The inverse boundary spectral problem for a hyperbolic equation with first order perturbation

M. Lassas
Rolf Nevanlinna -institute[1]

Abstract. *We study an inverse boundary spectral problem for the hyperbolic equation* $(\partial_t^2 - a(x)\partial_t - \Delta + p(x)\cdot\nabla + q(x))u(x,t) = 0$ *in a bounded domain in* $I\!R^d,\ d \geq 2$. *The corresponding time-harmonic equation* $(-\Delta + p\cdot\nabla + q - ia\lambda - \lambda^2)u = 0$ *can be written to a non-selfadjoint eigenvalue problem* $(A-\lambda)U = 0$. *We assume that the boundary spectral data, i.e., the eigenvalues and the boundary values of the generalized eigenfunctions of* A *are known. (This assumption is equivalent to that the singularities of the Neumann-to-Dirichlet mapping* $\Lambda_\lambda : \partial_n u\big|_{\partial\Omega} \mapsto u\big|_{\partial\Omega}$ *of the time-harmonic equation are known.) The main result is that the boundary spectral data determine* $a(x)$ *uniquely and* $p(x)$ *and* $q(x)$ *within a generalized gauge transformation.*
Keywords: Hyperbolic equations, Inverse problems, Inverse boundary spectral problems.
AMS-classification: 35R30, 35P25

1 Introduction and results

The inverse boundary spectral problem for the Schrödinger operator $\Delta + q$ is the following: Can the potential q be recovered from the boundary spectral data, that is, from the Dirichlet eigenvalues λ_j and the Neumann-boundary values $\frac{\partial}{\partial n}\phi_j\big|_{\partial\Omega}$

[1] University of Helsinki, P. O. Box 4(Yliopistonkatu 5), FIN-00014, Finland

of the normalized eigenfunctions ϕ_j. The problem for real q was solved in [14] by using exponentially growing solutions. This was generalized for a non-real q in [11] and later the analogous problem was studied for a general elliptic non-selfadjoint operator by the means of the boundary control method [10] (for the boundary control method, see e.g. [1], [2], [7], [9]). For the studies for hyperbolic inverse boundary problem closely related to the present topic, see [6] and [16].

In this paper we study the inverse boundary spectral problem for an operator pencil raising from a hyperbolic equation with an Euclidean wave-operator and a general first order term. Our approach is the following: From the boundary spectral data we reconstruct first the Neumann-to-Dirichlet mapping and transform the problem to a scattering problem. After this the operator is reconstructed by using the Radon-transform technique as in [15], [16] and [17].

We consider the hyperbolic equation

$$
\begin{aligned}
(1) \qquad (\frac{\partial^2}{\partial t^2} - ia\frac{\partial}{\partial t} - \Delta + \sum_{j=1}^{d} p_j \frac{\partial}{\partial x_j} + q)u(x,t) &= 0 \text{ in } \Omega \times IR_+, \\
\frac{\partial}{\partial n}u(x,t)|_{\partial\Omega\times IR_+} = F(x,t),\ u(x,t)|_{t=0} &= 0,\ u_t(x,t)|_{t=0} = 0.
\end{aligned}
$$

Here $\Omega \subset IR^d$, $d \geq 2$ is a connected C^∞-smooth domain with connected complement and $\Delta = \text{div grad}$ is the Laplacian. Moreover, we assume that the coefficient functions of the equation are complex valued functions satisfying $a, p_j, q \in C_0^\infty(\overline{\Omega})$ (Observe that they vanish at the boundary). By taking Fourier transform respect of time, we get the corresponding 'time-harmonic' equation

$$
(2) \qquad (-\Delta + \sum_{j=1}^{d} p_j \frac{\partial}{\partial x_j} + q + a\lambda - \lambda^2)u = 0,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = f
$$

where we assume that $f \in H^{1/2}(\partial\Omega)$ where $H^{1/2}(\partial\Omega)$ is the standard Sobolev space. We use the operators

$$
\Delta_N u = \Delta u,\ Bu = \sum_{j=1}^{d} p_j \frac{\partial}{\partial x_j}u + qu,\ A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\Delta_N + B & a \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}
$$

defined in the domains

$$
\begin{aligned}
\mathcal{D}(\Delta_N) &= \{u \in H^2(\Omega) \,:\, \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0\} \subset X := L^2(\Omega), \\
\mathcal{D}(B) &= H^1(\Omega),\ \mathcal{D}(A) = \mathcal{D}(\Delta_N) \times X \subset X \times X.
\end{aligned}
$$

Moreover, we define an operator pencil $R(\lambda) = -\Delta_N + B + a\lambda - \lambda^2$ and its adjoint pencil $R^*(\lambda) = -\Delta_N + B^* + \overline{a}\lambda - \lambda^2$.

Next we recall some properties of operator pencils (see e.g. [13]). First we linearize the pencil equation (2). Namely, the equations $R(\lambda)u = f$ and $R^*(\overline{\lambda})v = g$ are equivalent to

$$(A-\lambda)\begin{pmatrix} u \\ \lambda u \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad (A^* - \overline{\lambda})\begin{pmatrix} (\overline{\lambda} - \overline{a})u \\ u \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}.$$

The eigenvalues $\lambda_j \in \mathbb{C}$ of the operator A are called the eigenvalues of the pencil $R(\lambda)$. For $\lambda \neq \lambda_j$ the operator $R(\lambda)$ is invertible. A function x is called a root function (or a generalized eigenfunction) of the operator A corresponding to an eigenvalue λ_j if $(A - \lambda_j)^n x = 0$ with some $n \in \mathbb{Z}_+$. We denote by N_j and N_j^* the spaces of the root functions of A and A^* corresponding to the eigenvalues λ_j and $\overline{\lambda}_j$. Obviously N_j is orthogonal to $N_{j'}^*$ if $j \neq j'$. Since A defines in $H^1(\Omega) \times X$ an unbounded operator with smoothing inverse, one can show by using [13], Theorem 4.3 that the eigenvalues λ_j of $R(\lambda)$ form a discrete set, the spaces N_j are finite dimensional and the root functions of A (or A^*) span a dense set in $X \times X$. For asymptotics of the eigenvalues, see e.g. [13].

Let $\Phi_{jkl} = (\phi^1_{jkl}, \phi^2_{jkl})$ be the basis of the space N_j satisfying

$$(3) \quad (A - \lambda_j)\Phi_{jkl} \;=\; \Phi_{jk,l-1}, j = 1, \dots,\ k = 1, \dots, m_j,\ l = 1, \dots, n_{jk},$$

where we denote $\Phi_{jk,0} = 0$. This means that matrix of $A : N_j \to N_j$ respect of the basis Φ_{jkl} consists of Jordan blocks. Since $N_j \perp N_{j'}^*$ for $j \neq j'$ and the root functions span a dense set, we can choose for N_j^* the basis $\Psi_{jkl} = (\psi^1_{jkl}, \psi^2_{jkl})$ satisfying

$$(\Phi_{jkl}, \Psi_{j'k'l'}) = \delta_{j,j'}\delta_{k,k'}\delta_{l,n_{jk}+1-l'}.$$

Since N_j^* can be identified with the dual of N_j, one see by studying the matrix of $A^* : N_j^* \to N_j^*$ that

$$(4) \quad (A^* - \overline{\lambda_j})\Psi_{jkl} \;=\; \Psi_{jk,l-1}, j = 1, \dots,\ k = 1, \dots, m_j,\ l = 1, \dots, n_{jk}$$

where $\Psi_{jk,0} = 0$.

For selfadjoint inverse boundary spectral problem the boundary spectral data is defined to be the boundary values of the eigenfunctions. In our non-selfadjoint case the natural generalization is the following.

Definition 1.1 *The boundary spectral data (BSD) is the collection*

$$\{\lambda_j,\ \Phi_{jkl}\big|_{\partial\Omega},\ \Psi_{jkl}\big|_{\partial\Omega},\ j=1,\dots,\ k=1,\dots,m_j,\ l=1,\dots,n_{jk}\}$$

where λ_j *are the eigenvalues and* $\Phi_{jkl}\big|_{\partial\Omega}$*,* $\Psi_{jkl}\big|_{\partial\Omega}$ *are the Dirichlet-boundary values of the root functions of* A *and* A^*

To motivate Definition 1.1, we begin with the Neumann-to-Dirichlet mapping. When λ is not an eigenvalue, we define the mapping

$$\Lambda_\lambda : H^{1/2}(\partial\Omega) \ \ \to \ \ H^{3/2}(\partial\Omega),\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} \mapsto u\Big|_{\partial\Omega}$$

which maps the Neumann boundary value to the Dirichlet boundary value of the solution of the equation (2). We will see that the operator valued function $\lambda \mapsto \Lambda_\lambda$ is meromorphic function having poles at the eigenvalues of $R(\lambda)$. Near each eigenvalue λ_j we have a representation

$$\Lambda_\lambda = A_\lambda + \sum_{p=1}^{m_j} \frac{T_{jp}}{(\lambda-\lambda_j)^p}$$

where $\lambda \mapsto A_\lambda$ is analytic. The later part is equal to the singular part of the Laurent series and we call it the principal part of the singularity or simply the singularity of Λ_λ at λ_j. The singularity will be denoted by sing Λ_λ.

In the case $a = 0$ the boundary spectral data is known to be very natural concept. By [10] it can be reconstructed from the knowledge of the singularities of the boundary measurements, i.e., from the singularities of the boundary values of the Green's function $G(x,y,\lambda)$, $x,y\in\partial\Omega$, $\lambda\in\mathbb{C}$ which are equivalent to the the singularities of Λ_λ. In the dispersive case, we have the analogous results:

Lemma 1.1 *The BSD determines the singularities of the operator* Λ_λ*, i.e., the operators* T_{jp}*.*

Remark 1. The converse is also true, i.e., the singularities of Λ_λ determine BSD. Namely, in the proof of Lemma 1.1 we see that the kernels of the operators T_{jp} can be given as a sum of terms $\phi^1_{jkl}(x)\overline{\psi^2_{jkh}(y)}$. Thus $\phi^1_{jkl}\big|_{\partial\Omega}$ and $\psi^2_{jkh}\big|_{\partial\Omega}$ can be constructed from T_{jp} with the same method as in [10]. After this we can easily obtain $\phi^2_{jkl}\big|_{\partial\Omega}$ and $\psi^1_{jkh}\big|_{\partial\Omega}$ by using (3) and (4).

Next we consider two pencils $R_i(\lambda) = -\Delta + B^i + a^i\lambda - \lambda^2,\ i=1,2$ corresponding to functions $a^i(x), p^i_j(x), q^i(x) \in C^\infty_0(\overline{\Omega})$ and the corresponding Neumann-to-Dirichlet mappings Λ^1_λ and Λ^2_λ.

Theorem 1.2 *If $R_1(\lambda)$ and $R_2(\lambda)$ have the same BSD then $\Lambda^1_\lambda = \Lambda^2_\lambda$ for all $\lambda \in \mathbb{C}$.*

Let now $f \in C_0^\infty(\overline{\Omega}),\ \kappa = e^f$. Then $R(\lambda)^n\phi = 0$ imply $(\kappa R(\lambda)\kappa^{-1})^n(\kappa\phi) = 0$ and hence we see that BSD is invariant in the generalized gauge-transformation

$$-\Delta + B \mapsto \kappa(-\Delta + B)\kappa^{-1}. \tag{5}$$

Because of this we define the equivalence class of $-\Delta + B$ within the group of the generalized gauge-transformations (see [16]):

$$[-\Delta + B] = \{e^f(-\Delta + B)e^{-f} :\ f \in C_0^\infty(\overline{\Omega})\}.$$

By using Theorem 1.2 we will prove our main result:

Theorem 1.3 *The pencils $R_1(\lambda)$ and $R_2(\lambda)$ have the same BSD if and only if $a_1 = a_2$ and the operators $-\Delta + B^1$ and $-\Delta + B^2$ are the same within a generalized gauge transformation, i.e.,*

$$-\Delta + B^1 = e^f(-\Delta + B^2)e^{-f} \textit{ for some } f \in C_0^\infty(\overline{\Omega}, \mathbb{C}). \tag{6}$$

2 Singularities of Λ_λ

Here we use the extension Δ of Δ_N defined in the domain $\mathcal{D}(\Delta) = H^2(\Omega)$ and the extension $\tilde{A}$ of A with $\mathcal{D}(\tilde{A}) = \mathcal{D}(\Delta) \times L^2(\Omega)$. Similarly, $\tilde{R}(\lambda)$ is the pencil defined in the domain $\mathcal{D}(\Delta)$. First we observe that the equation (2) is equivalent to

$$(\tilde{A} - \lambda)U = 0,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = f,\ \text{where } U = \begin{pmatrix} u \\ \lambda u \end{pmatrix}. \tag{7}$$

Let $H_f = (h_f, 0)^t$ where $h_f \in H^2(\Omega)$ is a function for which $\frac{\partial}{\partial n} h_f\Big|_{\partial\Omega} = f$. Then (7) yields $U - H_f \in \mathcal{D}(A)$ and

$$U = H_f - (A - \lambda)^{-1}(\tilde{A} - \lambda)H_f. \tag{8}$$

Particularly, this shows that $\lambda \mapsto \Lambda_\lambda$ is analytic outside the eigenvalues. Let P_j be the Riesz projection of A corresponding to the eigenvalue λ_j, i.e., P_j is the projection into N_j along the space spanned by $N_{j'},\ j' \neq j$. Obviously

$$P_j x = \sum_{k=1}^{m_j} \sum_{l=1}^{n_{jk}} (x, \Psi_{jkl}) \Phi_{jk, n_{jk}+1-l}. \tag{9}$$

By [8], Theorem III 6.17, P_j defines an A-invariant non-orthogonal decomposition $X^2 = (1-P_j)X^2 \oplus P_j X^2$ such that the operator $A-\lambda : (1-P_j)X^2 \to (1-P_j)X^2$ is invertible for λ near λ_j. Next we prove that BSD determines the singularities of Λ_λ.

Proof. (of Lemma 1.1). We denote the solution of (7) by $U_f(\lambda)$. By using (8), $(1-P_j)U_f(\lambda)$ is analytic near λ_j and we see that

$$\text{(10)} \quad \text{sing}\, U_f(\lambda) = \text{ sing}\, P_j U_f(\lambda) = \text{ sing} \sum_{k,l} (U_f(\lambda), \Psi_{jkl}) \Phi_{jk,n_{jk}+1-l}.$$

Since $\Psi_{jkl} \in \mathcal{D}(A^*)$ we have $(\frac{\partial}{\partial n} - n\cdot p)\Psi^2_{jkl}\Big|_{\partial\Omega} = 0$ and thus by Green's formula,

$$0 = ((\tilde{A} - \lambda)U_f(\lambda), \Psi_{jkl}) = (U_f(\lambda), (A^* - \lambda)\Psi_{jkl}) - \int_{\partial\Omega} f\overline{\Psi^2_{jkl}}\, dS(x).$$

Thus the formula (4) yields

(11)
$$(U_f(\lambda), \Psi_{jkl}) = \frac{1}{\lambda_j - \lambda}\Big[\int_{\partial\Omega} f\overline{\Psi^2_{jkl}}\, dS(x) - (U_f(\lambda), \Psi_{jk,l-1})\Big],\ l = 1, \dots, n_{jk}$$

where $\Psi_{jk,l-1} = 0$ for $l = 1$. Equations (11) form recurrence relations from which the inner products $(U_f(\lambda), \Psi_{jkl})$ can be computed by using BSD. Since the positive Laurent coefficients of $(U_f(\lambda), \Psi_{jkl})$ at λ_j vanish, we see that BSD determines

$$\text{sing}\, \Lambda_\lambda f = \sum_{k,l} (U_f(\lambda), \Psi_{jkl}) \Phi_{jk,n_{jk}+1-l}\Big|_{\partial\Omega}.$$

□

3 From BSD to Λ_λ

Let $\mathfrak{S}_p$ be the space of the compact operators with s-numbers in ℓ^p (see [3]) and let $h_f \in H^2(\Omega)$ be a function depending continuously on $f \in H^{1/2}(\partial\Omega)$ and satisfying $\frac{\partial}{\partial n} h_f\Big|_{\partial\Omega} = f$. Then the equation (2) yields

$$(-\Delta_N + B + a\lambda - \lambda^2)(u_f - h_f) = -(-\Delta + B + a\lambda - \lambda^2)h_f$$

and thus

$$
\begin{aligned}
(12) \qquad u &= h_f - R(\lambda)^{-1}(-\Delta + B + a\lambda - \lambda^2)h_f, \\
R(\lambda)^{-1} &= (-\Delta_N + 1)^{-1}(I + T_0 + \lambda T_1 + \lambda^2 T_2)^{-1}
\end{aligned}
$$

where $T_0 = (B-1)(-\Delta_N+1)^{-1}$, $T_1 = a(-\Delta_N+1)^{-1}$ and $T_2 = -(-\Delta_N+1)^{-1}$ are compact operators. Since the eigenvalues of the selfadjoint operator Δ_N have asymptotics $j^{2/d}$, we see that $T_i \in \mathfrak{S}_p,\ p > d/2$ where d is the dimension of Ω. Next we denote by D_α the double cone $\{\lambda :\ |\arg\ \lambda| < \alpha \text{ or } |\arg\ (-\lambda)| < \alpha\}$. By [13], Lemma 18.8, for any $\alpha > 0$ there exists C_α and c_α such that

$$(13) \qquad ||(I + T_0 + \lambda T_1 + \lambda^2 T_2)^{-1}|| \le C_\alpha$$

for $\lambda \notin D_\alpha$ and $|\lambda| > c_\alpha$.

Since $(-\Delta_N + 1)^{-1} : L^2(\Omega) \to H^2(\Omega)$ is continuous, we have

$$(14) \qquad ||R(\lambda)^{-1}||_{X \to H^2} \le cC_\alpha,\ \lambda \notin D_\alpha,\ |\lambda| > c_\alpha.$$

Lemma 3.1 *Assume that two Neumann-to-Dirichlet mappings $\Lambda_\lambda^{(i)},\ i = 1, 2$ have the same singularities. Then $\Lambda_\lambda^{(1)} - \Lambda_\lambda^{(2)}$ is a second order polynomial as an operator-valued function of λ.*

Proof. By theory of Keldysh pencils (see [13], Lemma 18.5) there exist numbers $r_m^{(i)} \in I\!R_+, r_m^{(i)} \to \infty,\ i = 1, 2$ such that

$$||(I + T_0^{(i)} + \lambda T_1^{(i)} + \lambda^2 T_2^{(i)})^{-1}|| \le c\, exp(c|\lambda|^{4d+2})$$

when $|\lambda| = r_m^{(i)}$. We can assume that $m < r_m^{(i)} < m + 1$. Thus by (12) we have on the circles $|\lambda| = r_m$

$$||\Lambda_\lambda^{(i)}|| \le c\, exp(c|\lambda|^{4d+2}).$$

Now $\Lambda_\lambda^{(1)} - \Lambda_\lambda^{(2)}$ is an entire function of λ. By applying the maximum principle for analytic functions in the same way as in the proof of [11], Theorem 9.1, we see

$$(15) \qquad ||\Lambda_\lambda^{(1)} - \Lambda_\lambda^{(2)}|| \le c\, exp(c|\lambda|^{4d+2}),\ \lambda \in \mathbb{C}.$$

Let $\alpha < \pi/(4d + 2)$. For $\lambda \notin D_\alpha$ and $|\lambda| > c_\alpha$ we see from (12) and (14) that for $||\Lambda_\lambda^{(i)} f|| \le c|\lambda|^2$. Thus by using Pragmen-Lindelöf theorem [4] and (15) we see in an analogous way to [11], Theorem 9.7 that the analytic operator-valued function $\Lambda_\lambda^{(1)} - \Lambda_\lambda^{(2)}$ is a second order polynomial. ▫

Lemma 3.2 *Let $\Lambda_\lambda^{(i)},\ i = 1,2$ be two Neumann-to-Dirichlet mappings corresponding to pencils $R^{(i)}$. Then for any fixed f we have*

$$\lim_{t\to\infty} ||\Lambda_{it}^{(1)} - \Lambda_{it}^{(2)})f||_{H^{1/4}(\partial\Omega)} = 0,\ t \in I\!R_+$$

Proof. We note that by [13], Lemma 3.1 there can be only a finite number of eigenvalues in $iI\!R_+$ and thus the above limit is well defined.

We study the difference $\Lambda_\lambda^{(1)} f - \Lambda_\lambda^{(2)} f$ for $\lambda = it,\ t > c_0$ when c_0 is big enough and $t \to \infty$. Let $u^{(i)}(\lambda)$ be the solutions

$$\tilde{R}^{(i)}(\lambda)u^{(i)}(\lambda) = 0,\ \frac{\partial}{\partial n}u^{(i)}(\lambda)\Big|_{\partial\Omega} = f, \quad i = 1,2.$$

Then for $v(\lambda) = u^{(1)}(\lambda) - u^{(2)}(\lambda)$ we have

(16)
$$R^{(1)}(\lambda)v(\lambda) = (B^{(1)} - B^{(2)})u^{(2)}(\lambda) + \lambda(a^{(1)} - a^{(2)})u^{(2)}(\lambda),\ \frac{\partial}{\partial n}v(\lambda)\Big|_{\partial\Omega} = 0$$

We have by (12) for $\lambda = it$

$$(17)\quad \begin{aligned} u^{(2)}(\lambda) &= h_f - R(it)^{-1}(-\Delta + B + ait + t^2)h_f, \\ R(it)^{-1} &= (-\Delta_N + t^2)^{-1}\Big(I + B(-\Delta_N + t^2)^{-1} + ait(-\Delta_N + t^2)^{-1}\Big)^{-1}. \end{aligned}$$

By using the spectral representation of Δ_N and an interpolation argument as in [11], Lemma 4.5, one can easily show that

$$(18)\qquad ||(-\Delta_N + t^2)^{-1}||_{X\to H^s(\Omega)} \le ct^{2-s},\ 0 \le s \le 2.$$

Thus in the equation (17) all inverse operators exist and we get $||u^{(2)}(\lambda)||_{L^2(\Omega)} \le c$ and $||u^{(2)}(\lambda)||_{H^2} \le c|\lambda|^2$ when $\lambda = it$ is big enough. By using interpolation we get $||u^{(2)}(\lambda)||_{H^1(\Omega)} \le c|\lambda|$. Thus from formulas (16) and (18) it follows that

$$\begin{aligned} ||v(\lambda)||_X &\le ||R^{(1)}(\lambda)^{-1}||_{X\to X}\cdot (||B^{(1)} - B^{(2)}||_{H^1(\Omega)\to X}||u^{(2)}||_{H^1(\Omega)} \\ &\quad +|\lambda|\ ||a^{(1)} - a^{(2)}||_{X\to X}||u^{(2)}||_X) \\ &\le ct^{-1}. \end{aligned}$$

In the same way we see $||v(\lambda)||_{H^2} \le ct$. By using interpolation we get

$$(19)\qquad ||v(\lambda)||_{H^{3/4}(\Omega)} \le (ct)^{3/8}(ct^{-1})^{5/8} \le ct^{-1/8}.$$

Thus

$$||(\Lambda_\lambda^{(1)} - \Lambda_\lambda^{(2)})f||_{H^{1/4}(\Omega)} \le ||v(\lambda)||_{H^{3/4}(\Omega)} \le ct^{-1/8}.$$

$\square$

Now Theorem 1.2 follows immediately from Lemma 3.1 and Lemma 3.2.

4 From Λ_λ to the unknown coefficients

For the hyperbolic equation (1) we define the response operator

$$\mathcal{R} : \frac{\partial u}{\partial n}|_{\partial\Omega\times I\!R_+} \mapsto u|_{\partial\Omega\times I\!R_+}, \ \ \frac{\partial u}{\partial n}|_{\partial\Omega\times I\!R_+} \in C^\infty(\partial\Omega \times \overline{I\!R}_+).$$

Lemma 4.1 *The mappings $\Lambda_\lambda,\ \lambda \in \mathbb{C}$ determine the response operator $\mathcal{R}$.*

Proof. Let $u(x,t)$ be the solution of the initial value problem

$$(20) \qquad \begin{aligned} (\frac{\partial^2}{\partial t^2} - ia\frac{\partial}{\partial t} - \Delta + B)u(x,t) &= 0 \text{ in } \Omega \times I\!R_+, \\ \frac{\partial}{\partial n}u(x,t)|_{\partial\Omega\times I\!R_+} = F(x,t),\ u(x,t)|_{t=0} &= 0,\ u_t(x,t)|_{t=0} = 0. \end{aligned}$$

Clearly it is enough to find $u|_{\partial\Omega\times[0,T]}$ with arbitrary $T > 0$. Thus we can assume that the support of F is a compact subset of $\partial\Omega \times I\!R_+$. By standard energy estimates, the Laplace transform $\tilde{u}(\cdot\,,\xi) = \mathcal{L}_t(u(\cdot\,,t))(\xi)$ is a $H^2(\Omega)$-valued function defined in some half space Re $\xi > c_0$. Since the Laplace transform satisfies the time-harmonic equation

$$(\xi^2 - ia\xi - \Delta + B)\tilde{u}(x,\xi) \ = \ 0,\ \frac{\partial}{\partial n}\tilde{u}(x,\xi)\Big|_{\partial\Omega} = \tilde{F}(x,\xi),$$

one see by using inverse Laplace transform that

$$u(\cdot\,,t)\Big|_{\partial\Omega} = \frac{1}{2\pi i}\lim_{M\to\infty}\int_{\xi_0-iM}^{\xi_0+iM} e^{\xi t}\Lambda_\xi\tilde{F}(x,\xi)d\xi,\ Re\ \xi_0 > c_0.$$

Thus the mappings $\Lambda_\xi,\ \xi \in \mathbb{C}$ determine the response operator. $\square$

Next we study the wave equation

$$(21)\qquad \begin{aligned}(\frac{\partial^2}{\partial t^2}-ia\frac{\partial}{\partial t}-\Delta+B)u(x,t) &= 0,\ (x,t)\in IR^d\times IR_+,\\ u(x,0) &= f,\ u_t(x,0)=g.\end{aligned}$$

Particularly we are interested of the solutions corresponding to the initial data

(22)
$f(x)=\delta(-t_0-x\cdot\omega),\ \ g(x)=\delta^{(1)}(-t_0-x\cdot\omega),$ where $|\omega|=1$ and $t_0\ll 0$.

These solutions correspond to the incoming delta-waves $\delta(t-t_0-x\cdot\omega),\ t<0$. By [16], the solutions $u(x,t,\omega)$ of (21) with above initial data have the representations

$$u(x,t,\omega)=\sum_{j=0}^{N}u_k(x,\omega)\delta^{(-j)}(t-t_0-x\cdot\omega)\ \text{ mod } C^{N-1}$$

for every N where $\delta^{(-j)}(t)=t_+^{j-1}/(j-1)!$ for $j>0$. Furthermore, by [16],

Theorem 4.2 *The functions $u_0(x,\omega)$ and $u_1(x,\omega)$ in a domain $\{(x,\omega)\in IR^n\times S^{n-1}\mid x\cdot\omega>C\},\ C>0$ determine uniquely the function $a(x)$ and determine uniquely the equivalence class*

$$[-\Delta+B]=\{e^f(-\Delta+B)e^{-f}:\ f\in C_0^\infty(IR^n)\}.$$

Therefore it is enough to show that the Λ_λ-mappings determine the solutions of the equation (21) outside Ω with an appropriate incoming initial data.

Lemma 4.3 *The knowledge of the operators $\Lambda_\lambda,\ \lambda\in\mathbb{C}$ determine the function $u(x,t)|_{(IR^d\setminus\overline{\Omega})\times IR_+}$ corresponding to initial data (22)*

Proof. We start with the standard argument concerning the equivalence of the hyperbolic boundary value problem and the scattering problem. Assume that $R^{(1)}$ and $R^{(2)}$ have the same BSD. Let $T>0$, $f,g\in C^\infty(IR^d\setminus\Omega)$ and u_1 and u_2 be the solutions of the wave equations (21) with the coefficient functions corresponding to $R^{(1)}$ and $R^{(2)}$. Since the response operators $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ coincide, there exist a solution v for $R^{(1)}$ such that $v|_{\partial\Omega\times[0,T]}=u_2|_{\partial\Omega\times[0,T]}$ and $\frac{\partial}{\partial n}v|_{\partial\Omega\times[0,T]}=$

$\frac{\partial}{\partial n} u_2|_{\partial\Omega\times[0,T]}$. By defining $V(x,t) = u_2(x,t)$ for $(x,t) \in (I\!R^d \setminus \Omega) \times [0,T]$ and $V(x,t) = v(x,t)$ for $(x,t) \in \Omega \times [0,T]$ we see that V is the solution of the uniquely solvable initial value problem (21) for $R^{(1)}$ with initial data (f,g). This yields $V = u_1$ and thus $u_2 = u_1$ in $(I\!R^d \setminus \Omega) \times [0,T]$. Hence BSD determine uniquely the values of the solution in $(I\!R^d \setminus \Omega) \times [0,T]$.

Let now $d_j \in C_0^\infty(I\!R)$, $j = 1, 2, \dots$ be functions for which $d_j \to \delta$ in the space $H^{-1}(I\!R)$. Let $f_0 = \delta(-t_0 - x\cdot\omega)$, $g_0 = \delta^{(1)}(-t_0 - x\cdot\omega)$ and $f_j = d_j(-t_0 - x\cdot\omega)$, $g_j = d_j'(-t_0 - x\cdot\omega)$ and let u_j be the solution of the equation (21) corresponding to the initial data (f_j, g_j). Since $f_j \to f_0$ and $g_j \to g_0$ in $H_{loc}^{-d-2}(I\!R^d)$, it follows from [5], Lemma 23.2.1 and the finite speed of wave propagation that $u_j \to u_0$ in the space $C^1([-T,T], H^{-d-2}(V))$ for any $V \subset I\!R^d$. From this the claim follows. □

Finally, we prove our main result.

Proof.(of Theorem 1.3) If two operators are the same within a generalized gauge-transformation then their BSD coincide. Next we prove the converse. We have shown that if the BSD coincide then for the operators $-\Delta + B^i$ in $L^2(I\!R^d)$ we have

(23) $$-\Delta + B^1 = e^f(-\Delta + B^2)e^{-f} = -\Delta - 2e^f \bigtriangledown f \cdot \bigtriangledown - e^f \Delta f + B^2$$

with some $f \in C_0^\infty(I\!R^n)$. Since B^1 and B^2 are supported in Ω, we see that $\bigtriangledown f = 0$ outside Ω. Since the complement of Ω is connected, we see that supp $f \subset \overline{\Omega}$ which proves the claim. □

Acknowledgements. The author is grateful for prof. Y. V. Kurylev and Dr. Petri Ola for many valuable discussions. The research was partly supported by Finnish Academy project 37692.

References

[1] M. I. Belishev, On an approach to multidimensional inverse problems for the wave equation. *Dokl. Akad. Naum. SSSR* 297(1987), 711-736.

[2] M. I. Belishev and Y. V. Kurylev, To the reconstruction of a Riemannian manifold via its boundary spectral data (BC-method). *Comm. PDE* 17(1992), 767-804.

[3] I. Gohberg and M. Krein, *Introduction to the theory of linear nonselfadjoint operators*, American Mathematical Society, 1969.

[4] E. Hille, *Analytic function theory*, Vol 2, Ginn and Company, 1962.

[5] L. Hörmander, *The analysis of linear partial differential operators*, Vol. 3, Springer Verlag, 1985.

[6] V. Isakov and Z. Sun, Stability estimates for hyperbolic inverse problem. *Inverse Problems* 8(1992), 193-206.

[7] A. Katchalov and Y. Kurylev, Multidimensional inverse problem with incomplete boundary spectral data. *Comm. Partial Differential Equations* 23 (1998), 55-95.

[8] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, 1966.

[9] Y. V. Kurylev, An inverse boundary problem for the schrodinger operator with magnetic field. *J. Math. Phys* 36(1995), 2761-2776.

[10] Y. V. Kurylev and M. Lassas, The multidimensional Gelfand inverse problem for non-selfadjoint operators. *Inverse Problems* 13 (1997), 1495-1501

[11] M. Lassas, Non-selfadjoint inverse spectral problems and their applications to random bodies. *Ann. Acad. Sci. Fenn. dissertations* 103(1995).

[12] B. Ja. Levin, *Distribution of zeros of entire functions*, American mathematical society, Providence 1964.

[13] A. S. Markus, *Introduction to the spectral theory of operator pencils*, American Mathematical Society, 1988.

[14] A. Nachman, J. Sylvester and G. Uhlmann, An n-dimensional Borg-Levinson theorem. *Comm. Math. Phys.* 115(1988), 593-605.

[15] Rakesh and W. Symes, Uniqueness for an inverse problem for the wave equation, *Comm. PDE* 13(1988), 87-96.

[16] T. Shiota, An inverse problem for the wave equation with first order perturbation, *Amer. J. Math.* 107(1985), 241-251.

[17] P. Stefanov and G. Uhlmann, Stability estimates for the hyperbolic Dirichlet to Neumann map in ansiotropic media, preprint.