Gelf'and Inverse Problem for a Quadratic Operator Pencil

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1 Introduction and main result

In the paper we deal with an inverse problem for a quadratic operator pencil

$$\begin{aligned} A(\lambda)u &= a(x,D)u - i\lambda b_0 u - \lambda^2 u, && (1.1)\\ Bu &= \partial_\nu u - \sigma u|_{\partial M} = 0 && (1.2)\end{aligned}$$

on a differentiable compact connected manifold M, $\dim M = m \geq 1$, with non-empty boundary $\partial M \neq \emptyset$. Here $a(x,D)$ is a uniformly elliptic symbol

$$a(x,D) = -g^{-1/2}(\partial_j + b_j)g^{1/2}g^{jl}(\partial_l + b_l) + q,$$

where $[g^{jl}]_{j,l=1}^m$ defines a C^∞-smooth Riemannian metric and $b = (b_1, ..., b_m)$ and q are, correspondingly, C^∞-smooth complex-valued 1-form and function on M. σ is a C^∞-smooth complex-valued function on ∂M and ∂_ν stands for the normal derivative.

Let R_λ be the resolvent of $A(\lambda)$ defined in the domain $H^2_\sigma(M) = \{u \in H^2(M):\ Bu = 0\}$. The operator $R(\lambda)$ is meromorphic for $\lambda \in \mathbb{C}$ (see Sect. 3 and [1]). Denote by $R_\lambda(x,y)$ its Schwartz kernel. A natural analog of the Gel'fand inverse problem [2] is

Problem I Let ∂M and $R_\lambda(x,y)$; $\lambda \in \mathbb{C}$, $x,y \in \partial M$ be given. Do these data, called Gel'fand boundary spectral data, determine $(M, a(x,D), b_0, \sigma)$ uniquely?

We note that the determination of a Riemannian manifold (M,g) means the determination of a Riemannian manifold which is isometric to (M,g).

Remark 1: Let $\mathcal{G}_\lambda$ be the Neumann-to-Dirichlet map $\mathcal{G}_\lambda f := u^f_\lambda|_{\partial M}$ where

$$A(\lambda)u^f_\lambda = 0, \quad Bu^f_\lambda = f. \tag{1.3}$$

Then the Gel'fand boundary spectral data is equivalent to the knowledge of $\mathcal{G}_\lambda$ for all λ.

Remark 2: Problem I is equivalent to the inverse boundary problem for the dissipative wave equation. Namely, assume that $u^f = u^f(x,t)$ is the solution of the problem

$$u^f_{tt} + b_0 u^f_t + a(x,D)u^f = 0, \tag{1.4}$$

$$Bu^f|_{\partial M\times\mathbb{R}_+} = f; \quad u^f|_{t=0} = u^f_t|_{t=0} = 0 \tag{1.5}$$

where $f \in C(\mathbb{R}_+, L^2(\partial M))$. We define the response operator R^h,

$$R^h(f) := u^f|_{\partial M\times\mathbb{R}_+}. \tag{1.6}$$

Since the Fourier transform of $u^f(x,t)$ with respect to t is u^f_λ, we see that R^h determines the Gel'fand boundary spectral data and vice versa.

This hyperbolic inverse problem and its analogs were considered e.g. in [3], [4] and [5]. Paper [3] dealt with the inverse scattering problem, $M = \mathbb{R}^m$, with $g^{jl} = \delta^{jl}$. It was generalized in [4] onto the Gel'fand inverse boundary problem in a bounded domain in $\mathbb{R}^m$, $g^{jl} = \delta^{jl}$. In [5] the uniqueness of the reconstruction of the conformally Euclidean metric in $M \subset \mathbb{R}^m$ and the lower order terms (with some restrictions upon these terms) was proven for the geodesically regular domains M. The case $b_0 = 0$ and self-adjoint $a(x,D)$ was considered in [6] and [7].

In the paper we give the answer to Problem I assuming some geometric conditions upon (M,g). The main technique used is the boundary control (BC) method (see e.g. [8]) in its geometrical version [7], together with the technique elaborated in [9].

To use control theory, we assume the following geometrical condition (for details see [10]) which generalizes the condition that the rays of the geometrical optics hit the boundary transversally.

Definition 1.1 The Riemannian manifold (M,g) satisfies the Bardos-Lebeau-Rauch (BLR) condition if there is $t_* > 0$ and an open conic neighborhood $\mathcal{O}$ of the set of not-nondiffractive points $(x,\xi) \in T^*(M\times[0,t_*])$, $x \in \partial M$ such that any generalized bicharacteristic of the wave operator $\partial_t^2 - \Delta_g$ passes through a point $(x,\xi) \in T^*(M\times[0,t_*]) \setminus \mathcal{O}$, $x \in \partial M$.

The main result of the paper is:

Theorem 1.2 *Let $(\partial M; \mathcal{G}_\lambda, \lambda \in \mathbb{C})$ be the Gel'fand boundary spectral data for a quadratic operator pencil (1.1), (1.2). Assume that the corresponding Riemannian manifold (M,g) satisfies the BLR-condition. Then these data determine M and b_0 uniquely and also the equivalence class of the pair $(a(x,D),\sigma)$ with respect to the group of gauge transformations, that is, the class*

$$\{(\kappa a(x,D)\kappa^{-1}, \sigma + \partial_\nu(ln\kappa)) : \ \kappa \in C^\infty(M;\mathbb{C}), \ \kappa|_{\partial M} = 1, \ \kappa \neq 0 \ on \ \ M\}.$$

When $b_0 = 0$ Theorem 1.2 was proven in [13]. However the generalization of the methods of [13] to the quadratic operator pencils is far from straightforward which results, in particular, in a new sesquilinear form (4.9) corresponding to the wave equation (1.4).

2 Auxiliary constructions

In view of the gauge invariance we can assume that $\sigma = 0$. To transform the pencil equation to an eigenvalue problem, we define

$$U = (u, \lambda u)^t.$$

For pairs $P = (a,b)^t$ we denote $[P]^1 = a$ and $[P]^2 = b$. Then the pencil equation (1.1), (1.2) is equivalent to $(\mathcal{A} - \lambda)U = 0$ where $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$,

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ A_0 & 0 \end{pmatrix}, \quad \mathcal{A}_1 = \begin{pmatrix} 0 & 0 \\ a_1(x,D) & -ib_0 \end{pmatrix}.$$

Here $A_0 = -\Delta_g$ is the Laplace operator with Neumann boundary condition,

$$\mathcal{D}(A_0) = H^2_\nu(M) := \{u \in H^2(M) : \partial_\nu u|_{\partial M} = 0\}$$

and $a_1(x,D) = a(x,D) - A_0$ is a first order operator. Operators $\mathcal{A}_0, \mathcal{A}$ with

$$\mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A}) = H^2_\nu(M) \times L^2(M)$$

are closed in $\mathcal{H} = [L^2(M)]^2$. Using the transformation $\lambda \to \lambda + di$; $A_0 \to A_0 + d^2$, $d \in \mathbb{R}$ we can assume that

$$||A_0^{-1}|| < 1 \text{ and } ||a_1(x,D)A_0^{-3/4}|| < 1/2. \tag{2.1}$$

The adjoint operator, $\mathcal{A}^*$, is then

$$\mathcal{A}^* = \begin{pmatrix} 0 & A^* \\ I & i\bar{b}_0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}^*) = L^2(M) \times \mathcal{D}(A^*), \tag{2.2}$$

$$\mathcal{D}(A^*) = H^2_{\nu,b} := \{u \in H^2; \quad B^*u := \partial_\nu u - 2b_\nu u|_{\partial M} = 0\} \tag{2.3}$$

where $b_\nu = (\nu, b)$ is the conormal component of b.

Using A^* instead of A we define operators $\mathcal{A}_{ad}$ and $\mathcal{A}^*_{ad}$,

$$\mathcal{A}_{ad} = \begin{pmatrix} 0 & I \\ A^* & i\bar{b}_0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_{ad}) = H^2_{\nu,b}(M) \times L^2(M).$$

Our goal is to use eigenfunction expansions corresponding to $\mathcal{A}, \mathcal{A}^*$ and $\mathcal{A}_{ad}, \mathcal{A}^*_{ad}$. To this end we introduce operators the T_0 and $T = T_0 + T_1$ where

$$T_0 = \begin{pmatrix} 0 & A_0^{1/2} \\ A_0^{1/2} & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & 0 \\ A_0^{-1/4} a_1 A_0^{-1/4} & -iA_0^{-1/4} b_0 A_0^{1/4} \end{pmatrix} \tag{2.4}$$

$$\mathcal{D}(T) = \mathcal{D}(T_0) = [\mathcal{D}(A_0^{1/2})]^2 = [H^1(M)]^2. \tag{2.5}$$

By (2.1) T has a bounded inverse. We have

$$T_0 U = L^{-1}\mathcal{A}_0 L U, \quad TU = L^{-1}\mathcal{A}LU \quad \text{for} \tag{2.6}$$

$$L = \begin{pmatrix} A_0^{-1/4} & 0 \\ 0 & A_0^{1/4} \end{pmatrix}, \quad U \in \mathcal{D}(A_0^{3/4}) \times \mathcal{D}(A_0^{1/2}).$$

3 Abel-Lidskii expansion

The operator $T_0^{-1} \in \mathfrak{S}_p,\ p > m$ where $\mathfrak{S}_p$ is the Schatten-von Neumann class (see e.g. [11]). As T_1 is bounded $T = T_0 + T_1$ is a weak perturbation of T_0. Due to the general theory of weak perturbations of self-adjoint operators (see e.g. [1], Sect. 6.2-6.4) the spectrum $\sigma(T)$ of T is normal, that is, the spectrum of T consists of discrete eigenvalues with finite multiplicities.

Let $\beta > m$ be an even integer, $\tau > 0$ and Γ be a contour in $\mathbb{C}$, $\Gamma \cap \sigma(T) = \emptyset$. Denote by $P^{\beta}_{\Gamma,\tau}(T)$ the modified Riesz projector for T;

$$P^{\beta}_{\Gamma,\tau}(T) = -\frac{1}{2\pi i}\int_{\Gamma} e^{-\tau z^{\beta}}(T-z)^{-1}dz,$$

and by $P^{\beta}_{\Gamma,\tau}(T_0)$ the analogous projector for T_0.

Let Γ be a contour in $\mathbb{C}$ which consists of two segments Im $z = \pm a$, Re $z \in [-b, b]$, and four semi-axes Im $z = \pm c$Re z (see Fig. 1).

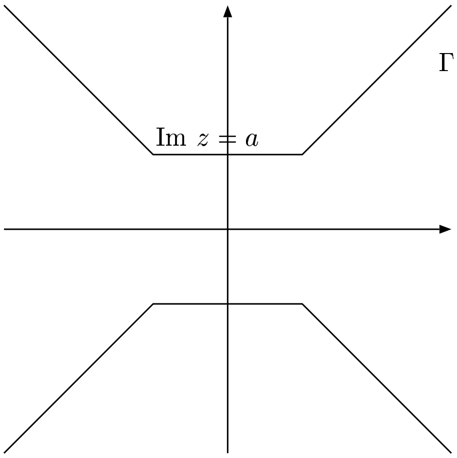

Figure 1: Contour Γ

By using asymptotics of the eigenvalues of T (see e.g. [1]), parameters a, b, c are chosen so that

i. $\sigma(T)$ lies inside Γ.

ii. Re $z^{\beta} \geq c_0|z^{\beta}|, c_0 > 0$ for $|\text{Im } z| \leq c|\text{Re } z|$.

Let $P^{\beta}_{N,\tau}(T)$ be the modified Riesz projections corresponding to the contours Γ_N which are obtained from Γ by cutting them by the vertical lines Re $z = \pm\alpha_N$ (see Fig. 2).

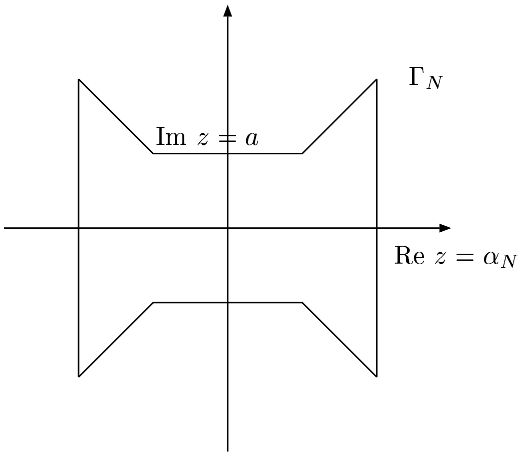

Figure 2: Contour Γ_N

Theorem 3.1 *(Abel-Lidskii convergence) There exist a series of real numbers $\alpha_N > 0,\ N = 1, 2, \ldots$, which depends only upon $\sigma(T)$ such that*

$$Y = \lim_{\tau\to+0}\lim_{N\to\infty} P^{\beta}_{N,\tau}(T)Y. \tag{3.1}$$

The convergence in (3.1) takes place in $[H^s(M)]^2$, $s \in [-1/2, 1/2]$ when $Y \in [H^s(M)]^2$ and in the graph norm of T^n when $Y \in \mathcal{D}(T^n), n = 1, 2, \ldots.$

Proof. Since $T_0 \in \mathfrak{S}_p, \quad p > m$ and T_1 is bounded the results of ([1], Sect. 6.2-6.4; see also [12]) show the existence of α'_N which depend upon $\sigma(T_0)$ and $\sigma(T)$ such that

$$\text{s-}\lim_{N\to\infty} P^{\beta}_{N,\tau}(T) = P^{\beta}_{\tau}(T),$$

where s-lim is the limit in the strong operator topology and $P^{\beta}_{N,\tau}$ and P^{β}_{τ} correspond to contours Γ_N and Γ, correspondingly. The proof of the strong convergence is based upon exponential estimates for $(T - z)^{-1}$ and $(T_0 - z)^{-1}$. However, since $P^{\beta}_{N,\tau}(T)$ remains intact under small deviations of α'_N it is possible to choose α_N independent of $\sigma(T_0)$. Now

$$(T_0 - z)^{-1} - (T - z)^{-1} \ = (T - z)^{-1}T_1(T_0 - z)^{-1}$$

and $||(T - z)^{-1}||_{s,s'} \le c|z|^{-1-s+s'}$, $s \le s'$ when z lies outside Γ and $||\cdot||_{s,s'}$ stands for the operator norm from $[H^s(M)]^2$ to $[H^{s'}(M)]^2$. Hence

$$P^{\beta}_{\tau}(T) - P^{\beta}_{\tau}(T_0) = -\frac{1}{2\pi i}\int_{\Gamma} e^{-\tau z^{\beta}}(T - z)^{-1}T_1(T_0 - z)^{-1}dz, \tag{3.2}$$

$$||(T - z)^{-1}T_1(T_0 - z)^{-1}||_s \le c_s|z|^{-3/2} \tag{3.3}$$

where $s \in [-1/2, 1/2]$ and z lies outside Γ. The integrand in right-hand side of (3.2) is analytic outside the spectra of T_0 and T. Hence we can change the parameter a which defines Γ without changing the right-hand side of (3.2). Thus by choosing $a = \tau^{-1/\beta}$ with τ sufficiently small, we see from (3.3) that (3.2) goes to zero when $\tau \to 0$. By spectral theory of the self-adjoint operators, $\text{s-lim}\, P_\tau^\beta(T_0) = I$. This implies the statement for $Y \in [H^s(M)]^2$.

The last part of the theorem follows from the case $s = 0$ since for $Y \in \mathcal{D}(T^n)$

$$T^n P_{N,\tau}^\beta(T)Y = P_{N,\tau}^\beta(T)T^n Y.$$

$\square$

Lemma 3.2 *Let $U = (u^1, u^2)^t \in H^1(M) \times L^2(M)$ or $U \in [C_0^\infty(M)]^2$. Then*

$$U = \lim_{\tau \to 0+} \lim_{N \to \infty} P_{N,\tau}^\beta(\mathcal{A})U,$$

where the convergence takes place in $H^1(M) \times L^2(M)$ when U lies in this space or in $C^N(\Omega)$ for any $N > 0$, $\Omega \subset\subset M$ when $U \in [C_0^\infty(M)]^2$.

Proof. By (2.6), $(T - z)^{-1}Y = L^{-1}(\mathcal{A} - z)^{-1}LY$ for $Y \in \mathcal{D}(A_0^{3/2}) \times \mathcal{D}(A_0^{1/2})$. Integrating this equation along Γ_N we come to the equation

$$P_{N,\tau}^\beta(T)Y = L^{-1}P_{N,\tau}^\beta(\mathcal{A})LY.$$

Since the modified Riesz projectors are finite-dimensional operators this equation remains valid for $Y \in \mathcal{D}(L)$. As $Y = L^{-1}U \in [H^{1/2}(M)]^2$ when $U \in H^1(M) \times L^2(M)$, Theorem 3.1 with $s = 1/2$ proves the statement for this case. As $L^{-1}[C_0^\infty(M)]^2 \subset \mathcal{D}(T^n)$ for any $n > 0$ and $\mathcal{D}(T^n) \subset [H^n(M)]^2$ this case also follows from Theorem 3.1 and the fact that L is a pseudodifferential operator of the order $1/2$. $\square$

Corollary 3.3 *Let $U \in L^2(M) \times H^1(M)$ or $U \in [C_0^\infty(M)]^2$. Then*

$$U = \lim_{\tau \to 0+} \lim_{N \to \infty} P_{N,\tau}^\beta(\mathcal{A}^*)U \tag{3.4}$$

where the convergence takes place in $L^2(M) \times H^1(M)$ and $C^N(\Omega)$ for any $N > 0, \Omega \subset\subset M$, respectively.

Proof. As $||(T^* - \bar{z})^{-1} - (T_0 - \bar{z})^{-1}||_s = ||(T - z)^{-1} - (T_0 - z)^{-1}||_{-s}$, estimate (3.3) remains valid for T^*, T_0 and $s = 1/2$ for z outside Γ. The same arguments as in Theorem 3.1 show that

$$Y = \lim_{\tau \to 0+} \lim_{N \to \infty} P_{N,\tau}^\beta(T^*)Y \quad \text{in} \quad [H^{1/2}(M)]^2.$$

As $Y = LU \in [H^{1/2}(M)]^2$ when $U \in L^2(M) \times H^1(M)$, (3.4) follows. The case $U \in [C_0^\infty(M)]^2$ is considered as in Lemma 3.2. $\square$

Using the representation

$$\mathcal{A}_{ad}^* = J\mathcal{A}J^{-1}, \quad \mathcal{A}^* = J^*\mathcal{A}_{ad}[J^*]^{-1}, \\ J\left[(u^1, u^2)^t\right] = (u^2 + ib_0 u^1, u^1)^t, \tag{3.5}$$

we come to

Corollary 3.4 *The statement of Lemma 3.2 is valid for $\mathcal{A}_{ad}^*$. The statement of Corollary 3.3 is valid for $\mathcal{A}_{ad}$.*

4 Root functions and boundary spectral data

Let $\mathcal{H} = [L^2(M)]^2$ and denote by $P_{\lambda_j}(\mathcal{A})$ the Riesz projector $P^0_{\Gamma_j,0}(\mathcal{A})$ where Γ_j encloses only one point λ_j of the spectrum. Let $\mu_j = \dim\, \mathcal{H}_j = \dim\, \mathcal{H}_j^*$ where $\mathcal{H}_j = P_{\lambda_j}(\mathcal{A})\mathcal{H}$ and $\mathcal{H}_j^* = P_{\bar{\lambda}_j}(\mathcal{A}^*)\mathcal{H}$. Moreover, let $r_j = \dim\, \mathrm{Ker}(\mathcal{A} - \lambda_j) = \dim\, \mathrm{Ker}(\mathcal{A}^* - \bar{\lambda}_j)$. Denote by

$$\Phi_{j,k,0} = (\phi^1_{j,k,0}, \phi^2_{j,k,0})^t, \ \Psi_{j,k,0} = (\psi^1_{j,k,0}, \psi^2_{j,k,0})^t, \ k = 1, \dots, r_j$$

the eigenvectors of $\mathcal{A}$ and $\mathcal{A}^*$ at eigenvalues λ_j and $\bar{\lambda}_j$, correspondingly, and by $n_{j,k}, \quad n_{j,1} \geq n_{j,2} \geq ... \geq n_{j,r_j}$ their partial null multiplicities. The total multiplicity of λ_j is denoted by $\mu_j = n_{j,1} + ... + n_{j,r_j}$. This means that there are vectors $\Phi_{j,k,l}, \Psi_{j,k,l}, l = 1, ..., n_{j,k}$, called the root functions associated with $\Phi_{j,k,0}, \Psi_{j,k,0}$, such that

$$(\mathcal{A} - \lambda_j)\Phi_{j,k,l} = \Phi_{j,k,l-1}, \quad (\mathcal{A}^* - \bar{\lambda}_j)\Psi_{j,k,l} = \Psi_{j,k,l-1}. \tag{4.1}$$

It is possible to choose $\Phi_{j,k,l}$ and $\Psi_{j,k,l}$ for $j = 1, 2, \dots, \ k = 1, \dots, r_j, \ l = 1, \dots, n_{j,k}$ so that

$$(\Phi_{j,k,l}, \Psi_{j',k',l'})_{\mathcal{H}} = \delta_{j,j'}\delta_{k,k'}\delta_{l,n_{j,k}-l'-1} \tag{4.2}$$

(see e.g. [13], Sect. 2 or [14], Sect. 1.2). The choice of $\Phi_{j,k,l}, \Psi_{j,k,l}$ with fixed j is non-unique. The group of admissible transformations $\mathcal{P}_j$ form a subgroup in $GL(\mu_j, \mathbb{C})$ defined by conditions (4.1), (4.2) (see e.g. [13], Sect. 2).

Let $U, V \in \mathcal{H}$. The sequences

$$\begin{aligned} \mathcal{F}(U) = \mathcal{U} \quad &:= \quad \{U_{j,k,l}\}_{j,k,l}, \ U_{j,k,l} = (U, \Psi_{j,k,n_{j,k}-l-1})_{\mathcal{H}}, \\ \mathcal{F}^*(V) = \mathcal{V}^* \quad &:= \quad \{V^*_{j,k,l}\}_{j,k,l}, \ V^*_{j,k,l} = (V, \Phi_{j,k,n_{j,k}-l-1})_{\mathcal{H}} \end{aligned}$$

are the Fourier transforms of U and N with respect to the dual basis of the eigenfunctions of $\mathcal{A}$ and $\mathcal{A}^*$, correspondingly. Using Lemma 3.2 and Corollary 3.4 we obtain

Corollary 4.1 *Let $U \in H^1(M)\times L^2(M)$ and $V \in L^2(M)\times H^1(M)$. Then their Fourier transforms $\mathcal{U}$ and $\mathcal{V}^*$ determine the inner product $(U,V) = (U,V)_{\mathcal{H}}$ uniquely.*

Due to relations (3.5), analogous results take place for $\mathcal{A}_{ad}, \mathcal{A}^*_{ad}$ with the bases

$$\tilde{\Psi}_{j,k,l} = J\Phi_{j,k,l}; \quad \tilde{\Phi}_{j,k,l} = (J^*)^{-1}\Psi_{j,k,l}. \tag{4.3}$$

The bases $\Phi_{j,k,l}, \Psi_{j,k,l}$ makes sense to the following

Definition 4.2 The boundary spectral data of the pencil (1.1), (1.2) is the collection

$$\{\partial M; \lambda_j, \phi^1_{j,k,l}|_{\partial M}, \psi^2_{j,k,l}|_{\partial M} : \ j = 1,2,...,k = 1,...,r_j, l = 1,...,n_{j,k}\} \tag{4.4}$$

.

Theorem 4.3 *The Gel'fand boundary spectral data determine boundary spectral data (4.4) to within the group $\mathcal{P}_j$ of the transformations of the biorthogonal bases which preserve properties (4.1), (4.2).*

Proof. Given $R_\lambda(x,y), x,y \in \partial M$ it is possible to find $u^f_\lambda|_{\partial M}$, where u^f_λ is the solution to (1.3). Consider $U^f_\lambda = (u^f_\lambda, \lambda u^f_\lambda)^t$. Then

$$(a-\lambda)U^f_\lambda = 0,$$

where a is an extension of A on $H^2(M)\times L^2(M)$;

$$a = \begin{pmatrix} 0 & I \\ a(x,D) & -ib_0 \end{pmatrix}, \ \mathcal{D}(a) = H^2(M)\times L^2(M).$$

Let $e \in H^2(M),\ \partial_\nu e|_{\partial M} = f$ and $E = (e,0)^t$. Then

$$U^f_\lambda = E - (\mathcal{A}-\lambda)^{-1}(a-\lambda)E.$$

U^f_λ is a meromorphic function of λ with possible singularities only at $\lambda_j \in \sigma(\mathcal{A})$ and $U^f_\lambda - P_{\lambda_j}(\mathcal{A})U^f_\lambda$ is analytic at λ_j. Clearly,

$$[P_{\lambda_j}(\mathcal{A})U^f_\lambda]^1|_{\partial M} = \sum_{k=1}^{r_j}\sum_{l=0}^{n_{j,k}-1} U^f_{j,k,l}(\lambda)\phi^1_{j,k,l}|_{\partial M}. \tag{4.5}$$

By Green's formula

$$\begin{aligned}(\lambda-\lambda_j)(U^f_\lambda, \Psi_{j,k,n_{j,k}-l-1})_{\mathcal{H}} \quad &= \quad \int_{\partial M} f\,\psi^2_{j,k,n_{j,k}-l-1}|_{\partial M} dS - \\ &\quad -(U^f_\lambda, \Psi_{j,k,n_{j,k}-l-2})_{\mathcal{H}},\end{aligned} \tag{4.6}$$

where $\Psi_{j,k,-1} = 0$. Formulae (4.5), (4.6) show that $P_{\lambda_j}(\mathcal{A})U_\lambda^f$ is pure singular at λ_j, i.e. all its positive Laurent coefficients vanish. Thus (4.5) is exactly the singular part of the Laurent expansion of $R_\lambda f$ at λ_j. Hence by means of equation (4.5) (with different f's) it is possible to find all $\lambda_j \in \sigma(\mathcal{A}) = \sigma(A(\lambda))$ as well as the boundary values $\phi^1_{j,k,l}|_{\partial M}$, $\psi^2_{j,k,l}|_{\partial M}$ to within a linear transformation preserving (4.1), (4.2) (for details see e.g. [13], Sect. 3). $\square$

Let $u^f(x,t)$ be the solution to (1.4), (1.5) and $v^g(x,s)$ be the solution to the initial-boundary value problem

$$v^g_{ss} - \bar{b}_0 v^g_s + a^*(x,D)v^g = 0, \tag{4.7}$$

$$B^* v|_{\partial M \times \mathbb{R}_+} = g; \quad v^g|_{s=0} = v^g_s|_{s=0} = 0 \tag{4.8}$$

which is associated with $\mathcal{A}_{ad}$. Let

$$U^f(t) = \begin{pmatrix} u^f(t) \\ iu^f_t(t) \end{pmatrix}, \quad V^g(s) = \begin{pmatrix} v^g(s) \\ iv^g_s(s) \end{pmatrix}.$$

Then

$$U^f_t + i\mathcal{A}U^f = 0, \quad V^g_s + i\mathcal{A}_{ad}V^g = 0.$$

Lemma 4.4 *For any $f, g \in L^2(\partial M \times \mathbb{R}_+)$ the boundary spectral data $\{\lambda_j$, $\phi^1_{j,k,l}|_{\partial M}$, $\psi^2_{j,k,l}|_{\partial M}\}$ determine the Fourier coefficients $\mathcal{F}U^f(t)$ and*

$$\mathcal{F}_{ad}V^g(s) = \mathcal{V}^g_{ad}(s) = \{(V^g(s), \widetilde{\Psi}_{j,k,n_{j,k}-l-1})_{\mathcal{H}}\}_{jkl}.$$

Proof. Integration by parts together with the second of relations (4.1) yields that

$$\begin{aligned} i\partial_t(U^f(t), \Psi_{j,k,n_{j,k}-l-1}) &= \lambda_j(U^f(t), \Psi_{j,k,n_{j,k}-l-1}) + (U^f(t), \Psi_{j,k,n_{j,k}-l-2}) + \\ &+ \int_{\partial M} f(t)\psi^2_{j,k,n_{j,k}-l-1}|_{\partial M} dS. \end{aligned}$$

As $U^f|_{t=0} = 0$ this equation proves Lemma for $U^f(t)$. Taking into account (4.3), the same considerations prove Lemma for $V^g(s)$. $\square$

Corollary 4.5 *Let $f, g \in L^2(\partial M \times \mathbb{R}_+)$. Given boundary spectral data (4.4) and $t, s \geq 0$ it is possible to evaluate*

$$\begin{aligned} &(U^f(t), J^*V^g(s)) = \\ &= i\int_M [u^f_t(x,t)\overline{v^g(x,s)} - u^f(t)\overline{v^g_s(x,s)} + b_0(x)u^f(x,t)\overline{v^g(x,s)}]dx. \end{aligned} \tag{4.9}$$

Proof. The statement is an immediate corollary of the fact that $U^f(t) \in H^1(M) \times L^2(M)$, $J^*V^g(s) \in L^2(M) \times H^1(M)$, Lemma 3.2, Corollary 3.3, formula (3.5), and Lemma 4.4. $\square$

5 Reconstruction of (M, g)

Denote by $\mathcal{L}^s$, $s \in \mathbb{R}$ the subspace in $H^{s+1}(M) \times H^s(M)$ of the functions which satisfy natural compatibility conditions for the hyperbolic problem (1.4), (1.5) (see e.g [15]) and by $\mathcal{L}^s_{ad}$ the analogous subspace for (4.7), (4.8).

Theorem 5.1 *[10] Let (M, g) satisfies the BLR-condition. Then*

$$\{U^f(T): \ f \in H^s_0(\partial M, [0,T])\} = \mathcal{L}^s, \quad T > t_*, \ s \geq -1/2.$$

Corollary 5.2 *Let (M, g) satisfies the BLR-condition. Then boundary spectral data (4.4) determine $\mathcal{F}(\mathcal{L}^s)$ and $\mathcal{F}_{ad}(\mathcal{L}^s_{ad})$ for $s \geq -1/2$.*

Proof. The statement follows from Lemma 4.4 and Theorem 4.3. □

Let $\Gamma \subset M$ be open, $t \geq 0$. Denote by $d(\cdot\,,\cdot\,)$ the distance in M and

$$M(\Gamma, t) = \{x \in M : d(x, \Gamma) \leq t\}.$$

Lemma 5.3 *Let $\mathcal{U} \in \mathcal{F}(\mathcal{L}^s)$, $s \geq 0$, $\mathcal{U} = \mathcal{F}U$. Then for any $\Gamma \subset \partial M$ and $t_0 \geq 0$ boundary spectral data (4.4) determine whether $m_g(supp\ (U) \cap M(\Gamma, t)) = 0$ or not. Analogous statement takes place for $\mathcal{V}_{ad}$.*

Here m_g is the Riemannian measure on (M, g).

Proof. Consider $\mathcal{U}(t) = \{U_{j,k,l}(t)\}$ where

$$\frac{d}{dt} U_{j,k,l}(t) + i\lambda_j U_{j,k,l}(t) + iU_{j,k,l+1}(t) = 0, \tag{5.1}$$

$$U_{j,k,l}(0) = U_{0;j,k,l}, \tag{5.2}$$

and $\{U_{0;j,k,l}\} = \mathcal{U}_0 = \mathcal{F}U_0$, $U_0 \in \mathcal{L}^s$. Then there is $U(t) = (u^1(t), u^2(t))$ such that $\mathcal{U}(t) = \mathcal{F}U(t)$ and U satisfies the wave equation

$$U_t(t) + i\mathcal{A}U(t) = 0, \quad U(0) = U_0.$$

Moreover, $\mathcal{U}(t) \in \mathcal{F}(\mathcal{L}^s)$. When $s \geq 0$ Lemma 3.2 and trace theorem show that

$$u^1(t)|_{\partial M} = \lim_{\tau \to 0+} \lim_{N \to \infty} [P^\beta_{N,\tau}(\mathcal{A})U(t)]^1|_{\partial M}, \tag{5.3}$$

where the convergence takes place in $L^2(\partial M)$. In view of the Holmgren-John theorem [16], the fact that $m_g(\text{supp }(U) \cap M(\Gamma, t)) = 0$ is equivalent to the fact that

$$\text{supp }(u^1|_{\partial M \times \mathbb{R}}) \cap (\Gamma \times [-t_0, t_0]) = \emptyset. \tag{5.4}$$

However, $\phi^1_{j,k,l}|_{\partial M}$ are known so that equations (5.1) - (5.3) determine $u^1(t)|_{\partial M}$. □

Corollary 5.4 *Let $\Gamma \subset \partial M, t_0 \geq 0$ and $s \geq 0$. Then boundary spectral data (4.4) determine subspaces*

$$\mathcal{F}(\mathcal{L}^s(\Gamma, t_0)),\ \mathcal{F}([\mathcal{L}^s(\Gamma, t_0)]^c) \text{ and } \mathcal{F}_{ad}(\mathcal{L}^s_{ad}(\Gamma, t_0)),\ \mathcal{F}_{ad}([\mathcal{L}^s_{ad}(\Gamma, t_0)]^c),$$

where

$$\mathcal{L}^s(\Gamma, t_0) = \{U \in \mathcal{L}^s :\ supp\ (U) \subset cl(M(\Gamma, t_0))\},$$
$$[\mathcal{L}^s(\Gamma, t_0)]^c = \{U \in \mathcal{L}^s : supp\ (U) \subset cl(M \setminus M(\Gamma, t_0)\}$$

and analogous definitions are valid for $\mathcal{L}^s_{ad}(\Gamma, t_0)$ and $[\mathcal{L}^s_{ad}(\Gamma, t_0)]^c$.

Proof. By Lemma 5.3, boundary spectral data (4.4) determine $[\mathcal{L}^s(\Gamma, t_0)]^c$ and $[\mathcal{L}^s_{ad}(\Gamma, t_0)]^c$. Since $U \in \mathcal{L}^s(\Gamma, t_0)$ if and only if $(U, J^*V) = 0$ for all $V \in [\mathcal{L}^s_{ad}(\Gamma, t_0)]^c$ the remaining part of Corollary 5.4 follow from Corollary 4.5. $\square$

Using intersections and unions of the sets described in Corollary 5.4 we obtain

Corollary 5.5 *Let $\Gamma_i \subset \partial M, t_i^+ > t_i^- \geq 0,\ i = 1, ..., I$. Denote by M_I the set*

$$M_I = \bigcap_{i=1}^{I} (M(\Gamma, t_i^+) \setminus M(\Gamma, t_i^-)). \tag{5.5}$$

Then boundary spectral data (4.4) determine whether $m_g(M_I) = 0$ or not.

Corollary 5.5 is the basic analytic tool in the reconstruction of (M, g). For this end introduce $\mathcal{R} : M \to L^\infty(\partial M)$ which maps $x \in M$ to the boundary distance function

$$\mathcal{R}(x) : y \mapsto r_x(y) = d(x, y), \quad y \in \partial M.$$

It is shown in [7] that when the set $\mathcal{R}(M) \subset L^\infty(\partial M)$ is given, it is possible to define a Riemannian structure on $\mathcal{R}(M)$ such that $\mathcal{R} : M \to \mathcal{R}(M)$ is an isometry.

Theorem 5.6 *Boundary spectral data (4.4) of operator pencil (1.1), (1.2) which satisfies the BLR-condition determine (M, g) uniquely.*

Proof. In view of the above remark about isometry between (M, g) and $\mathcal{R}(M)$, it is sufficient to show that the boundary spectral data determine $\mathcal{R}(M)$. Choose $\varepsilon > 0$ and a collection of $\Gamma_i, i = 1, ..., I(\varepsilon)$ such that $\mathrm{diam}(\Gamma_i) \leq \varepsilon$, $\bigcup \Gamma_i = \partial M$. Let

$$p = (p_1, ..., p_{I(\varepsilon)}), \quad p_i \in \mathbb{Z}_+, \quad t_i^+ = (p_i + 1)\varepsilon, \quad t_i^- = (p_i - 1)\varepsilon. \tag{5.6}$$

Denote by $M_I(p)$ the set M_I (see (5.5)) with $t_i^\pm$ of form (5.6). For every p such that $m_g(M_I(p)) > 0$ define a piecewise constant function r_p by setting $r_p(y) = p_i\varepsilon$ when $y \in \Gamma_i$. Let $\mathcal{R}_\varepsilon(M)$ be the collection of these functions. Then

$$\mathrm{dist}_M(\mathcal{R}_\varepsilon(M), \mathcal{R}(M)) \leq 3\varepsilon$$

where $\mathrm{dist}_M(\Omega, \widetilde{\Omega})$ stands for the Hausdorff distance between arbitrary subsets $\Omega, \widetilde{\Omega}$ in $L^\infty(\partial M)$. Taking $\varepsilon \to 0$ we construct $\mathcal{R}(M)$. By [7] the set $\mathcal{R}(M)$ can be considered as a Riemannian manifold which is isometric to (M, g). $\square$

6 Reconstruction of the lower-order terms

Let $\mathcal{U} = \mathcal{F}(U)$, $\mathcal{V} = \mathcal{F}_{ad}(V)$, for $U, V \in \mathcal{L}^0$. We define an inner product

$$(\mathcal{U}, \mathcal{V}) = \sum_{j,k,l} \mathcal{U}_{j,k,l} \mathcal{V}_{j,k,n_{j,k}-l+1} = (J \sum_{j,k,l} \mathcal{U}_{j,k,l} \Phi_{j,k,l}, \sum_{j,k,l} \mathcal{V}_{j,k,l} \widetilde{\Phi}_{j,k,l})$$

where the sums are understood as limits in the Abel-Lidskii sense.

Let x_0 be an interior point of M and sets $M(\varepsilon) \subset M$,

$$M_I(\varepsilon) \longrightarrow x_0 \quad \text{when} \quad \varepsilon \to 0. \tag{6.1}$$

Consider a family $\mathcal{V}(\varepsilon) \in \mathcal{F}_{ad}(\mathcal{L}^0)$, $\varepsilon > 0$, such that

$$\mathrm{supp}\ (V(\varepsilon)) \subset \mathrm{cl}(M_I(\varepsilon)), \quad \mathcal{V}^\varepsilon = \mathcal{F}_{ad}(V(\varepsilon)), \tag{6.2}$$

and for any $\mathcal{U} \in \mathcal{F}(\mathcal{L}^s)$, $s < m/2 < s+1$ there is a limit $\mathcal{W}^{x_0}(\mathcal{U})$,

$$\mathcal{W}^{x_0}(\mathcal{U}) = \lim_{\varepsilon \to 0} (\mathcal{U}, \mathcal{V}(\varepsilon)).$$

Such families exist, indeed it is sufficient to take C_0^∞-approximations to $(0, \delta(\cdot - x_0))^t$ where δ is the Dirac distribution. On the other hand,

$$(\mathcal{U}, \mathcal{V}(\varepsilon)) = (U, J^* V(\varepsilon))_{\mathcal{H}}.$$

Since the limit exists for every $\mathcal{U} \in \mathcal{F}(\mathcal{L}^s)$ and, in particular, for every $U \in [C_0^\infty(M)]^2$, the Banach-Steinhaus theorem imply that there is $W^{x_0} \in [D'(M)]^2$ such that

$$\lim_{\varepsilon \to 0} (\mathcal{U}, \mathcal{V}(\varepsilon)) = W^{x_0}(U) = (J^* W^{x_0}, U)_{\mathcal{H}}.$$

By (6.1) $\mathrm{supp}\ (W^{x_0}) \subset \{x_0\}$. Moreover, as the limit exists for all $U \in \mathcal{L}^s, s < m/2 < s+1$, we must have $W^{x_0} = (0, \kappa(x_0)\delta(\cdot - x_0))^t$ for some $\kappa(x_0) \in \mathbb{C}$.

Lemma 6.1 *Let boundary spectral data (4.4) of operator pencil (1.1), (1.2) be given, (M, g) satisfies the BLR-condition. Then for any s, $s < m/2 < s+1$, and any $r_{x_0} \in \mathcal{R}(M)$ (corresponding some $x_0 \in M$) it is possible to construct a family $\mathcal{V}^{x_0}(\varepsilon)$, $\varepsilon > 0$ which has a limit $\mathcal{W}^{x_0} : \mathcal{F}(\mathcal{L}^s) \to \mathbb{C}$. Moreover, there is $\kappa(x_0)$ such that*

$$\mathcal{W}^{x_0}(\mathcal{U}) \quad = \quad \kappa(x_0) u^1(x_0) \tag{6.3}$$

when $\mathcal{U} = \mathcal{F}(u^1, u^2)^t$. Finally, the families $\mathcal{V}^{x_0}(\varepsilon)$, $x_0 \in M$, can be constructed in such a way that

$$\kappa \in C^\infty(M), \quad \kappa|_{\partial M} = 1, \quad \kappa \neq 0 \quad \text{on } M. \tag{6.4}$$

Proof. To prove the lemma it is sufficient to show the existence of families $\mathcal{V}^{x_0}(\varepsilon)$, $x_0 \in M$, such that their limits $\mathcal{W}^{x_0}$ satisfy the following conditions:

i. For any $x_0 \in M$ there is $\mathcal{U}$ such that $\mathcal{W}^{x_0}(\mathcal{U}) \neq 0$,

ii. $\mathcal{W}^{x_0}(\mathcal{U}) \in C^\infty(M)$ when $\mathcal{U} \in \mathcal{F}(\bigcap_{s>0} \mathcal{L}^s)$.

iii. $\lim_{x\to x_0} \mathcal{W}^x(\mathcal{U}) = u^1(x_0)$ when $x_0 \in \partial M$ and $\mathcal{U} \in \mathcal{F}(\mathcal{L}^s)$.

To prove the existence of such $\mathcal{V}^{x_0}(\varepsilon)$ we can take adjoint Fourier transforms of some smooth approximations to $(0, \delta(\cdot - x_0))^t$. On the other hand, conditions i.-iii. may be verified algorithmically due to Lemma 4.4, Corollary 4.5, Corollary 5.2, Lemma 5.3 and Lemma 3.2. □

Note that Lemma 6.1 shows the existence of the function κ but its values are unknown.

Corollary 6.2 *Boundary spectral data (4.4) of a pencil (1.1),(1.2) with (M, g) satisfying the BLR-condition determine the functions $\kappa(x)\phi^1_{j,k,l}(x)$, $j = 1, 2, \dots$, $k = 1, \dots, r_j$ $l = 1, \dots, n_{j,k}$, where the function κ satisfies relations (6.4).*

Proof. Obviously $\Phi_{j,k,l} \in \mathcal{L}^s$ for any s. By applying Lemma 6.1 to these functions, we can find the values

$$\kappa(x_0)\phi^1_{j,k,l}(x_0) = \mathcal{W}^{x_0}(\mathcal{E}_{(j,k,l)})$$

where $\mathcal{E}_{(j,k,l)}$ is the sequence with 1 at the (j, k, l)-place and 0 otherwise. □

The functions $\kappa\phi^1_{j,k,l}$ are the root functions for the pencil $A_\kappa(\lambda)$:

$$\begin{aligned} A_\kappa(\lambda_j)(\kappa\phi^1_{j,k,l}) &= a_\kappa(x, D)(\kappa\phi^1_{j,k,l}) - i\lambda_j b_0(\kappa\phi^1_{j,k,l}) - \lambda_j^2(\kappa\phi^1_{j,k,l}) \quad (6.5) \\ &= \kappa\phi^1_{j,k,l-1} \\ B_\kappa(\kappa\phi^1_{j,k,l}) &= (\partial_\nu(\kappa\phi^1_{j,k,l}) - \sigma_\kappa(\kappa\phi^1_{j,k,l}))|_{\partial M} = 0. \quad (6.6) \end{aligned}$$

Here $A_\kappa(\lambda)$ is the pencil corresponding the pair

$$a_\kappa(x, D) = \kappa a(x, D)\kappa^{-1}, \quad \sigma_\kappa = \sigma + \partial_\nu[\ln \kappa],$$

which is the the gauge transformation of $a(x, D)$ and $\sigma(x)$.

Lemma 6.3 *When κ satisfies (6.4), the functions $\kappa\phi^1_{j,k,l}$, $j = 1, 2, \dots$, $k = 1, \dots, r_j$, $l = 1, \dots, n_{j,k}$ determine $a_\kappa(x, D), b_0(x)$ and $\sigma_\kappa|_{\partial M}$.*

Proof. From Lemma 3.2 we see that the finite linear combinations of $\kappa\Phi_{j,k,l} = (\kappa\phi^1_{j,k,l}, \lambda_j\kappa\phi^1_{j,k,l})^t$ are dense in $[C^N(\Omega)]^2$ for any $N \geq 0$, $\Omega \subset\subset M$. In particular, when x_0 is an interior point of M the vectors

$$(\kappa(x_0)\phi^1_{j,k,l}(x_0), \nabla(\kappa\phi^1_{j,k,l})(x_0), \lambda_j\kappa(x_0)\phi^1_{j,k,l}(x_0)) \in \mathbb{C}^{m+2}$$

span $\mathbb{C}^{m+2}$. Then equations (6.5) determine a_κ and b_0.

On the other hand, for any $y \in \partial M$ there is $\phi^1_{j,k,l}$ such that $\phi^1_{j,k,l}(y) \neq 0$. Hence equations (6.6) determine σ_κ. $\square$

Theorem 1.2 is now a corollary of Lemma 6.3, Lemma 6.2 and properties (6.4) of κ.

7 Remarks

i. The BLR-condition is always satisfied for $M \subset \mathbb{R}^m$ when $g^{jl} = \delta^{j,l}$ or is its C^1-small perturbation (see e.g. [10] and [17]).

ii. In particular the results of the paper are always valid for $m = 1$ even when the Gel'fand boundary spectral data are prescribed only at one boundary point.

iii. Using the non-stationary variant of the BC-method (see e.g. [8] and [18]) it is possible to prove an analog of Theorem 1.2 for the dynamic inverse problem. In this case the data is the response operator $R^h(t)$ of form (1.6) for the problem (1.4), (1.5) given for $0 \le t \le t_*$ where t_* is the same as in Definition 1.1.

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