Inverse problem for a random potential

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Abstract: We study an inverse problem for the two-dimensional random Schrödinger equation $(\Delta+q+k^2)u(x,y,k)=0$. The potential $q(x)$ is assumed to be a Gaussian random function that defines a Markov field. We outline the proof of the following result: The backscattered field, obtained from a single realization of the random potential q, determines uniquely the principal symbol of the covariance operator. The analysis is carried out by combining methods of harmonic and microlocal analysis to stochastic methods.

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1 Introduction

We consider the Schrödinger equation with outgoing radiation condition

$$\begin{cases} (\Delta - q + k^2)u = \delta_y, & \text{in } \mathbb{R}^2 \\ \left(\frac{\partial}{\partial r} - ik\right) u(x) = o(|x|^{-1/2}) \end{cases} \tag{1}$$

where the potential q is a random function supported in a compact domain D. The wave u is decomposed to two parts

$$u = u_0(x,y,k) + u_s(x,y,k),$$

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where $u_s(x,y,k)$ is the scattered field and

$$u_0(x,y,k) = \Phi_k(x-y) = \frac{i}{4} H_0^{(1)}(k|x-y|)$$

is the incident field corresponding a point source at y. We shall assume that $y \in U$, where the domain $U \subset \mathbb{R}^2 \setminus \overline{D}$, called the *measurement domain*, is bounded, convex, and has positive distance to the support of q. As $q = q(x,\omega)$ is random, also the scattered field is a random variable that we sometimes emphasize by writing $u_s(x,y,k) = u_s(x,y,k,\omega)$. Here ω denotes an element of the probability space Ω. The inverse problem is to determine the parameter functions describing the random process $q(x,\omega)$, e.g., from the energy of the scattered wave $|u_s(x,y,k,\omega_0)|^2$ corresponding to the realization of the potential $q(x,\omega_0)$.

In applied literature the measured data is often assumed to coincide with the averaged data like $\mathbb{E}\,|u_s(x,y,k,\omega)|^2$. This corresponds to the case when the measurements could be done on many independent samples of the scatterer and these measurements could be averaged. Often the scatterer does not change during the time when the measurements are done. This suggests that one should study the problem how much information of the potential a realization of the scattered field carries.

A related approach for the scattering from a random media is the study of the multi-scale asymptotics of the scattered field. In this case the approximations made can be justified when the frequency k and the spatial frequency of the scatterer have appropriate magnitudes. This type of asymptotic analysis has been studied by Papanicolaou and others in various cases, see e.g. [14], [15], [3], [5].

In this paper our purpose is to apply techniques developed in deterministic inverse problems to stochastic inverse problems. Our approach is completely rigorous applying no approximations whatsoever. In [12] it is shown that the mean values (over the frequency k) of the energy $|u_s(x,y,k,\omega_0)|^2$, obtained from a single realization $q(z,\omega_0)$ determine almost surely the principal symbol of the covariance operator of the random function $q(x,\omega)$. We summarize in this note the basic elements of the proof.

1.1 Set-up of the model for the random potential

Fix a bounded simply connected domain $D \subset \mathbb{R}^2$. We assume that the potential q is a generalized Gaussian field supported in D. Recall, that this includes that q is a measurable map from the probability space Ω to the space of distributions $\mathcal{D}'(\mathbb{R}^2)$ such that for all $\phi \in C_0^\infty(\mathbb{R}^2)$ the mapping $\Omega \ni \omega \mapsto \langle q(\omega), \phi\rangle$ is a Gaussian random variable. We will assume that the probability measure space $(\Omega, \mathcal{F}, \mathcal{P})$ is complete. Then distribution of q is determined by the expectation $\mathbb{E}\, q$ and the covariance operator $C_q : C_0^\infty(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2)$ defined by

$$\langle \psi_1, C_q \psi_2\rangle = \mathbb{E}\left(\langle q - \mathbb{E}\, q, \psi_1\rangle \langle q - \mathbb{E}\, q, \psi_2\rangle\right). \tag{2}$$

Let $k_q(x, y)$ be the Schwartz kernel of the covariance operator C_q. We call $k_q(x, y)$ the covariance function of q. Then, in the sense of generalized functions, (2) reads as

$$k_q(x, y) = \mathbb{E}\left((q(x) - \mathbb{E}\, q(x))(q(y) - \mathbb{E}\, q(y))\right).$$

In many situations it is natural to assume that the covariance function $k_q(x, y)$ is smooth outside the diagonal (reader can consider as example the Brownian motion, the Levy-Brownian field in $\mathbb{R}^n$ [1] or a free Gaussian field [8]). As this is typical property of Schwartz kernels of pseudodifferential operators, we introduce the following definition.

Definition 1.1 A generalized Gaussian random field q on $\mathbb{R}^2$ is said to be microlocally isotropic (of order 2) in D, if the realizations of q are almost surely supported on the domain D and its covariance operator C_q is a classical pseudodifferential operator having the principal symbol $\mu(x)|\xi|^{-2}$ where $\mu \in C_0^\infty(D)$ is non-negative.

Important examples of the micro-locally isotropic fields of order two are considered in subsection 1.3.

We call μ the *micro-correlation strength of* q. In the above case the covariance function $k_q(z_1, z_2)$ is locally integrable for fixed z_2 and has the asymptotics

$$k_q(z_1, z_2) = -\mu(z_2) \log|z_1 - z_2| + f(z_1, z_2)$$

where f is locally bounded. Hence the function $\mu(z)$ describes the strength of the singularity of k_q near the diagonal, and determines approximately the radius of the set $\{z_1 : \ k_q(z_1, z_2) > M\}$ with a given large bound M. Thus micro-correlation strength $\mu(z)$ is closely related to local correlation length of the random field.

Definition 1.2 Given $\omega \in \Omega$, the measurement is the pointwise limit

$$m(x, y, \omega) = \lim_{K \to \infty} \frac{1}{K-1} \int_1^K k^4 |u_s(x, y, k, \omega)|^2 dk. \tag{3}$$

A special case is the backscattering measurement

$$m(x, x, \omega) = \lim_{K \to \infty} \frac{1}{K-1} \int_1^K k^4 |u_s(x, x, k, \omega)|^2 dk. \tag{4}$$

The measurement is an average over all frequencies implying that it is not sensitive to measurement errors. For example, the white noise error in the measurement, is filtered out with frequency averaging. Moreover, the measurement uses information only from the amplitude (not the phase) of the scattered field. It is truly a non-trivial fact that the above definition gives a well-defined, finite and non-zero quantity. That this is so, is a part of our main result.

1.2 The result

In [12] it is shown:

Theorem 1.3 *Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain, $U \subset \mathbb{R}^2 \setminus \overline{D}$ be bounded and convex domain and q be a microlocally isotropic Gaussian random field of order two in D. Then*

- **(i)** *For any $x, y \in U$ the measurement $m(x, y, \omega)$ is well-defined (that is, the limit in* (3) *exist almost surely).*

- **(ii)** *There exists a continuous deterministic function $m_0(x, y)$ such that for any $x, y \in U$ it holds $m(x, y, \omega) = m_0(x, y)$ almost surely. In particular, the function $m_0(x, x)$ is almost surely determined from the backscattering data $m(x, x, \omega)$.*

- **(iii)** *The backscattering data $n_0(x) := m_0(x, x)$, $x \in U$ uniquely determines the micro-correlation strength $\mu(z)$, $z \in D$. Moreover, there is a linear operator T such that*

$$T(n_0) = \mu \in C_0^\infty(D).$$

We stress that the above result allows us to determine the principal structure of the covariance from measurements from a single realization of the potential only! Property (ii) in Theorem 1.3 is sometimes called statistical stability, c.f. [5].

1.3 Examples of microlocally isotropic fields

Assume that the potential q is a localization of a generalized Gaussian field, that is, $q = \chi Q$, where $\chi \in C_0^\infty(D)$ and Q is a centered (i.e., $\mathbb{E}\, Q = 0$) generalized Gaussian field on $\mathbb{R}^2$. The reason for introducing the cutoff χ is to avoid the possible effects arising from discontinuity at the boundary ∂D.

To get a more concrete structure we will assume further that Q has additionally a Markov structure. Below we will repeat the definition of such fields. The basic properties of generalized Markov fields can be found from [17]. The definition of Markov fields mimics the situation where physical particles in a lattice have no long-distance interactions, i.e., only neighboring particles have direct interaction. Assume that $S_1 \subset D$ is an open set. We set $S_2 = D \setminus \overline{S}_1$ and $S_\epsilon = \{x \in D :\ d(x, \partial S_1) \le \varepsilon\}$, $\varepsilon > 0$, a collar neighborhood of the boundary ∂S_1 in S_1. Intuitively the Markov property means that the influence from the inside to the outside must pass through the collar.

Definition 1.4 *A generalized random field Q on $\mathbb{R}^2$ satisfies the Markov property if for any S_1, S_2 and S_ε as described above, and $\varepsilon > 0$ small enough, the conditional expectations satisfy*

$$\mathbb{E}\,(h \circ Q(\psi)|\mathcal{B}(S_\varepsilon)) = \mathbb{E}\,(h \circ Q(\psi)|\mathcal{B}(S_\varepsilon \cup S_1))$$

for any complex polynomial h and for any test function $\psi \in C_0^\infty(S_2)$.

Here $\mathcal{B}(S_j)$ is the σ-algebra generated by the random variables $Q(\phi)$, $\phi \in C_0^\infty(S_j)$, $j = 1, 2$, and $\mathcal{B}(S_\varepsilon)$ is defined respectively.

The Markov property has dramatic implications to the structure of the field Q and especially to its covariance operator C_Q. Under minor additional conditions (c.f. [17]), the inverse operator $(C_Q)^{-1}$ must be a local operator: it cannot increase the support of a test function. By a well-known theorem of J. Peetre [16] $(C_Q)^{-1}$ must be a linear partial differential operator. As C_Q is non-negative operator, $(C_Q)^{-1}$ has to be of even order. Let us assume that $(C_Q)^{-1}$ is a non-degenerate elliptic operator of 2nd order, has smooth coefficients, and its principal part is positive and rotationally symmetric (i.e. isotropic). This implies that

$$(C_Q)^{-1} = P(z, D_z) = -\sum_{j,k=1}^{2} \frac{\partial}{\partial z^j} a(z) \frac{\partial}{\partial z^k} + b(z),$$

where $a(z) > 0$ and $b(z)$ are smooth real functions in $\mathbb{R}^2$. Then the field Q is micro-locally isotropic of order two as C_Q is a pseudodifferential operator with an isotropic principal symbol. To motivate the assumption that the order of $(C_Q)^{-1}$ is two, let us consider the case where $(C_Q)^{-1}$ would be of fourth order or higher, with smooth coefficients. Then the realizations of q would lie in the Sobolev class $W^{s,p}_{comp}(\mathbb{R}^2)$ for all $s < 1$ and $1 < p < \infty$. In particular, they would be Hölder continuous. As our aim is to consider non-smooth potentials, the second order case is most interesting. An important example of such random fields is obtained by the free Gaussian fields, which appear in two dimension quantum field theory (c.f. [8]). This correspond to choices $a(z) = 1$, $b(z) = 0$. More complicated examples can be constructed easily.

Finally, the covariance operator C_q of the potential q has the kernel

$$k_q(z_1, z_2) = \chi(z_1) k_Q(z_1, z_2) \chi(z_2).$$

This implies that q is microlocally isotropic of order two in D and has the micro-correlation strength function $\mu(z) = \chi(z)^2\, a(z)^{-1}$.

2 Outline of the proof of Theorem 1.3

In section we sketch the proofs of the main results, often in quite heuristic manner. Complete and rigorous proofs are given in [12]. Because of notational simplicity, in the proof of Theorem 1.3 we will assume that $\mathbb{E}\, q = 0$. This assumption can easily be dispensed with.

2.1 The stochastic potential

By our assumption, the covariance operator of the potential has principal symbol $\mu(x)|\xi|^{-2}$ and thus it's Schwartz kernel has similar singularity as the inverse of the weighted Laplace operator. Hence, as we are working in the plane one can show that

$$C(x, y) = c_0(x, y) \log |x - y| + \text{ bounded terms.}$$

The blow up of the covariance in the diagonal has the slightly unwelcome consequence that the realizations of the potential almost surely are not functions or even measures. We are thus dealing with honest distributional potentials, which makes life harder, but also more interesting.

However, the potential q just barely fails to be a function. To see this, let us denote by J_δ ($\delta \in \mathbb{R}$) the standard Bessel potential operator defined by the Fourier transform as $J_\delta f := \mathcal{F}^{-1}((1+|\xi|^2)^{\delta/2}\widehat{f}(\xi))$. The covariance operator of the field $J_\delta q$ equals $J_\delta C J_\delta$, which turns out to have a uniformly bounded kernel for any $\delta > 0$. For such covariance operators it is known, as we are dealing with a Gaussian field, that the realizations lie in L^p_{loc} almost surely. This yields for q the following result.

Theorem 2.1 *Almost surely $q \in W^{-\epsilon,p}_{comp}(D)$ for all $\varepsilon > 0$ and $1 < p < \infty$.*

This fact is crucial for the success of the subsequent analysis of our problem.

2.2 Existence and uniqueness for solutions of the scattering problem

To study the stochastic scattering problem we immediately face the following problem: how to get the existence and properties of the solution for the Schrödinger equation with a distributional potential. It turns out that this can be accomplished for a deterministic non-smooth potential $q_0 \in W_0^{-\epsilon,p}(D)$, assuming that ε is small enough, and the obtained results are of independent interest. Here, $W^{s,p}(\mathbb{R}^n)$ is the Sobolev space of functions f with $(1-\Delta)^{s/2} f \in L^p$. The space $W^{s,p}(D)$ is the space of restrictions $f|_D$ of $f \in W^{s,p}(\mathbb{R}^n)$ and $W_0^{-\epsilon,p}(D)$ is the closure of $C_0^\infty(D)$ in $W^{s,p}(\mathbb{R}^n)$.

The direct scattering theory from a potential that is in weighted L^2 spaces is classical (c.f. [2],[4]). For the L^p scattering theory the key tool is the unique continuation of the solution. Jerison and Kenig showed in [10] that the strong unique continuation principle for L^p-potentials in $\mathbb{R}^n$ holds for $p \geq n/2$ and fails for $p < n/2$ in dimensions $n > 2$. In dimension two the unique continuation holds in a space of functions that is close to L^1 [10]. Below in Lemma 2.3 we state a positive result for negative index Sobolev spaces.

The (deterministic) scattering problem we are considering in this subsection is

$$\begin{cases} (\Delta - q_0 + k^2)u = \delta_y \\ \left(\frac{\partial}{\partial r} - ik\right) u(x) = o(|x|^{-1/2}) \end{cases} \tag{5}$$

where the potential $q_0 \in W_0^{-\epsilon,p'}(D)$, $p^{-1} + (p')^{-1} = 1$, $1 < p < 2$. We can show that the problem (5) is equivalent to the Lippmann-Schwinger equation

$$u(x) = u_0(x) - \int_{\mathbb{R}^2} \Phi_k(x-y) q_0(y) u(y) dy. \qquad (6)$$

Dealing with this equation, the first task is to understand the product $q_0 u$ in both of the above equations. For that end we seek u from the space $W^{2p,\epsilon}_{loc}(\mathbb{R}^2)$. This is taken care by the following result

Lemma 2.2 *Let $1 < p < \infty$ and $\epsilon = \varepsilon(p) > 0$ be small enough. Assume that $u \in W^{\epsilon,2p}_{loc}(\mathbb{R}^n)$ and $q_0 \in W_0^{-\epsilon,p'}(D)$. Then the product $q_0 u$ is well-defined as an element of $W_0^{-\epsilon,\widetilde{p}}(D)$, where $\widetilde{p} = \frac{2p}{2p-1}$ and*

$$||q_0 u||_{W_0^{-\epsilon,\widetilde{p}}(D)} \leq c||q_0||_{W_0^{-\epsilon,p'}(D)} ||u||_{W^{\epsilon,2p}(D)}. \qquad (7)$$

The proof of the lemma is essentially a duality argument combined with the known pointwise multiplication results for Sobolev spaces $W^{s,p}$, that in turn are proven by paraproduct techniques, see e.g. [18].

In order to show that (6) has a unique solution $u \in W^{\epsilon,2p}_{loc}(\mathbb{R}^2)$ we need some kind of the unique continuation principle of solutions. For completeness we formulate the corresponding lemma in dimension n. Roughly speaking, it says that if u is a compactly supported solution of the Schrödinger equation with $q_0 \in W_0^{-\epsilon,r}(D)$, $r > n/2$ and ϵ is small then u must vanish identically. It appears to the authors that this result could also be obtained as a special case of D. Tataru's and H. Koch's recent unique continuation results based on L^p Carleman estimates [11]. However, [12] contains a a direct and simple proof.

Lemma 2.3 (Unique continuation principle into an interior domain) *Assume that $p' \in (n/2, \infty)$, together with $0 < \epsilon < \frac{n}{4}(\frac{2}{n} - \frac{1}{p'})$. Let $q_0 \in W^{-\epsilon,p'}_{comp}(\mathbb{R}^n)$. If $u \in W^{\epsilon,2p}_{loc}(\mathbb{R}^n)$ is compactly supported and satisfies the Schrödinger equation*

$$(\Delta - q_0 + k^2)u = 0$$

in the weak sense, then $u = 0$ identically.

The proof of this result applies estimates for the Faddeev Green's function in negative index Sobolev spaces. The unique continuation principle enables one to modify slightly the classical proof (see e.g. [6, Chapter 8]) of uniqueness and existence of solutions to the Lippmann-Schwinger equation.

Theorem 2.4 *For $q_0 \in W^{-\epsilon,p'}_{comp}(\mathbb{R}^n)$, with $n \geq 2$, $p' \in (n/2, \infty)$, and $0 < \epsilon < \frac{n}{4}(\frac{2}{n} - \frac{1}{p'})$, the scattering problem (5) is equivalent to the Lippmann-Schwinger-equation* (6) *and has a unique solution $u \in W^{\epsilon,2p}_{loc}(\mathbb{R}^n)$.*

2.3 Ergodicity of the first term of the Born series

By iterating the Lippmann-Schwinger equation, one can formally represent the scattered field $u_s = u - u_0$ as the Born series,

$$u_s(x, y, k) = u_1(x, y, k) + u_2(x, y, k) + \dots \tag{8}$$

where $u_0(x, y, k) = \Phi_k(x - y)$ is the incident field and $u_{n+1} = (\Delta + k^2 + i0)^{-1}(qu_n)$. The operator $(\Delta + k^2 + i0)^{-1}$ has the kernel $\Phi_k(x - y)$. The main part of our work consists of analyzing the effect of different terms in this development.

In this subsection we analyze stochastic behaviour of the first term

$$u_1(x, y, k) = \int_D \Phi_k(x - z)q(z)\Phi_k(z - y)\, dz, \tag{9}$$

and calculate its contribution to the measurement. As rigorous considerations are quite technical, we give here a very heuristic explanation.

The first step is to replace the Hankel functions Φ_k by their leading order asymptotics for large argument values:

$$\Phi_k(x) \sim c\frac{e^{ik|x|}}{\sqrt{k|x|}}, \quad \text{as } k|x| \to \infty.$$

We show that substituting the leading order term to u_1 yields the leading order in (9). One should recall here that $x, y \in U$ and $z \in D$ where there is a positive distance between the domains D and U. The result reads as

$$u_1(x, y, k) \sim \frac{c}{k} \int_D \frac{e^{ik(|x-z|+|z-y|)}q(z)}{\sqrt{|z - y||z - x|}}\, dz \quad \text{as } k \to \infty. \tag{10}$$

Here the integration is understood in the sense of distributions. Moreover, the above asymptotics is true only in the *mean sense* as we are dealing with a stochastic quantity.

Recall that the measurement involves the limit of $\frac{1}{K-1}\int_1^K |u(x, y, k)|^2 k^2 dk$ as $K \to \infty$ and we want to show that this limit exists *almost surely*, not just in the mean sense. In order to understand the behaviour of this limit we need to understand better the stochastic process $k^2 u_1(x,y,k)$. We restrict ourselves to the case of backscattering, i.e., $x=y$. Thus for fixed $x \in U$

$$k^2 u_1(x,x,k) = cX(k) + o(1), \quad \text{as } k \to \infty$$

where

$$X(k) := k \int_D \frac{e^{2ik|x-z|} q(z)}{|x-z|}\, dz. \tag{11}$$

Denote by $\mathcal{F}$ the one dimensional Fourier transform. Then

$$X(k) = k \int_{\mathbb{R}} e^{2ikt} \widetilde{q}(t)\, dt = k\, \mathcal{F}\widetilde{q}(2k), \quad \text{where } \widetilde{q}(t) = \frac{1}{t} \int_{|z-x|=t} q(z) d|z|. \tag{12}$$

In order to understand just the basic stochastic nature of the process $\widetilde{q}$ we approximate circles $|z-x|=t$ by lines. This corresponds to the case where the center of the circle x is very far from D where q is supported. Thus, instead of $\widetilde{q}$ and $X(k)$ we consider integrals $\widetilde{Q}(t)$ over lines (omitting the factor $\frac{1}{t}$) and corresponding approximation $X_{\widetilde{Q}}(k)$,

$$\widetilde{Q}(t) = \int_{\mathbb{R}} q(t, z_2)\, dz_2, \quad X_{\widetilde{Q}}(k) = k \mathcal{F} \widetilde{Q}(2k).$$

To simplify these motivating calculations, we consider the case $D = [0,\pi]^2$ and

$$C_q(x,y) = \log(1/|x-y|) \quad \text{for } x, y \in [0,\pi]^2,$$

which again is a crude approximation: Keep in mind that in the final analysis we are, of course, interested in exact quantitative estimates and the dependence of $X(k)$ on μ and on the point x.

In the above approximation the covariance of $\widetilde{Q}$ is now obtained by integrating $C_q(x,y)$ in the variables x_2 and y_2. Let us write

$$C_{\widetilde{Q}}(t_1,t_2) = -\frac{1}{2} \int_0^\pi \int_0^\pi \log((t_1-t_2)^2 + (x_2-y_2)^2)\, dx_2 dy_2.$$

A computation shows that the $C_{\widetilde{Q}}(t_1,t_2)$ is again smooth outside the diagonal where the singularity has the form

$$C_{\widetilde{Q}}(t_1,t_2) = c_0 |t_1 - t_2| + \text{ smoother terms.}$$

But this is a reminiscent of the one dimensional Green's function of the Laplace operator, which has the same leading singularity. This observation gives rise to our next approximation: we shall assume for a while that $C_{\widetilde{Q}}(t_1,t_2)$ is the Dirichlet-Green's function of the operator $(d/dt)^2$ on the interval $[0,\pi]$. In this case the eigenfunctions of $C_{\widetilde{Q}}$ can be explicitly written down

$$C_{\widetilde{Q}}(t_1,t_2) = \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{1}{j^2}\sin(jt_1)\sin(jt_2). \tag{13}$$

One may now directly check that the representation

$$\widetilde{Q}(t) = \sqrt{\frac{2}{\pi}}\sum_{j=1}^{\infty}\frac{Y_j}{j}\sin(jt), \tag{14}$$

where the $Y_j \sim N(0,1)$ are independent normalized Gaussian random variables, leads to (13). As we are interested in the quantity $X_{\widetilde{Q}}(k) = k\mathcal{F}\widetilde{Q}(2k)$, one is tempted to guess that (at least for integer values of k) the major contribution to the Fourier transform comes from the k:th term, that is

$$X_{\widetilde{Q}}(k) = Y_k + \text{ terms with minor influence.}$$

Especially, we expect that the covariance of the Gaussian, zero-mean random variable $X(k)$ does not depend very much on k and that (as Y_k:s are independent), $X_{\widetilde{Q}}(k_1)$ and $X_{\widetilde{Q}}(k_2)$ should show up some amount of independence for large $|k_1-k_2|$. In more exact terms we prove that the random variables $X_{\widetilde{Q}}(k_1)$ and $X_{\widetilde{Q}}(k_2)$ are *asymptotically independent* as $|k_1-k_2|\to\infty$.

The previous discussion demonstrated what kind of behaviour one can expect for $X(k)$. In order to describe a somewhat more rigorous treatment of $X(k)$ it is advantageous to observe that the kernel of the covariance of the Fourier-transform is simply obtained by taking the two-dimensional Fourier-transform $\widehat{C}_{\widetilde{Q}}$ of the covariance $C_{\widetilde{Q}}(t_1,t_2)$ and observing that

$$\mathbb{E}\left(X_{\widetilde{Q}}(k_1)\overline{X_{\widetilde{Q}}(k_2)}\right) = k_1k_2\widehat{C}_{\widetilde{Q}}(2k_1,-2k_2) \tag{15}$$

Recall that in our approximation $C_{\widetilde{Q}}$ corresponds to a Green's function of the Laplace operator. If it were the fundamental solution in the whole line, the Fourier transform would satisfy $\widehat{C}_{\widetilde{Q}}(\xi_1,\xi_2)\sim|\xi_1-\xi_2|^{-2}\delta_0(\xi_1+\xi_2)$, with a suitable interpretation for the singularity. As we are actually dealing with compactly supported covariances, a better approximation is obtained by convolving the above with a smooth function so that

$$\widehat{C}_{\widetilde{Q}}(\xi_1,\xi_2) \sim \frac{1}{1+|\xi_1-\xi_2|^2}\psi(\xi_1+\xi_2),$$

where ψ is a rapidly decreasing Schwartz function. Hence

$$\mathbb{E}\,(X_{\widetilde{Q}}(k_1)\overline{X_{\widetilde{Q}}(k_2)}) \sim k_1k_2\frac{1}{1+|k_1+k_2|^2}\psi(k_1-k_2) \tag{16}$$

This approximative formula yields immediately that $\mathbb{E}\,|X_{\widetilde{Q}}(k)|^2 \sim c_1$ as $k\to\infty$, where $c_1\in(0,\infty)$ is a constant independent of k. Here we use the fact that for two random variables from the same Gaussian field the independence is directly measured by the mutual covariance. Since ψ is rapidly decreasing the formula (16) yields that $X_{\widetilde{Q}}(k_1)$ and $X_{\widetilde{Q}}(k_2)$ are asymptotically independent as $|k_1-k_2|\to\infty$.

In [12] we give rigorous proof for the above approximative and partly heuristic argument. Moreover, we show that at the limit $k\to\infty$ there exists a linear dependency between the covariance on x and the function μ. More exactly, we prove

Proposition 2.5 *For any $n\geq 1$, $|k_1|,|k_2|>1$, and $x,y\in U$ it holds that*

$$|\mathbb{E}\,(u_1(x,y,k_1)\overline{u_1(x,y,k_2)})| \leq C_n(1+|k_1|+|k_2|)^{-4}(1+|k_1-k_2|)^{-n}, \quad n>0 \tag{17}$$

for any $n\geq 1$, $|k_1|,|k_2|>1$, and $x,y\in U$. Moreover, for $x=y$ there is the asymptotics

$$\lim_{k\to\infty}\mathbb{E}\,k^4|u_1(x,x,k)|^2 = \frac{1}{2}\int_{\mathbb{R}^2}\frac{\mu(z)}{|z-x|^2}\,dz. \tag{18}$$

In the backscattering case $x=y$ the proof of the above proposition can be based on the above heuristic argument. Of course one cannot anymore approximate circles by lines, and the weight factor must be kept in the formulas, which makes the argument fairly technical. This approach has the advantage that one can somewhat dispense with the smoothness assumptions on the function μ. However, this approach can not easily be extended to the case $x\neq y$.

In [12] we adopted another approach which uses pseudodifferential calculus. Namely, after the legitimate approximation (10) one is lead to consider oscillatory integrals with certain type of phase functions:

$$\begin{aligned}&\mathbb{E}\left(u_1(x,y,k_1)\overline{u_1(x,y,k_2)}\right) \sim \\ &\frac{C}{k_1k_2}\int_{\mathbb{R}^2} \frac{\exp\big(ik_1\phi(z_1,x,y) - ik_2\phi(z_2,x,y)\big)\, C_q(z_1,z_2)}{|x-z_1|^{\frac{1}{2}}|z_1-y|^{\frac{1}{2}}|x-z_2|^{\frac{1}{2}}|z_2-y|^{\frac{1}{2}}}\, dz_1dz_2.\end{aligned}$$

Above $\phi(z,x,y) = |x-z| + |z-y|$. As C_q is the kernel of a pseudodifferential operator of order -2, it turns out that one may apply calculus of conormal distributions (see e.g. [9, Chapter 18.2]). Especially, in proper coordinates above integral can be viewed as Fourier transform of kernels of suitable pseudodifferential operators.

The above proposition is the key that enables us to determine exactly the contribution of the first Born term to the measurement. Recall from basic probability theory the famous law of large numbers: It states that for integrable, independent, and identically distributed random variables X_k, $k = 1, \ldots$, it holds that $\lim_{n\to\infty}(1/n)\sum_{k=1}^n X_k = \mathbb{E}\, X_1$ almost surely. In our case $X_k = k^2|u(x,x,k)|^2$, the summation is replaced by integration, and the random variables are only asymptotically independent. Still the ergodicity takes place and one obtains from the classical Cramer-Leadbetter theorem ([7]) the following result.

Theorem 2.6 *Let X_t, $t \geq 0$ be a real valued stochastic process with continuous paths. Assume that for some positive constants $E_0, c, \varepsilon > 0$ the condition*

$$|\mathbb{E}\,(X_t - E_0)(X_{t+r} - E_0)| \leq c(1+r)^{-\varepsilon},$$

holds for all $t, r \geq 0$. Then almost surely

$$\frac{1}{K}\int_1^K X_t dt \to E_0 \quad \text{as} \quad K \to \infty.$$

We obtain as an almost immediate corollary of Proposition 2.5 and Theorem 2.6 the following result

Proposition 2.7 *For any $x, y \in U$ the finite limit*

$$\lim_{K\to\infty} \frac{1}{K-1}\int_1^K k^4|u_1(x,y,k,\omega)|^2\, dk$$

exists almost surely, and is deterministic, i.e. it depends only on $x, y \in U$. Moreover, in the case $y = x$ we have almost surely that

$$\lim_{K\to\infty} \frac{1}{K-1} \int_1^K k^4 |u_1(x,x,k,\omega)|^2 \, dk = \frac{1}{2} \int_{\mathbb{R}^2} \frac{\mu(z)}{|z-x|^2} \, dz.$$

2.4 Higher order terms in the Born series

After Proposition 2.7 we need to show that only the first term finally contributes to the outcome of the measurement. Let us write

$$u_s = u - u_0 = u_1 + u_2 + u_R$$

so that u_R contain higher order terms in the Born series. For the remainder term we may apply the following deterministic result:

Proposition 2.8 *There are $\varepsilon_0 > 0$ and $p_0 > 1$ such that if $q_0 \in W_0^{p,s}(D)$ with $p > p_0$, $s > -\varepsilon_0$, then for $k \geq k_0 = k_0(q_0)$ the Born series* (8) *converges for any $x, y \in U$ to the solution $u(x, y, k)$. Moreover, for any $\delta > 0$ there is $C = C(q_0, \delta)$ such that*

$$\sup_{x,y\in U} |u_R(x,y,k)| \leq C k^{-5/2+\delta} \quad \textit{for } \ k \geq k_0.$$

The proof of this result applies the pointwise multiplier results for Sobolev spaces and norm estimates for the operator with kernel $\Phi_k(x-y)$, used already in Section 2.2. For scattering from our stochastic potential $q = q(x,\omega)$ we obtain by the above proposition for any $x, y \in U$ that

$$u_R(x,y,k,\omega) = o(k^{-2}) \quad \text{as } \ k \to \infty \tag{19}$$

almost surely. Thus the remainder term plays no role in the measurement.

Our final task is to consider the second term $u_2(x, y, k, \omega)$ of the Born series. Unfortunately one cannot apply the same approach as with the remainder term u_R or as with the first term u_1, and the estimates for u_j as $k \to \infty$ are most difficult in case of $j = 2$. Here we are dealing with the integral

$$u_2(x,y,k,\omega) = \int_D \int_D \Phi_k(x-z_1) q(z_1,\omega) \Phi_k(z_1-z_2) q(z_2,\omega) \Phi_k(z_2-y) \, dz_1 dz_2.$$

As a stochastic variable, u_2 is no more Gaussian since it is bilinear in $q(z_1, \omega)$ and $q(z_2, \omega)$. In fact, the random variable u_2 belongs to the union of the first and second order Wiener chaos decomposition (see e.g. [13]). To analyze (20), the domain $D \times D$ can be foliated to surfaces $\Gamma_t = \{(z_1, z_2) \in D \times D : A(z_1, z_2) = t\}$ where $A(z_1, z_2) = |z_1| + |z_1 - z_2| + |z_2 - y|$. After certain nontrivial approximations and a Fourier analysis argument one can reduce estimation of (20) to the problem of finding estimates for the covariances of the integrals

$$S(t) := \int_{\Gamma_t} q(z_1)q(z_2)(|z_1 - z_2||z_1||z_2 - y|)^{-1/2}|\nabla A(z_1, z_2)|^{-1}\, d\mathcal{H}^3(z_1, z_2)$$

where $d\mathcal{H}^3$ is the 3-Hausdorff measure. By this manner one obtains the following result:

Proposition 2.9 *For all $x, y \in U$ it holds almost surely that*

$$\lim_{K \to \infty} \frac{1}{K-1} \int_1^K |k^2 u_2(x, y, k, \omega)|^2 dk = 0.$$

2.5 Conclusion: a problem of integral geometry

Let us collect together results that we have obtained so far. A simple application of Cauchy-Schwartz inequality together with Propositions 2.7, 2.8, and 2.9 yields that almost surely the measurement is well-defined, and for any $x \in U$ the equality

$$\lim_{K \to \infty} \frac{1}{K-1} \int_1^K k^4 |u_s(x, x, k, \omega)|^2\, dk = n_0(x) := \frac{1}{2} \int_{\mathbb{R}^2} \frac{\mu(z)}{|z - x|^2}\, dz \qquad (20)$$

holds almost surely. This yields the first part of Theorem 1.3.

We still have to show that the integrals on the right hand side of (20) uniquely determine the coefficient $\mu(z)$. The function $n_0(x)$ is defined in the open set U that has a positive distance to the compact support of the function μ. By repeated differentiation of $n_0(x)$ with respect to x we see that $n_0(x)$ also determines the integrals of $\mu(z)$ against the powers $|x - z|^{-2n}$, $n \geq 1$. A density argument yields that we may thus determine integrals of μ over all circles having center in U. Hence we are reduced to a classical problem of integral geometry. This Radon type problem is well known to have a unique solution and hence μ is uniquely determined. This finishes the proof of Theorem 1.3. $\square$

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