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# Hyperbolic inverse problem with data on a part of the boundary

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ABSTRACT. We consider an inverse problem for a second order hyperbolic initial boundary value problem on a compact Riemannian manifold $M$ with boundary. Let $\Gamma \subset \partial M$ be an open set. Assume that we know the Cauchy data on $\Gamma \times \mathbb{R}_+$ of the solutions to the wave equations which vanish at $t = 0$ and have Dirichlet data supported on $\Gamma \times \mathbb{R}_+$. We show that under some geometric assumptions it is possible to determine manifold $M$ and the wave operator to within the group of the generalized gauge transformations.

## 1. Introduction and main ideas.

In the paper we study an inverse problem for the hyperbolic initial boundary value problem

$$\begin{aligned} u_{tt} + bu_t + a(x, D)u &= 0 \text{ in } M \times \mathbb{R}_+, \\ u|_{\partial M \times \mathbb{R}_+} &= f; \quad u|_{t=0} = u_t|_{t=0} = 0 \end{aligned} \tag{1.1}$$

on a compact connected $C^\infty$-Riemannian manifold $M, \dim M = m \geq 1$, with metric $g = (g^{jl})_{j,l=1}^m$ and non-empty boundary $\partial M$. The operator $a(x, D)$ is a first order perturbation of the Laplace-Beltrami operator $-\Delta_g$,

$$a(x, D) = -\Delta_g + P + q. \tag{1.2}$$

Here $b$ is a smooth complex-valued function on $M$, $P$ is a complex-values smooth vector field, $P = p^l \partial_l$ and $q$ is a smooth complex-valued function on $M$.

The symbol $a(x, D)$ is, in general, not formally symmetric. We note that in local coordinates $a(x, D)$ can be also written in another familiar form

$$a(x, D) = -g^{-1/2}(\partial_j + \tilde{P}_j)g^{1/2}g^{jk}(\partial_k + \tilde{P}_k) + \tilde{q} \tag{1.3}$$

where $g = \det(g_{ij})$, the covector field $\tilde{P}_j$ and scalar function $\tilde{q}$ represent the magnetic potential and the potential, correspondingly. It can easily be seen that by an appropriate choice of the metric $g$ any second-order elliptic partial differential operator having real valued coefficients in principal part can be written in form (1.2) or (1.3).

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We denote by $H^s(A)$ the Sobolev space of functions on $A$ and by $H^s_0(B)$, $B \subset A$ we denote the space of functions $u \in H^s(A)$ for which supp $(u) \subset B$. For $x \in \partial M$, $\nu = \nu_x$ stands for the normal unit vector to $\partial M$ in metric $g$ at point $x$. By [**12**] the following mapping is well defined:

DEFINITION 1.1. We define the response operator

$$R : H^1_0(\Gamma \times \mathbb{R}_+) \to L^2_{\text{loc}}(\overline{\Gamma \times \mathbb{R}_+})$$

by

$$R(f) = \partial_\nu u^f|_{\partial M \times \mathbb{R}_+},$$

where $u^f$ is the solution of the problem (1.1).

We will consider $\Gamma$ as a Riemannian manifold $(\Gamma, g_\Gamma)$ where $g_\Gamma$ is the metric inherited from $(M, g)$. In this paper we consider the following question:

**Problem 1** Do $(\Gamma, g_\Gamma)$ and $R$ determine $(M, a(x, D), b)$ uniquely?

In the following we call the triple $\{\Gamma, g_\Gamma, R\}$ the dynamical boundary data and abbreviate it by DBD.

In the paper we give a solution to this problem assuming that, in the general case, the Riemannian manifold $(M, g)$ satisfies some geometric conditions.

Our method of the reconstruction of the unknown Riemannian manifold is based on two main ideas. The first idea is the use of the Holmgren-John unique continuation theorem to obtain controllability results. The second idea is the use of boundary quasinorms. The existence of such quasinorms is, in general, not known. In the paper, we will present two important cases when such quasinorms exists. First we define these quasinorms.

DEFINITION 1.2. A function $(f, t) \mapsto Q_t(f)$ is a boundary quasinorm, if there are functions $c_1(t)$ and $c_2(t)$ and and time $t_e > 0$ such that for any $f \in H^1_0(\Gamma \times \mathbb{R}_+)$ and $t > t_e$

$$c_1(t)Q_t(f) \le ||u^f(\cdot, t)||^2_{H^1(M)} + ||u^f_t(\cdot, t)||^2_{L^2(M)} \le c_2(t)Q_t(f). \tag{1.4}$$

Before proving the existence of boundary quasinorms for some special cases we formulate a geometric condition (for details see [**1**]) which generalizes the condition that the geodesics on $(M, g)$ and their reflections from the boundary hit $\Gamma$ transversally during time $[0, t_*]$.

DEFINITION 1.3. $(M, g, \Gamma)$ satisfies the Bardos-Lebeau-Rauch condition with time $t_*$ if there is $t_* > 0$ and an open conic neighborhood $\mathcal{O}$ of the set of the not-nondiffractive points $(x, t, \xi, \omega) \in T^*(M \times [0, t_*])$, $x \in \Gamma$ such that any generalized bicharacteristic of the wave operator $\partial_t^2 - \Delta_g$ passes through a point of $(x, t, \xi, \omega) \in T^*(M \times [0, t_*]) \setminus \mathcal{O}$, $x \in \partial M$.

LEMMA 1.4. *Let the system (1.1) satisfy one of the following conditions:*

i. *System (1.1) is selfadjoint. This means that $b = 0$ and that $a(x, D)$ is selfadjoint operator.*
ii. *The triple $(M, g, \Gamma)$ satisfies Bardos-Lebeau-Rauch condition at time $t_*$.*

*Then this system possess a boundary quasinorm $Q_t$ with $t_e = 0$ in the case i. and $t_e = t_*$ in the case ii.*

In the following we call a system (1.1) satisfying condition i. a self-adjoint system. Particularly, in this case $\tilde{P}_j$ is purely imaginary and $\tilde{q}$ is real.

Before stating our main results we would like to make some general comments. It is well-known that a boundary inverse problem for an operator (1.1) does not have a unique solution. Indeed, by replacing $a(x,D)$ by $a_\kappa(x,D)$

$$a_\kappa(x,D) = \kappa a(x,D)\kappa^{-1}, \tag{1.5}$$

where $\kappa|_\Gamma = 1$, $\kappa \neq 0$ on $M$ we do not change $R$. Thus the best we can hope to recover is the equivalence class of $a(x,D)$ with respect to the generalized gauge transformations, namely the set

$$[a(x,D)] := \{\kappa a(x,D)\kappa^{-1} : \ \kappa \in C^\infty(M;\mathbb{C}),\ \kappa|_\Gamma = 1,\ \kappa \neq 0 \text{ on } M\}.$$

This set forms an orbit of the group $G$ of the generalized gauge transformations which acts on the space of elliptic operator s on $M$. The above observations lead to the following reformulation of Problem 1:

**Problem 1'** Do $(\Gamma, g_\Gamma)$ and $R$ determine $(M,g)$ and the equivalence class $[a(x,D)]$ uniquely?

The above hyperbolic inverse problem and its analogs were considered in several papers. The case of the Euclidean metric $g^{jl} = \delta^{jl}$ and $M = \mathbb{R}^m$ was considered in [**16**]. The corresponding inverse boundary spectral problem in a domain $M \subset \mathbb{R}^m$ and $\Gamma = \partial M$ was studied in [**13**]. An analogous problem with data measured on a part of the boundary and $g^{ij} = \delta^{ij}$ was considered in [**6**].

In [**15**] the uniqueness of the reconstruction of $a(x,D)$ with conformally Euclidean metric and lower order terms (with some restrictions upon these terms) was proven for geodesically regular domains $M \subset \mathbb{R}^m$.

In the anisotropic case the main results were obtained for the spectral analog of Problem 1'. The inverse boundary spectral problem for a self-adjoint $a(x,D)$ was considered in [**7**], [**8**], and for the non-self-adjoint $a(x,D)$ with, however, $b=0$, in [**9**]. A particular case of the hyperbolic inverse problem with $b=0$, $a(x,D) = -\Delta_g$ and $\Gamma = \partial M$ was considered in [**3**]. The present work is based on paper [**10**] of the authors where an analogous problem was studied for the Gel'fand inverse boundary spectral problem and the paper [**11**] where the data was given on the whole boundary.

The main tool in this paper is the BC-method (see e.g. [**2**], [**4**] and [**7**]-[**11**] for its generalizations for systems of form (1.1), (1.2)). Particularly, we use here the geometrical formulation of the Boundary Control method [**8**] and exact controllability results [**1**].

The main result of the paper is the following:

THEOREM 1.5. *Assume that there is a boundary quasinorm $Q_t$ for equation (1.1). Let $(\Gamma, g_\Gamma)$ and the operator $R$ be given. Then these data determine $(M,g)$, $b$, and the equivalence class $[a(x,D)]$ uniquely.*

Combining this theorem with Lemma 1.4 we obtain

THEOREM 1.6. *Assume that either the system (1.1) is selfadjoint or the Bardos-Lebeau-Rauch condition is valid. Then $(\Gamma, g_\Gamma)$ and the operator $R$ determine $(M,g)$, $b$, and the equivalence class $[a(x,D)]$ uniquely.*

At the end of this section, we explain what we mean by the reconstruction of a Riemannian manifold $(M,g)$. Since a manifold is an 'abstract' collection of coordinate patches we construct a representative of an equivalence class of the isometric Riemannian manifolds or a metric space $D$ which is isometric to $(M, g)$. After constructing $D$ one can take any coordinate patch and construct the vector field $P$ and the potential $q$ in local coordinates.

## 2. Bilinear products.

Let us define an operator $R^T$ which corresponds to the measurements on a finite time interval $[0, T]$. Precisely, let $R^T : H_0^1(\Gamma \times [0, T]) \to L^2(\Gamma \times [0, T])$ be the operator

$$R^T(f) = \partial_\nu u^f|_{\partial M \times [0,T]},$$

where $u^f$ is the solution of the problem (1.1).

We also consider the adjoint system for (1.1). Let $v^g(x, t)$ be the solution of the adjoint initial-boundary value problem,

$$\begin{aligned} v_{tt}^g + \bar{b} v_t^g + a^*(x, D) v^g &= 0 \text{ in } M \times \mathbb{R}_+, \\ v^g|_{\partial M \times \mathbb{R}_+} = g; \quad v^g|_{t=0} &= v_t^g|_{t=0} = 0 \end{aligned} \tag{2.1}$$

where $a^*(x, D)$ is the formal adjoint of $a(x, D)$. We denote

$$U^f(t) = \begin{pmatrix} u^f(x,t) \\ u_t^f(x,t) \end{pmatrix}, \ V^g(t) = \begin{pmatrix} v^g(x,t) \\ v_t^g(x,t) \end{pmatrix}, \ J \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} u^2 + bu^1 \\ u^1 \end{pmatrix}$$

and the inner product in $L^2(M)^2$ by $(\cdot\,,\cdot)_{L^2}$. The corresponding distribution duality is denoted similarly.

For the adjoint equation we define the response operator $R_* : H_0^1(\Gamma \times \mathbb{R}_+) \to L^2_{\text{loc}}(\overline{\Gamma \times \mathbb{R}_+})$,

$$R_*(g) = \partial_\nu v + \overline{P_\nu} v|_{\Gamma \times \mathbb{R}_+},$$

where $P_\nu$ is the normal part of the vector field $P$.

LEMMA 2.1. *The response operator $R_*$ is determined by $R$.*

PROOF. It is enough to show that the response operator $R^T$ on a finite time interval determines the corresponding operator $R_*^T$ for the adjoint operator.

Let $(Y^t) f(s) = f(t - s)$ be the reflection in time and $e$ be the solution of the backward wave equation

$$\begin{aligned} e_{tt} - \bar{b} e + a^*(x, D) e &= 0 \text{ in } M \times [0, T], \\ e|_{\partial M \times [0,T]} &= h; \quad e|_{t=T} = e_t|_{t=T} = 0. \end{aligned}$$

Part integration together with initial and final conditions yield that

$$\begin{aligned} 0 &= \int_0^T \int_M \left( (u_{tt}^f + b u_t^f + a(x, D) u^f) \bar{e} - u^f (\overline{e_{tt} - \bar{b} e_t + a^*(x, D) e}) \right) dm_g dt \\ &= \int_0^T \int_{\partial M} \left( B u^f \, \bar{e} - u^f \, \overline{B^* e} \right) dS_x dt = \int_0^{t_0} \int_{\partial M} \left( R^T f \bar{h} - f \, \overline{B^* e} \right) dS_x dt \end{aligned}$$

where we use the notation $B^* e = \partial_\nu e + \overline{P_\nu} e|_\Gamma$ and $m_g$ is the measure on $(M, g)$. Since $f$ is arbitrary and $R^T f$ is known we can determine $B^* e|_{\partial M \times [0,T]} = R^T h$ for arbitrary $h \in H_0^1(\partial M \times [0, T])$ which determines $R_*^T = Y^T (R^T)^* Y^T$. □

Next we introduce a bilinear product of waves $U^f(t)$ and $V^g(s)$,

$$(J\,U^f(t), V^g(s))_{L^2} = \int_M [(u_t^f + bu^f)(x,t)\overline{v^g(x,s)} + u^f(x,t)\overline{v_s^g(x,s)}]dm_g(x).$$

The importance of this product is based upon the following lemma.

LEMMA 2.2. *Let $R$ be known. Then it is possible to evaluate the bilinear product $(J\,U^f(t), V^g(s))$ for any $f, g \in H_0^1(\Gamma \times \mathbb{R}_+)$ and $t, s \geq 0$.*

PROOF. By part integration

$$\begin{aligned} &(\partial_t - \partial_s)\big(JU^f(t), V^g(s)\big) \\ &= \int_M [(u_{tt}^f(t) + bu_t^f(t))\,\overline{v^g(s)} - u^f(t)\overline{(v_{ss}^g(s) + \bar{b}v_s^g(s))}]\,dm_g(x) \\ &= \int_{\partial M} [R^T f(t)\,\overline{g(s)} - f(t)\overline{R_*^T g(s)}]\,dS_x. \end{aligned} \tag{2.2}$$

As $R$ and $R_*$ are known, all the functions in the last integral are known. Hence (2.2) is a differential equation along the characteristic $t + s =$ constant. Furthermore,

$$\big(JU^f(0), V^g(s)\big)_{L^2} = \big(JU^f(t), V^g(0)\big)_{L^2} = 0$$

due to initial conditions (1.1), (2.1).

Equations (2.2) together with the above initial condition indicates the possibility to find the bilinear product $(J\,U^f(t), V^g(s))_{L^2}$. □

We apply the previous lemma particularly for $t = s$.

## 3. Controllability results.

As it was mentioned in Introduction, the reconstruction of the manifold is based upon two main ideas, the controllability results and the possibility to find the bilinear products of the waves via dynamical boundary data. In its turn, controllability takes place for sufficiently large times determined by the critical time $\tau$,

$$\tau = \max\,(t_e, 2\max d(x,\Gamma)), \tag{3.1}$$

where $d$ is the distance function on $(M, g)$. Precisely, the following the controllability result is valid.

THEOREM 3.1. *Let $t_0$, $t_1$, and $T$ satisfy the condition*

$$t_0 > 0, \quad t_1 > t_0 + \tau, \; T > t_1. \tag{3.2}$$

*Then the set*

$$\{U^f(T): \; f \in H_0^1(\Gamma \times [t_0, t_1])\}$$

*is dense in $H_0^1(M) \times L^2(M)$.*

PROOF. Since $t_1 - t_0 > \tau$, the assertion follows from Tataru's Holmgren-John theorem in the same way as in [**11**]. □

Next we prove Lemma 1.4.

PROOF. In the case i. the proof given below is analogous to the proof given in [**11**], Theorem 1.1. Note that in the case i. the covector field $\tilde{P}_j$ is purely imaginary.

It is enough to consider smooth $f$:s. Let $H$ be the energy function,

$$\begin{aligned} H(u^f,t) &= \int_M [g^{jk}(\partial_j u^f(t) + \tilde{P}_j u^f(t))\overline{(\partial_k u^f(t) + \tilde{P}_k u^f(t))} \\ &\quad + u^f_t(t)\overline{u^f_t(t)} + qu^f(t)\overline{u^f(t)}]m_g(x). \end{aligned}$$

Integrating by parts as in [**11**] we see that

$$\partial_t H(u^f,t) = \int_{\partial M} [f_t(t)\overline{R_* f(t)} + Rf(t)\overline{f_t(t)}]dS_x.$$

Since $H(u^f,0)=0$ the knowledge of $R$ and $R_*$ makes possible to find $H(u^f,t)$ for any $t>0$. Consider now $N(u^f,t,s), t,s>0$,

$$N(u^f,t,s) = \int_M u^f(t)\overline{u^f(s)}dm_g(x).$$

Then using equation and boundary conditions (1.1) we have that

$$N_{tt} - N_{ss} = \int_{\partial M} [Rf(t)\overline{f(s)} - f(t)\overline{R_* f(s)}]dS_x. \tag{3.3}$$

Since $N|_{t=0} = N_t|_{t=0} = N|_{s=0} = N_s|_{s=0} = 0$ due to the initial conditions (1.1) the knowledge of $R$ and $R_*$ makes possible to find $N(u^f,t,s)$ for any $t,s>0$. In particular, taking $t=s$ we see that $R$ determines $||u^f(t)||_{L^2(M)}$ for any $t$. Since the quasinorm $Q_t(f)$,

$$Q_t(f) = H(u^f,t) + CN(u^f,t,t),$$

where $C>0$ is sufficiently large, satisfies relations (1.4), Lemma 1.4 for the case i. is proven with $t_e=0$.

For the non-selfadjoint case ii. we first find the set $I_\Gamma$ of the boundary sources $f \in H^1_0(\Gamma\times[0,t])$ for which $U^f(t)=0$, $t>t_*$. Indeed, by Theorem 3.1 and its analog for $V^g(T)$ such sources $f$ are determined by the condition

$$(JU^f(t), V^h(t))_{L^2} = 0$$

for all $h \in H^1_0(\Gamma\times[0,t])$. Now by [**1**] the Bardos-Lebeau-Rauch condition yields that the mapping $f \mapsto U^f(T)$ is an isomorphism from $H^1_0(\Gamma\times[0,t])/I_\Gamma$ onto $H^1_0(M)\times L^2(M)$. Thus we can define

$$Q_T(f) = \inf_{h\in I_\Gamma} ||f-h||_{H^1_0(\Gamma\times[0,T])}.$$

□

## 4. Constructions in collar neighborhood of Γ.

In order to reconstruct a collar neighborhood of $\Gamma$ we start with a proper set of admissible sequences of boundary sources. In the following, let $T > t_1+\tau$. Let

$$\mathcal{F} = \{(f_j)_{j=1}^\infty: \ f_j \in C^\infty_0(\Gamma\times[t_0,t_1]),\ \lim_{i,j\to\infty} E(f_i - f_j, T) = 0\}.$$

We denote the above sequences by $\underline{f} = (f_j)_{j=1}^\infty$. Clearly the mapping $U: f \mapsto U^f(T)$ transforms the sequences $(f_j) \in F$ to Cauchy sequences $(U^{f_j}(T))$ in the space $H^1_0(M)\times L^2(M)$. Hence we define

$$U^{\underline{f}}(T) = \lim_{j\to\infty} U^{f_j}(T)$$

where the limit is consider in $H_0^1(M) \times L^2(M)$. We equip $\mathcal{F}$ with a quasinorm

$$||\underline{f}||^2_{\mathcal{F}} = \lim_{j\to\infty} E(f_j, T).$$

When we identify all $\underline{f}$ and $\underline{h}$ for which $||\underline{f} - \underline{h}||_{\mathcal{F}} = 0$, the mapping

$$\mathcal{U} : \mathcal{F} \to H_0^1(M) \times L^2(M) \tag{4.1}$$

becomes an isomorphism. We denote by $u^{\underline{f}}(x,t)$, $t > T$, the solution of the initial-boundary

$$\begin{aligned} u_{tt} + bu_t + a(x, D)u &= 0 \text{ in } M\times]T, \infty[, \\ u|_{\partial M\times]T,\infty[} = 0; \quad u|_{t=T} &= u^{\underline{f}}(T), \quad u_t|_{t=T} = u_t^{\underline{f}}(T). \end{aligned} \tag{4.2}$$

For $\underline{f}, \underline{h} \in \mathcal{F}$ we define a pairing

$$\langle \underline{f}, \underline{h} \rangle = (J\, U^{\underline{f}}(T), V^{\underline{h}}(T))_{L^2(M)\times L^2(M)}.$$

As we have observed before, we can compute these pairings by using DBD.

For open $\omega \subset \Gamma$ and $t \geq 0$ we introduce

$$M(\omega, t) = \{x \in M : d(x, \omega) \leq t\}$$

and

$$\mathcal{F}(\omega, t) = \{\underline{f} \in \mathcal{F} : \text{ supp } (U^{\underline{f}}(T)) \subset M(\omega, t)\},$$

$$\mathcal{F}^c(\omega, t) = \{\underline{f} \in \mathcal{F} : \text{ supp } (U^{\underline{f}}(T)) \subset \text{cl}(M \setminus M(\omega, t))\}$$

and $\mathcal{F}_{\text{ad}}(\omega, t), \mathcal{F}^c_{\text{ad}}(\omega, t)$ be the analogous sets for the adjoint equation.

Our next goal is to find these sets by using DBD.

LEMMA 4.1. *Let $\underline{f} \in \mathcal{F}$. Then for any open $\omega \subset \Gamma, t > 0$ DBD determine whether*

$$m_g(supp\ (U^{\underline{f}}(T)) \cap M(\omega, t)) = 0$$

*or not. Analogous statement takes place for the adjoint solutions $V^g(T)$.*

PROOF. Obviously we can assume $t < \tau$. We note that by the definition of $t_1$ and $\mathcal{F}$, for $\underline{f} = (f_j)$ we have $f_j(x, t) = 0$ for $t > t_1$ yielding that $u^{\underline{f}}|_{\partial M\times[T-\tau,T+\tau]} = 0$. If

$$m_g(\text{supp } (U^{\underline{f}}(T)) \cap M(\omega, t)) = 0 \tag{4.3}$$

then the finite velocity of the wave propagation implies that

$$\partial_\nu u^{\underline{f}}|_{\omega\times[T-t,T+t]} = 0 \text{ and } u^{\underline{f}}|_{\omega\times[T-t,T+t]} = 0. \tag{4.4}$$

On the other hand, by Tataru's Holmgren-John theorem [**17**] equation (4.4) implies relation (4.3). The statement of Lemma for the wave $U^{\underline{f}}$ now follows from the fact that DBD determines

$$\partial_\nu u_{\underline{f}}|_{\partial M\times[t_1,\infty[} = \lim_{j\to\infty} Rf_j|_{[t_1,\infty]}.$$

The claim for the adjoint solutions is obtained analogously. □

LEMMA 4.2. *Let $\omega \subset \Gamma$ and $t \geq 0$. Then DBD determine $\mathcal{F}(\omega, t)$, $\mathcal{F}^c(\omega, t)$ and $\mathcal{F}_{ad}(\Gamma, t)$, $\mathcal{F}^c_{ad}(\Gamma, t)$.*

PROOF. By Lemma 4.1 DBD determine $\mathcal{F}^c(\omega,t)$ and $\mathcal{F}^c_{\rm ad}(\omega,t)$. For $\underline{f}\in\mathcal{F}$ we have $\underline{f}\in\mathcal{F}(\omega,t)$ if and only if

$$(J\,U^{\underline{f}}(T),V^{\underline{h}}(T))=0$$

for all $\underline{h}\in\mathcal{F}^c_{\rm ad}(\omega,t)$. Hence by Lemma 2.2 we can determine $\mathcal{F}(\omega,t)$. The space $\mathcal{F}_{\rm ad}(\omega,t)$ can be constructed analogously. □

Let $y\in\Gamma$, $\varepsilon>0$. Denote by $\omega_y(\varepsilon)$ an open subset of $\Gamma$,

$$\omega_y(\varepsilon)=B(y,\varepsilon)\cap\Gamma,$$

where $B(y,\varepsilon)$ is the ball on $\partial M$. We define the compact sets

$$X(y,t,\varepsilon)=\mathrm{cl}\,(M(\omega_y(\varepsilon),t+\varepsilon)\backslash M(\Gamma,t-\varepsilon))$$

where $\varepsilon>0$ and

$$X(y,t)=\bigcap_{\varepsilon>0}X(y,t,\varepsilon).$$

In the next lemma we present some generalizations of the considerations in [**3**].

LEMMA 4.3. *The set $X(y,t)$ is either empty or contains only one point $x=\exp_y(t\nu_y)$. The later is true if and only if the normal geodesic from $y$ is a minimal geodesic from $x$ to $\Gamma$. Moreover it is possible to determine using DBD where $X(y,t)$ is empty or not.*

PROOF. Consider point $x$ in the intersection of all sets $X(y,t,\varepsilon)$, $\varepsilon>0$. Clearly $d(x,y)=t$. Moreover, the shortest geodesic from $y$ to $x$ has to be normal, i.e. $x=\exp_y(t\nu_y)$. Indeed, otherwise we see by short-cut arguments that $x\in M(\Gamma,t-\varepsilon)$ for sufficiently small $\varepsilon$. Thus $X(y,t)$ can contain not more then one point.

Since $X(y,t,\varepsilon)$ are compact their intersection is non-empty if and only if the intersection of any finite number of these sets is non-empty. However, $X(y,t,\varepsilon_1)\subset X(y,t,\varepsilon_2)$ when $\varepsilon_1>\varepsilon_2$. Hence, $X(y,t)\neq\emptyset$ if and only if $X(y,t,\varepsilon)\neq\emptyset$ for any $\varepsilon>0$. Clearly when the normal geodesic from $y$ to $x$ is shortest, i.e. $d(x,y)=d(x,\Gamma)$ then

$$K(x_\varepsilon,\varepsilon/2)\subset X(y,t,\varepsilon),$$

where $x_\varepsilon=\exp_y((t-\varepsilon/2)\nu_y)$ and $K(x,r)$ denotes a ball in $M$ with center in $x$ and radius $r$. On the other hand, when $t>d(\exp_y(t\nu_y),\Gamma)$ then $X(y,t,\varepsilon)=\emptyset$ for sufficiently small $\varepsilon$.

Therefore we can distinguish between the cases $X(y,t,\varepsilon)=\emptyset$ and $X(y,t,\varepsilon)\neq\emptyset$ by looking at

$$\mathcal{F}(X(y,t,\varepsilon))=\mathcal{F}(\omega_y(\varepsilon),t+\varepsilon)\cap\mathcal{F}^c(\Gamma,t-\varepsilon).$$

Indeed, $X(y,t,\varepsilon)=\emptyset$ if and only if $\mathcal{F}(X(y,t,\varepsilon))\neq\{0\}$. Hence the claim follows from Lemma 4.2. □

DEFINITION 4.4. Let $L_\Gamma$ be the set of those points $x\in M$ for which there is a unique shortest geodesic $\gamma$ to $\Gamma$ which is normal to $\Gamma$ and let $M_\Gamma$ be the interior of $L_\Gamma$. We denote by $y=y(x)$ the point where $\gamma$ intersects $\Gamma$ and by $s(x)$ the length of this geodesic. By $S_\Gamma$ we denote the set of the points $(y(x),s(x))\in\Gamma\times\mathbb{R}_+$, $x\in M_\Gamma$.

We note that the normal geodesic mapping

$$E:(y,t)\mapsto\exp_y(t\nu_y)$$

is $C^\infty$-diffeomorphism from $S_\Gamma$ to $M_\Gamma$.

Let $(y,s)\in\Gamma\times\mathbb{R}_+$. Then $(y,s)\in S_\Gamma$ if and only if there is $t>s$ such that $X(y,t)\neq\emptyset$. Hence Lemma 4.3 implies the following result.

THEOREM 4.5. *DBD determine the set* $S_\Gamma\subset\Gamma\times\mathbb{R}_+$.

## 5. Reconstruction of the metric and the operator.

Let $n\geq 0$ be an integer, $(y,t)\in S_\Gamma$. Let $\varepsilon\in]0,t[$ and denote by $\mathcal{F}_n(X(y,t,\varepsilon))$ those $\underline{f}\in\mathcal{F}$ for which

$$U^{\underline{f}}(T)\in H_0^{n+1}(X(y,t,\varepsilon))\times H_0^n(X(y,t,\varepsilon)).$$

THEOREM 5.1. *For integer $n\geq 0$ and $t>0$, $\varepsilon\in]0,t[$ the DBD determine the sets $\mathcal{F}_n(X(y,t,\varepsilon))$.*

PROOF. By Lemma 4.2, we can find $\mathcal{F}_0(X(y,t,\varepsilon))$.
Assume now that the assertion is true for a given $n$. Let

$$\underline{f}=(f_j)\in\mathcal{F}_n(X(y,t,\varepsilon))$$

yielding that $\operatorname{supp}(U^{\underline{f}}(T))\subset X(y,t,\varepsilon)$ and define a translation operator $H_\rho f(x,t)=f(x,t+\rho),\ \rho>0$. Moreover, let $\underline{f}=(f_j)\in\mathcal{F}$ and $t>T$. Then for any $\underline{h}\in\mathcal{F}_{\text{ad}}$

$$\begin{aligned}
&\lim_{j\to\infty}\partial_t\langle H_{t-T}f_j,\underline{h}\rangle|_{t=T}\\
=\;&\lim_{j\to\infty}\partial_t\int_M[(u_t^{f_j}+bu^{f_j})(x,t)\overline{v^{\underline{h}}(x,T)}+u^{f_j}(x,t)\overline{v_t^{\underline{h}}(x,T)}]dx|_{t=T}\\
=\;&\lim_{j\to\infty}\int_M[-a(x,D)u^{f_j}(x,T)\overline{v^{\underline{h}}(x,T)}+u_t^{f_j}(x,T)\overline{v_t^{\underline{h}}(x,T)}]dx.
\end{aligned}$$

Since $V^{\underline{h}}(T)\in H_0^1(M)\times L^2(M)$ and $U^{f_j}(T)\to U^{\underline{f}}(T)$ in this space we see that

$$\begin{aligned}
&\lim_{j\to\infty}\int_M[-a(x,D)u^{f_j}(x,T)\overline{v^{\underline{h}}(x,T)}+u_t^{f_j}(x,T)\overline{v_t^{\underline{h}}(x,T)}]dx\\
=\;&\int_M[-a(x,D)u^{\underline{f}}(x,T)\overline{v^{\underline{h}}(x,T)}+u_t^{\underline{f}}(x,T)\overline{v_t^{\underline{h}}(x,T)}]dx.
\end{aligned}$$

Since the mapping (4.1) is isomorphism and $a(x,D)$ is elliptic, we see that $\underline{f}\in\mathcal{F}_{n+1}$ if and only if there is $\underline{p}\in\mathcal{F}_n$ such that

$$(u_t^{\underline{f}}(T),-a(x,D)u^{\underline{f}}(T))=(u^{\underline{p}}(T),u_t^{\underline{p}}(T)).$$

Thus, we have $\underline{f}\in\mathcal{F}_{n+1}((X(y,t,\varepsilon))$ if and only if there is $\underline{p}\in\mathcal{F}_n$ such that for all $\underline{h}\in\mathcal{F}$

$$\lim_{j\to\infty}\partial_t\langle H_{t-T}f_j,\underline{h}\rangle|_{t=T}=\langle\underline{p},\underline{h}\rangle.$$

Hence we can find the set $\mathcal{F}_{n+1}$ and the assertion follows by induction. □

Next the construct sequences converging to delta-distributions.

Let $(y,t)\in S_\Gamma$ and let $x=\exp_y(t\nu_y)$ be the unique point in $X(y,t)$. Also, let $n$ be the integer for which $n<m/2\leq n+1$ and $\varepsilon_0\in]0,t[$. Consider a family of functions $g(\varepsilon)\in\mathcal{F},\ \varepsilon>0$ such that

i. supp $(V^{g(\varepsilon)}(T))\subset X(y,t,\varepsilon)$.

ii. For any $\underline{f}\in\mathcal{F}_n(X(y,t,\varepsilon_0))$ there exists a limit

$$\mathcal{W}^x(\underline{f})=\lim_{\varepsilon\to 0}\langle\underline{f},g(\varepsilon)\rangle.$$

We note that such families exists. Indeed, it is sufficient to take $V^{g(\varepsilon)}(T)$ to be $C_0^\infty(X(y,t,\varepsilon))^2$ -approximations to $(0,\delta(\cdot - x_0))$. Moreover, for a given sequence $g(\varepsilon)$ we can verify using Theorem 5.1 and Lemma 2.2, if conditions i. and ii. are satisfied. Thus we can construct a sequence $g(\varepsilon)$ with properties i. and ii.

Let us study these sequences more closely. First, since mapping (4.1) is isomorphism, for any $\phi \in C_0^\infty(X(y,t,\varepsilon_0))^2$ there is $\underline{f} \in \mathcal{F}_n(X(y,t,\varepsilon_0))$ such that $U^{\underline{f}}(T) = \phi$. Hence the property ii. implies that there is a distribution $W^x \in \mathcal{D}'(X(y,t,\varepsilon_0))^2$ such that

$$\lim_{\varepsilon\to 0} V^{g(\varepsilon)}(T) = W^x. \tag{5.1}$$

The property i. implies that

$$\text{supp }(W^x) \subset \{x\}$$

and hence $W_x$ has to be a finite sum of the derivates of delta-distribution at $x$. Finally, by ii. the limit

$$\lim_{\varepsilon\to 0}\langle \underline{f}, g(\varepsilon)\rangle = \lim_{\varepsilon\to 0}(J\,U^{\underline{f}}(T), V^{g(\varepsilon)}(T))_{L^2}$$

exists for every $\underline{f} \in \mathcal{F}_n(X(y,t,\varepsilon_0))$, or equivalently, for every

$$U^{\underline{f}}(T) \in H_0^{n+1}(X(y,t,\varepsilon_0)) \times H_0^n(X(y,t,\varepsilon_0)).$$

Due to the choice of $n$ this implies that there is a constant $\kappa(x)$ such that

$$W^x = \begin{pmatrix} 0 \\ \kappa(x)\delta(\cdot - x)\end{pmatrix}. \tag{5.2}$$

Using the above construction we define an operator

$$\mathcal{W}^x : \mathcal{F} \to \mathbb{C},\ \ \underline{f} \mapsto (J\,U^{\underline{f}}(T), W^x)_{L^2}.$$

Some properties of this operator are described in the following lemmas.

LEMMA 5.2. *Let DBD be given. Then for any $x \in M_\Gamma$ it is possible to construct functions $g(\varepsilon) = g_x(\varepsilon)$ such that for corresponding operators $\mathcal{W}^x$,*

$$\mathcal{W}^x(f) = \kappa(x)u^f(x,T), \quad f \in C_0^\infty(\Gamma \times \mathbb{R}_+).$$

*Moreover it is possible to construct $g_x(\varepsilon)$ in such a way that $\kappa : M_\Gamma \to \mathbb{C}$ satisfies the conditions*

$$\kappa \in C^\infty(M_\Gamma \cup \Gamma), \quad \kappa|_\Gamma = 1, \quad \kappa \neq 0 \quad on\ M_\Gamma. \tag{5.3}$$

PROOF. Consider a family of boundary sources $g_x(\varepsilon), \varepsilon > 0, x \in M_\Gamma$ such that the corresponding $\mathcal{W}^x$ satisfy the following conditions

iii. $\mathcal{W}^x \neq 0$ for any $x \in M_\Gamma$.

iv. For any $f \in C_0^\infty(\Gamma \times \mathbb{R}_+)$ and $y \in \Gamma$

$$\lim_{x\to y} \mathcal{W}^x(f) = f(y,T).$$

v. The function $x \mapsto \mathcal{W}^x(f)$ is in $C^\infty(M_\Gamma \cup \Gamma)$ when $f \in C_0^\infty(\Gamma \times \mathbb{R}_+)$.

As we already know such sequences exist. On the other hand, we can verify conditions iii. -v. for a given family $g_x(\varepsilon)$ in local coordinates on $S_\Gamma$ by means of Lemma 2.2. However, since $E : S_\Gamma \cup \Gamma \times \{0\} \to M_\Gamma \cup \Gamma$ is a diffeomorphism these conditions are satisfied also on $M_\Gamma$. □

LEMMA 5.3. *Let DBD be given. These data determine the mapping $\mathcal{F} \to \kappa(E(y,s))\mathcal{U}^{\underline{f}}(E(y,s),t)$ where $(y,s) \in s_\Gamma$ and $t > T$.*

PROOF. For $\underline{f} = f \in C_0^\infty(\Gamma \times \mathbb{R}_+)$ the statement follows from Lemma 2.2 and formula (5.2). Hence the assertion follows from the continuous dependence of the solution $u^{\underline{f}}(\cdot\,, t) \in H_0^1(M)$ on $\underline{f} \in \mathcal{F}$. □

We want to emphasize that we do not know $\kappa(x)$ and, henceforth, can not reconstruct $u^f(x,t)$ using Lemma 5.3. However, we have the following theorem:

THEOREM 5.4. *By using DBD we can construct a metric $\tilde{g}$ on $S_\Gamma$ such that the space $(s_\Gamma, \tilde{g})$ is isometric to $(M_\Gamma, g)$.*

PROOF. Consider the point $(y,s) \in S_\Gamma$ and the corresponding point $x = E(y,s) \in M_\Gamma$. By Lemma 5.3 it is possible to find $\kappa(x')\mathcal{U}^{\underline{f}}(x',t)$ for any $x' = E(y',s')$, $(y',s') \in s_\Gamma$ and $t > T$. Let $V$ be a sufficiently small neighborhood of $x$ so that for any $z \in V$ a unique shortest geodesic $\gamma_{x,z}$ which connects $x$ and $z$ lies in $M_\Gamma$. Let us take $z \in V$ and $d_0 > 0$. We consider the wave produced by an initial state $U^{\underline{f}}(T)$

$$\text{supp } (U^{\underline{f}}(T)) \subset X(y,s,\varepsilon).$$

where $\underline{f} \in \mathcal{F}_0(X(y,s,\varepsilon))$ for some $\varepsilon > 0$.

Firstly, if $d(x,z) > d_0$, then due to the finite velocity of the wave propagation the wave $u^{\underline{f}}$ vanishes near $z$ for all

$$t \in ]T, T + d_0 - \text{diam } (X(y,s,\varepsilon)[.$$

Since diam $(X(y,s,\varepsilon)) \to 0$ when $\varepsilon \to 0$ then $u^{\underline{f}} = 0$ for sufficiently small $\varepsilon$ near $z$ for all $t \in ]T, T + d_0[$. Furthermore, the converse is also true. Indeed, let $\gamma_{x,z}$ be the shortest geodesic from $x$ to $z$ with length $d(x,y)$. For any $\varepsilon > 0$ there is $u^{\underline{f}}$ with supp $(u^{\underline{f}}(T)) \subset X(y,s,\varepsilon)$ such that

$$(x, \gamma'_x) \in WF(U^{\underline{f}}(T)$$

where $WF(U)$ stands for the wave-front set of $U$ and $\gamma'_x$ for the tangent vector to $\gamma$ at $x$. Using the well-known results about the propagation of singularities for the wave equation we see that $z \in$ sing supp $(U^{\underline{f}}(T + d(x,z)))$. Hence if $d_0 < d(x,z)$ then for any $\varepsilon > 0$ there is $\underline{f} \in \mathcal{F}_0(X(y,s,\varepsilon))$ such that $u^{\underline{f}}$ does not vanish identically near $\{z\} \times ]T, T + d_0[$. Thus $d(x,z)$ is the supremum of those $d_0 > 0$ for which $u^{\underline{f}}(t) = 0$ in the vicinity of $z$ for all $t \in ]T, T + d_0[$ and $\underline{f} \in \mathcal{F}_0(X(y,s,\varepsilon))$ with sufficiently small $\varepsilon$. As $\kappa \neq 0$ we can find the distances between the points $x = E(y,s)$ and $x' = E(y',s')$ for all sufficiently close $(y,s), (y',s') \in S_\Gamma$. Having found these local distances, we can construct the metric tensor $\tilde{g}$ as is done e.g. at **[11]**. □

Thus $(S_\Gamma, \tilde{g})$ can be identified with $(M_\Gamma, g)$ as a metric space. With this identification, we have constructed the highest order terms of $a(x,D)$ on $M_\Gamma$.

The end of this section is devoted to the construction of the lower-order terms in $a(x,D)$.

LEMMA 5.5. *Let $e^{\underline{f}}(x,t)$ be the functions*

$$e^{\underline{f}}(x,t) = \kappa(x,t) u^{\underline{f}}(x,t), \quad x \in M_\Gamma, t > T,$$

*where $\underline{f} \in \mathcal{F}$ and $\kappa$ is defined by (5.1). Then these functions determine $a_\kappa(x,D)$ on $M_\Gamma$.*

PROOF. The functions $e^{\underline{f}}(x,t) = \kappa(x)u^{\underline{f}}(x,t)$ are the solutions of the equation

$$e^{\underline{f}}_{tt} + be^{\underline{f}}_{t} + a_\kappa(x,D)e^{\underline{f}} = 0. \tag{5.4}$$

For any $x_0 \in M_\Gamma$ consider the vectors

$$\left(e^{\underline{f}}(x_0,T), \partial_j(e^{\underline{f}}(x_0,T)), \partial_k\partial_l(e^{\underline{f}}(x_0,T))), e^{\underline{f}}_t(x_0,T)\right)_{j,k,l=1}^m$$

where $\partial_j$ stands for the derivatives with respect to some local coordinates in $M_\Gamma$. These vectors are defined for $\underline{f} \in \mathcal{F}_n$ with sufficiently large $n$ and due to the isomorphism (4.1) they span the space $\mathbb{C}^{(m^2+3m+4)/2}$ when $\underline{f}$ varies in $\mathcal{F}_n$. Hence equation (5.4) may be used to determine $a_\kappa(x,D)$. □

Theorem 5.4 and Lemma 5.5 imply that we have constructed $(S_\Gamma, \tilde{g})$ which is isometric to $(M_\Gamma, g)$ and the operator $a(x,D)$ modulo generalized gauge-transformations.

## 6. From local constructions to global reconstruction.

In this section we provide a proof of the main result, Theorem 1.5, and make some additional remarks about the necessity of assumption made.

PROOF. (of Theorem 1.5) It is known that the Schwartz kernel $R(x,y,t,s)$ of operator $R$ is $\partial_{\nu(x)}\partial_{\nu(y)}G(x,y,t-s)$ where $G(x,y,t)$ is a appropriate Green's function. Precisely, if we fix $y$ then $(x,t) \mapsto G(x,y,t)$ is Green's function of problem (1.1) with $a_\kappa$ instead of $a$, i.e.,

$$\begin{aligned}(\partial_t^2 + b(x)\partial_t + a_\kappa(x,D_x))G(x,y,t) &= \delta_{(y,0)}(x,t) \text{ in } M\times\mathbb{R},\\ G(\cdot,y,\cdot)|_{\partial M\times\mathbb{R}} = 0; \quad G(x,y,t)|_{t<0} &= G_t(x,y,t)|_{t<0} = 0.\end{aligned} \tag{6.1}$$

The function $\partial_{\nu(y)}G(x,y,t), y \in \Gamma$ is a solution of the wave equation (6.1) with a known source term and boundary values. Moreover, since $b(x), a_\kappa(x,D)$ for $x \in M_\Gamma$ are already found, the Holmgren-John uniqueness theorem [**17**] implies that $\partial_{\nu(y)}G(x,y,t)$ is uniquely determined in $M_\Gamma \times \mathbb{R}_+$ in terms of its Cauchy data on $\Gamma \times \mathbb{R}_+$ which is known.

Let us now fix $x \in M_\Gamma$. Then the function $(y,t) \mapsto \overline{G(x,y,t)}$ is the Green's function of the wave equation,

$$\begin{aligned}(\partial_t^2 - \overline{b(y)}\partial_t + a_\kappa^*(y,D_y))\overline{G(x,y,t)} &= \delta_{(x,0)}(y,t) \text{ in } M\times\mathbb{R},\\ \overline{G(x,\cdot,\cdot)}|_{\partial M\times\mathbb{R}} = 0; \ \overline{G(x,y,t)}|_{t<0} &= \overline{G_t(x,y,t)}|_{t<0} = 0.\end{aligned}$$

Since the Cauchy data of $\overline{G(x,\cdot,\cdot)}$ on $\Gamma\times\mathbb{R}$ are known the same arguments show that $G(x,y,t)$ is uniquely determined in $M_\Gamma\times M_\Gamma\times\mathbb{R}$.

Let now $U \subset M_\Gamma$ be an open domain in $M_\Gamma$ with smooth geodesically convex boundary. Consider the initial-boundary value problem

$$\begin{aligned}e_{tt} + be_t + a_\kappa(x,D)e &= F \text{ in } M\times\mathbb{R}_+,\\ e|_{\partial M\times\mathbb{R}_+} &= f; \quad e|_{t=0} = e_t|_{t=0} = 0.\end{aligned} \tag{6.2}$$

Here supp $(f) \subset \Gamma\times\mathbb{R}_+$ and supp $(F) \subset U\times\mathbb{R}_+$ and we assume that $f \in C_t^\infty(\Gamma\times\mathbb{R}_+)$ and $F \in C_t^\infty(U\times\mathbb{R}_+)$ where $C_t^\infty$ is the set of locally bounded (with respect to $t$) $C^\infty$- functions which are equal to 0 near $t=0$. Since Green's function $G(x,y,t)$ is known everywhere in $M_\Gamma$ we can find arbitrary solutions $e(x,t)$ of (6.2) for $(x,t) \in M_\Gamma\times\mathbb{R}_+$ and arbitrary $f$ and $F$.

Consider the manifolds $M_1 = M\backslash U$ and $\Gamma_1 = \Gamma \cup \partial U$. The restrictions of the above functions $e(x,t)$ onto $M_1 \times \mathbb{R}_+$ contains all solutions to the initial-boundary value problem

$$(6.3) \qquad \begin{aligned} e_{tt} + be_t + a(x,D)e &= 0 \text{ in } M_1 \times \mathbb{R}_+ \\ e|_{\partial(M_1\times\mathbb{R}_+)} &= \tilde{f}; \quad e|_{t=0} = e_t|_{t=0} = 0 \end{aligned}$$

where supp $(\tilde{f}) \subset \Gamma_1 \times \mathbb{R}_+$. Indeed, we can take any smooth continuation of the solution of problem (6.3) and obtain a solution of problem (6.2) with appropriate $F$ and $f$.

Hence we can find solutions $e(x,t)$ of equation (6.3) for $x \in M_\Gamma\backslash U$, $t \in \mathbb{R}_+$ for all $\tilde{f} \in C_t^\infty(\Gamma_1 \times \mathbb{R}_+)$ and, henceforth, to find $\partial_\nu e|_{\Gamma_1\times\mathbb{R}_+}$ of these functions. Clearly we can then find the response operator $R^1$ for $(M_1, g, \Gamma_1)$. We note that the Bardos-Lebeau-Rauch condition is valid for $M_1$ with $t_*$, if it is valid for the original manifold $M$ with the time $t_*$.

By iterating this procedure finitely many times we can reconstruct $(M,g)$ and the equivalence class of $a(x,D)$. Hence theorem 1.5 is proven.

□

As a final remark we note that Theorem 1.5 remains valid under weaker assumptions than those maid in the paper and give some generalizations:

i. The assumption that $g_\Gamma$ is known may be omitted and we can assume that we just know $\Gamma$ and $R$.
ii. The assumption that $R$ is given for all times $t$ may be changed into a weaker assumption that $R^T$ is known for $0 < T < 2\tau + \varepsilon$ for arbitrary small $\varepsilon$.
iii. The assumption that $a(x,D)$ is symmetric in $L^2(M, m_g)$ may be weakened to the assumption that $a(x,D)$ is symmetric in $L^2(M,m)$ where $m$ is some smooth measure on $M$.
iv. The problem studied in this paper is equivalent to the corresponding spectral problem. Indeed, by knowing the eigenvalues $\lambda_j$ of the system (1.1) and the boundary values of the corresponding generalized eigenfunctions $\psi_j$ on $\Gamma$, it is possible to find the operator $R$. This relation has been studied more closely in [**10**]. Thus the presented results apply for the inverse boundary spectral problem when the data is known only on a part of the boundary.

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