \to M$ joining these points, $\gamma(0) = p_0,\ \gamma(1) = p_1$, such that the length of γ is equal to $d_g(p_0, p_1)$, and all inner points of γ belong to $M \setminus \partial M$.

Theorem 2.1 *Let (M, g) be a connected Riemannian manifold with convex boundary. Then the C^∞-jet of the metric g at the boundary is uniquely determined by the boundary distance function d_g in the following sense. If ∂M is convex with respect to another metric g' on M, then the equality $d_g = d_{g'}$ implies the existence of a diffeomorphism $\varphi : M \to M$ which is the identity on the boundary, $\varphi|_{\partial M} = Id$, and such that the metrics g and $g'' = \varphi^* g'$ satisfy the following: In any local coordinate system $(x^1, \dots, x^n)$ defined in a neighborhood of a boundary point, we have $D^\alpha g|_{\partial M} = D^\alpha g''|_{\partial M}$ for every multi-index α.*

This result was proven by Michel in two dimensions [13] and for $|\alpha| \le 2$ in [12]. Theorem 2.1 has the following corollary:

Theorem 2.2 *Let M be a compact real-analytic manifold with a real-analytic boundary. Let g, g' be two real-analytic Riemannian metrics on M such that M is simple with respect to the metrics g and g'. Assume $d_g = d_{g'}$. Then there exists a real-analytic diffeomorphism $\psi : M \to M$ with $\psi|_{\partial M} = Id$ such that $g = \psi^* g'$.*

The proof of Theorem 2.2 follows readily from Theorem 2.1 and Theorem C(a) of [11].

Proof of Theorem 2.1. Assume that the metrics g and g' satisfy the hypothesis of the theorem. Then g and g' induce the same metric on ∂M, i.e., for every point $p \in \partial M$ and for every vectors $\xi, \eta \in T_p(\partial M)$,

$$\langle \xi, \eta \rangle_g = \langle \xi, \eta \rangle_{g'}, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle_g$ is the inner product with respect to g.

Let us recall the definition of the boundary exponential map $\exp_{\partial M}$. Given $p \in \partial M$, let $\nu(p) \in T_pM$ be the unit inner normal vector to the boundary with respect to the metric g. The map $\exp_{\partial M}(p,t) = \exp_p(t\nu(p))$ is defined for sufficiently small $t \ge 0$ and maps some neighborhood of the set $\partial M \times \{0\}$ in $\partial M \times \mathbf{R}^+$ diffeomorphically onto some neighborhood of the boundary ∂M. Let $\exp'_{\partial M}$ denote the same map with respect to the metric g'. The map $\varphi = \exp_{\partial M} \circ (\exp'_{\partial M})^{-1}$, which we are considering in a small neighborhood of ∂M, can be extended to a diffeomorphism $\varphi : M \to M$ which is the identity on the boundary. Let $g'' = \varphi^* g'$. Then there exists a neighborhood $U \subset M$ of the boundary ∂M such that if $\exp_{\partial M}(p,t) \in U$ for a point $p \in \partial M$ and $0 \le t \le t_0$, then $\exp_{\partial M}(p,t) = \exp''_{\partial M}(p,t)$ for $0 \le t \le t_0$. In particular, $\nu(p) = \nu''(p)$.

To simplify notations, we will denote the above-constructed metric g'' by g' again. In other words, we can assume without loss of generality the initial metrics g and g' to satisfy $\exp_{\partial M}(p,t) = \exp'_{\partial M}(p,t)$ for an arbitrary point $p \in \partial M$ and for sufficiently small $t > 0$. We will show that, for such two metrics, the equality $d_g = d_{g'}$ implies that the C^∞-jets of the metrics at the boundary are the same.

We define
$$f = g - g'.$$
Let $\gamma : [0,1] \to M$ be a shortest geodesic of the metric g joining two boundary points, $p_0 = \gamma(0) \in \partial M,\ p_1 = \gamma(1) \in \partial M$; and let $\gamma' : [0,1] \to M$ be the shortest geodesic of the metric g' joining the same points, $p_0 = \gamma'(0),\ p_1 = \gamma'(1)$. We will prove the inequalities
$$If(\gamma) = \int_0^1 f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt \le 0, \tag{2.2}$$
$$I'f(\gamma') = \int_0^1 f_{ij}(\gamma'(t))\dot\gamma'^i(t)\dot\gamma'^j(t)\,dt \ge 0. \tag{2.3}$$
The integrands above are written in local coordinates but it is easy to see that they are independent of the choice of coordinates. Since γ' is the shortest geodesic of the metric g',
$$d^2_{g'}(p_0,p_1) = \int_0^1 g'_{ij}(\gamma'(t))\dot\gamma'^i(t)\dot\gamma'^j(t)\,dt \le \int_0^1 g'_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt.$$
This implies the inequality
$$If(\gamma) = \int_0^1 g_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt - \int_0^1 g'_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt \le$$
$$\le d^2_g(p_0,p_1) - d^2_{g'}(p_0,p_1) = 0$$
which proves (2.2). The inequality (2.3) is proved in the same way by changing the roles of g and g'.

We fix a point $p \in \partial M$ and introduce semigeodesic coordinates $(x^1,\dots,x^n) = (y^1,\dots,y^{n-1},z)$ in a neighborhood $U \subset M$ of the point such that the boundary is determined by the equation $z = 0,\ z \ge 0$ in U, and the length element ds_g of the metric g is given by
$$ds_g^2 = g_{\alpha\beta}dy^\alpha dy^\beta + dz^2$$

in these coordinates. In this and subsequent formulas, Greek indices vary from 1 to $n-1$; summation from 1 to $n-1$ means over repeated Greek indices. The coordinate lines $y=$ const are geodesics of the metric g orthogonal to the boundary. Therefore the same coordinate system will be also semigeodesic for the metric g', i.e., the length element $ds_{g'}$ of the metric g' is given by

$$ds^2_{g'} = g'_{\alpha\beta}dy^\alpha dy^\beta + dz^2.$$

Therefore, the tensor field $f = g - g'$ has components $f_{\alpha\beta} = g_{\alpha\beta} - g'_{\alpha\beta}$ and $f_{in} = 0, i = 1, ..., n$.

Using induction on k, we will prove that

$$\left.\frac{\partial^k f_{\alpha\beta}}{\partial z^k}\right|_{z=0} = 0 \tag{2.4}$$

for all k, α, β. For $k = 0$, (2.4) is valid because of (2.1). Assume that (2.4) holds for all k satisfying $0 \le k < l$, but is not valid for $k = l$. We choose a point $p_0 \in \partial M$ and vector $\xi_0 \in T_{p_0}(\partial M)$, $|\xi_0| = 1$, such that

$$\frac{\partial^l f_{\alpha\beta}}{\partial z^l}(p_0)\xi_0^\alpha\xi_0^\beta \neq 0.$$

Without loss of generality, let us assume that

$$\frac{\partial^l f_{\alpha\beta}}{\partial z^l}(p_0)\xi_0^\alpha\xi_0^\beta > 0.$$

Then there is a neighborhood $V \subset TM$ of the point (p_0, ξ_0) such that

$$\frac{\partial^l f_{\alpha\beta}}{\partial z^l}(p)\xi^\alpha\xi^\beta > 0 \tag{2.5}$$

for all $(p, \xi) \in V$. The inequality (2.5) holds also for all points (p, ξ) belonging to the conic neighborhood

$$CV = \{(p, \xi) \in TM \mid \xi \neq 0,\ (p, \xi/|\xi|) \in V\}$$

of the point (p_0, ξ_0).

Developing in Taylor series we have

$$f_{\alpha\beta}(y, z) = \frac{1}{l!}\frac{\partial^l f_{\alpha\beta}}{\partial z^l}(y, 0)z^l + o(z^l).$$

This combined with (2.5) implies the inequality

$$f_{\alpha\beta}(p)\xi^\alpha\xi^\beta > 0 \tag{2.6}$$

for all $(p, \xi) \in CV'$, $p \notin \partial M$, where $V' \subset V$ is some neighborhood of (p_0, ξ_0).

Let $\delta : (-\varepsilon, \varepsilon) \to \partial M$ be a smooth curve starting at the point p_0 in the direction ξ_0, i.e., $\delta(0) = p_0$ and $\dot\delta(0) = \xi_0$. Let $\gamma : [0, 1] \to M$ be the shortest geodesic of the metric g joining the points p_0 and $\delta(\tau)$, for sufficiently small $\tau > 0$, i.e., $\gamma(0) = p_0$ and $\gamma(1) = \delta(\tau)$. The point $(\gamma(t), \dot\gamma(t)/|\dot\gamma(t)|)$ tends to (p_0, ξ_0) uniformly in $t \in [0, 1]$ as $\tau \to 0$.

In particular, all points $(\gamma(t),\dot\gamma(t))$ $(0\le t\le 1)$ belong to CV' for a sufficiently small $\tau>0$. Therefore (2.6) gives

$$f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)=f_{\alpha\beta}(\gamma(t))\dot\gamma^\alpha(t)\dot\gamma^\beta(t)>0 \quad\text{for}\quad 0<t<1.$$

This implies the inequality

$$\int\limits_0^1 f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt>0$$

which contradicts (2.2). This finishes the proof of Theorem 2.1.

Using Theorem 2.1, we will reduce Theorem 1.1 to the particular case that the domain D is a ball of sufficiently large radius. We denote by $B^n_\rho=\{x\in\mathbf{R}^n \mid |x|\le\rho\}$ the closed ball of radius $\rho>0$ centered at the origin. We will show that Theorem 1.1 follows from the following special case.

Lemma 2.3 *Let $g=(g_{ij})$ and $g'=(g'_{ij})$ be two Riemannian C^∞-metrics on $\mathbf{R}^n$. Assume the metrics coincide outside the ball $B^n_{\rho/2n}$,*

$$g_{ij}(x)=g'_{ij}(x)\quad\textit{for}\quad x\notin B^n_{\rho/2n}, \tag{2.7}$$

and are the Euclidean metric outside the ball $B^n_{\rho/n}$,

$$g_{ij}(x)=g'_{ij}(x)=\delta_{ij}\quad\textit{for}\quad x\notin B^n_{\rho/n}. \tag{2.8}$$

Assume that

$$d_g=d_{g'}$$

where the boundary distance functions are defined on the ball $B^n_{\rho/n}$. Then for every $K>0$ and $\rho>0$, there exists $\varepsilon=\varepsilon(K,\rho,n)>0$ such that if

$$\|R_g\|_{C^k(\mathbf{R}^n,g)}\le K, \tag{2.9}$$

$$\kappa^+(B^n_\rho,g)<1/3, \tag{2.10}$$

$$\|g'_{ij}-\delta_{ij}\|_{C^l(\mathbf{R}^n,e)}<\varepsilon \tag{2.11}$$

with $k=[n/2]+18$ and $l=18$, then there exists a C^∞-diffeomorphism $\varphi:\mathbf{R}^n\to\mathbf{R}^n$ which is the identity outside $B^n_{\rho/n}$,

$$\varphi(x)=x\quad\textit{for}\quad x\notin B^n_{\rho/n}, \tag{2.12}$$

and such that $\varphi^*g=g'$.

Proof of Theorem 1.1. Let the metrics g and g' satisfy the hypotheses of Theorem 1.1 with $k=l-2=[n/2]+18$. Using Theorem 2.1, we can assume that all derivatives of the functions g_{ij} and g'_{ij} coincide on ∂D.

We choose ρ such that $D\subset B^n_{\rho/2n}$. We now extend the metric g' to the whole space $\mathbf{R}^n$ in such a way that it coincides with the Euclidean metric outside the ball $B^n_{\rho/n}$ and satisfies the inequality

$$\|g'_{ij}-\delta_{ij}\|_{C^l(\mathbf{R}^n,e)}<\varepsilon'. \tag{2.13}$$

Such extension is clearly possible with some $\varepsilon' = \varepsilon'(\varepsilon, \rho, n)$ that goes to zero as ε does. We also extend the metric g to $\mathbf{R}^n$ by defining $g = g'$ outside D. The constructed metric g satisfies (2.9) and (2.10).

If ε in (1.3) is sufficiently small, the hypersurface ∂D is strictly convex with respect to the metric g'. The hypersurface ∂D is also strictly convex with respect to the metric g because the Taylor series of the metrics g and g' coincide on ∂D.

We now show that the boundary distance functions of the manifolds $(B^n_{\rho/n}, g)$ and $(B^n_{\rho/n}, g')$ are the same. Given points $p, q \in \partial B^n_{\rho/n}$, let $\gamma : [0,1] \to B^n_{\rho/n}$ be the shortest geodesic of the metric g joining these points. Since D is convex with respect to g, the geodesic γ intersects D in a finite number of segments, i.e., $\gamma(t) \in D$ for $0 < \tau_i \le t \le \tau'_i < 1$ and $\gamma(\tau_i), \gamma(\tau'_i) \in \partial D$, where $1 \le i \le m$ and $\tau'_i < \tau_{i+1}$. We replace each of $\gamma|_{[\tau_i,\tau'_i]}$ of the curve γ with the shortest geodesic of the metric g' joining the points $\gamma(\tau_i), \gamma(\tau'_i) \in \partial D$. Let γ' be the curve obtained by these replacements. Then the g'-length of γ' is equal to the g-length of γ because the boundary distance functions of (D, g) and (D, g') are the same. Therefore we have proved the inequality $d_{g'}(p,q) \le d_g(p,q)$. The converse inequality is proved in the same way.

Assuming Lemma 2.3, we obtain a diffeomorphism $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ satisfying (2.12) and such that $\varphi^* g = g'$. Since φ is the identity outside $B^n_{\rho/n}$ and the metrics g and g' coincide outside D, φ is the identity outside D. Therefore φ transform D onto itself and is the identity on ∂D. This finishes the proof of Theorem 1.1.

3 Semigeodesic coordinates

In this section it is shown that if the metrics g, g' have a special form and satisfy the conditions of Lemma 2.3 then in fact the metric g is also close to the Euclidean metric. Then Lemma 2.3 follows for this class of metrics using the result of [19].

The special form of the metrics will be obtained using semigeodesic coordinates. It is natural to consider such coordinates in a rectangular domain. Therefore we introduce the notation $J^n_\rho = \{x = (x^1, \dots, x^n) \in \mathbf{R}^n \mid -\rho \le x^i \le \rho\} \subset \mathbf{R}^n$. One of the coordinates, say x^n, will be distinguished in semigeodesic coordinates. Therefore we will use the notation $x = (y, z)$ for points of $\mathbf{R}^n$ with $y \in \mathbf{R}^{n-1}$ and $z \in \mathbf{R}$.

We will show that Lemma 2.3 follows from the following

Lemma 3.1 *For every $K > 0$ and $\rho > 0$, there exists $\varepsilon > 0$ such that the following statement is valid. Let g and g' be two Riemannian C^∞-metrics on the cube J^n_ρ whose length elements ds_g and $ds_{g'}$ have the form*

$$ds_g^2 = g_{\alpha\beta}(y,z)dy^\alpha dy^\beta + dz^2, \tag{3.1}$$

$$ds_{g'}^2 = g'_{\alpha\beta}(y,z)dy^\alpha dy^\beta + dz^2 \tag{3.2}$$

where $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are smooth functions on J^n_ρ such that

$$g_{\alpha\beta}(x) = g'_{\alpha\beta}(x) \quad \text{for} \quad x \notin J^n_{\rho/n}, \tag{3.3}$$

Assume that the boundary distance functions for the manifolds (B^n_ρ, g) and (B^n_ρ, g') are the same. Assume the metrics satisfy the inequalities

$$\|R_g\|_{C^k(J^n_\rho, g)} \le K, \tag{3.4}$$

$$\kappa^+(B^n_\rho, g) < 1/3, \tag{3.5}$$

$$\|g'_{\alpha\beta} - \delta_{\alpha\beta}\|_{C^l(J^n_\rho, e)} < \varepsilon \tag{3.6}$$

with $k = [n/2] + 18$ and $l = 16$. Assume also the boundary conditions

$$g_{\alpha\beta}|_{\mathcal{P}} = \delta_{\alpha\beta} \tag{3.7}$$

$$g_{\alpha\beta}|_{z \le -\rho/n} = \delta_{\alpha\beta} \tag{3.8}$$

to be satisfied, where $\mathcal{P}$ is the part of the boundary of the cube J^n_ρ, $\mathcal{P} = \{(y^1, \dots, y^{n-1}, z) \in \partial J^n_\rho : \ |y^\gamma| = \rho$ for some $\gamma \le n-1\}$. Then the functions $g_{\alpha\beta}$ satisfy the estimate

$$\|g_{\alpha\beta} - \delta_{ij}\|_{C^p(J^n_\rho, e)} < \delta \tag{3.9}$$

where $p = 12$ and $\delta = \delta(\varepsilon, K, \rho, n)$ tends to zero when ε goes to zero.

Proof of Lemma 2.3. Let two metrics g and g' satisfy the hypotheses of Lemma 2.3 with $k = [n/2] + 18$ and $l = 18$, and let the boundary distance functions of $(B^n_{\rho/n}, g)$ and $(B^n_{\rho/n}, g')$ coincide. First of all, this implies that these manifolds have the same lens structure, see Lemma 4.8.6 of [18] or Lemma 2.1 of [19]. This means the following: Given a point $p \in \partial B^n_{\rho/n}$ and a vector $0 \ne \xi \in \mathbf{R}^n$, let $\gamma(t)$ (resp. $\gamma'(t)$) be the geodesic of the metric g (resp. of the metric g') satisfying the initial conditions $\gamma(0) = p$, $\dot\gamma(0) = \xi$ (resp. $\gamma'(0) = p$, $\dot\gamma'(0) = \xi$). If $\gamma([0,a]) \subset B^n_{\rho/n}$ and $\gamma(a) \in \partial B^n_{\rho/n}$ (resp. $\gamma'([0,a']) \subset B^n_{\rho/n}$ and $\gamma'(a) \in \partial B^n_{\rho/n}$), then $a = a'$, $\gamma(a) = \gamma'(a')$, and $\dot\gamma(a) = \dot\gamma'(a')$.

Note also that condition (2.10) implies simplicity of the metric g. This follows from Theorem XI.5.1 of [9] by setting $m(t) = t$ in this theorem.

Let us construct an embedding $\varphi' : J^n_\rho \to \mathbf{R}^n$ in such a way that the length element $ds_{g'_1}$ of the metric $g'_1 = \varphi'^* g'$ is given by

$$ds^2_{g'_1} = g'_{\alpha\beta}(y,z) dy^\alpha dy^\beta + dz^2. \tag{3.10}$$

This means that the standard coordinates of $\mathbf{R}^n$ constitute a semigeodesic coordinate system for the metric g'_1. Given $y \in J^{n-1}_\rho$, let $\gamma'_y : [-\rho, \rho] \to \mathbf{R}^n$ be the geodesic of the metric g' which is determined by the initial conditions

$$\gamma'_y(-\rho) = (y, -\rho), \quad \dot\gamma'_y(-\rho) = (0,1).$$

(2.11) implies that, for a sufficiently small $\varepsilon > 0$, the transform

$$\varphi'(y,z) = \gamma'_y(z) \tag{3.11}$$

maps the cube J^n_ρ diffeomorphically onto the domain

$$\{(y,z) \mid -\rho \le y^\alpha \le \rho, \ -\rho \le z \le \gamma'_y(\rho)\} \tag{3.12}$$

which is close to J^n_ρ and, in particular, contains the cube $J^n_{\rho/n}$. The transform φ' sends a vertical straight line $y = \text{const}$, $z = t$ to the geodesic $\gamma'_y(t)$. This means that the metric $g'_1 = \varphi'^* g'$ has the form (3.10). By (2.11), the transform φ' is C^{l-1}-close to the identity. This implies the estimate

$$\|g'_{\alpha\beta} - \delta_{\alpha\beta}\|_{C^{l-2}(J^n_\rho, e)} < \varepsilon' \tag{3.13}$$

with some $\varepsilon' = \varepsilon'(\varepsilon, l, \rho, n)$ which tends to zero as ε goes to zero.

Because of (2.8), the geodesic $\gamma'_y(t)$ coincides with the vertical straight line (y,t) until this line hits to the ball $B^n_{\rho/n}$. In particular, if $|y| > \rho/n$ then $\gamma'_y(t) = (y,t)$ for all $t \in [-\rho, \rho]$. Hence, we have $\varphi'(y,z) = (y,z)$ for (y,z) satisfying either $z \le -\rho/n$ or $|y| \ge \rho/n$. This implies that the functions $g'_{\alpha\beta}$ satisfy the boundary condition

$$g'_{\alpha\beta}\Big|_{\mathcal{P}} = \delta_{\alpha\beta} \tag{3.14}$$

and the condition

$$g'_{\alpha\beta}\Big|_{z \le -\rho/n} = \delta_{\alpha\beta}. \tag{3.15}$$

Now, we construct a similar embedding $\varphi : J^n_\rho \to \mathbf{R}^n$ for the metric g. Given $y \in J^n_\rho$, by $\gamma_y : [-\rho, \rho]$ we denote the geodesic of the metric g which is determined by the initial conditions

$$\gamma_y(-\rho) = (y, -\rho), \quad \dot\gamma_y(-\rho) = (0,1).$$

By the above remark on the lens spaces $(B^n_{\rho/n}, g)$ and $(B^n_{\rho/n}, g')$ and condition (2.7), the geodesics γ_y and γ'_y coincide outside $B^n_{\rho/2n}$. In particular, the transform

$$\varphi(y,z) = \gamma_y(z) \tag{3.16}$$

maps the cube J^n_ρ onto the same domain (3.12). Furthermore the maps (3.11) and (3.16) coincide on $J^n_\rho \setminus J^n_{\rho/n}$.

For a sufficiently small ε' in (3.13), every geodesic $\gamma' : [-\rho, \rho] \to \mathbf{R}^n$ is the shortest path, in the metric g', from the hyperplane $P = \{(y, -\rho) \mid y \in \mathbf{R}^{n-1}\} \subset \mathbf{R}^n$ to the point $\gamma'(\rho)$. This implies, since the boundary distance functions of $(B^n_{\rho/n}, g)$ and $(B^n_{\rho/n}, g')$ are the same, that every geodesic $\gamma_y : [-\rho, \rho] \to \mathbf{R}^n$ is the shortest path, in the metric g, from the hyperplane P to the point $\gamma_y(\rho)$.

Let us show that (3.16) is a one-to-one map. Suppose not. Then the geodesics γ_{y_1} and γ_{y_2} have a point in common for some $y_1, y_2 \in J^{n-1}_\rho$, $y_1 \ne y_2$. This point belongs to $B^n_{\rho/2n}$ because, outside $B^n_{\rho/2n}$, the geodesics γ_{y_1} and γ_{y_2} coincide with the disjoint curves γ'_{y_1} and γ'_{y_2} respectively. So, let $\gamma_{y_1}(t_1) = \gamma_{y_2}(t_2)$ for some $t_1, t_2 \in [-\rho/n, \rho/n]$. Let, for instance, $t_1 \le t_2$ (otherwise we change the roles of y_1 and y_2). Consider the broken geodesic $\gamma = \gamma_{y_1}\big|_{[-\rho, t_1]} \cup \gamma_{y_2}\big|_{[t_2, \rho]}$ joining the points $(y_1, -\rho) \in P$ and $\gamma_{y_2}(\rho)$. The length of γ is equal to $(t_1 + \rho) + (\rho - t_2) \le 2\rho$. Since γ has a nonzero angle at the point $\gamma_{y_1}(t_1) = \gamma_{y_2}(t_2)$, it is not a shortest way from $(y_1, -\rho)$ to $\gamma_{y_2}(\rho)$. Therefore the distance from $\gamma_{y_2}(\rho)$ to P is less than 2ρ. This contradicts to the statement of the previous paragraph.

A similar argument shows that the transform (3.16) is a diffeomorphism of the cube J^n_ρ onto the domain (3.12). The metric $g_1 = \varphi^* g$ has the form

$$ds^2_{g_1} = g_{\alpha\beta}(y,z)dy^\alpha dy^\beta + dz^2. \tag{3.17}$$

Since the transforms φ and φ' coincide outside $J^n_{\rho/n}$, the functions $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ satisfy the condition (3.3). Furthermore, by (3.13)–(3.15), the functions $g_{\alpha\beta}$ satisfy the boundary conditions (3.7)–(3.8).

Using exactly the same argument as in the corresponding paragraph of the previous section we show that the boundary distance functions for the manifolds (B^n_ρ, g_1) and (B^n_ρ, g'_1) are the same.

We have thus proven that the metrics g_1 and g_1' satisfy all hypotheses of Lemma 3.1. Assuming this lemma, we obtain the estimate (3.9) with some δ that can be made arbitrarily small. Since $l \geq 14$ and $p = 12$, inequalities (3.13) and (3.9) show that the metrics g_1 and g_1' are in a small C^{12}-neighborhood of the Euclidean metric. By using the main result of [19], we conclude that these two metrics are isometric via an isometry which is the identity on ∂B^n_ρ. The same is valid for the original metrics $g = (\varphi^{-1})^* g_1$ and $g' = (\varphi'^{-1})^* g_1'$.

4 Estimates for a metric tensor in semigeodesic coordinates through the curvature tensor

Fix $n \geq 2$ and $\rho > 0$. We recall that the points of $\mathbf{R}^n$ are denoted by $x = (y, z)$ with $y \in \mathbf{R}^{n-1}$ and $z \in \mathbf{R}$. We denote by $\mathcal{M}(\rho, n)$ the set of all Riemannian metrics, on the cube J^n_ρ, whose length element ds_g has the form

$$ds_g^2 = g_{\alpha\beta}(y, z) dy^\alpha dy^\beta + dz^2, \tag{4.1}$$

where $g_{\alpha\beta}$ are C^∞-functions satisfying the boundary conditions

$$g_{\alpha\beta}|_{\mathcal{P}} = g_{\alpha\beta}|_{z=-\rho} = \delta_{\alpha\beta}, \qquad \left.\frac{\partial^l g_{\alpha\beta}}{\partial z^l}\right|_{z=-\rho} = 0 \quad (l = 1, 2, \dots), \tag{4.2}$$

$$\det\,(g_{\alpha\beta})|_{z=\rho} \geq 1/2, \tag{4.3}$$

where $\mathcal{P}$ is the same as in Lemma 3.1.

For an integer $k \geq 0$ and $K > 0$, we denote by $\mathcal{M}(k, K, \rho, n)$ the subset of $\mathcal{M}(\rho, n)$ consisting of metrics g whose curvature tensor R_g satisfies the inequality

$$\|R_g\|_{C^k(J^n_\rho, g)} \leq K.$$

Let $\varphi(g; x)$ be a smooth function of $x \in J^n_\rho$ depending also on a metric $g \in \mathcal{M}(\rho, n)$. We say that the function φ is *k-bounded* if the function

$$\sup_{x \in J^n_\rho} |\varphi(g; x)|$$

is bounded on the set $\mathcal{M}(k, K, \rho, n)$ for every K.

The goal of the present section is to prove the following

Theorem 4.1 *For a metric $g \in \mathcal{M}(\rho, n)$ whose length element has form (4.1), the partial derivatives $D^\gamma_x g_{\alpha\beta}$ are $|\gamma|$-bounded for every n-multiindex γ. The same holds for the partial derivatives $D^\gamma_x g^{\alpha\beta}$ of the inverse matrix $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$.*

Corollary 4.2 *The norms $\|\cdot\|_{C^k(J^n_\rho, g)}$ and $\|\cdot\|_{C^k(J^n_\rho, e)}$ are equivalent in the following sense. For every metric $g \in \mathcal{M}(k, K, \rho, n)$ and for every tensor field T of rank m on J^n_ρ, the inequalities*

$$C^{-1}\|T\|_{C^k(J^n_\rho, e)} \leq \|T\|_{C^k(J^n_\rho, g)} \leq C\|T\|_{C^k(J^n_\rho, e)}$$

hold with some positive constant C depending only on k, K, ρ, n, and m. The same is valid for the H^k-norms.

To prove Theorem 4.1 we will need two auxiliary results. The first one is obvious:

Lemma 4.3 *If a function $0 \le f : \mathbf{R}^+ \to \mathbf{R}$ satisfies the inequality*

$$f'(t) \le C_1 f(t) + C_2$$

for some nonnegative constants C_1 and C_2, then

$$f(t) \le e^{C_1 t}(f(0) + C_2 t).$$

The second needed result is related to Jacobi fields:

Lemma 4.4 *Let $\gamma : [-\rho, \rho] \to M$ be a unit speed geodesic in a Riemannian manifold (M, g). If a vector field $Y(t)$ along γ satisfies the inequality*

$$|\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y(t)|_g \le C_1|Y(t)|_g + C_2 \tag{4.4}$$

for some nonnegative constants C_1 and C_2, then the estimates

$$|Y(t)|_g^2 \le C_3\left(|Y(-\rho)|_g^2 + |\nabla_{\dot\gamma}Y(-\rho)|_g^2\right) + C_4,$$

$$|\nabla_{\dot\gamma}Y(t)|_g^2 \le C_3\left(|Y(-\rho)|_g^2 + |\nabla_{\dot\gamma}Y(-\rho)|_g^2\right) + C_4$$

hold for $t \in [-\rho, \rho]$ with some constants C_3 and C_4 depending only on C_1, C_2, and ρ.

Proof. We define

$$f(t) = |Y(t)|_g^2 + |\nabla_{\dot\gamma}Y(t)|_g^2.$$

Using (4.4), we obtain

$$\begin{aligned}
f'(t) &= 2\langle Y, \nabla_{\dot\gamma}Y\rangle_g + 2\langle \nabla_{\dot\gamma}Y, \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y\rangle_g \\
&\le 2\left(|\nabla_{\dot\gamma}\nabla_{\dot\gamma}Y|_g + |Y|_g\right)|\nabla_{\dot\gamma}Y|_g \\
&\le 2\Big((C_1+1)|Y|_g + C_2\Big)|\nabla_{\dot\gamma}Y|_g \\
&\le \Big((C_1+1)|Y|_g + C_2\Big)^2 + |\nabla_{\dot\gamma}Y|_g^2 \\
&\le \Big((C_1+1)^2+1\Big)\Big(|Y|_g^2 + |\nabla_{\dot\gamma}Y|_g^2\Big) + \Big((C_1+1)^2+1\Big)C_2^2.
\end{aligned}$$

We have thus proved the inequality

$$f'(t) \le C_5 f(t) + C_6$$

with $C_5 = (C_1+1)^2 + 1$ and $C_6 = ((C_1+1)^2+1)\,C_2^2$. Applying Lemma 4.3, we obtain the estimate $f(t) \le e^{C_5(t+\rho)}\Big(f(-\rho) + C_6(t+\rho)\Big)$ which implies the Lemma.

Lemma 4.5 *Let a metric $g \in \mathcal{M}(\rho, n)$ be of the form (4.1) and $\partial_i = \frac{\partial}{\partial x^i}$ $(1 \le i \le n)$ be the coordinate vector fields. For every indices $1 \le i_1, \ldots, i_k, j \le n$, the functions*

$$\Big|\nabla_{\partial_{i_1}} \ldots \nabla_{\partial_{i_k}}\partial_j\Big|_g \quad \textit{and} \quad \Big|\nabla_{\partial_n}\nabla_{\partial_{i_1}} \ldots \nabla_{\partial_{i_k}}\partial_j\Big|_g$$

are k-bounded.

Proof. The vector field ∂_j is a Jacobi vector field along every geodesic $y = \text{const}$. In other words, ∂_j satisfies the equation

$$\nabla_{\partial_n}\nabla_{\partial_n}\partial_j + R(\partial_j, \partial_n)\partial_n = 0 \tag{4.5}$$

with $R = R_g$. By (4.2), this vector field also satisfies the initial conditions

$$|\partial_j|_{z=-\rho} = 1, \quad \underbrace{\nabla \dots \nabla}_{m} \partial_j|_{z=-\rho} = 0$$

for every $m > 0$.

Since $|\partial_n|_g = 1$, the Jacobi equation (4.5) implies the inequality

$$|\nabla_{\partial_n}\nabla_{\partial_n}\partial_j|_g \le |R|_g \cdot |\partial_j|_g \le \|R\|_{C^0(M,g)} \cdot |\partial_j|_g.$$

Applying Lemma 4.4, we get 0-boundedness of the functions $|\partial_j|_g$ and $|\nabla_{\partial_n}\partial_j|_g$.

Differentiating (4.5) with respect to ∂_i, we obtain

$$\nabla_{\partial_i}\Big(\nabla_{\partial_n}\nabla_{\partial_n}\partial_j + R(\partial_j, \partial_n)\partial_n\Big) = 0. \tag{4.6}$$

Since

$$\nabla_{\partial_i}\nabla_{\partial_n} - \nabla_{\partial_n}\nabla_{\partial_i} = R(\partial_i, \partial_n), \quad \nabla_{\partial_i}\partial_n = \nabla_{\partial_n}\partial_i, \tag{4.7}$$

the equation (4.6) can be transformed to the following one:

$$\nabla_{\partial_n}\nabla_{\partial_n}(\nabla_{\partial_i}\partial_j) + R(\nabla_{\partial_i}\partial_j, \partial_n)\partial_n =$$

$$= -\Big((\nabla_{\partial_i}R)(\partial_j, \partial_n)\partial_n + (\nabla_{\partial_n}R)(\partial_i, \partial_n)\partial_j +$$

$$+R(\nabla_{\partial_n}\partial_i, \partial_n)\partial_j + R(\partial_j, \nabla_{\partial_n}\partial_i)\partial_n + 2R(\partial_i, \partial_n)\nabla_{\partial_n}\partial_j + R(\partial_j, \partial_n)\nabla_{\partial_n}\partial_i\Big).$$

The right-hand side of the latter relation is 1-bounded. Therefore it implies the inequality

$$|\nabla_{\partial_n}\nabla_{\partial_n}(\nabla_{\partial_i}\partial_j)|_g \le \|R\|_{C^0(M,g)} \cdot |\nabla_{\partial_i}\partial_j|_g + C$$

which is valid for $g \in \mathcal{M}(1, K, \rho, n)$ with some constant C independent of g. Applying Lemma 4.4, we get the 1-boundedness of the functions $|\nabla_{\partial_i}\partial_j|_g$ and $|\nabla_{\partial_n}\nabla_{\partial_i}\partial_j|_g$.

Now, we prove the statement of Lemma 4.5 by using induction on k. Applying the operator $\nabla_{\partial_{i_1}} \dots \nabla_{\partial_{i_k}}$ to the Jacobi equation (4.5) gives

$$\nabla_{\partial_{i_1}} \dots \nabla_{\partial_{i_k}}\Big(\nabla_{\partial_n}\nabla_{\partial_n}\partial_j + R(\partial_j, \partial_n)\partial_n\Big) = 0.$$

Using the commutator formula (4.7), the latter equation can be transformed into:

$$\nabla_{\partial_n}\nabla_{\partial_n}(\nabla_{\partial_{i_1}} \dots \nabla_{\partial_{i_k}}\partial_j) + R(\nabla_{\partial_{i_1}} \dots \nabla_{\partial_{i_k}}\partial_j, \partial_n)\partial_n = T, \tag{4.8}$$

where T is some tensor field expressed in terms of $\underbrace{\nabla \dots \nabla}_{s} R$, $\underbrace{\nabla \dots \nabla}_{t} \partial_m$, and $\nabla_{\partial_n} \underbrace{\nabla \dots \nabla}_{t} \partial_m$ with $0 \le s \le k$ and $0 \le t \le k-1$. By the induction hypothesis, $|T|_g$ is k-bounded. Therefore (4.8) implies the inequality

$$|\nabla_{\partial_n}\nabla_{\partial_n}(\nabla_{\partial_{i_1}} \dots \nabla_{\partial_{i_k}}\partial_j)|_g \le \|R\|_{C^0(M,g)} \cdot |\nabla_{\partial_{i_1}} \dots \nabla_{\partial_{i_k}}\partial_j|_g + C$$

which is valid for $g \in \mathcal{M}(k, K, \rho, n)$ with some constant C independent of g. Applying Lemma 4.4, we get the statement of Lemma 4.5.

The statement of Theorem 4.1 on partial derivatives of the functions $g_{\alpha\beta}$ follows from Lemma 4.5 using that

$$D^\gamma g_{\alpha\beta} = D^\gamma \langle \partial_\alpha, \partial_\beta \rangle_g = \sum_{k+l=|\gamma|} c_{\alpha\beta}^{\gamma i_1 \dots i_k j_1 \dots j_l} \langle \nabla_{\partial_{i_1}} \dots \nabla_{\partial_{i_k}} \partial_\alpha, \nabla_{\partial_{j_1}} \dots \nabla_{\partial_{j_l}} \partial_\beta \rangle_g.$$

The second statement of Theorem 4.1 follows from the first one using the following

Lemma 4.6 *Under the hypotheses of Theorem 4.1, the function* $\left(\det (g_{\alpha\beta})\right)^{-1}$ *is 0-bounded.*

Proof. We use the following fact [16], page 44: the function $h(y,z) = \left(\det (g_{\alpha\beta})\right)^{1/(n-1)}$ satisfies the inequality

$$\frac{\partial^2 h}{\partial z^2} \le -\frac{\mathrm{Ric}\,(\partial_n, \partial_n)}{n-1} h$$

which implies

$$\frac{\partial^2 h}{\partial z^2} \le K h \tag{4.9}$$

for $g \in \mathcal{M}(0, K, \rho, n)$.

We fix a point y and consider h as a function of z. Let z_0 be a point such that

$$h(z_0) = \min h(z). \tag{4.10}$$

By (4.3), we can assume that $-\rho < z_0 < \rho$ and $h'(z_0) = 0$. The function

$$f(z) = \begin{cases} (h'(z))^2 + (h(z))^2 & \text{if } \ h'(z) \ge 0, \\ (h(z))^2 & \text{if } \ h'(z) < 0 \end{cases}$$

is a C^1-function on $[-\rho, \rho]$. If $h'(z) \ge 0$, then we conclude using (4.9)

$$f'(z) = 2h'h'' + 2hh' \le (K+1)(h'^2 + h^2) = (K+1) f(z).$$

If $h'(z) < 0$, then $f'(z) < 0$. In both cases we have the inequality

$$f'(z) \le (K+1) f(z).$$

Applying Lemma 4.3, we obtain

$$h(\rho) \le e^{(K+1)(\rho - z_0)} h(z_0) \le e^{2\rho(K+1)} h(z_0).$$

The latter inequality, together with (4.3), implies

$$h(z_0) \ge e^{-2\rho(K+1)} h(\rho) \ge 2^{1/(1-n)} e^{-2\rho(K+1)}.$$

Because of (4.10), this means that the function $\det (g_{\alpha\beta})$ is bounded from below by some positive constant depending only on K, ρ, and n.

5 Proof of Lemma 3.1

After our preparations, we can use results from integral geometry to prove Lemma 3.1. Let g and g' be two Riemannian metrics on J^n_ρ satisfying the hypotheses of Lemma 3.1. In particular, the length elements of these metrics are given by formulas (3.1) and (3.2) respectively, and the boundary distance functions of the manifolds (B^n_ρ, g) and (B^n_ρ, g') coincide.

By Corollary 4.2, the norms $\|T\|_{C^k(B^n_\rho,g)}$ and $\|T\|_{H^k(B^n_\rho,g)}$ are equivalent to the norms

$$\|T\|_{C^k(B^n_\rho,e)} = \sum_{|\gamma|\le k} \sup_{x\in B^n_\rho} |D^\gamma T(x)|_e \quad \text{and} \quad \|T\|^2_{H^k(B^n_\rho,e)} = \sum_{|\gamma|\le k} \int\limits_{B^n_\rho} |D^\gamma T(x)|^2_e \, dx$$

respectively. In what follows, we use the notation $\|T\|_{C^k(B^n_\rho)}$ ($\|T\|_{H^k(B^n_\rho)}$) that means either $\|T\|_{C^k(B^n_\rho,g)}$ ($\|T\|_{H^k(B^n_\rho,g)}$) or $\|T\|_{C^k(B^n_\rho,e)}$ ($\|T\|_{H^k(B^n_\rho,e)}$).

We denote by $\Omega B^n_\rho \subset B^n_\rho \times \mathbf{R}^n$ the unit sphere bundle over the ball B^n_ρ and by

$$\partial_+\Omega B^n_\rho = \{(p,\xi) \mid p \in \partial B^n_\rho,\ \xi \in \mathbf{R}^n,\ |\xi|_g = 1, \langle \nu(p), \xi\rangle_g \ge 0\},$$

the part of the boundary of ΩB^n_ρ consisting of unit outward vectors that are tangent to B^n_ρ at points of the boundary; here $\nu(p)$ is the outer vector normal to the boundary with respect to the metric g. This part is a compact manifold with boundary diffeomorphic to the product $\Omega^{n-1} \times B^{n-1}_1$ of a sphere and a ball. Fixing a diffeomorphism $\partial_+\Omega B^n_\rho \to \Omega^{n-1} \times B^{n-1}_1$, the norms $\|\cdot\|_{C^k(\partial_+\Omega B^n_\rho)}$ and $\|\cdot\|_{H^k(\partial_+\Omega B^n_\rho)}$ are defined.

We recall from [18] that, given a smooth tensor field $f = (f_{jk})$ on B^n_ρ, the ray transform of f is the function $If \in C^\infty(\partial_+\Omega B^n_\rho)$ defined by

$$If(p,\xi) = \int\limits_{\tau_-(p,\xi)}^{0} f_{jk}(\gamma_{p,\xi}(t))\dot\gamma^j_{p,\xi}(t)\dot\gamma^k_{p,\xi}(t)\, dt, \tag{5.1}$$

where $\gamma_{p,\xi} : [\tau_-(p,\xi), 0] \to B^n_\rho$ is the maximal geodesic of the metric g satisfying the initial conditions $\gamma_{p,\xi}(0) = p,\ \dot\gamma_{p,\xi}(0) = \xi$ and such that $\gamma_{p,\xi}(\tau_-(p,\xi)) \in \partial B^n_\rho$.

Integrating the differential equation for the geodesics of the metric g

$$\ddot\gamma^i_{p,\xi} = -\Gamma^i_{jk}\dot\gamma^j_{p,\xi}\dot\gamma^k_{p,\xi},$$

where Γ^i_{jk} are the Christoffel symbols of the metric g, gives

$$\dot\gamma^i_{p,\xi}(\tau_-(p,\xi)) - \xi^i = \int\limits_{\tau_-(p,\xi)}^{0} \Gamma^i_{jk}(\gamma_{p,\xi}(t))\dot\gamma^j_{p,\xi}(t)\dot\gamma^k_{p,\xi}(t)\, dt.$$

Comparing the latter equality with (5.1), we have that

$$If^{(i)}(p,\xi) = \dot\gamma^i_{p,\xi}(\tau_-(p,\xi)) - \xi^i, \tag{5.2}$$

where $f^{(i)}$ is the symmetric tensor field with the components $(f^{(i)})_{jk} = \Gamma^i_{jk}$ in the standard coordinate system of $\mathbf{R}^n$.

If $\gamma'_{p,\xi} : [\tau_-(p,\xi), 0] \to B^n_\rho$ is the geodesic of the metric g' satisfying the same initial conditions

$$\gamma'_{p,\xi}(0) = p, \quad \dot\gamma'_{p,\xi}(0) = \xi,$$

then, since the boundary distance functions for the metrics g and g' are the same as well as their lens structures, we get

$$\dot\gamma_{p,\xi}(\tau_-(p,\xi)) = \dot\gamma'_{p,\xi}(\tau_-(p,\xi)) \tag{5.3}$$

On the other hand we have by (3.6),

$$\|\dot\gamma'_{p,\xi}(\tau_-(p,\xi)) - \xi\|_{C^{l-1}(\partial_+\Omega B^n_\rho)} < \varepsilon_1 \tag{5.4}$$

with some $\varepsilon_1 = \varepsilon_1(\varepsilon, l, \rho)$ which tends to zero as ε goes to zero. Combining (5.2)–(5.4), we obtain

$$\|If^{(i)}\|_{C^{l-1}(\partial_+\Omega B^n_\rho)} < \varepsilon_1. \tag{5.5}$$

Next we use some notions of tensor analysis introduced in [18], namely, the inner derivative d_g and divergence δ_g with respect to the metric g. Here we use only the inner derivative of covector fields $v = (v_i)$ and divergence of symmetric covariant 2-tensor fields $h = (h_{ij})$ which are defined in local coordinates as

$$(d_g v)_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i), \quad (\delta_g h)_i = g^{jk}\nabla_k h_{ij}.$$

Here ∇ is the covariant derivative with respect to the metric g. Every symmetric tensor field $h = (h_{ij})$ on B^n_ρ can be represented in the form

$$h = \tilde h + d_g v, \quad \delta_g \tilde h = 0, \quad v_i|_{\partial B^n_\rho} = 0.$$

The summands of this decomposition are called the solenoidal part and potential part of h respectively. The existence and uniqueness of this decomposition is proved in Section 3.3 of [18].

Returning to the proof of Lemma 3.1, we decompose the field $f^{(i)}$ into its potential and solenoidal parts

$$f^{(i)} = \tilde f^{(i)} + d_g v^{(i)}, \quad \delta_g \tilde f^{(i)} = 0, \quad v^{(i)}|_{\partial B^n_\rho} = 0. \tag{5.6}$$

By Theorem 3.3.2 of [18] the solenoidal part of a tensor field depends continuously on the field in Sobolev norms, i.e., we have

$$\|\tilde f^{(i)}\|_{H^s(B^n_\rho)} \le C(g)\|f^{(i)}\|_{H^s(B^n_\rho)}$$

for every $s \ge 1$. The constant $C(g)$ depends on the metric g but can be chosen uniformly on the metrics involved into Lemma 3.1. Namely we have

Lemma 5.1 *For a metric $g \in \mathcal{M}(k, K, \rho, n)$ and smooth symmetric tensor field f of second rank on B^n_ρ, let*

$$f = \tilde f + d_g v \quad in \quad B^n_\rho, \quad \delta_g \tilde f = 0 \quad in \quad B^n_\rho, \quad v|_{\partial B^n_\rho} = 0 \tag{5.7}$$

be the decomposition of f into its solenoidal and potential parts. Then we have

$$\|\tilde f\|_{H^s(B^n_\rho)} \le C\|f\|_{H^s(B^n_\rho)} \tag{5.8}$$

for $s \le k-2$ with some constant C depending on s, k, K, ρ, n but not on g.

The proof of this lemma is presented in the next section. We continue proving Lemma 3.1 using Lemma 5.1. Our idea is to show that the solenoidal part of $f^{(i)}$ is small. Therefore, the Christoffel symbols $\Gamma^{(i)} = (\Gamma^i_{jk})$ with any i are approximately potential tensor fields. However, since the Christoffel symbols have special structure, we can show that they have to be approximately zeros. This will further imply that the metric g has to be near Euclidean one. To carry out this plan we start with estimating the solenoidal part of $f^{(i)}$.

By Theorem 4.3.3 of [18], we have the estimate

$$\|\tilde{f}^{(i)}\|^2_{L_2(B^n_\rho)} \leq C_1 \left(\|f^{(i)}\|_{H^1(B^n_\rho)} \cdot \|If^{(i)}\|_{L_2(\partial_+\Omega B^n_\rho)} + \|If^{(i)}\|^2_{H^1(\partial_+\Omega B^n_\rho)} \right). \tag{5.9}$$

As it can be seen from the proof in [18] the constant C_1 depends only on the constant C from (5.8) for $s = 2$, and on the C^1-jet of the metric g on ∂B^n_ρ. Moreover, the latter is a continuous dependence. Since the C^1-jet of the metric g on ∂B^n_ρ coincides with the one of the metric g', and g' is close to the Euclidean metric; we can consider C_1 in (5.9) as an universal constant depending only on K, ρ, and n.

By Theorem 4.1, the tensor field $(f^{(i)})_{jk} = \Gamma^i_{jk}$ satisfies the estimate

$$\|f^{(i)}\|_{H^1(B^n_\rho)} \leq K_1, \tag{5.10}$$

where the constant K_1 depends only on K from (3.4), ρ, and n. Inequalities (5.9) and (5.10) imply the estimate

$$\|\tilde{f}^{(i)}\|^2_{L_2(B^n_\rho)} \leq C_1 \left(K_1 \|If^{(i)}\|^2_{L_2(\partial_+\Omega B^n_\rho)} + \|If^{(i)}\|^2_{H^1(\partial_+\Omega B^n_\rho)} \right)$$

which, together with (5.5), gives

$$\|\tilde{f}^{(i)}\|_{L_2(B^n_\rho)} \leq \varepsilon_2 := \left(C_1(K_1\varepsilon_1 + \varepsilon_1^2) \right)^{1/2}. \tag{5.11}$$

Now, we use the interpolation of Sobolev spaces [1] to obtain the inequality for tensor fields on B^n_ρ

$$\|\tilde{f}^{(i)}\|_{H^j(B^n_\rho)} \leq C_2 \|\tilde{f}^{(i)}\|^{(s-j)/s}_{L_2(B^n_\rho)} \cdot \|\tilde{f}^{(i)}\|^{j/s}_{H^s(B^n_\rho)} \tag{5.12}$$

for $0 \leq j \leq s$, where the constant C_2 depends only on j, s, ρ, and n. Besides this, by the Sobolev embedding theorem, we have the estimate

$$\|\tilde{f}^{(i)}\|_{C^r(B^n_\rho)} \leq C_3 \|\tilde{f}^{(i)}\|_{H^j(B^n_\rho)} \tag{5.13}$$

for $j > n/2 + r$ with some constant C_3 dependent only on j, r, ρ and n. Combining (5.12) and (5.13), we obtain

$$\|\tilde{f}^{(i)}\|_{C^r(B^n_\rho)} \leq C_4 \|\tilde{f}^{(i)}\|^{(s-j)/s}_{L_2(B^n_\rho)} \cdot \|\tilde{f}^{(i)}\|^{j/s}_{H^s(B^n_\rho)}. \tag{5.14}$$

Let $s = j + 1$. We conclude using (5.11) and (5.14)

$$\|\tilde{f}^{(i)}\|_{C^r(B^n_\rho)} \leq C_4 \varepsilon_2^{1/(j+1)} \|\tilde{f}^{(i)}\|^{j/2}_{H^{j+1}(B^n_\rho)}. \tag{5.15}$$

By Theorem 4.1, for $j + 1 \leq k - 1$, we have

$$\|f^{(i)}\|_{H^{j+1}(B^n_\rho)} \leq K_1 \tag{5.16}$$

with the right-hand side K_1 depending only on the constant K as in (3.4), ρ, and n. Combining (5.8), (5.15) and (5.16) gives

$$\|\tilde{f}^{(i)}\|_{C^r(B^n_\rho)} < \varepsilon_3 := C_4 K_1^{j/2} \varepsilon_2^{1/(j+1)}. \tag{5.17}$$

For a fixed K in (3.4), the right-hand side ε_3 of (5.17) can be made arbitrary small by choosing sufficiently small ε in (3.6).

The final part of the proof is to deal with the potential part in (5.6), namely

$$d_g v^{(i)} = f^{(i)} - \tilde{f}^{(i)}. \tag{5.18}$$

We consider (5.18) as a system of equations with unknowns $v^{(i)}$. By (5.17), the second term on the right-hand side of (5.18) is C^r-small. Note that (5.18) is an overdetermined system. Indeed, for a given i, (5.18) consists of $n(n+1)/2$ equations for n components of the covector field $v^{(i)}$. This overdeterminacy, together with the homogeneous boundary condition

$$v^{(i)}|_{\partial B^n_\rho} = 0 \tag{5.19}$$

will allow us to prove the smallness of the solution $v^{(i)}$ as well as the smallness of $f^{(i)}$. We will prove the smallness by distinguishing some subsystems, of the system (5.18), which can be considered as self-closed systems of ordinary differential equations on a segment $y = \mathrm{const}$, $z \in [-(\rho^2 - |y|^2)^{1/2}, (\rho^2 - |y|^2)^{1/2}]$. Unfortunately, the boundary condition (5.19) is not adapted to this method because (5.19) does not imply the corresponding boundary condition for the derivatives: $D^\gamma_y v^{(i)}|_{\partial B^n_\rho} = 0$. Therefore we will start by deriving another boundary condition which is more suitable to our plan.

Let

$$\partial^-_n J^n_{\rho/n} = \{x = (y, z) \in \mathbf{R}^n \mid y \in J^{n-1}_{\rho/n},\ z = -\rho/n\} \subset \mathbf{R}^n$$

be one of the faces of the cube $J^n_{\rho/n}$ which is considered as a submanifold of $\mathbf{R}^n$. We will prove the estimate

$$\|v^{(i)}|_{\partial^-_n J^n_{\rho/n}}\|_{C^{r-1}(\partial^-_n J^n_{\rho/n})} < \varepsilon_4 \tag{5.20}$$

with some $\varepsilon_4 = \varepsilon_4(\varepsilon, r, \rho, n)$ tending to zero as ε goes to zero. In what follows, we will use (5.20) as the new boundary (more exactly, initial) condition instead of (5.19).

Let $\gamma : [a, b] \to B^n_\rho$ be a geodesic of the metric g. By (5.18) we have

$$\frac{d}{dt}\left(v^{(i)}_j(\gamma(t))\dot{\gamma}^j(t)\right) = \nabla_k v^{(i)}_j \cdot \dot{\gamma}^j \dot{\gamma}^k = \left(dv^{(i)}\right)_{jk} \dot{\gamma}^j \dot{\gamma}^k = (\Gamma^i_{jk} - \tilde{f}^{(i)}_{jk})\dot{\gamma}^j \dot{\gamma}^k.$$

Integrating the latter equality, we obtain

$$v^{(i)}_j(\gamma(b))\dot{\gamma}^j(b) - v^{(i)}_j(\gamma(a))\dot{\gamma}^j(a) = \int\limits_a^b \left(\Gamma^i_{jk} - \tilde{f}^{(i)}_{jk}\right)(\gamma(t))\dot{\gamma}^j(t)\dot{\gamma}^k(t)\,dt. \tag{5.21}$$

Given a point $x \in B^n_\rho$ and vector $0 \neq \xi \in \mathbf{R}^n$, let $\gamma_{x,\xi} : [\tau_-(x,\xi), 0] \to B^n_\rho$ be the maximal geodesic determined by the initial conditions $\gamma_{x,\xi}(0) = x$, $\dot{\gamma}_{x,\xi}(0) = \xi$. Then $\gamma_{x,\xi}(\tau_-(x,\xi)) \in \partial B^n_\rho$. Using (5.19) and (5.21) gives

$$\xi^j v^{(i)}_j(x) = \int\limits_{\tau_-(x,\xi)}^0 \left(\Gamma^i_{jk} - \tilde{f}^{(i)}_{jk}\right)(\gamma_{x,\xi}(t))\dot{\gamma}^j_{x,\xi}(t)\dot{\gamma}^k_{x,\xi}(t)\,dt. \tag{5.22}$$

Let now $x \in \partial_n^- J^n_{\rho/n}$ and the vector $\xi \in \mathbf{R}^n$ be such that $\xi^n > 0$. Then the geodesic $\gamma_{x,\xi}$ lies completely in the closure of $J^n_\rho \setminus J^n_{\rho/n}$ where the metrics g and g' coincide. By (3.6) and (5.17), the integrand of (5.22) is C^r-small. Besides this, the geodesic $\gamma_{x,\xi}$ and the integration limit $\tau_-(x,\xi)$ are C^r-close to the same quantities with respect to the Euclidean metric, and therefore they are C^r-bounded by some universal constant. Differentiating (5.22), we conclude

$$D^\gamma_x v^{(i)}_l(x) = D^\gamma_x \frac{\partial}{\partial \xi^l}\left(\xi^j v^{(i)}_j(x)\right) = D^\gamma_x \frac{\partial}{\partial \xi^l} \int\limits_{\tau_-(x,\xi)}^{0} \left(\Gamma^i_{jk} - \tilde f^{(i)}_{jk}\right)(\gamma_{x,\xi}(t))\dot\gamma^j_{x,\xi}(t)\dot\gamma^k_{x,\xi}(t)\, dt.$$

The right-hand side of the latter equality can be estimated, for $|\gamma| \le r-1$, by $C_5(\varepsilon + \varepsilon_3)$ where ε and ε_3 are as in (3.6), (5.17) and $C_5 = C_5(r,\rho,n)$. This proves (5.20).

To investigate the system (5.18), we need also the following

Lemma 5.2 *Consider the Cauchy problem for the linear system of ordinary differential equations*

$$\left.\begin{aligned} &\frac{\partial u}{\partial z} + A(y,z)u = f(y,z) \\ &u|_{z=-\rho/n} = u_0(y) \end{aligned}\right\} \tag{5.23}$$

whose coefficient and right-hand side depend smoothly on $x = (y,z) \in J^n_{\rho/n}$, and the initial condition depends smoothly on $y \in J^{n-1}_{\rho/n}$. Here $u = (u_1,\dots,u_m)$, $f = (f_1,\dots,f_m)$, $A = (a_{ij})^m_{i,j=1}$. If the inequalities

$$\|A\|_{C^l(J^n_{\rho/n})} \le C, \quad \|f\|_{C^l(J^n_{\rho/n})} < \delta, \quad \|u_0\|_{C^l(J^{n-1}_{\rho/n})} < \delta$$

hold, then we have for the solution to (5.23)

$$\|u\|_{C^l(J^n_{\rho/n})} < C_1\delta$$

with some constant C_1 depending only on C, l, ρ, and n.

Proof. By Lemma IV.4.1 of [9], the following estimate holds for the solution to problem (5.23):

$$|u(y,z)| \le \left(|u_0(y)| + \left|\int\limits_{-\rho/n}^{z} f(y,t)\,dt\right|\right)\exp\left(\int\limits_{-\rho/n}^{z}\|A(y,t)\|\,dt\right)$$

that implies the statement of the lemma in the case of $l = 0$. The general case follows easily by induction on l.

In coordinates the system (5.18) takes the form

$$\frac{1}{2}\left(\nabla_j v^{(i)}_k + \nabla_k v^{(i)}_j\right) = \Gamma^i_{jk} - \tilde f^{(i)}_{jk}, \tag{5.24}$$

where

$$\nabla_j v^{(i)}_k = \frac{\partial v^{(i)}_k}{\partial x^j} - \Gamma^l_{jk} v^{(i)}_l.$$

We will consider the system in the cube $J^n_{\rho/n}$ and will use the initial condition (5.20).

Because of (3.1), the Christoffel symbols satisfy

$$\Gamma^n_{jn} = \Gamma^i_{nn} = 0, \tag{5.25}$$

$$\Gamma^n_{\alpha\beta} = -\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial z}. \tag{5.26}$$

First, we set $i = j = k = n$ in (5.24) and obtain

$$\frac{\partial v^{(n)}_n}{\partial z} = -\tilde{f}^{(n)}_{nn}.$$

By (5.17) and (5.20), the right hand side of the latter equation and the initial value $v^{(n)}_n|_{z=-\rho/n}$ are C^{r-1}-small. Applying Lemma 5.1, we obtain the estimate

$$\|v^{(n)}_n\|_{C^{r-1}(J^n_{\rho/n})} < C_5\varepsilon_5 := C_5(\varepsilon_3 + \varepsilon_4), \tag{5.27}$$

where ε_3 and ε_4 are as in (5.17) and (5.20) respectively, and $C_5 = C_5(r, \rho, n)$.

Next, we set $j = \alpha,\ i = k = n$ in (5.24). This gives

$$\frac{\partial v^{(n)}_\alpha}{\partial z} - \Gamma^\beta_{\alpha n} v^{(n)}_\beta = -2\tilde{f}^{(n)}_{\alpha n} - \nabla_\alpha v^{(n)}_n.$$

The right hand sides of these equations and the initial values $v^{(n)}_\alpha|_{z=-\rho/n}$ are C^{r-2}-small by (5.17) and (5.27), and the coefficients $\Gamma^\beta_{\alpha n}$ are C^{r-2}-bounded by Theorem 4.1. Applying Lemma 5.1 we conclude

$$\|v^{(n)}_\alpha\|_{C^{r-2}(J^n_{\rho/n})} < C_6\varepsilon_5 \tag{5.28}$$

with some $C_6 = C_6(r, K, \rho, n)$.

We have thus proven the C^{r-2}-smallness of the covector field $v^{(n)}$. We now set $i = n, j = \alpha, k = \beta$ in (5.24). This gives

$$\frac{1}{2}\left(\nabla_\alpha v^{(n)}_\beta + \nabla_\beta v^{(n)}_\alpha\right) = \Gamma^n_{\alpha\beta} - \tilde{f}^{(n)}_{\alpha\beta}$$

which implies the C^{r-3}-smallness of $\Gamma^n_{\alpha\beta}$:

$$\|\Gamma^n_{\alpha\beta}\|_{C^{r-3}(J^n_{\rho/n})} < C_7\varepsilon_5 \tag{5.29}$$

with some $C_7 = C_7(r, K, \rho, n)$.

Finally, we consider (5.26) as a differential equation in $g_{\alpha\beta} - \delta_{\alpha\beta}$:

$$\frac{\partial(g_{\alpha\beta} - \delta_{\alpha\beta})}{\partial z} = -2\Gamma^n_{\alpha\beta}.$$

By (5.29), the right-hand side of the equation is C^{r-3}-small. The homogeneous initial condition

$$\left[D^\gamma_y(g_{\alpha\beta} - \delta_{\alpha\beta})\right]_{z=-\rho/n} = 0$$

is valid for every $(n-1)$-multiindex γ because of (3.8). Applying Lemma 5.1, we obtain the estimate

$$\|g_{\alpha\beta} - \delta_{\alpha\beta}\|_{C^{r-3}(J^n_{\rho/n})} < C_8\varepsilon_5 \tag{5.30}$$

with some $C_8 = C_8(r, K, \rho, n)$.

By (3.3), $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ coincide in $J^n_\rho \setminus J^n_{\rho/n}$. Therefore (3.6) and (5.30) imply that

$$\|g_{\alpha\beta} - \delta_{\alpha\beta}\|_{C^{r-3}(J^n_\rho)} < \delta$$

with some $\delta = \delta(\varepsilon, K, \rho, n)$ tending to zero as ε goes to zero. This estimate coincides with (3.9) for $p = r - 3$. This finishes the proof.

6 Proof of Lemma 5.1

Let (M, g) be a compact Riemannian manifold with boundary, and f be a symmetric tensor field of second rank on M. There exist uniquely determined tensor field $\tilde{f}$ and covector field v on M such that

$$f = \tilde{f} + d_g v, \quad \delta_g \tilde{f} = 0, \quad v|_{\partial M} = 0. \tag{6.1}$$

We recall how the existence and uniqueness of the decomposition (6.1) are proved in Section 3.3 of [18]. Let us assume the existence of the decomposition (6.1). By applying the operator δ_g to the first of these equalities we conclude that the potential v satisfies the boundary value problem

$$\delta_g d_g v = \delta_g f, \quad v|_{\partial M} = 0.$$

Conversely, if we would have proven the existence of a solution to the boundary value problem

$$\delta_g d_g v = h, \quad v|_{\partial M} = 0 \tag{6.2}$$

and the stability estimate

$$\|v\|_{H^{s+1}(M,g)} \le C(g, s)\|h\|_{H^{s-1}(M,g)}, \tag{6.3}$$

then, setting $h = \delta_g f$ and $\tilde{f} = f - d_g v$, we obtain the existence of the decomposition (6.1) as well as the stability estimate

$$\|\tilde{f}\|_{H^s(M,g)} \le C_1(g, s)\|f\|_{H^s(M,g)}. \tag{6.4}$$

As is shown in [18], (6.2) is an elliptic problem with trivial kernel and cokernel. By general results of elliptic theory, the stability estimate (6.3) is valid for every s. In [18] this fact is proved for a C^∞-metric. The same proof is valid, however, in the case of a C^k-metric g for $s \le k - 2$.

Proof of Lemma 5.1. The proof is by contradiction. A more constructive, but somewhat more technical proof can be given by using estimates for elliptic systems, see e.g. [4]

Let us assume that the Lemma is not valid. Then there exists a sequence $g^{(m)} \in \mathcal{M}(k, K, \rho, n)$ $(m = 1, 2, \dots)$ of metrics and a sequence $f^{(m)}$ of smooth tensor fields on B^n_ρ such that for the corresponding decompositions

$$f^{(m)} = \tilde{f}^{(m)} + d_{g^{(m)}} v^{(m)} \quad \text{in} \quad B^n_\rho, \quad \delta_{g^{(m)}} \tilde{f}^{(m)} = 0 \quad \text{in} \quad B^n_\rho, \quad v^{(m)}|_{\partial B^n_\rho} = 0 \tag{6.5}$$

we have the inequalities

$$\|\tilde{f}^{(m)}\|_{H^s(B^n_\rho)} \geq \frac{1}{m}\|f^{(m)}\|_{H^s(B^n_\rho)}. \tag{6.6}$$

We can assume that $\|f^{(m)}\|_{H^{s+1}(B^n_\rho)} = 1$. There exists a subsequence of $f^{(m)}$ converging in $H^s(B^n_\rho)$. Without loss of generality, we can assume the initial sequence to have this property:

$$f^{(m)} \to f \quad \text{in} \quad H^s(B^n_\rho) \quad \text{as} \quad m \to \infty. \tag{6.7}$$

By Theorem 4.1, $\mathcal{M}(k, K, \rho, n)$ is a precompact set in $\mathcal{M}(k-1, K, \rho, n)$ if the latter set is considered with the C^{k-1}-topology. Actually, this fact is a particular case of the Gromov–Cheeger compactness theory. Therefore there is a subsequence of $g^{(m)}$ converging in the C^{k-1}-topology to some C^{k-1} metric g. Without loss of generality, we can assume the initial sequence to satisfy

$$(g^{(m)}_{\alpha\beta}) \to (g_{\alpha\beta}), \quad (g^{(m)}_{\alpha\beta})^{-1} \to (g_{\alpha\beta})^{-1} \quad \text{in} \quad C^{k-1}(J^n_\rho) \quad \text{as} \quad m \to \infty. \tag{6.8}$$

The coefficients of the operator $\delta_{g^{(m)}} d_{g^{(m)}}$ depend on the C^2-jets of the matrices $(g^{(m)}_{\alpha\beta})$ and $(g^{(m)}_{\alpha\beta})^{-1}$. Therefore (6.8) implies that

$$\delta_{g^{(m)}} d_{g^{(m)}} = \delta_g d_g + L^{(m)}, \tag{6.9}$$

where $L^{(m)}$ is a second order differential operator whose coefficients tend to zero in the C^{k-3}-norm as $m \to \infty$. This implies that, for every $\varepsilon > 0$, we have the estimate

$$\|L^{(m)}h\|_{H^{s-1}(B^n_\rho)} < \varepsilon\|h\|_{H^{s+1}(B^n_\rho)} \quad (s \leq k-2) \tag{6.10}$$

for every smooth tensor field h and for sufficiently large m. Similarly, we can write

$$d_{g^{(m)}} = d_g + l^{(m)} \tag{6.11}$$

for some first order operator $l^{(m)}$ satisfying the estimate

$$\|l^{(m)}h\|_{H^s(B^n_\rho)} < \varepsilon\|h\|_{H^{s+1}(B^n_\rho)} \quad (s \leq k-2) \tag{6.12}$$

for sufficiently large m.

Applying the operator $\delta_{g^{(m)}}$ to the first of the equations (6.5), we obtain

$$\delta_{g^{(m)}} d_{g^{(m)}} v^{(m)} = \delta_{g^{(m)}} f^{(m)}.$$

Using (6.9), we can rewrite the latter equation in the form

$$\delta_g d_g v^{(m)} = \delta_{g^{(m)}} f^{(m)} - L^{(m)} v^{(m)}.$$

By (6.3), the latter equation, together with the boundary condition $v^{(m)}|_{\partial B^n_\rho)} = 0$, implies the estimate

$$\|v^{(m)}\|_{H^{s+1}(B^n_\rho)} \leq C\|\delta_{g^{(m)}} f^{(m)} - L^{(m)} v^{(m)}\|_{H^{s-1}(B^n_\rho)} \tag{6.13}$$

with some constant C independent of m. For $s \leq k-2$, we estimate the right-hand side of (6.13) using (6.10). We conclude

$$\|\delta_{g^{(m)}} f^{(m)} - L^{(m)} v^{(m)}\|_{H^{s-1}(B^n_\rho)} < \|\delta_{g^{(m)}} f^{(m)}\|_{H^{s-1}(B^n_\rho)} + \|v^{(m)}\|_{H^{s+1}(B^n_\rho)}.$$

Combining the latter estimate with (6.13) gives

$$(1 - C\varepsilon)\|v^{(m)}\|_{H^{s+1}(B^n_\rho)} \leq C\|\delta_{g^{(m)}} f^{(m)}\|_{H^{s-1}(B^n_\rho)}.$$

For sufficiently large m, $1 - C\varepsilon \geq 1/2$, and therefore

$$\|v^{(m)}\|_{H^{s+1}(B^n_\rho)} \leq 2C\|\delta_{g^{(m)}} f^{(m)}\|_{H^{s-1}(B^n_\rho)}. \tag{6.14}$$

By (6.7) and (6.8), the right-hand side of (6.14) tends to $2C\|\delta_g f\|_{H^{s-1}(B^n_\rho)}$ as $m \to \infty$. Thus, for m sufficiently large we have

$$\|v^{(m)}\|_{H^{s+1}(B^n_\rho)} \leq 2C\|\delta_g f\|_{H^{s-1}(B^n_\rho)} + 1 \leq C_1\|f\|_{H^s(B^n_\rho)} \tag{6.15}$$

with some constant C_1 independent of m.

Using (6.11), we obtain from (6.15),

$$\|d_{g^{(m)}} v^{(m)}\|_{H^s(B^n_\rho)} \leq \|d_g v^{(m)}\|_{H^s(B^n_\rho)} + \|l^{(m)} v^{(m)}\|_{H^s(B^n_\rho)} \leq C_2\|v^{(m)}\|_{H^{s+1}(B^n_\rho)} \leq C_3\|f\|_{H^s(B^n_\rho)} \tag{6.16}$$

with some C_3 independent of m.

Finally, from (6.5) and (6.16) we conclude

$$\|\tilde{f}^{(m)}\|_{H^s(B^n_\rho)} \leq \|f^{(m)}\|_{H^s(B^n_\rho)} + \|d_{g^{(m)}} v^{(m)}\|_{H^s(B^n_\rho)} \leq (C_3 + 1)\|f\|_{H^s(B^n_\rho)}$$

which contradicts (6.6). This finishes the proof of the Lemma.

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