Uniqueness for a wave propagation inverse problem in a half space

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Abstract

This paper considers an inverse problem for wave propagation in a perturbed, dissipative half-space. The perturbation is assumed to be compactly supported. This paper shows that in dimension three, the perturbation is uniquely determined by knowledge of the Dirichlet-to-Neumann map on an open subset of the boundary.

1 Introduction

In this paper, we consider an inverse problem for wave propagation modeled by the equation

$$(\nabla^2 + q(x))u(x) = 0, \tag{1}$$

where x is in $\mathbf{R}^3$ and q can be complex-valued. This equation arises in the propagation of electromagnetic [1], [4] and acoustic [7], [10] waves.

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We consider the problem in the half-space $x_3 < 0$. In the lower half-space, $q(x)$ differs from the constant q_- only in a region of compact support. Here the imaginary part of $q(x)$ and q_- are assumed to be positive.

For this wave propagation problem we consider an inverse boundary value problem for the lower half-space. In particular, we specify Dirichlet data on the top surface:

$$u|_{x_3=0} = f, \tag{2}$$

together with an outgoing radiation condition in the lower half-space. We assume that the boundary data f is in the Sobolev space $H^{1/2}$ defined by

$$H^s = \{f : \int |\hat{f}(\xi)|^2 (1 + |\xi|^2)^{s/2} d\xi < \infty\}.$$

For boundary data in this space, it is known that the boundary value problem (1), (2), together with radiation conditions, has a unique solution [1].

Thus the normal derivative $\partial u / \partial x_3$ on the surface $x_3 = 0$ is uniquely determined. The mapping from $H^{1/2}$ to $H^{-1/2}$

$$\Lambda : u|_{x_3=0} \mapsto \frac{\partial u}{\partial x_3}\Big|_{x_3=0} \tag{3}$$

is called the Dirichlet-to-Neumann map. Such maps have been used a great deal recently in the study of inverse problems [15], [9], [13], [11], [14]. Knowledge of the Dirichlet-to-Neumann map is equivalent, in a certain sense, to scattering data [1].

The inverse boundary value problem is to determine $q(x)$ in the lower half-space from knowledge of Λ. In the case in which the Dirichlet-to-Neumann map is defined on the boundary of a compact region, it is known [15] that knowledge of Λ uniquely determines $q(x)$. The purpose of this paper is to extend this result to the half-space geometry. For the case in which q is purely real, the half-space uniqueness question was studied in the unpublished manuscripts [17] and [5]. Here we consider the complex case.

2 The theorem and proof

We consider two equations of the form (1). We denote the two qs by q_1 and q_2, and we will use subscripts to denote the corresponding associated solutions, Green's functions, Dirichlet-to-Neumann maps, etc..

We assume that the general region of space containing the possible perturbations q_1 and q_2 is known. We denote by B a large open set containing the supports of q_1 and q_2. We also assume that the perturbation is strictly contained in the lower half-space, so that the closure of B does not intersect the boundary $x_3 = 0$.

The proof of the theorem will make use of the Dirichlet Green's function, which can be defined as follows by the method of images. We use a tilde to denote the image point, so that $\tilde{y}$ is the point obtained by reflecting y across the $y_3 = 0$ plane. First we recall that the free-space outgoing Green's function corresponding to the medium parameters in the lower half-space is

$$G_0(x,y) = \frac{e^{i\sqrt{q_-}|x-y|}}{4\pi|x-y|}. \tag{4}$$

For a source located at the image point $\tilde{y}$ we define the function $\tilde{G}_0(x,y) = G_0(x,\tilde{y})$. Thus the Green's function of the homogeneous half-space is defined by $G_0^D(x,y) = G_0(x,y) - \tilde{G}_0(x,y)$. This function satisfies the Dirichlet boundary condition $G_0^D(x,y)|_{x_3=0} = 0$ and the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} |x|\left(\frac{\partial}{\partial|x|} - ik\right)G_0^D(x,y) = 0. \tag{5}$$

Thus we can define the perturbed free-space Green's function as the unique solution of the Lippmann-Schwinger equation

$$G^D(x,y) = G_0^D(x,y) + \int G_0^D(x,z)(q(z) - q_-)G^D(z,y)\,dz. \tag{6}$$

By construction, this Dirichlet Green's function satisfies (in the distribution sense)

$$(\nabla^2 + q(x))G^D(x,y) = -\delta(x-y) \quad \text{in} \quad \mathbf{R^3_-}. \tag{7}$$

Moreover, simple calculations show that this G^D satisfies the boundary condition

$$G^D(x,y)|_{x_3=0} = 0, \tag{8}$$

and the Sommerfeld radiation condition (5). Moreover, general facts about Green's functions [3] imply that G^D also satisisfies the reciprocity relation

$$G^D(x,y) = G^D(y,x). \tag{9}$$

Proposition 2.1 *The kernel of the Dirichlet-to-Neumann map* Λ *is*

$$\frac{-\partial}{\partial x_3}\frac{\partial}{\partial y_3} G^D(x,y)|_{x_3=0,y_3=0}. \tag{10}$$

Proof. We apply the divergence theorem and radiation condition (5) to $\int (G^D \nabla^2 u - u \nabla^2 G^D) = u$, where u denotes a solution of (1), (2), with radiation conditions in the lower half-space. □

We denote by $\mathbf{R}^{\mathbf{3}}_{-}$ the lower half-space $\{x : x_3 < 0\}$, and by A the set $\mathbf{R}^{\mathbf{3}}_{-} \backslash \overline{\mathbf{B}}$. In the following we say that $\Lambda_1 = \Lambda_2$ on some open subset Γ of the boundary $x_3 = 0$ if the kernels of the operators coincide on $\Gamma \times \Gamma$, i.e.,

$$\frac{\partial}{\partial x_3}\frac{\partial}{\partial y_3} G_1^D(x,y)\Big|_{x_3=0,y_3=0} = \frac{\partial}{\partial x_3}\frac{\partial}{\partial y_3} G_2^D(x,y)\Big|_{x_3=0,y_3=0} \tag{11}$$

for x and y in Γ.

Theorem 2.1 *Suppose the set* B *containing the supports of* $q_1 - q_-$ *and* $q_2 - q_-$ *is strictly contained in the lower half-space. If* $\Lambda_1 = \Lambda_2$ *on some open subset* Γ *of the boundary* $x_3 = 0$*, then* $q_1 = q_2$.

Proof. The proof involves a series of five lemmas. The first two show that knowledge of the Dirichlet-to-Neumann map suffices to determine the Dirichlet Green's function outside the perturbation in q. The third and fourth lemmas establish a version of the Green's theorem identity that is often used for uniqueness arguments. The last lemma, which is roughly based on the uniqueness proof in [8], shows that linear combinations of the Dirichlet Green's functions can be used to approximate the Sylvester-Uhlmann solutions.

Lemma 2.2 *Suppose the hypotheses of the theorem are satisfied. Then* $\partial_{y_3} G_1^D(x,y) = \partial_{y_3} G_2^D(x,y)$ *for* y *in* Γ *and* x *in* A.

Proof. For a fixed y in Γ, we denote by $F_i(x)$ the function $\partial_{y_3} G_i^D(x,y)$ for $i = 1, 2$. Each F_i satisfies the Helmholtz equation $(\nabla^2 + q_-)F_i(x) = 0$ in the set A away from the perturbation. Because $\Lambda_1 = \Lambda_2$ on Γ, from (11) we have that the normal derivatives of F_i coincide there: $\partial_{x_3} F_1(x) = \partial_{x_3} F_2(x)$ for x on

Γ. Moreover, by taking y_3 derivatives of both sides of the boundary condition (8) we see that for x on Γ, $F_1(x) = F_2(x) = 0$. Thus $F_1 - F_2$ satisfies the unperturbed Helmholtz equation outside $\overline{B}$, and has zero Cauchy data on Γ. The Holmgren uniqueness theorem [16] implies that F_1 and F_2 are identical in a neighborhood of Γ.

From this the claim follows from the unique continuation principle [2]. □

Lemma 2.3 *Suppose the hypotheses of the theorem are satisfied. Then* $G_1^D(x,y) = G_2^D(x,y)$ *for* x *in* A *and* y *in* $A\backslash\{x\}$.

Proof. Let x in A be fixed. Then $G_i^D(x,y)$, $i = 1,2$ satisfies the Helmholtz equation $(\nabla^2 + q_-)G_i^D(x,\cdot) = 0$ in the set $A\backslash\{x\}$. By the boundary condition (8) and reciprocity, $G_i^D(x,y) = 0$ for y in Γ. By Lemma 2.2, $\partial_{y_3}G_1^D(x,y) = \partial_{y_3}G_2^D(x,y)$ for y in Γ. Thus G_1^D and G_2^D satisfy the homogeneous Helmholtz equation in $A\backslash\{x\}$ and have the same Cauchy data on Γ. Again, by the Holmgren uniqueness theorem $G_1^D(x,\cdot)$ and $G_2^D(x,\cdot)$ are identical in a neighborhood of Γ. Thus by the unique continuation principle they coincide everywhere outside $\overline{B} \cup \{x\}$. □

Lemma 2.4 *For* x *in* B, *and* y *and* z *in* A,

$$\int_B (q_i - q_j)(x)G_i^D(x,y)G_j^D(x,z)dx$$
$$= \int_{\partial B} \big(G_i^D(x,y)\partial_{\nu_x}G_j^D(x,z) - G_j^D(x,z)\partial_{\nu_x}G_i^D(x,y)\big)dS_x,$$

where ∂_{ν_x} *denotes differentiation with respect to the outward unit normal to* B *and* $i,j = 1,2$.

Proof. This follows from the divergence theorem. □

Lemma 2.5 *Suppose the hypotheses of the theorem are satisfied. Then for* x *in* B, *and* y *and* z *in* A,

$$\int_B (q_1 - q_2)(x)G_1^D(x,y)G_2^D(x,z)dx = 0. \tag{12}$$

Proof. From Lemma 2.4 with $i = 1, j = 2$ we have

$$\int_B (q_1 - q_2)(x) G_1^D(x,y) G_2^D(x,z) dx$$
$$= \int_{\partial B} \big(G_1^D(x,y) \partial_{\nu_x} G_2^D(x,z) - G_2^D(x,z) \partial_{\nu_x} G_1^D(x,y) \big) dS_x. \quad (13)$$

However, by Lemma 2.3, $G_1^D(x,y) = G_2^D(x,y)$ for $x, y \in A$. Hence the Dirichlet and Neumann boundary values of G_1^D and G_2^D coincide. By applying Lemma 2.4 in the case $i = j$ see that the right hand side of (13) vanishes and thus we obtain (12). $\square$

Let next A_1 be an open set, $\overline{A}_1 \subset A$ and define $U = \text{span}\,\{G^D(\cdot,y)|_B : y \in A_1\}$ and $V = \{v \in H^2(B) : (\nabla^2 + q)v = 0\}$. We consider U and V as non-closed subspaces of $L^2(B)$.

Lemma 2.6 *The set U is a dense subset of V in the relative $L^2(B)$-topology.*

Proof. First, U is certainly a subset of V, because by elliptic regularity, $G^D(\cdot,y)$ is smooth in B when the source y is outside $\overline{B}$. To show that U is dense in V, it is enough to show that any $f \in \overline{V}$ that is orthogonal to U in $L^2(B)$ must be zero (see, for example, [12], section 53). Accordingly, we consider an $f \in \overline{V}$ that is orthogonal to U, i.e., an f for which $\int_B G^D(x,y)\overline{f(x)}dx = 0$ for all $y \in A_1$. Let

$$w(y) = -\int_B G^D(x,y)\overline{f(x)}dx, \quad y \in \mathbf{R}^3_-.$$

The assumption on the orthogonality of f is precisely that w is identically zero in A_1. By reciprocity (9) and (7), $(\nabla^2 + q)w = \tilde{f}$ where $\tilde{f}$ is the zero-continuation of $\overline{f}$ to $\mathbf{R}^3_-$. Since $w = 0$ in A_1 and $\tilde{f} = 0$ in A, the unique continuation principle [2] yields that w must vanish everywhere in A. In particular, the Dirichlet and Neumann boundary values of w vanish on ∂B. Thus for any $v \in V$, the boundary terms vanish in the second equality of

$$\int_B v\overline{f} = \int_B v(\nabla^2 + q)w = \int_B w(\nabla^2 + q)v = 0.$$

The third equality follows from the definition of V. Since $f \in \overline{V}$, this yields $f = 0$ in B. $\square$

Let $v_1, v_2 \in V$. Since formula (12) is true for all functions in U, Lemma 2.6 and the Schwarz inequality yield

$$\int_B (q_1 - q_2) v_1 v_2 \, dx = 0. \tag{14}$$

Thus we can conclude the proof of the theorem by using the following standard arguments (see [15]).

For any complex vector $\zeta \in \mathbf{C}^3$ such that $\zeta \cdot \zeta = 0$ the equations $(\nabla^2 + q_i)u = 0,\ i = 1,2$ have so-called Sylvester-Uhlmann solutions (see Appendix A) of the form

$$\psi_i(x, \zeta) = e^{i\zeta \cdot x}(1 + O(1/|\zeta|)).$$

Let now $\xi \in \mathbf{R}^3$, $\zeta_1 \in \mathbf{C}^3$ and define $\zeta_2 = \xi - \zeta_1$. By substituting $v_i = \psi_i(x, \zeta_i)$ into formula (14) we get

$$\lim_{|\zeta_1| \to \infty} \int_B (q_1 - q_2) \psi_1(x, \zeta_1) \psi_2(x, \zeta_2) \, dx = \int_B (q_1 - q_2) e^{i\xi \cdot x} \, dx = 0.$$

Thus the Fourier transform of $q_1 - q_2$ vanishes identically and $q_1 = q_2$. $\square$

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References

[1] M. Cheney and D. Isaacson, "Inverse Problems for a Perturbed Dissipative Half-Space", Inverse Problems 11, 856–888 (1995).

[2] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, Springer, New York, 1992.

[3] J. W. Dettman, Applied Complex Variables, Dover, 1965.

[4] J.D. Jackson, Classical Electrodynamics, 2nd Ed., Wiley, New York, 1975.

[5] G. Eskin and J. Ralston, "Inverse coefficient problems in perturbed halfspaces", preprint, 1997.

[6] L.D. Faddeev, “Growing solutions of the Schrödinger equation”, Dokl. Akad. Nauk SSSR 165, 514–517 (1965); translated Sov. Phys. Dokl. 10, 1033.

[7] A. Kirsch, “An Introduction To The Mathematical Theory Of Inverse Problems”, Springer, New York, 1996.

[8] R. Melrose, “Geometric Scattering Theory”, Cambridge University Press 1995.

[9] A. Nachman, “Reconstructions from boundary measurements,” *Annals of Math.* **128** (1988), pp. 531-556.

[10] F. Natterer and F. Wübbeling, “A propagation–backpropagation method for ultrasound tomography”, Inverse Problems 11, 1225–1232 (1995).

[11] P. Ola, L. Päivärinta, and E. Somersalo, “An inverse boundary value problem in electrodynamics”, Duke Math. J. 70 (1993) 617–653.

[12] G.F. Simmons, “Introduction to Topology and Modern Analysis”, McGraw-Hill, New York, 1963.

[13] E. Somersalo, M. Cheney, D. Isaacson, and E. Isaacson, “Layer Stripping: A Direct Numerical Method for Impedance Imaging,” Inverse Problems, 7 (1991) 899–926 ; 8 (1992), 493.

[14] E. Somersalo, “Layer stripping for time-harmonic Maxwell’s equations with fixed frequency”, Inverse Problems 10 (1994) 449–466.

[15] J. Sylvester and G. Uhlmann, “A uniqueness theorem for an inverse boundary value problem in electrical prospection”, Comm. Pure Appl. Math. 39 (1986) 91-112; “A global uniqueness theorem for an inverse boundary value problem in electrical prospection”, Ann. of Math., 125 (1987) 153-169; “Inverse boundary value problems at the boundary—continuous dependence”, Comm. Pure Appl. Math. 41 (1988) 197–221.

[16] F. Trèves, “Basic Linear Partial Differential Equations”, Academic Press, New York, 1975.

[17] R. Weder, “Uniqueness theorem for an inverse problem in geophysics”, preprint, Universidad Nacional Autónoma de México.

A Appendix: Sylvester-Uhlmann Solutions

Because our proof of Theorem 2.1 relies on the Sylvester-Uhlmann solutions [15], we include here a brief outline of the construction of these solutions.

The key is to use a different Green's function in the Lippmann-Schwinger equation (6). In particular, we replace the usual outgoing unperturbed Green's function G_0 by the Faddeev Green's function [6]

$$G_\zeta(x) = e^{ix\cdot\zeta} g_\zeta(x) \tag{15}$$

where

$$g_\zeta(x) = \frac{1}{(2\pi)^3} \int \frac{e^{ix\cdot\xi}}{\xi^2 + 2\zeta\cdot\xi} d\xi. \tag{16}$$

Here ζ is a three-dimensional complex vector satisfying $\zeta \cdot \zeta = 0$.

A variety of estimates are available that exhibit the decay of g_ζ for large $|\zeta|$. One such large-$|\zeta|$ estimate is [9]

$$\|g_\zeta * f\|_{-\delta} \leq \frac{c}{|\zeta|} \|f\|_\delta \tag{17}$$

for $\delta > 1/2$, where the subscripts indicate the weighted norm

$$\|f\|_\delta = \Big(\int (1 + |x|)^\delta |f(x)|^2 dx \Big)^{1/2} \tag{18}$$

This estimate suffices for proving that the Lippmann-Schwinger equation has a unique solution ψ satisfying

$$\|e^{-ix\cdot\zeta} \psi(x,\zeta) - 1\|_{-\delta} \leq \frac{c}{|\zeta|} \tag{19}$$