Hyperbolic inverse boundary value problem and time-continuation of the non-stationary Dirichlet-to Neumann map

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Abstract. *Let M be a compact Riemannian manifold M with non-empty boundary ∂M. In the paper we consider an inverse problem for the second order hyperbolic initial-boundary value problem $u_{tt}+bu_t+a(x,D)u=0$ in $M\times\mathbb{R}_+$, $u|_{\partial M\times\mathbb{R}_+}=f$; $u|_{t=0}=u_t|_{t=0}=0$. Our goal is to determine $(M,g), b$ and $a(x,D)$ from the knowledge of the non-stationary Dirichlet-to-Neumann map (the hyperbolic response operator), R^T with sufficiently large $T\geq 0$. The response operator R^T is the map $f\to u^f_\nu|_{\partial M\times[0,T]}$ where u^f_ν is the normal derivative of the solution of the initial-boundary value problem.*

More specifically, we show that

(i) It is possible to determine R^t for any $t\geq 0$ if we know R^T for sufficiently large T and some geometric condition upon the geodesic behaviour on (M,g) is satisfied.

(ii) It is then possible to determine (M,g) and b uniquely and the elliptic operator $a(x,D)$ modulo generalized gauge transformations.

1. Introduction and main result.

In the paper we study an inverse problem for the hyperbolic initial-boundary value problem

$$u_{tt}+bu_t+a(x,D)u=0 \text{ in } M\times\mathbb{R}_+ \tag{1.1}$$

$$u|_{\partial M\times\mathbb{R}_+}=f;\quad u|_{t=0}=u_t|_{t=0}=0;\quad f\in H^1_0(\partial M\times\mathbb{R}_+) \tag{1.2}$$

on a compact connected C^∞-Riemannian manifold $M, \dim M=m\geq 1$, with metric $g=(g^{jl})^m_{j,l=1}$. The manifold is assumed to have a C^∞-smooth non-empty boundary ∂M. The operator $a(x,D)$ is a first-order perturbation of the Laplace operator $-\Delta_g$ on (M,g),

$$a(x,D)=-\Delta_g+P+Q. \tag{1.3}$$

Here P is a complex valued C^∞-vector field which in local coordinates $x = (x^1, ..., x^m)$ on M has the form $P = P^l \partial_l$ while Q and b are complex valued C^∞-functions on M. The symbol $a(x,D)$ is, in general, not formally symmetric. Later in the paper we refer to the case (1.1), (1.2) with $b(x) \neq 0$ and $a(x,D)$ of form (1.3) as to a "generic case". We also study the "selfadjoint case",

$$a(x,D) = a^*(x,D), \quad b(x) = 0, \tag{1.4}$$

when the results are quite different from the generic case. Here the adjoint symbol $a^*(x,D)$ is considered with respect to a suitable $L^2(M, d\mu)$ norm with $d\mu = \eta dm_g(x)$, where η is a positive C^∞-function on M and dx is the Riemannian volume on (M,g),

$$dm_g(x) = g^{1/2} dx^1 \cdots dx^m.$$

In this case

$$a(x,D) = \eta^{-1} g^{-1/2} (\partial_j + ip_j) g^{1/2} \eta g^{jl} (\partial_l + ip_l) + q, \tag{1.5}$$

with real p_j and q.

Remark. Any second order uniformly elliptic symbol with real principal part can be written in form (1.3) and, in the self-adjoint case, (1.5).

By $H^s(A)$ we denote the Sobolev space of functions on A and by $H^s_0(\partial M \times [0,t])$ the space of $u \in H^s(\partial M \times \mathbb{R})$ with supp $u \in \partial M \times [0,t]$. We denote by ν the unit normal vector to ∂M with respect to g and define the boundary operator $Bu = \partial_\nu u - P_\nu u|_{\partial M \times [0,T]}$, where ∂_ν and $P_\nu = (\nu, P)_g$ are the normal derivative and the normal component of P, correspondingly.

Definition 1. *Let $T > 0$. The response operator $R^T : H^1_0(\partial M \times [0,T]) \to L^2(\partial M \times [0,T])$ is given by the formula,*

$$R^T(f) = \partial_\nu u^f - P_\nu u^f|_{\partial M \times [0,T]},$$

where u^f is the solution of the problem (1.1), (1.2).

In the following, we call the pair $\{\partial M, R^T\}$ the dynamical boundary data, corresponding to problem (1.1), (1.2), and abbreviate it by DBD.

The operator R^T can be represented by means of Green's function, $G = G(x,t,y,s)$, of (1.1), (1.2),

$$(\partial_t^2 + b\partial_t + a(x, D_x))G(x,t,y,s) = \delta_{y,s}(x,t) \text{ in } M \times \mathbb{R}_+$$

$$G(x,t,y,s)|_{(x,t)\in\partial M\times\mathbb{R}_+}=0;\quad G(x,t,y,s)|_{t=0}=G_t(x,t,y,s)|_{t=0}=0.$$

Indeed, the Schwartz kernel $S(x,t,y,s)$ of the operator R^T is then

$$S(x,t,y,s)=\partial_{\nu(x)}\partial_{\nu(y)}G(x,t,y,s)|_{x,y\in\partial M}.$$

Hence the knowledge of response operator R^T is equivalent to the knowledge of the Cauchy data of $G(x,t,y,s)$ on the lateral boundary $\{(x,t,y,s)\in\partial M\times[0,T]\times\partial M\times[0,T]\}$.

In the paper, we consider two problems:

Problem I. *Let ∂M and R^T, with some $T>0$, be given. Do these data determine the operator R^t with any $t>0$?*

Problem II. *Let ∂M and R^T, with some $T>0$, be given. Do these data determine $(M,a(x,D),b)$ uniquely?*

The Problem I is equivalent to the problem of the unique continuation of Green's function along the lateral boundary:

Problem I'. *Let ∂M and the Cauchy data of Green's function $G(x,t,y,s)$ on the boundary cylinder $(\partial M\times[0,T])^2$, with some $T>0$, be given. Do these data determine the Cauchy data of Green's function on the whole lateral boundary $(\partial M\times\mathbb{R}_+)^2$?*

There are at least for two reasons why it is important to solve Problem I without solving the inverse problem. First, there is a point of view related to the applications. Since the methods of solving inverse problems are usually unstable, it is often efficient to first generate data on a larger time interval without doing constructions inside the manifold and then use the new, larger set of data to solve the inverse problem. Second, many methods require boundary data to be known for all $t,s\geq 0$. Particularly, the inverse boundary spectral problems, that is the inverse problems for elliptic operators with a variable spectral parameter, can be considered as the hyperbolic inverse boundary-value problems with data given on the time interval $\mathbb{R}_+$. Typical examples of this kind are quantum mechanical or acoustic inverse problems, when measurements are made at many energy/frequency levels. Thus Problem I gives an opportunity to continue boundary data and then to solve the inverse problem by a method which is the most suitable for the studied case. Particularly, the method which we use to solve Problem II, requires the knowledge of the boundary data on a large time interval and we first continue R^T onto $t>T$.

In Problem II, we consider how the boundary data can be used to reconstruct the unknown manifold and the wave operator on it. Physically speaking, Problem II is analogous to the question of finding the speed of the wave propagation and other physical parameters inside an unknown body by making measurements on its boundary. This type of problems has been studied quite extensively and numerous references can be found in monograph [7].

Particularly, inverse boundary-value Problem II for operators on manifolds or, more precisely, its spectral analogue was considered in [8],[9] and [10] for the self-adjoint case, $a(x, D) = a^*(x, D)$. The non-selfadjoint case, $a^*(x, D) \neq a(x, D)$ with, however, $b = 0$, was studied in [12] and for a quadratic operator pencil in [13].

When $b \neq 0$, the known results concern mainly with the Euclidean case $M \subset \mathbb{R}^m$, $g^{ij} = \delta^{ij}$. The scattering analogue of the inverse initial-boundary value problem was considered in [18]. For the case, when boundary data are prescribed only on a part of the boundary, we refer to [7]. The inverse boundary spectral problem where one knows the generalized eigenvalues and the boundary values of the generalized eigenfunctions of the operator pencil corresponding to wave equation (1.1), was considered in [15].

At last, in [17] the uniqueness of the reconstruction of a conformally Euclidean metric $g^{jl}(x) = \sigma(x)\delta^{jl}$ in $M \subset \mathbb{R}^m$ and of some lower-order terms (with further restrictions upon these terms) was proven in the geodesically regular case.

The assumption $b \neq 0$ changes the nature of the problem quite dramatically because the term bu_t is responsible for the dispersion of the waves governed by equation (1.1). Ideologically, this difference in the nature of the problem implies that, to solve the inverse problem with $b \neq 0$, we need R^T with $T > 2t_*$ where t_* is the exact controllability time (see Definition 2). On the other hand, when $b = 0$ it is sufficient to know R^T when $T > t_*$ (see Theorem 1.2). Technically, this results is based on the consideration of the indefinite form $(JU^f(t), V^g(s))$ (see e.g. Lemma 3.3) when $b \neq 0$, rather then semi-definite energy-type forms which was used in the self-adjoint case and also non-self-adjoint case with $b = 0$ (see e.g. [9], [10]and [12]).

This paper is based on the Boundary Control method introduced in [2] (see also [3]). Particularly, we use here the geometrical formulation of the Boundary Control method (see [11]) together with exact controllability results [1].

To formulate our results for Problems I and II we first consider how large time T should be and impose some geometric conditions upon (M, g) necessary for generic case.

To answer positively to problem I, it is clear that the waves sent from the boundary at time $t = 0$ should be able to reach all points inside M and return back to the boundary before time $t = T$. Hence, in the self-adjoint case we should assume that $T > 2\rho$, where $\rho = max\{\text{dist}(x, \partial M) : \ x \in M\}$ is the geodesic radius of M. On the other hand, in generic case we should know R^T for larger T and, moreover, should impose the following geometrical condition (for details see [1]), which generalizes the condition that the rays of the geometrical optics hit the boundary transversally.

Definition 2. *(M, g) satisfies the Bardos-Lebeau-Rauch condition if there is $t_* > 0$ and an open conic neighborhood $\mathcal{O}$ of the set of the not-nondiffractive points $(x, t, \xi, \omega) \in T^*(M \times [0, t_*])$, $x \in \partial M$ such that any generalized bicharacteristic of the wave operator $\partial_t^2 - \Delta_g$ passes through a point of $(x, t, \xi, \omega) \in T^*(M \times [0, t_*]) \setminus \mathcal{O}$, $x \in \partial M$.*

(In the future, we refer to t_* as to the exact controllability time.)

We can formulate now the first main result of the paper.

Theorem 1.1. *Assume that*

i. In generic case, the Riemannian manifold (M, g) satisfies the Bardos-Lebeau-Rauch condition with exact controllability time t_* and R^T is known for $T > 2t_*$.

ii. In the self-adjoint case R^T is known for $T > 2\rho$.

Then these data determine uniquely R^t for any $t > 0$.

Moreover, in Section 4 we give a corollary of Theorem 1.1 for the case of the inverse problem with one measurement. Namely, we show that there is $f \in H^1_{loc}(\partial M \times \mathbb{R}_+)$ such that the function $\partial_\nu u^f|_{\partial M \times \mathbb{R}_+}$ determines R^t for any $t \geq 0$.

Returning to Problem II, we first note that it is well known that, in general, Problem II has a negative answer since generalized gauge transformations preserve the boundary data. This means that, by replacing $a(x, D)$ by $a_\kappa(x, D)$,

$$a_\kappa(x, D) = \kappa a(x, D) \kappa^{-1}, \tag{1.6}$$

where $\kappa|_{\partial M} = 1$, $\kappa \neq 0$ on M, we do not change R^T. Thus, the best we can hope, is to recover the equivalence class of $a(x, D)$ with respect to the generalized gauge transformations, namely the set

$$[a(x, D)] := \{\kappa a(x, D) \kappa^{-1} : \ \kappa \in C^\infty(M; \mathbb{C}), \ \kappa|_{\partial M} = 1, \ \kappa \neq 0 \text{ on } M\}.$$

We formulate now the second main result of the paper.

Theorem 1.2. *In generic case, let Riemannian manifold (M, g) satisfy the Bardos-Lebeau-Rauch condition with exact controllability time t_*. Let ∂M and R^T, with $T > 2t_*$, be given. Then these data determine M, b and the equivalence class $[a(x, D)]$ uniquely.*

The analog of this result for the inverse boundary spectral problem in the self-adjoint case was proven in [5], [9], [10] (see also [4] for the dynamic inverse problem with $a(x,D) = -\Delta_g$ when the group of gauge transformations is trivial).

Example 1. Let us consider a general 1-dimensional wave equation for $u = u(y,t)$,

$$u_{tt} - \tilde{c}^2(y)u_{yy} + \tilde{b}(y)u_t + \tilde{p}(y)u_y + \tilde{q}(y)u = 0, \quad \text{in} \quad (x,t) \in [0,\tilde{l}] \times \mathbb{R}_+.$$

Introducing new coordinates $x = x(y),\ dx/dy = \tilde{c}(y)$, we reduce this equation to

$$u_{tt} - u_{xx} + bu_t + pu_x + qu = 0, \quad \text{in} \quad (x,t) \in [0,l] \times \mathbb{R}_+. \tag{1.7}$$

As the wave speed is equal to 1 for (1.7), $\rho = l/2$ and (1.7) satisfies the Bardos-Lebeau-Rauch condition with $t_* = l$. We consider equation (1.7) with initial and boundary conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad u(0,t) = f_0(t), \quad u(l,t) = f_1(t).$$

Then the dynamical boundary data is given by the mapping

$$(f_0(t),\ f_1(t)) \mapsto (u_x(0,t), \quad -u_x(l,t)),$$

where $f_0(t), f_1(t) \in H_0^1([0,T])$ are arbitrary. These data correspond to the measurements which one makes at the boundary on the time-interval $[0,T]$.

The results presented in this paper imply that the boundary measurements on a time interval $[0,T], T > 2l$ determine all measurements at any time interval $[0,t]$.

Furthermore, one can construct uniquely the function $b(x)$ and the equivalence class

$$\{(p_\kappa,\, q_\kappa): \quad \kappa(x) \neq 0,\ \kappa(0) = \kappa(1) = 1\},$$

where

$$p_\kappa = p - 2\partial_x(\kappa^{-1}), \quad q_\kappa = -\partial_x^2(\kappa^{-1}) + p\partial_x(\kappa^{-1}) + q.$$

To find p, q uniquely one needs further *a priori* information about their behaviour. For instance, if the coefficient $p(x)$ is given, one can reconstruct functions $b(x)$ and $q(x)$ uniquely. Similar examples are considered in more detail in [8].

At the end of this section, we explain what we mean by the reconstruction of a Riemannian manifold (M,g). Since a manifold is an 'abstract' collection of coordinate patches, we construct a representative of an equivalence class of the isometric Riemannian manifolds, i.e. metric space X which is isometric to (M,g). After constructing X, one can take any suitable local coordinates and construct the vector field P and the potential q in these local coordinates.

2. Continuation of data in the selfadjoint case.

In this section we consider Problem I for the initial-boundary value problem

$$u^f_{tt} + a(x,D)u^f = 0 \ \text{ in } M \times \mathbb{R}_+,$$

$$u^f|_{\partial M \times \mathbb{R}_+} = f; \quad u^f|_{t=0} = u^f_t|_{t=0} = 0,$$

where $a(x,D) = a^*(x,D)$ with respect to a suitable smooth measure $d\mu = \eta dx$. We point out that we do not assume that the Bardos-Lebeau-Rauch condition is valid.

By λ_j and ϕ_j we denote the Dirichlet eigenvalues and the normalized eigenfunctions of the operator $a(x,D)$ in $L^2(M, d\mu)$. Also, choose r satisfying $\rho < r < T/2$.

We start with a well-known result on approximate controllability.

Lemma 2.1. *The pairs* $(u^f(2r), u^f_t(2r))$, $f \in C_0^\infty(\partial M \times [0,2r])$ *are dense in* $H^1_0(M) \times L^2(M)$.

Proof. For the sake of simplicity we prove the statement for the case $a(x,D) = -\Delta_g + q$, with real q, and $d\mu = dx$. Necessary changes in the general case may be found e.g. in [10], [11].

Assume that a pair

$$(\psi, -\phi) \in (H^1_0(M) \times L^2(M))' = H^{-1}(M) \times L^2(M)$$

satisfy the duality

$$(u^f(2r), \psi)_{(H^1_0, H^{-1})} + (u^f_t(2r), -\phi)_{L^2} = 0$$

for all $f \in C_0^\infty(\partial M \times [0,2r])$. Let

$$e_{tt} - \Delta_g e + q\, e = 0 \text{ in } M \times [0,2r], \tag{2.1}$$

$$e|_{\partial M} = 0; \quad e|_{t=2r} = \phi; \ e_t|_{t=2r} = \psi.$$

By part integration

$$0 = \int_{M\times[0,2r]} [u^f \overline{(e_{tt} - \Delta_g e + qe)} - (u^f_{tt} - \Delta_g u^f + qu^f)\overline{e}]\, d\mu\, dt =$$

$$= \int_M (u^f_t(2r)\,\overline{\phi} - u^f(2r)\,\overline{\psi})\, d\mu + \int_{\partial M}\int_0^{2r} f\,\overline{\partial_\nu e}\, dS_x\, dt = \int_{\partial M}\int_0^{2r} f\,\overline{\partial_\nu e}\, dS_x\, dt$$

for all $f \in C_0^\infty(\partial M \times [0,2r])$. This yields that

$$e|_{\partial M \times [0,2r]} = \partial_\nu e|_{\partial M \times [0,2r]} = 0.$$

Since by (2.1) $e \in \mathcal{D}'(]0,2r[, H_0^1(M))$, Tataru's Holmgren-John uniqueness theorem (see e.g [19], [20]) is applicable and so that $e(r) = e_t(r) = 0$. Hence $e = 0$ identically on $M \times [0,2r]$ and thus $\phi = \psi = 0$. □

Consider a bilinear form

$$E(u^f, u^g, t) =$$

$$\int_M [((\nabla + ip)u^f(t), (\nabla + ip)u^g(t))_g + u_t^f(t)\, \overline{u_t^g(t)} + q\, u^f(t)\, \overline{u^g(t)}]\, d\mu(x),$$

where $p = (p_1, ..., p_m)$ and, as usual, $(a,b)_g = g^{jl} a_j \overline{b_l}$, $j,l = 1,..,m$. Denote $E(u^f, t) = E(u^f, u^f, t)$.

Lemma 2.2. *Operator R^t determines $E(u^f, u^g, t)$ for $f, g \in C_0^\infty(\partial M \times [0,t])$.*

Proof. By part integration

$$\frac{\partial}{\partial t} E(u^f, t) =$$

$$2 \int_M [((\nabla + ip)u_t^f(t), (\nabla + ip)u^f(t))_g + u_{tt}^f(t)\, \overline{u_t^f(t)} + q\, u^f(t)\, \overline{u_t^f(t)}]\, d\mu(x) =$$

$$2 \int_M [a(x,D)u^f(t) + u_{tt}^f(t)]\, \overline{u_t^f(t)}\, d\mu(x) + 2 \int_{\partial M} u_t^f(t) \overline{\partial_\nu u^f(t)}\, dS_x$$

$$= 2 \int_{\partial M} f_t(t)\, \overline{R^t f(t)}\, dS_x.$$

Since by initial conditions $E(u^f, 0) = 0$, we can determine $E(u^f, t)$. Since $4E(u^f, u^g, t) = E(u^{f+g}, t) - E(u^{f-g}, t)$, this proves the assertion. □

Next we show that we can continue data without solving the inverse problem.

Proof. (of Theorem 1.1 in the selfadjoint case) It is sufficient to show that R^T determines $R^t f$ for any $f \in C_0^\infty(\partial M \times [0,2r])$ and any $t \geq 0$.

Let $\varepsilon = (T - 2r)/2$ and $t_0 = 2r + \varepsilon$. By Lemma 2.1 there are $f_n \in C_0^\infty(\partial M \times [0,2r])$ such that

$$\lim_{n \to \infty} (u^{f_n}(2r), u_t^{f_n}(2r)) = (u^f(t_0), u_t^f(t_0)) \tag{2.2}$$

in $H_0^1(M) \times L^2(M)$-topology. We want to show that (2.2) is valid if and only if

$$\lim_{n \to \infty} E(u^{g_n}, t_0) = 0, \tag{2.3}$$

$$\lim_{n\to\infty} ||R^{t_0+\varepsilon} g_n||_{L^2(\partial M\times[t_0,t_0+\varepsilon])} = 0, \tag{2.4}$$

and for every $h \in C_0^\infty(\partial M \times [0,2r])$

$$\lim_{n\to\infty} E(u^{g_n}, u^h, t_0) = 0, \tag{2.5}$$

where $g_n(t) = f(t) - f_n(t-\varepsilon)$. Since the direct problem depends continuously on initial data ([8], [14]), we see that (2.2) obviously yields (2.3)-(2.5). Thus assume that (2.3)-(2.5) are valid. We use the eigenfunction expansion $u^{g_n}(t_0) = \sum_j a_j^n \phi_j$ and $u^h(t_0) = \sum_j b_j \phi_j$. Then by (2.3)

$$\lim_{n\to\infty} \left(\sum_{j=0}^{\infty} \lambda_j (a_j^n)^2 + ||u_t^{g_n}||_{L^2}^2 \right) = 0. \tag{2.6}$$

Let $0 \le j_- \le j_+$ be such that $\lambda_j < 0$ for $j \le j_-$ and $\lambda_j > 0$ for $j \ge j_+$ and let P be the orthogonal projection in $H_0^1(M)$ onto the space of the eigenfunctions corresponding $\lambda_j = 0$, $j_- < j \le j_+$.

Using these notations, we rewrite (2.6) in the following form

$$\sum_{j\le j_-} -\lambda_j (a_j^n)^2 = \sum_{j\ge j_+} \lambda_j (a_j^n)^2 + ||u_t^{g_n}(t_0)||_{L^2(M)}^2 + o(1) \tag{2.7}$$

where $o(1)$ goes to zero when $n \to \infty$.

First we show that $a_j^n \to 0$ for $j \le j_-$. Indeed, assume that there is $k \le j_-$ such that $a_k^n \not\to 0$. By choosing a subsequence, the sign of a_j^n with $j \le j_-$ depends only upon j. Moreover, without loss of generality we can assume that $a_j^n \ge 0$, $j \le j_-$.

Since $(u^h(t_0), u_t^h(t_0))$ are dense in $H_0^1(M) \times L^2(M)$ we can choose h such that its Fourier coefficients (b_j) satisfy $b_j = \delta_{j\le j_-} + c_j$ where $||(\lambda_j^{1/2} c_j)||_{\ell^2} < \varepsilon$ and also $||u_t^h(t_0)||_{L^2(M)} < \varepsilon$, $\varepsilon \in]0, \frac{1}{2}[$. Then (2.5) yields that

$$\sum_{j\le j_-} -\lambda_j a_j^n (1 + c_j) = \sum_{j\ge j_+} \lambda_j a_j^n c_j + (u_t^{g_n}(t_0), u_t^h(t_0))_{L^2(M)} + o(1).$$

Hence by (2.7)

$$\sum_{j\le j_-} -\lambda_j a_j^n (1 + c_j) \le \tag{2.8}$$

$$\le \Big(\sum_{j\ge j_+} \lambda_j (a_j^n)^2 \Big)^{\frac{1}{2}} \Big(\sum_{j\ge j_+} \lambda_j (c_j)^2 \Big)^{\frac{1}{2}} + ||u_t^{g_n}(t_0)||\, ||u_t^h(t_0)|| + o(1)$$

$$\leq \varepsilon\Big(\sum_{j\geq j_+} \lambda_j (a_j^n)^2\Big)^{\frac{1}{2}} + \varepsilon ||u_t^{g_n}(t_0)||_{L^2} + o(1) \leq \sqrt{2}\varepsilon\big(\sum_{j\leq j_-} -\lambda_j (a_j^n)^2)^{1/2} + o(1).$$

On the other hand, there is $C > 0$ which is independent of ε such that

$$\sum_{j\leq j_-} -\lambda_j a_j^n (1 + c_j) \geq C\Big(\sum_{j\leq j_-} -\lambda_j (a_j^n)^2\Big)^{\frac{1}{2}}.$$

But for some $k \leq j_-$, $a_k^n \not\to 0$. This leads to a contradiction with (2.8).

Thus we have proven that $a_j^n \to 0$ for all $j \leq j_-$. By (2.6), this implies that $(1 - P)u^{g_n}(t_0) \to 0$ in $H_0^1(M)$ and $u_t^{g_n}(t_0) \to 0$ in $L^2(M)$.

The solution of the initial-boundary value problem and also $\partial_\nu u|_{\partial M\times[0,t]}$ depends continuously on initial data (e.g [8], [14]), so that

$$\lim_{n\to\infty} ||R^T g_n - \sum_{\lambda_j=0} a_j^n \partial_\nu \phi_j|_{\partial M}||_{L^2(\partial M\times[t_0,T])} = 0.$$

At last, the linear independence of $\partial_\nu \phi_j|_{\partial M}$ implies that (2.4) can only be valid if $a_j^n \to 0$ for $j_- < j < j_+$.

Thus (2.2) and (2.3)-(2.5) are equivalent.

We can now use Lemma 2.2 to construct f_n which satisfy conditions (2.2). The functions $y_n(t) = u^{f_n}(t)$ for $t \in [2r, T]$ are the solutions of the initial value problem

$$y_{tt}^n + a(x, D)y^n = 0 \text{ in } M \times [2r, T]$$

$$y^n|_{\partial M\times[2r,T]} = 0; \quad y^n|_{t=2r} = u^{f_n}(2r); \; y_t^n|_{t=2r} = u_t^{f_n}(2r).$$

However, $y(t) = u^f(t + \varepsilon)$ satisfies the same equation with initial data

$$y|_{t=2r} = u^f(t_0), \; y_t|_{t=2r} = u_t^f(t_0).$$

Then it follows from (2.2) and continuous dependence of solutions on the initial data ([8], [14]) that

$$\lim_{n\to\infty} \partial_\nu y^n|_{\partial M\times[2r,T]} = \partial_\nu y|_{\partial M\times[2r,T]}$$

in L^2-topology. Since we know $y_n(t)|_{\partial M\times[2r,T]} = (R^T f_n)(t), \; t \in [2r, T]$ we can determine $R^{T+\varepsilon} f$.

By iterating the above consideration, we get the assertion. □

3. Continuation of data and uniqueness results in the non-selfadjoint case.

In this section we study the inverse problem for the initial-boundary value problem in generic case

$$u_{tt}^f + bu_t^f + a(x,D)u^f = 0 \text{ in } M \times \mathbb{R}_+ \tag{3.1}$$

$$u^f|_{\partial M\times\mathbb{R}_+} = f; \quad u|_{t=0} = u_t|_{t=0} = 0; \quad f \in H_0^1(\partial M \times \mathbb{R}_+), \tag{3.2}$$

where $a(x,D)$ is of form (1.3) and (M,g) satisfies the Bardos-Lebeau-Rauch condition. We use the notations

$$U^f(t) := \begin{pmatrix} u^f(x,t) \\ u_t^f(x,t) \end{pmatrix} \in L^2(M)^2, \quad J\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} u^2 + bu^1 \\ u^1 \end{pmatrix}. \tag{3.3}$$

and denote the inner product in $[L^2(M)]^2$ with respect to the Riemannian volume (1.4) by $(\cdot\,,\cdot)$.

3.1 Adjoint equation.

Let $v^g(x,s)$ be the solution to the adjoint initial-boundary value problem,

$$v_{tt}^g + \bar{b}v_t^g + a^*(x,D)v^g = 0 \text{ in } M \times \mathbb{R}_+, \tag{3.4}$$

$$v^g|_{\partial M\times\mathbb{R}_+} = g; \quad v^g|_{t=0} = v_t^g|_{t=0} = 0. \tag{3.5}$$

We denote

$$V^g(t) = \begin{pmatrix} v^g(x,t) \\ v_t^g(x,t) \end{pmatrix}. \tag{3.6}$$

For the adjoint equation we define the response operator $R_*^T : H_0^1(\partial M \times [0,T]) \to L^2(\partial M \times [0,T])$,

$$R_*^T(g) = B^* v^g, \quad B^* v := \partial_\nu v|_{\partial M\times[0,T]}. \tag{3.7}$$

Lemma 3.1. *For any $t_0 > 0$ R^{t_0} determines $R_*^{t_0}$.*

Proof. Let $f, h \in H_0^1(\partial M \times [0,t_0])$ and let e^h be the solution of the backward wave equation

$$e_{tt}^h - \bar{b}e_t^h + a^*(x,D)e^h = 0 \text{ in } M \times [0,t_0], \tag{3.8}$$

$$e^h|_{\partial M\times[0,t_0]} = h; \quad e^h|_{t=t_0} = e_t^h|_{t=t_0} = 0. \tag{3.9}$$

Notice that for $h(t)=g(t_0-t)$ we have $e^h(t)=v^g(t_0-t)$. Part integration together with initial and final conditions (3.2), (3.9) yield that

$$0=\int_0^{t_0}\int_M \left((u^f_{tt}+bu^f_t+a(x,D)u^f)\overline{e^h}-u^f\,(\overline{e^h_{tt}-\bar{b}e^h_t+a^*(x,D)e^h})\right)dm_g(x)dt$$

$$=\int_0^{t_0}\int_{\partial M}\left(Bu^f\,\overline{e^h}-u^f\,\overline{B^*e^h}\right)dS_x dt=\int_0^{t_0}\int_{\partial M}\left(R^{t_0}f\overline{h}-f\,\overline{B^*e^h}\right)dS_x dt.$$

Since f is arbitrary and $R^{t_0}f$ is known, we can determine $B^*e^h|_{\partial M\times[0,t_0]}$ for each $h\in H^1_0(\partial M\times[0,t_0])$, i.e. to find $R^{t_0}_*$. □

3.2 Controllability results and continuation of R^T.

We denote by $\mathcal{L}^s, s\in\mathbb{R}$ the subspace of functions in $H^{s+1}_0(M)\times H^s(M)$ which satisfy the natural boundary compatibility conditions for the hyperbolic problem (3.1), (3.2) for $t\notin$ supp f (see e.g [8], [16]) and by $\mathcal{L}^s_{\rm ad}$ the analogous subspace for (3.4), (3.5).

We use the following exact controllability result.

Theorem 3.2. [1] *Assume that (M,g) satisfies the Bardos-Lebeau-Rauch condition. Then*

$$\{U^f(t_1):\ f\in H^{s+1}_0(\partial M\times[0,t_0])\}=\mathcal{L}^s,\quad t_1\geq t_0>t_*, s\geq 0,$$

where t_* is the exact controllability time.

The analogous result is valid for the adjoint equation.

Lemma 3.3. *Assume that we know R^T for some $T\geq 0$. Then for any $f,g\in H^1_0(\partial M\times[0,T])$, $t+s\leq T$ we can evaluate*

$$(J\,U^f(t),V^g(s))=$$

$$=\int_M[u^f_t(x,t)\overline{v^g(x,s)}+u^f(t)\overline{v^g_s(x,s)}+b(x)u^f(x,t)\overline{v^g(x,s)}]dm_g(x).$$

Proof. By part integration

$$(\partial_t-\partial_s)\big(JU^f(t),V^g(s)\big)=\int_M[(u^f_{tt}+bu^f_t)\,\overline{v^g}-u^f\,(\overline{v^g_{tt}+\bar{b}v^g_t})]\,dm_g(x)\ =$$

$$=\int_{\partial M}[Bu^f(t)\overline{v^g(s)}-u^f(t)\overline{B^*v^g(s)}]\,dS_x$$

$$= \int_{\partial M} [R^T f(t)\, \overline{g(s)} - f(t) \overline{R_*^T g(s)}]\, dS_x. \tag{3.10}$$

As R^T and R_*^T are known, all the functions in the last integral are known. Hence (3.10) is a differential equation along the characteristic $t + s = \text{const}$. Furthermore,

$$\big(JU^f(0), V^g(s)\big) = \big(JU^f(t), V^g(0)\big) = 0$$

due to initial conditions (3.2), (3.5). Equation (3.10) together with the above initial conditions indicates the possibility to find $(J\,U^f(t), V^g(s))$. □

Next we prove that, in the generic case, R^t can be determined for all $t > 0$.

Proof. (of Theorem 1.1) Let $\varepsilon < T/2 - t_*$, $T_0 = T/2$. We will first prove that when R^T and R_*^T are given, it is possible to find $R^{T+\varepsilon}$ and $R_*^{T+\varepsilon}$.

Clearly it is sufficient to determine $R^{T+\varepsilon} f$ for any $f \in H_0^1(\partial M \times [0, T_0])$. As $T_0 - \varepsilon > t_*$ then by Theorem 3.2 there is $\tilde{f} \in H_0^1(\partial M \times [0, T_0 - \varepsilon])$ for which

$$U^f(T_0) = U^{\tilde{f}}(T_0 - \varepsilon).$$

Moreover, this function can be found by choosing $\tilde{f}$ which satisfies the following equation,

$$(J\,U^f(T_0), V^g(T_0)) = (J\,U^{\tilde{f}}(T_0 - \varepsilon), V^g(T_0))$$

for all $g \in H_0^1(\partial M \times [0, T_0])$.

Let now $F \in H_0^1(\partial M \times [0, T])$ be the function

$$F(x,t) = \tilde{f}(x,t) \text{ for } t \in [0, T_0 - \varepsilon], \quad F(x,t) = 0 \text{ for } t \in]T_0 - \varepsilon, T].$$

Let $\phi = u^f|_{t=T_0}$ and $\psi = u_t^f|_{t=T_0}$. Since u^f solves the equation

$$u_{tt}^f + bu_t^f + a(x,D)u^f = 0 \text{ in } M \times [T_0, T+\varepsilon],$$

$$u^f|_{\partial M \times [T_0, T+\varepsilon]} = 0; \quad u^f|_{t=T_0} = \phi, \; u_t^f|_{t=T_0} = \psi$$

and u^F solves the equation

$$u_{tt}^F + bu_t^F + a(x,D)u^F = 0 \text{ in } M \times [T_0 - \varepsilon, T]$$

$$u^F|_{\partial M \times [T_0 - \varepsilon, T]} = 0; \quad u^F|_{t=T_0-\varepsilon} = \phi; \; u_t^F|_{t=T_0-\varepsilon} = \psi,$$

we see that

$$u^f(t+\varepsilon) = u^F(t) \text{ for } t \in [T_0 - \varepsilon, T].$$

Hence we get

$$R^{T+\varepsilon} f(\cdot\,,t) = R^T F(\cdot\,,t-\varepsilon) \text{ for } t \in [T_0, T+\varepsilon].$$

Since by assertion $(R^T F)(\cdot\,,t)$ for $t \le T$ is known, we reconstruct $R^{T+\varepsilon}$. The claim follows similarly for $R_*^{T+\varepsilon}$.

By iteration the above procedure with fixed T_0, we reconstruct $R^{T+n\varepsilon}$, $n = 0,1,2,\dots$. This proves Theorem 1.1. □

Analogously to Lemma 3.3, we can now obtain

Corollary 3.4. *Assume that DBD are given for $T > 2t_*$. Then for any $f, g \in H_0^1(\partial M \times \mathbb{R}_+)$ and $t, s > 0$ we can evaluate $(J\,U^f(t), V^g(s))$.*

3.4 Construction of the boundary distance functions.

Let $r_x(y), x \in M$ be the boundary distance functions

$$r_x(y) = d(x,y), \quad y \in \partial M.$$

We define a mapping $\mathcal{R} : M \to L^\infty(\partial M)$ by setting

$$\mathcal{R}(x) = r_x.$$

We are going to show that we can reconstruct the set $\mathcal{R}(M) = \{r_x : \ x \in M\}$.

In the standard Boundary Control method one constructs the projections to the spaces of the Fourier coefficients of the functions $L^2(A),\ A \subset M$. Inspired by this we consider the Dirichlet boundary value f as a control parameter of $U^f(T)$ and define the following spaces.

Definition 3. *Let $H \subset \mathcal{L}^s$ be a lineal, $s \ge 0$ and let*

$$T > 2\max\{diam\ M, t_*\},$$

where diam M is the geodesic diameter of (M,g). We define the control sets $\mathcal{H}^s(H)$ for H by

$$\mathcal{H}^s(H) = \{\ f \in H_0^{s+1}(\partial M \times [0,T/2]) :\ U^f(T) \in H\},$$

$$\mathcal{H}^s_{\rm ad}(H) = \{g \in H_0^{s+1}(\partial M \times [0,T/2]) :\ V^g(T) \in H\}.$$

Let $\Gamma \subset M$ be open, $t_0 \ge 0$. Denote

$$M(\Gamma, t_0) = \{x \in M : d(x,\Gamma) \le t_0\}. \tag{3.11}$$

We remark that since $M(\Gamma, t_0) = M$ for $t_0 \ge$ diam M, it is sufficient to consider $t_0 \le$ diam $M < T/2$.

Definition 4. *For $s \geq 0$ let*

$$\mathcal{L}^s(\Gamma, t_0) = \{U \in \mathcal{L}^s : \textit{ supp } U \subset cl(M(\Gamma, t_0))\},$$

$$[\mathcal{L}^s(\Gamma, t_0)]^c = \{U \in \mathcal{L}^s : \textit{ supp } U \subset cl(M \setminus M(\Gamma, t_0))\}$$

and analogous sets $\mathcal{L}^s_{\mathrm{ad}}(\Gamma, t_0), [\mathcal{L}^s_{\mathrm{ad}}(\Gamma, t_0)]^c$.

Our next goal is to find the control sets for the above subsets of $\mathcal{L}^s$.

Lemma 3.5. *Let R^T be given for $T > 2t_*$. Then for any $f \in H_0^{s+1}(\partial M \times [0, T/2]), s \geq 0$ and any $\Gamma \subset \partial M, t_0 \in [0, T/2]$ it is possible to determine whether*

$$m_g(\textit{supp } U^f(T) \ \cap M(\Gamma, t_0)) = 0$$

or not. Here m_g is the Riemannian measure on (M, g).

Analogous statement takes place for the adjoint solutions $V^g(T)$.

Proof. By Theorem 1.1 we can assume that we know R^t for any $t \geq 0$. Particularly, we know $R^{3T/2}$.

Recall that in the assertion $f(x,t) = 0$ for $t > T/2$. Thus, if

$$m_g(\text{supp } U^f(T) \cap M(\Gamma, t_0)) = 0$$

then, by the finite velocity of the wave propagation,

$$Bu^f|_{\Gamma \times [T - t_0, T + t_0]} = 0 \text{ and } f|_{\Gamma \times [T - t_0, T + t_0]} = 0.$$

On the other hand, by Tataru's Holmgren-John theorem [19], the converse is also true. Since $Bu^f|_{\partial M \times [0, 3T/2]} = R^{3T/2} f$ is known, the statement follows. The claim for adjoint solutions follows from Lemma 3.1. □

Corollary 3.6. *Let $\Gamma \subset \partial M, t_0 \geq 0$ and $s \geq 0$. Then DBD determine lineals $\mathcal{H}^s(\mathcal{L}^s(\Gamma, t_0))$, $\mathcal{H}^s([\mathcal{L}^s(\Gamma, t_0)]^c)$ and $\mathcal{H}^s_{\mathrm{ad}}(\mathcal{L}^s_{\mathrm{ad}}(\Gamma, t_0))$, $\mathcal{H}^s_{\mathrm{ad}}([\mathcal{L}^s_{\mathrm{ad}}(\Gamma, t_0)]^c)$.*

Proof. By Theorem 3.2,

$$\{U^f(T) : \ f \in H_0^{s+1}(\partial M \times [0, T/2])\} = \mathcal{L}^s.$$

Thus by Lemma 3.5 DBD determine $\mathcal{H}^s[\mathcal{L}^s(\Gamma, t_0)]^c$ and $\mathcal{H}^s_{\mathrm{ad}}[\mathcal{L}^s_{\mathrm{ad}}(\Gamma, t_0)]^c$.

For $f \in H_0^{s+1}(\partial M \times [0, T/2])$ we have $f \in \mathcal{H}^s(\mathcal{L}^s(\Gamma, t_0))$ if and only if

$$(J\, U^f(T), V^g(T)) = 0$$

for all $g \in \mathcal{H}^s_{\mathrm{ad}}[\mathcal{L}^s_{\mathrm{ad}}(\Gamma, t_0)]^c$. Hence we can determine $\mathcal{H}^s(\mathcal{L}^s(\Gamma, t_0))$.

The case $\mathcal{H}^s_{\mathrm{ad}}(\mathcal{L}^s_{\mathrm{ad}}(\Gamma, t_0))$ can be considered analogously. □

Corollary 3.7. *Let $\Gamma_i \subset \partial M, t_i^+ > t_i^- \geq 0;\ i = 1, ..., I$. Denote by M_I the set*

$$M_I = \bigcap_{i=1}^{I} (M(\Gamma_i, t_i^+) \setminus M(\Gamma_i, t_i^-)). \tag{3.12}$$

Then DBD determine whether $m_g(M_I) = 0$ or not.

Proof. Using intersections of sets described in Corollary 3.6 we find whether $\mathcal{L}^s$ contains functions supported in the closure of M_I. That kind of functions exists if and only if $m_g(M_I) \neq 0$. □

Corollary 3.7 is the basic analytic tool in the reconstruction of $\mathcal{R}(M)$.

Theorem 3.8. *The response operator R^T with $T > 2t_*$ determines $R(M)$ uniquely.*

Proof. For every $\varepsilon > 0$ we choose a collection $\Gamma_i \subset \partial M, i = 1, ..., I(\varepsilon)$ such that $\mathrm{diam}(\Gamma_i) \leq \varepsilon$, $\cup\Gamma_i = \partial M$. Let

$$p = (p_1, ..., p_{I(\varepsilon)}), \quad p_i \in \mathbb{Z}_+, \quad t_i^+ = (p_i + 1)\varepsilon, \quad t_i^- = (p_i - 1)\varepsilon. \tag{3.13}$$

Denote by $M(\varepsilon, p)$ the set M_I (see (3.12)) with $t_i^\pm$ of form (3.13). For every p we define a piecewise constant function $r_p \in L^\infty(\partial M)$ by setting $r_p(y) = p_i\varepsilon$ when $y \in \Gamma_i$. Using Corollary 3.7 we define whether $m_g(M(\varepsilon, p)) > 0$ or not and introduce the set

$$\mathcal{R}_\varepsilon(M) = \{r_p :\ p \in \mathbb{Z}_+^{I(\varepsilon)} \text{ such that } m_g(M(\varepsilon, p)) > 0\} \subset L^\infty(\partial M).$$

As $||r_x - r_p|| < 2\varepsilon + \max \mathrm{diam}\ (\Gamma_i)$ when $x \in M(\varepsilon, p)$, then

$$\mathrm{dist}_H(\mathcal{R}_\varepsilon(M), \mathcal{R}(M)) \leq 3\varepsilon.$$

Here $\mathrm{dist}_H(\Omega, \widetilde{\Omega})$ is the Hausdorff distance between subsets $\Omega, \widetilde{\Omega} \in L^\infty(\partial M)$. When $\varepsilon \to 0$ we find the set $\mathcal{R}(M) \subset L^\infty(\partial M)$ as the limit of $\mathcal{R}_\varepsilon(M)$. □

Let $\mathcal{R}(M) \subset L^\infty(\partial M)$ be given. It is shown in [11] that it is possible to uniquely define a Riemannian structure on $\mathcal{R}(M)$ such that $\mathcal{R} : M \to \mathcal{R}(M)$ is an isometry. In this paper, we use another method to reconstruct (M, g) and also b and $[a(x, D)]$. This method is based upon an approximation of δ-functions. We start with the following result (see [11]).

Lemma 3.9. *$\mathcal{R}(M) \subset L^\infty(\partial M)$ is homeomorphic to M.*

Proof. Obviously, $\mathcal{R}$ is continuous. Assume that $r_x = r_y,\ x, y \in M$. If $z \in \partial M$ is a nearest point to x, r_x achieves the minimum $h = \min_{z' \in \partial M} r_x(z')$ at z. Thus x lies on the normal geodesic from z and $x = \exp_z(h\nu)$, exp being the standard exponential map on TM. The same holds for y and hence $\mathcal{R} : M \to \mathcal{R}(M)$ is one-to-one. By definition it is onto. Since M is compact, $\mathcal{R}$ is a homeomorphism. □

3.5. Reconstruction of the Riemannian metric and the operator.

Let $f, g \in H_0^{s+1}(\partial M \times [0, T/2]),\ s \geq 0$. We define a bilinear form

$$\langle f, g\rangle \ = (J\,U^f(T), V^g(T)).$$

Let

$$\mathcal{R}(\varepsilon, p) = \mathcal{R}(M(\varepsilon, p)),\ \ \varepsilon > 0,\ p \in \mathbb{Z}_+^{I(\varepsilon)}. \tag{3.14}$$

Here $M(\varepsilon, p)$ is defined as in the proof of Theorem 3.8, i.e. $\mathcal{R}(\varepsilon, p)$ is the set of all boundary distance functions r_x with $x \in M(\varepsilon, p) \subset M$.

Let $r_{x_0} \in \mathcal{R}(M \backslash \partial M)$. Then for any ε there exists $p_\varepsilon \in \mathbb{Z}_+^{I(\varepsilon)}$ such that $x_0 \in M(\varepsilon, p_\varepsilon)$ and

$$\mathcal{R}(\varepsilon, p_\varepsilon) \longrightarrow \{r_{x_0}\} \text{ when } \varepsilon \to 0,$$

i.e. the Hausdorff distance between the above sets goes to 0 when $\varepsilon \to 0$. By Lemma 3.9, this yields that

$$M(\varepsilon, p_\varepsilon) \longrightarrow \{x_0\} \text{ when } \varepsilon \to 0. \tag{3.15}$$

Denote by $g(\varepsilon),\ \varepsilon > 0$ a family of functions in $H_0^1(\partial M \times [0, T/2])$ such that

i. supp $V^{g(\varepsilon)}(T) \subset \mathrm{cl}\ (M(\varepsilon, p_\varepsilon))$.

ii. For any $f \in H_0^{s+1}(\partial M \times [0, T/2]),\ s < m/2 < s + 1$ there exists a limit

$$\mathcal{W}^{x_0}(f) = \lim_{\varepsilon \to 0} \langle f, g(\varepsilon)\rangle.$$

Such families exist, indeed it is sufficient to take $V^{g(\varepsilon)}$ to be C_0^∞ -approximations to $(0, \delta(\cdot - x_0))$. On the other hand, assume that for every $f \in H_0^{s+1}(\partial M \times [0, T/2])$ the limit

$$\lim_{\varepsilon \to 0}\langle f, g(\varepsilon)\rangle = \lim_{\varepsilon \to 0}(J\,U^f(T), V^{g(\varepsilon)}(T))$$

exists. Then by Banach-Steinhaus theorem there is $W^{x_0} \in (\mathcal{L}^s)' \subset H_0^{s+1}(M)' \times H_0^s(M)'$ such that

$$\lim_{\varepsilon \to 0} \langle f, g(\varepsilon) \rangle = (J U^f(T), W^{x_0}),$$

where the right hand side is understood in the distribution sense. Assumption i. together with (3.15) imply that supp $(W^{x_0}) \subset \{x_0\}$. Since any distribution supported in a point is a finite sum of derivatives of the delta-distribution, and since $W^{x_0} \in H_0^s(M)' \times H_0^{s+1}(M)'$, $s < m/2 < s+1$, it follows that there is a constant $\kappa(x_0)$ that

$$W^{x_0} = \begin{pmatrix} 0 \\ \kappa(x_0)\delta(\cdot - x_0) \end{pmatrix}.$$

Lemma 3.10. *Let (M, g) satisfies the Bardos-Lebeau-Rauch condition and R^T is given for $T > 2t_*$, where t_* is the exact controllability time. Then it is possible to construct functions $g(\varepsilon)$ such that*

$$\mathcal{W}^{x_0}(f) = \kappa(x_0) u^f(x_0, t), \quad f \in H_0^{s+1}(\partial M \times [0, T/2]),\, t \geq 0,\, s < m/2 < s+1$$

and

$$\kappa \in C^0(M), \quad \kappa|_{\partial M} = 1, \quad \kappa \neq 0 \quad \text{on } M. \tag{3.16}$$

Proof. To prove the statement is sufficient to show that for any $r_{x_0} \in \mathcal{R}(\mathrm{int}(M))$ it is possible to find a family $g_{x_0}(\varepsilon), \varepsilon > 0$ such that the corresponding $\mathcal{W}^{x_0}$ satisfy i., ii. and the following conditions

iii. $\mathcal{W}^{x_0} \neq 0$ for any $x_0 \in M$.

iv. The function $r_{x_0} \mapsto \mathcal{W}^{x_0}(f)$ has a continuous extension to $\mathcal{R}(M)$ when $f \in C_0^\infty(\partial M \times [0, T])$.

v. For $f \in C_0^\infty(\partial M \times [0, T])$ and $x_1 \in \partial M$

$$\lim_{x_0 \to x_1} \mathcal{W}^{x_0}(f) = f(x_1, T).$$

As we already know such sequence exists. Indeed, we can take functions $g_{x_0}(\varepsilon)$ such that $V^{g_{x_0}(\varepsilon)}(T)$ are smooth approximations to $(0, \delta(\cdot - x_0))^t$. On the other hand, Corollary 3.4 makes possible to algorithmically verify conditions i.-v. □

Corollary 3.11. *Let DBD and $r_{x_0} \in \mathcal{R}(M)$ be given. These data determine $\kappa(x) u^f(x_0, t)$ for any $t > 0$ and $f \in H_0^1(\partial M \times \mathbb{R}_+)$.*

Proof. The statement follows from Corollary 3.4 and Lemma 3.10. □

We want to emphasize that we do not know $\kappa(x)$ and, henceforth, can not reconstruct $u^f(x, t)$. However, we have the following

Theorem 3.12. *The DBD determines a metric E on $\mathcal{R}(M)$ such that the metric space $(\mathcal{R}(M), E)$ is isometric to (M, g).*

Proof. Let $r_x, r_y \in \mathcal{R}(int(M))$ and let $\mathcal{R}(\varepsilon, p_\varepsilon)$ (see (3.14)) be a sequence satisfying

$$\mathcal{R}(\varepsilon, p_\varepsilon) \to \{r_x\}$$

when $\varepsilon \to 0$. We denote $h_\varepsilon = \text{diam } M(\varepsilon, p_\varepsilon)$. By Corollary 3.7, we can construct the set

$$X(\varepsilon) = \{f \in H_0^{s+1}(\partial M \times [0, T/2]) : \text{ supp } U^f(T) \subset \text{cl}(M(\varepsilon, p_\varepsilon))\}. \tag{3.17}$$

Let $\tau > 0$. Assume that $d(x, y) > \tau$. Then due to the finite velocity of the wave propagation and the fact that $h_\varepsilon \to 0$, there is ε_0 such that for $\varepsilon < \varepsilon_0$ we have:

(A) There is a neighborhood N of y such that for any $f \in X(\varepsilon)$

$$U^f|_{N\times]T, T+\tau[} = 0.$$

Using Lemma 3.5 we can check if the property (A) is satisfied.

Let now $s(r_x, r_y)$ be the supremum of all $\tau > 0$ for which the property (A) is satisfied with some $\varepsilon > 0$. Then

$$s(r_x, r_y) \geq d(x, y). \tag{3.18}$$

On the other hand, assume that x and y are so near to each other that $d(x, y) < d(x, \partial M)/2$ and there is an unique minimal geodesic $\gamma(t) = \exp_x(tv)$ from x to y. Let $\tau > d(x, y)$. Then for every $\varepsilon > 0$ there is a solution $(u^f(x, T), 0)$, $f \in X(\varepsilon)$ such that $(x, T, v, 1) \in T^*(M \times \mathbb{R}_+)$ is in the wavefront set of u^f. By the standard theory of propagation of singularities,

$$\text{singsupp } u^f \cap \{y\} \times]T, T + \tau[\neq \emptyset.$$

Thus the function u^f can not vanish in any neighborhood of $y \times]T, T + \tau[$ and the property (A) is not satisfied with any ε. Thus $s(r_x, r_y) \leq d(x, y)$. Hence, for y sufficiently close to x, $s(r_x, r_y) = d(x, y)$.

Define the distance

$$E(r_x, r_y) := \inf\{\sum_{j=0}^{l} s(r_{y_j}, r_{y_{j+1}}) : \; x_0 = x, \; y_l = y, \; y_j \in \text{int } (M), \; l \geq 1\}.$$

For any curve $\gamma \subset \mathrm{int}(M)$, we see that the E-length of $\mathcal{R}(\gamma)$ is equal to the length of γ. Hence $E(r_x, r_y) = d(x, y)$ for any $x, y \in \mathrm{int}(M)$. By continuation of E onto $\mathcal{R}(\partial M)$ we obtain $(\mathcal{R}(M), E)$ which is isometric to (M, g). □

Thus $(\mathcal{R}(M), E)$ can be identified with (M, g) as a metric space. In order to construct local coordinates on $\mathcal{R}(M)$, we can first use the distance E to construct geodesics. By using triangular comparison theorems, we can then find the angles between intersecting geodesics. This defines normal coordinates near any $r_x \in \mathcal{R}(M)$ and, henceforth, the differentiable structure on $\mathcal{R}(M)$.

Using this structure, we can go back to Lemma 3.10 and demand (see iv. in the proof) that $\kappa \in C^\infty(M)$.

Lemma 3.13. *The functions $e^f(x,t) = \kappa(x) u^f(x,t)$, $x \in M$, $t \geq 0$, where $f \in H_0^{s+1}(\partial M \times [0, T/2])$ and $\kappa \in C^\infty(M)$ is of form (3.16), determine $a_\kappa(x, D)$ and $b(x)$.*

Proof. The functions $e^f(x,t) = \kappa(x) u^f(x,t)$ are the solutions of the initial-boundary value problem (see (1.6))

$$e^f_{tt} + b e^f_t + a_\kappa(x, D) e^f = 0, \tag{3.19}$$

$$e^f|_{\partial M \times \mathbb{R}_+} = f; \quad e^f|_{t=0} = e^f_t|_{t=0} = 0.$$

However, Theorem 3.2 implies that for any $x_0 \in \mathrm{int}(M)$ the vectors

$$\left(e^f(x_0, T), \partial_j(e^f(x_0, T)), \partial_k \partial_l(e^f(x_0, T))), e^f_t(x_0, t)\right)_{j,k,l=1}^m$$

with different $f \in C_0^\infty(\partial M \times [0, T])$, span the space $\mathbb{C}^{(m^2+3m+4)/2}$. Hence equation (3.19) may be used to determine b and $a_\kappa(x, D)$.

Theorem 1.2 is proven. □

4. Results for one measurement and further remarks.

In the first part of this section we analyse the possibility of the reconstruction of the response operator R^{t_0} using only one measurement.

Theorem 4.1. *For any $t_0 > 0$ there is $f \in H^1_{loc}(\partial M \times \mathbb{R}_+)$, $f|_{t=0} = 0$, such that $\partial_\nu u^f|_{\partial M \times \mathbb{R}_+}$ determines R^{t_0}.*

Proof. Our main tool is the consequence of energy inequality (see e.g. [8], [14]),

$$||\partial_\nu u^f||_{L^2(\partial M \times [0,t])} \leq c_0 e^{c_1 t} ||f||_{H_0^1(\partial M \times [0,t])}, \quad f \in H_0^1(\partial M \times [0, t]), \tag{4.1}$$

where c_0 and c_1 are independent of t.

For $t_0 > 0$, let $(f_j : j = 1, \dots)$ be an ortonormal basis of $H_0^1(\partial M \times [0, t_0])$. Let $g_n, n = 1, 2, ...$ be a sequence where each f_j occurs infinitely many times. Consider

$$f(x,t) = \sum_{n=1}^{\infty} e^{cn^2} g_n(x, t - nt_0)$$

with $c > c_1 t_0$ where c_1 is the constant in (4.1). Assume that $\partial_\nu u^f|_{\partial M \times \mathbb{R}_+}$ is known. By inequality (4.1) we see that

$$||e^{-cn^2} \partial_\nu u^f(x, t + nt_0)|_{\partial M \times [0,t_0]} - (R^{t_0} g_n)(x,t)||_{L^2} \leq c' \, n e^{-c_1 n t_0}.$$

As $ne^{-c_1 n t_0} \to 0$ when $n \to \infty$, this shows that we can determine all $R^{t_0} f_j, \ j = 1, 2, \dots$. □

Corollary 4.2. *Let, in generic case, (M, g) satisfy the Bardos-Lebeau-Rauch condition. There is $f \in H^1_{loc}(\partial M \times \mathbb{R}_+)$, $f|_{t=0} = 0$, such that $\partial_\nu u^f|_{\partial M \times \mathbb{R}_+}$ determines M, b and the equivalence class $[a(x, D)]$ uniquely.*

In the self-adjoint case the Bardos-Lebeau-Rauch condition is not necessary.

We conclude the paper with several remarks:

i. The Bardos-Lebeau-Rauch condition is always satisfied for $M \subset \mathbb{R}^m$ with the metric $g^{jl} = \delta^{j,l}$ or its C^1-small perturbations (see e.g. [1], [21]);

ii. In the case $b = 0$ but $a(x, D) \neq a^*(x, D)$ an analog of Theorem 1.1 states that given R^T for $T > t_*$ determines R^t for all t. Indeed, in this case we can use a sesquilinear form $u_t^f(t)\overline{v^g(t)} - u^f(t)\overline{v_t^g(t)}$. Then an analog of lemma 3.3 states that given R^T it is possible to find the value of this form for $t \leq T$. Further proof of Theorem 1.1 (with $T > t_*$ instead of $T > 2t_*$) follows as in §3.

iii. The present work remains open the question what is the minimum time T needed to reconstruct the manifold and the operator. Indeed, in the case $b = 0$, as we have just shown, $T > t_*$ is sufficient. In the selfadjoint case $T > 2\rho$ is sufficient where ρ is the geodesic radius of (M, g), $\rho \leq t_*/2$. Moreover, it is known that in the one-dimensional case, when $2\rho = t_*$, the case $b \neq 0$ does need time $T > 2t_*$.

v. Clearly the considerations of the paper remain valid for (M, g) satisfying the Bardos-Rauch-Lebeau conditions for a part of boundary $\Gamma \subset \partial M$.

iv. Corollary 1.3 remains open in the question if there is $f \in H_0^1(\partial M \times \mathbb{R}_+)$, that is, a boundary source with finite energy which determines R^T. By modifying the proof of Corollary 1.3 we see that this is true if $c_1 < 0$ in inequality (4.1).

vi. Instead the boundary operator $B = \partial_\nu - P_\nu$ we can use $B = \partial_\nu - \beta$, where β is an arbitrary complex-valued C^∞-function on ∂M.

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