On the existence and convergence of the solution of PML equations

Matti Lassas* and Erkki Somersalo

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1 Introduction

One of the central problems in the numerical approximation of acoustic and electromagnetic scattering solutions is the implementation of the radiation condition at infinity. In obstacle scattering methods, the boundary integral equation methods are attractive since the solutions satisfy automatically the radiation condition. When the obstacle boundary is complicated or the material parameters are not constants, the finite element methods (FEM) and the finite difference schemes (FD) are often found more convenient to implement. These methods require an efficient way of terminating the mesh so that the artificial boundary gives no spurious reflections. One popular approach is to use absorbing boundary conditions. These boundary conditions are typically local approximations of a non–local pseudodifferential operator equation corresponding to an exact absorbing condition. In recent years, there has been a growing interest in a scheme known as a Perfectly Matched Layer method (PML), suggested in [1] by Bérenger. The idea is to surround the scatterer by a fictious layer of absorbing medium that has the remarkable property of being perfectly reflectionless. In this work, we consider the questions of solvability of the Bérenger equations and approximating properties of the corresponding solutions. The main result (Theorem 2.1) is that with certain assumptions of the absorption coefficient, the Bérenger system truncated to a finite domain is solvable for all wave numbers, and the solution is close to the true scattering solution in the vicinity of the scatterer.

For simplicity, we consider here only one special obstacle scattering problem, although the essential parts of the work are more generally applicable. We may e.g.

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consider more general boundary conditions or potential scattering. More precisely, we consider electromagnetic scattering by an infinite cylindrical obstacle. Let $\Omega \subset \mathbb{R}^2$ denote the cross section of the cylinder. We assume that Ω is bounded, simply connected and has a smooth boundary curve. The electromagnetic field is assumed to have transverse electric polarization, i.e., the magnetic field is parallel to the axis of the cylinder. The field is assumed to be homogeneous in the direction of the cylinder. With the further assumption of harmonic time dependence, the scattered magnetic field can be written as

$$\vec{H} = u(r,\theta)e^{-i\omega t}\vec{e}_z,$$

where (r,θ) are the polar coordinates in the plane, $\omega > 0$ is the angular frequency and $\vec{e}_z$ is the polarization vector parallel to the axis of the scatterer. Outside Ω, the amplitude u satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0, \quad k^2 = \omega^2 \epsilon_0 \mu_0, \tag{1}$$

and the Sommerfeld radiation condition at infinity, i.e.,

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{2}$$

uniformly in all directions. We assume here that the obstacle is perfectly conducting, implying the Neumann boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g, \tag{3}$$

the function g being determined by the incoming wave.

Notations: In the following, we denote by $H^s(X)$, $X \subset \mathbb{R}^2$, the usual L^2–based Sobolev space on the set X.

We need the double layer potentials as a tool. Let $\Psi(x,y,k)$ be the fundamental solution of the Helmholtz equation,

$$\Psi(x,y,k) = \frac{i}{4}H_0^{(1)}(k|x-y|), \quad \mathrm{Im}\, k \geq 0.$$

We introduce the following notation: Let Γ be smooth, closed and bounded curve in $\mathbb{R}^2$ and $X \subset \mathbb{R}^2$, $\Gamma \cap X = \emptyset$. If ψ is a function defined on Γ, we write

$$D_{\Gamma,X}(k)\psi(x) = \int_\Gamma \frac{\partial \Psi}{\partial n_y}(x,y,k)\psi(y)dS(y), \quad x \in X.$$

Similarly, if the field point x is on the surface Γ, we write

$$D_\Gamma(k)\psi(x) = \int_\Gamma \frac{\partial \Phi}{\partial n_y}(x,y,k)\psi(y)dS(y), \quad x \in \Gamma.$$

In the sequel, the term *scattering solution* means the unique function $u_{\rm sc}$ in $\mathbb{R}^2 \setminus \overline{\Omega}$ with $u_{\rm sc}\big|_{B\setminus\overline{\Omega}} \in H^1(B \setminus \overline{\Omega})$ for each open ball B, $\overline{\Omega} \subset B \subset \mathbb{R}^2$, that satisfies the Helmholtz equation (1), the radiation condition (2) at infinity and the boundary condition (3) with $g \in H^{-1/2}(\partial\Omega)$. For the unique solvability of the problem, we refer to [5].

2 Perfectly Matched Layer

In this section we review first the PML method suggested originally by Bérenger in [1]. Our starting point is the article [2], where the authors analyze the Bérenger equations in cylindrical coordinates.

Assume that $B_1 = B(R_1)$ is a disc of radius $R_1 > 0$ centered at the origin such that $\overline{\Omega} \subset B_1$. Let $\sigma = \sigma(r) \in C^1$ be a fictious absorption coefficient with the properties

$$\sigma(r) = 0 \text{ for } r \le R_1, \quad \sigma(r) > 0 \text{ for } r > R_1, \quad \lim_{r\to\infty} \int_{R_1}^{r} \sigma(t)dt = \infty. \tag{4}$$

We denote by $\tilde{r}$ the complex radius defined as

$$\tilde{r} = \tilde{r}(r) = \begin{cases} r, & r \le R_1, \\ r\left(1 + \dfrac{i}{\omega r}\displaystyle\int_{R_1}^{r} \sigma(t)dt\right) = r\beta(r), & r > R_1. \end{cases}$$

The idea in the Bérenger equation is to continue the scattering solution analytically, replacing the true radius r by the complexified radius $\tilde{r}$. In [2], the Bérenger solution u_B in $\mathbb{R}^2 \setminus \overline{\Omega}$ is defined as the solution of the system

$$\begin{aligned} \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial u_B}{\partial \tilde{r}}\right) + \frac{1}{\tilde{r}^2}\frac{\partial^2 u_B}{\partial \theta^2} + k^2 u_B &= 0 \text{ in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ \frac{\partial u_B}{\partial n}\bigg|_{\partial\Omega} &= g \in H^{-1/2}(\partial\Omega), \\ |u_B| \text{ is uniformly bounded in } R^2 \setminus \overline{\Omega}. \end{aligned} \tag{5}$$

By denoting

$$\frac{\partial \tilde{r}}{\partial r} = 1 + \frac{i}{\omega}\sigma(r) = \alpha(r),$$

the complexified Helmholtz equation in (5) can be written equivalently as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\beta}{\alpha} r \frac{\partial u_B}{\partial r}\right) + \frac{\alpha}{\beta}\frac{1}{r^2}\frac{\partial^2 u_B}{\partial \theta^2} + k^2\alpha\beta u_B = 0. \tag{6}$$

This form of the equation will be used in the sequel. As in [2], we write the above equation occasionally in a coordinate free form as

$$(\nabla \cdot A\nabla + \alpha\beta k^2)u = 0, \tag{7}$$

where A is a matrix. The form of A in Cartesian coordinates was given in [2] as

$$A = \begin{pmatrix} \frac{\beta}{\alpha}\cos^2\theta + \frac{\alpha}{\beta}\sin^2\theta & \cos\theta\sin\theta\left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right) \\ \cos\theta\sin\theta\left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right) & \frac{\beta}{\alpha}\sin^2\theta + \frac{\alpha}{\beta}\cos^2\theta \end{pmatrix}. \tag{8}$$

Thus, the equation (7) is formally similar to the equation satisfied by the magnetic field in an orthotropic medium. However, the matrix A is non-physical.

The following proposition is the contents of Theorem 1 of [2].

Proposition 2.1 *The Bérenger system (5) has a unique solution, and*

$$u_B\Big|_{B_1\setminus\overline{\Omega}} = u_{\rm sc}\Big|_{B_1\setminus\overline{\Omega}}.$$

This theorem shows that the boundary of the PML medium at $r = R_1$ is perfectly reflectionless. The next step to replace the infinite PML layer by a layer of finite thickness to make the system (5) computationally feasible. For later purposes, we deviate here from the discussion in [2] and impose an extra condition on the absorption coefficient. More precisely, let $R_2 > R_1$. We assume that σ is chosen such that

$$\sigma(r) = \frac{1}{R_2}\int_{R_1}^{R_2} \sigma(t)dt = \sigma_0, \text{ as } r \geq R_2. \tag{9}$$

This choice of σ implies that for $r \geq R_2$,

$$\alpha(r) = 1 + \frac{i}{\omega}\sigma_0 = \alpha_0,$$

and more importantly,

$$\begin{aligned} \beta(r) &= 1 + \frac{i}{r\omega}\int_{R_1}^{r} \sigma(t)dt \\ &= 1 + \frac{i}{r\omega}\left(\int_{R_1}^{R_2} \sigma(t)dt + (r - R_2)\sigma_0\right) \\ &= 1 + \frac{i}{\omega}\sigma_0 = \alpha_0. \end{aligned}$$

The crux of this choice is that the Bérenger equation (7) for $r > R_2$ reduces to

$$\Delta u_B + (\alpha_0 k)^2 u_B = 0, \tag{10}$$

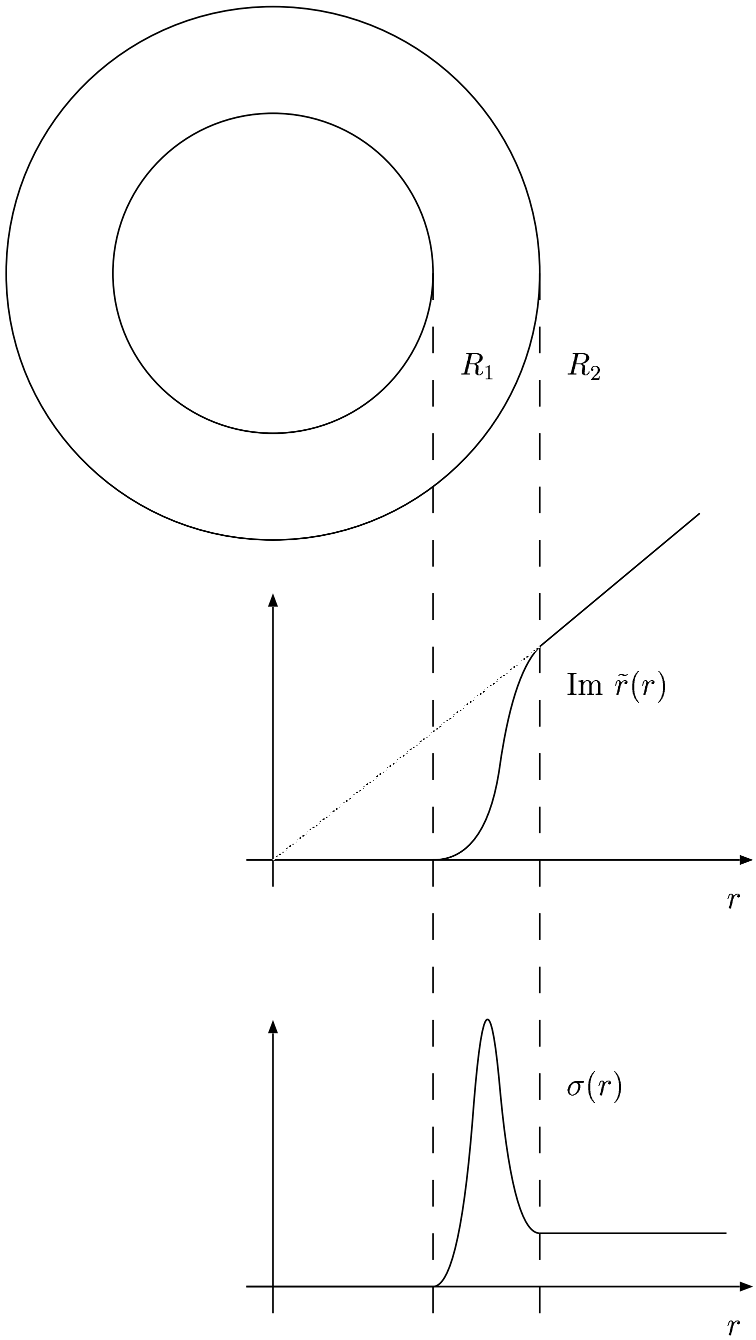

Figure 1: The functions $\sigma(r)$ and Im $\tilde{r}(r)$ when the conditions (4) and (9) are satisfied. Inside B_1 the Helmholtz equation is satisfied with a real wave number and outside B_2 with a complex wave number. It is crucial that Im $\tilde{r}(r)$ coincides to the line Im $\alpha_0 r$ with large r.

i.e., u_B satisfies the Helmholtz equation with a complex wave number $\alpha_0 k \in \mathbb{C}_{++} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0, \operatorname{Re} z > 0\}$. Our aim is to replace the condition of uniform boundedness at infinity by an equivalent near field condition. To this end, consider the exterior Dirichlet problem in $\mathbb{R}^2 \setminus \overline{B}_2$: Find a function w uniformly bounded near infinity, satisfying the equation (10) in $\mathbb{R}^2 \setminus \overline{B}_2$ and the boundary condition

$$w\Big|_{\partial B_2} = f \in H^{1/2}(\partial B_2).$$

This problem can be solved by writing w as

$$w = D_{\partial B_2, \mathbb{R}^2 \setminus \overline{B}_2}(\alpha_0 k)\psi.$$

In the discussion that follows, the wave number in the double layer potential is assumed to be $\alpha_0 k$. For brevity we shall suppress the explicit dependence it in the sequel. Since the interior Neumann problem in B_2 with the wave number $\alpha_0 k$ has only the trivial solution, we may deduce by the standard argument (see e.g. [3], Theorem 3.17) that the density $\psi \in H^{1/2}(\partial B_2)$ is obtained from the Dirichlet condition by

$$\psi = (\frac{1}{2} + D_{\partial B_2})^{-1} f.$$

Especially, the Bérenger solution u_B in $\mathbb{R}^2 \setminus \overline{B}_2$ satisfies

$$u_B\Big|_{\mathbb{R}^2 \setminus \overline{B}_2} = D_{\partial B_2, \mathbb{R}^2 \setminus \overline{B}_2}(\frac{1}{2} + D_{\partial B_2})^{-1}(u_B\Big|_{\partial B_2}).$$

Let $R_3 > R_2$. By taking the trace of u_B on the surface ∂B_3 we get the equation

$$u_B\Big|_{\partial B_3} = P(u_B\Big|_{\partial B_2}), \tag{11}$$

where the operator $P : H^{1/2}(\partial B_2) \to H^{1/2}(\partial B_3)$ is defined as

$$P = D_{\partial B_2, \partial B_3}(\frac{1}{2} + D_{\partial B_2})^{-1}. \tag{12}$$

Observe that the operator P, being a smoothing operator, is compact by Rellich compactness theorem. The operator P is closely related to the double surface radiation condition defined in the report [4]. We prove the following equivalence result.

Lemma 2.1 *The restriction of the Bérenger solution u_B to the set $B_3 \setminus \overline{\Omega}$ is the unique solution in $H^1(B_3 \setminus \overline{\Omega})$ that satisfies the system*

$$\begin{aligned} (\nabla \cdot A\nabla + \alpha\beta k^2)u &= 0 \text{ in } B_3 \setminus \overline{\Omega}, \\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} &= g \in H^{-1/2}(\partial\Omega), \\ u\Big|_{\partial B_3} &= P(u\Big|_{\partial B_2}). \end{aligned} \tag{13}$$

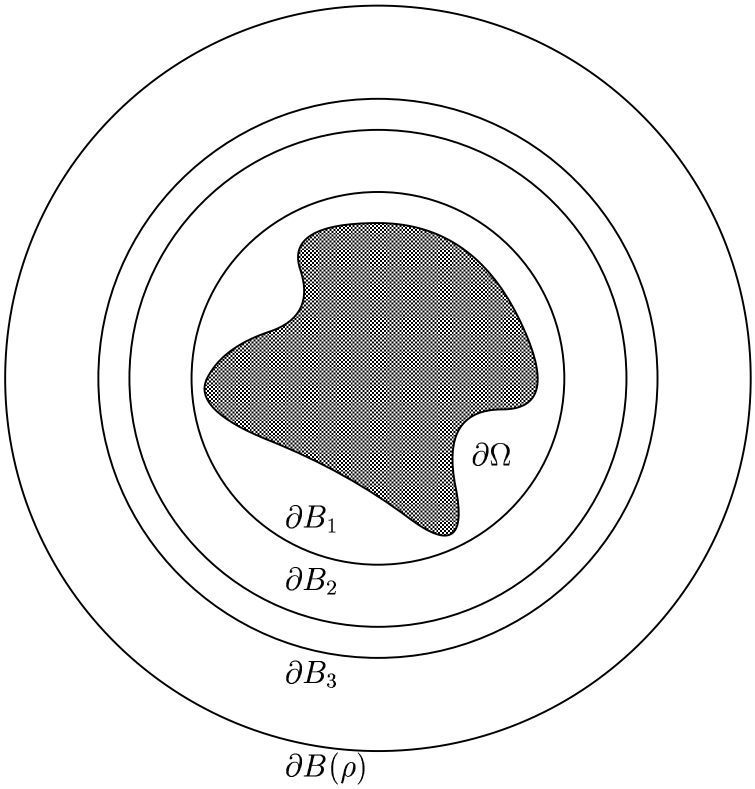

Figure 2: Between $\partial\Omega$ and ∂B_1 Helmholtz equation is satisfied with a real wave number, between ∂B_1 and ∂B_2 we have a non-physical equation and outside ∂B_2 the wave number is complex. The double-surface boundary operator P maps $u\Big|_{\partial B_2}$ to $u\Big|_{\partial B_3}$. Later in the equation (19) the domain is truncated with the Dirichlet boundary condition on $\partial B(\rho)$.

Proof: From the derivation of the operator P above, the existence of a solution is clear, since the Bérenger solution satisfies the system.

To show the uniqueness, assume that $w \in H^1(B_3 \setminus \overline{\Omega})$ is a solution of the system. Define a function w' in the exterior domain $\mathbb{R}^2 \setminus \overline{B}_2$ by the equation

$$w' = D_{\partial B_2, \mathbb{R}^2 \setminus \overline{B}_2} (\frac{1}{2} + D_{\partial B_2})^{-1} (w\big|_{\partial B_2}).$$

Then $w'\big|_{\partial B_j} = w\big|_{\partial B_j}$, $j = 2, 3$, and since $(\alpha_0 k)^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $B_3 \setminus \overline{B}_2$, we have $w'\big|_{B_3 \setminus \overline{B}_2} = w\big|_{B_3 \setminus \overline{B}_2}$. Let W be given by

$$W(x) = \begin{cases} w(x), & x \in B_3 \setminus \overline{\Omega}, \\ w'(x), & x \in \mathbb{R}^2 \setminus \overline{B}_3. \end{cases}$$

Evidently, W is uniformly bounded far away from the scatterer, it satisfies the equation (7) plus the boundary condition $\partial W / \partial n\big|_{\partial \Omega} = g$. By the uniqueness of the solution of the Bérenger system, we must therefore have $W = u_B$ and so especially $w = u_B\big|_{B_3 \setminus \overline{\Omega}}$. $\square$

The previous results asserts that the Bérenger solution can be found by solving the system (13) in a bounded domain. From the practical point of view, the following approximation result is found useful.

Lemma 2.2 *Assume that* $P_\varepsilon : H^{1/2}(\partial B_2) \to H^{1/2}(\partial B_3)$ *is an operator with the property*

$$\|P_\varepsilon - P\| < \varepsilon, \tag{14}$$

the norm being the uniform operator norm of the space $\mathcal{B}(H^{1/2}(\partial B_2), H^{1/2}(\partial B_2))$ *of bounded linear operators. Consider the system (13) with* P *replaced by* P_ε*. For* $\varepsilon > 0$ *small enough, the system has a unique solution* $u_\varepsilon \in H^1(B_3 \setminus \overline{\Omega})$*, and we have*

$$\|u_B - u_\varepsilon\|_{H^1(B_3 \setminus \overline{\Omega})} < C\varepsilon$$

for some positive constant C.

Proof: We start by writing an equivalent weak form of the system (13). Let $R : H^{1/2}(\partial B_3) \to H^1(B_3 \setminus \overline{\Omega})$ denote a right inverse of the trace mapping $u \mapsto u\big|_{\partial B_3}$. We choose R so that $\operatorname{supp} R\varphi \cap \overline{B}_2 = \emptyset$ for all $\varphi \in H^{1/2}(\partial B_3)$. Assume for a while that u satisfies the system (13). If w is defined as $w = u - RP(u\big|_{\partial B_2})$, we see that $w\Big|_{\partial B_2} = u\Big|_{\partial B_2}$ and w satisfies

$$
(15) \qquad \begin{aligned}
(\nabla \cdot A\nabla + \alpha\beta k^2)w &= Fw \text{ in } B_3 \setminus \overline{\Omega}, \\
w\Big|_{\partial B_3} &= 0, \\
\frac{\partial w}{\partial n}\Big|_{\partial\Omega} &= g,
\end{aligned}
$$

where

$$
Fw = -(\nabla \cdot A\nabla + \alpha\beta k^2)RP(w\Big|_{\partial B_2}).
$$

Conversely, if w satisfies the system (15), a solution of the original system (13) is obtained by $u = w + RP(w\Big|_{\partial B_2})$.

Let us denote by $H_0^1(B_3 \setminus \overline{\Omega})$ the space

$$
H_0^1(B_3 \setminus \overline{\Omega}) = \{u \in H^1(B_3 \setminus \overline{\Omega}) \mid u\Big|_{\partial B_3} = 0\}.
$$

Note that Fw vanishes near the boundary $\partial\Omega$ and by definition, $Fw \in H^{-1}(B_3 \setminus \overline{\Omega})$, so we may deduce that $Fw \in (H_0^1(B_3 \setminus \overline{\Omega}))'$, the dual of $H_0^1(B_3 \setminus \overline{\Omega})$. Since P is compact, also the mapping

$$
F : H_0^1(B_3 \setminus \overline{\Omega}) \to (H_0^1(B_3 \setminus \overline{\Omega}))'
$$

is compact.

The equivalent weak form to the problem (15) is to find $w \in H_0^1(B_3 \setminus \overline{\Omega})$ satisfying

$$
(16) \qquad \langle A\nabla w, \overline{\nabla v}\rangle - k^2\langle \alpha\beta w, \overline{v}\rangle + \langle Fw, \overline{v}\rangle = \langle g, \overline{v}\Big|_{\partial\Omega}\rangle_{\partial\Omega}
$$

for all $v \in H_0^1(B_3 \setminus \overline{\Omega})$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ denote the distribution duality over $B_3 \setminus \overline{\Omega}$ and $\partial\Omega$, respectively. The inner product in $H_0^1(B_3 \setminus \overline{\Omega})$ is denoted as

$$
(u, v) = \langle \nabla u, \overline{\nabla v}\rangle + \langle u, \overline{v}\rangle.
$$

Let $\mathcal{A}$ be the operator

$$
\langle A\nabla w, \overline{\nabla v}\rangle + \langle w, \overline{v}\rangle = (\mathcal{A}w, v), \quad \mathcal{A} : H_0^1(B_3 \setminus \overline{\Omega}) \to H_0^1(B_3 \setminus \overline{\Omega}),
$$

defined by the Riesz representation theorem. It is shown in [2] that the matrix (8) defines a positive definite quadratic form and thus the operator is strongly coercive, i.e.

$$
|\langle A\nabla w, \overline{\nabla w}\rangle + \langle w, \overline{w}\rangle| \geq \delta\|w\|^2_{H^1(B_3 \setminus \overline{\Omega})}
$$

with some constant $\delta > 0$. Consequently, by Lax–Milgram lemma, $\mathcal{A}^{-1}$ exists and is continuous. Similarly, let $\mathcal{J}$ and $\mathcal{K}$ denote the operators defined through

$$\begin{aligned} \langle Fw, \overline{v}\rangle &= (\mathcal{J}Fw, v), \quad \mathcal{J} : (H_0^1(B_3 \setminus \overline{\Omega}))' \to H_0^1(B_3 \setminus \overline{\Omega}), \\ \langle(\alpha\beta k^2 + 1)w, \overline{v}\rangle &= (\mathcal{K}w, v), \quad \mathcal{K} : H_0^1(B_3 \setminus \overline{\Omega}) \to H_0^1(B_3 \setminus \overline{\Omega}), \end{aligned}$$

the existence being guaranteed by the Riesz representation theorem. Further, we define the function $\mathcal{G} \in H_0^1(B_3 \setminus \overline{\Omega})$ by the formula

$$\langle g, \overline{v}\big|_{\partial\Omega}\rangle_{\partial\Omega} = (\mathcal{G}, v).$$

With these notations, the equation (16) is equivalent to the equation

$$(\mathcal{A} - \mathcal{K} + \mathcal{J}F)w = \mathcal{G}. \tag{17}$$

Obviously, $\mathcal{K}$ is compact, and the compactness of P and consequently F guarantees that the operator above is a Fredholm operator of index zero. To prove the injectivity, assume that w_0 satisfies the equation (17) with $\mathcal{G} = 0$. Then w_0 satisfies the system (15) with $g = 0$, so the function $u_0 = w_0 + RP(w_0\big|_{\partial B_2})$ satisfies the system (13) with $g = 0$. By the previous lemma, u_0 is the Bérenger solution with vanishing Neumann data, so $u_0 = 0$. This implies that also $w_0 = 0$, proving the injectivity. Thus, equation (17) is solvable and we can write

$$w = (\mathcal{A} - \mathcal{K} + \mathcal{J}F)^{-1}\mathcal{G}.$$

Consider next the system (13) with P replaced by P_ε. The same argument as above yields an equation

$$(\mathcal{A} - \mathcal{K} + \mathcal{J}F_\varepsilon)w_\varepsilon = \mathcal{G}, \tag{18}$$

where

$$F_\varepsilon = -(\nabla \cdot A\nabla + \alpha\beta k^2)RP_\varepsilon(\,\cdot\,\big|_{\partial B_2}).$$

Applying $(\mathcal{A} - \mathcal{K} + \mathcal{J}F)^{-1}$ on both sides of (18), we see that it is equivalent to the equation

$$(I + (\mathcal{A} - \mathcal{K} + \mathcal{J}F)^{-1}\mathcal{J}(F_\varepsilon - F))w_\varepsilon = w.$$

Since $\|F - F_\varepsilon\| < C\varepsilon$, this equation is solvable by Neumann series for ε small enough and we get the estimate

$$||w - w_\varepsilon||_{H^1} < C\varepsilon.$$

Finally, defining $u_\varepsilon = w_\varepsilon + RP_\varepsilon(w_\varepsilon\big|_{\partial B_1})$ that satisfies the system (13) with P_ε and noting that

$$||u_B - u_\varepsilon|| \le ||w - w_\varepsilon|| + ||R(P_\varepsilon - P)w|| + ||RP(w - w_\varepsilon)||,$$

we get the claim. $\square$

Our aim is to show that when the PML layer is truncated to a bounded region, the resulting boundary value problem is equivalent to a system (13) with P replaced by an approximating operator. Let $\rho > R_3$. Consider the following mixed boundary value problem: Find $\tilde{u}_B = \tilde{u}_B(r, \theta, \rho) \in H^1(B(\rho) \setminus \overline{\Omega})$ satisfying

$$
\begin{aligned}
(\nabla \cdot A\nabla + \alpha\beta k^2)\tilde{u}_B &= 0 \text{ in } B(\rho) \setminus \overline{\Omega}, \\
\frac{\partial \tilde{u}_B}{\partial n}\Big|_{\partial\Omega} &= g \in H^{-1/2}(\partial\Omega), \qquad (19) \\
\tilde{u}_B\Big|_{\partial B(\rho)} &= 0.
\end{aligned}
$$

We have chosen the Dirichlet boundary condition on the outer boundary surface. As in [2], one could use other type of conditions (e.g. Neumann or an impedance condition) as well. The forthcoming analysis would be essentially similar. To find an operator that propagates the Dirichlet boundary data on ∂B_2 of the possible solution of the system (19) to the corresponding Dirichlet data on ∂B_3, we prove the following lemma. Below, the wave number in the layer potentials is $\alpha_0 k$.

Lemma 2.3 *Assume that $\tilde{u}_B \in H^1(B(\rho) \setminus \overline{\Omega})$ satisfies the system (19). Then*

$$\tilde{u}_B\Big|_{\partial B_3} = \tilde{P}(\tilde{u}_B\Big|_{\partial B_2}),$$

where the operator $\tilde{P} = \tilde{P}(\rho) : H^{1/2}(\partial B_2) \to H^{1/2}(\partial B_3)$ is given as

$$\tilde{P} = \Big(D_{\partial B_2, \partial B_3} + D_{\partial B(\rho), \partial B_3}\Pi\Big)\Big(\frac{1}{2} + D_{\partial B_2} + D_{\partial B(\rho), \partial B_2}\Pi\Big)^{-1},$$

where $\Pi : H^{1/2}(\partial B_2) \to H^{1/2}(\partial B(\rho))$ is given by

$$\Pi = (\frac{1}{2} - D_{\partial B(\rho)})^{-1} D_{\partial B_2, \partial B(\rho)}$$

Conversely, if $\tilde{u}_B$ satisfies

$$
\begin{aligned}
(\nabla \cdot A\nabla + \alpha\beta k^2)\tilde{u}_B &= 0 \text{ in } B(\rho) \setminus \overline{\Omega}, \\
\frac{\partial \tilde{u}_B}{\partial n}\Big|_{\partial\Omega} &= g \in H^{-1/2}(\partial\Omega), \qquad (20) \\
\tilde{u}_B\Big|_{\partial B_3} &= \tilde{P}(\tilde{u}_B\Big|_{\partial B_2}),
\end{aligned}
$$

then it satisfies also the system (19).

Proof: Consider the Dirichlet boundary value problem of finding $w \in H^1(B(\rho) \setminus \overline{B}_2)$ satisfying

$$
(21) \qquad \begin{aligned}
(\Delta + (\alpha_0 k)^2) w &= 0 \text{ in } B(\rho) \setminus \overline{B}_2, \\
w\Big|_{\partial B_2} &= f \in H^{1/2}(\partial B_2), \\
w\Big|_{\partial B(\rho)} &= 0.
\end{aligned}
$$

Since $\alpha_0 k \in \mathbb{C}_+$, the above problem has a unique solution that can be found by writing

$$
w = D_{\partial B_2, B(\rho)\setminus \overline{B}_2} \psi + D_{\partial B(\rho), B(\rho)\setminus \overline{B}_2} \varphi,
$$

where the densities $\psi \in H^{1/2}(\partial B_2)$ and $\varphi \in H^{1/2}(\partial B(\rho))$ are determined by the boundary conditions as

$$
\begin{aligned}
(\frac{1}{2} + D_{\partial B_2}) \psi + D_{\partial B(\rho), \partial B_2} \varphi &= f, \\
D_{\partial B_2, \partial B(\rho)} \psi - (\frac{1}{2} - D_{\partial B(\rho)}) \varphi &= 0,
\end{aligned}
$$

or

$$
\begin{aligned}
\psi &= (\frac{1}{2} + D_{\partial B_2} + D_{\partial B(\rho), \partial B_2} \Pi)^{-1} f, \\
\varphi &= (\frac{1}{2} - D_{\partial B(\rho)})^{-1} D_{\partial B_2, \partial B(\rho)} \psi = \Pi \psi.
\end{aligned}
$$

The existence of the bounded inverses in the formulas above follow by a standard Fredholm argument. By substitution, we find that

$$
w\Big|_{\partial B_3} = (D_{\partial B_2, \partial B_3} + D_{\partial B(\rho), \partial B_3} \Pi)(\frac{1}{2} + D_{\partial B_2} + D_{\partial B(\rho), \partial B_2} \Pi)^{-1} f.
$$

Since $\tilde{u}_B$ satisfies the problem (21) with $f = \tilde{u}_B\Big|_{\partial B_2}$, we get the first part of the claim.

Conversely, assume that $\tilde{u}_B$ satisfies the system (20). All we need to show in this case is that $\tilde{u}_B\Big|_{\partial B(\rho)} = 0$. To this end, define $w \in H^1(B(\rho) \setminus \overline{B}_2)$ by

(22)
$$
w = (D_{\partial B_2, B(\rho)\setminus \overline{B}_2} + D_{\partial B(\rho), B(\rho)\setminus \overline{B}_2} \Pi)(\frac{1}{2} + D_{\partial B_2} + D_{\partial B(\rho), \partial B_2} \Pi)^{-1} (\tilde{u}_B\Big|_{\partial B_2}).
$$

By definition, we observe that

$$
w\Big|_{\partial B_j} = \tilde{u}_B\Big|_{\partial B_j}, \quad j = 2, 3,
$$

and since w and $\tilde{u}_B$ both satisfy the Helmholtz equation with the same complex wave number in $B_3 \setminus \overline{B}_2$, we have

$$w\Big|_{B_3\setminus\overline{B}_2} = \tilde{u}_B\Big|_{B_3\setminus\overline{B}_2}.$$

By the unique continuation principle of elliptic equations, we must therefore have $w = \tilde{u}_B\Big|_{B(\rho)\setminus\overline{B}_2}$. By substituting the operator Π in equation (22) and going to the boundary $\partial B(\rho)$, we see that $\tilde{u}_B\Big|_{\partial B(\rho)} = w\Big|_{\partial B(\rho)} = 0$, which completes the proof. □.

The previous lemma states that the truncated Bérenger problem can be replaced by one with an appropriate operator $\tilde{P}$ that propagates the solution from the surface ∂B_2 to the surface ∂B_3. The next lemma shows that the operator $\tilde{P}$ is an approximation of the operator P.

Lemma 2.4 *The operator $\tilde{P} = \tilde{P}(\rho)$ has the property*

$$\lim_{\rho\to\infty} \|\tilde{P}(\rho) - P\| = 0,$$

where the norm denotes the uniform operator norm in $\mathcal{B}(H^{1/2}(\partial B_2), H^{1/2}(\partial B_3))$.

Proof: We assume that $\rho > 1 + R_3$. Clearly, by the definition of the operator $\tilde{P}$, it is sufficient to show that

$$\lim_{\rho\to\infty} D_{\partial B(\rho),\partial B_j}\Pi(\rho) = 0$$

in the uniform operator topology of $\mathcal{B}(H^{1/2}(\partial B_2), H^{1/2}(\partial B_j))$ for $j = 2, 3$. First we estimate the operator $H : D_{\partial B(\rho),B(\rho)}(\frac{1}{2} - D_{\partial B(\rho)})^{-1}$ that maps the function $f \in H^{1/2}(\partial B(\rho))$ to the solution $v \in H^1(B(\rho))$ of the Dirichlet problem

$$\begin{aligned} (\Delta + (\alpha_0 k)^2)v &= 0 \text{ in } B(\rho), \\ v|_{\partial B(\rho)} &= f. \end{aligned}$$

Clearly, by scaling we get for $V = v(\rho\,\cdot\,)$ an equivalent equation

$$\begin{aligned} (\Delta + \rho^{-2}(\alpha_0 k)^2)V &= 0 \text{ in } B(1), \\ V|_{\partial B(1)} &= f(\rho\,\cdot\,). \end{aligned}$$

in the unit disc $B(1)$. Since $(\alpha_0 k)^2$ has a non-vanishing imaginary component, we see by using standard arguments that for $\rho > 1$

$$||V||_{H^1(B(1))} \leq C\rho^2||f(\rho\,\cdot\,)||_{H^{1/2}(\partial B(1))}.$$

By scaling again we get

$$\begin{aligned} ||f(\rho\,\cdot\,)||_{H^{1/2}(\partial B(1))} &\leq C\rho^{1/2}||f||_{H^{1/2}(\partial B(\rho))}, \\ ||v||_{H^1(B(\rho))} &\leq C\rho||V||_{H^1(B(1))} \end{aligned}$$

which yields

$$||v||_{H^1(B(\rho))} \leq C\rho^{7/2}||f||_{H^{1/2}(\partial B(\rho))}$$

and thus $||H|| \leq C\rho^{7/2}$.

To estimate $D_{\partial B_2, \partial B(\rho)}$, we use the asymptotics of the Hankel function $H_0^{(1)}$, giving

$$|D^j H_0^{(1)}(z)| \leq \frac{C}{\sqrt{|z|}}|e^{iz}|, \ j \leq 2$$

for z sufficiently large and $-\pi < \arg z < \pi$. By definition, the operator $D_{\partial B_2, \partial B(\rho)}$ is an integral on ∂B_2 of the normal derivate of $H_0^{(1)}(k\alpha_0|x-y|)$ where $x \in \partial B_2$ and $y \in \partial B(\rho)$. Thus the above estimate gives us point wise estimates for the kernel yielding

$$||D_{\partial B_2, \partial B(\rho)}||_{H^{1/2}(\partial B_2), H^{1/2}(\partial B(\rho))} \leq C\sqrt{\rho}e^{-k(\rho - R_2)\mathrm{Im}\,\alpha_0}.$$

By combining the previous estimates and using the trace mapping $T_j : u \mapsto u|_{B_j}$, we see that

$$\begin{aligned} ||D_{\partial B(\rho), \partial B_j}\Pi(\rho)|| &\leq ||T_j||\,||H||\,||D_{\partial B_2, \partial B(\rho)}|| \\ &\leq C\rho^4 e^{-k(\rho - R_2)\mathrm{Im}\,\alpha_0}. \end{aligned}$$

Thus, since $\mathrm{Im}\,\alpha_0 > 0$, the operators decay exponentially and the claim is proved. □

We are ready to gather the results together to the main theorem of the paper.

Theorem 2.1 *Assume that the fictious absorption coefficient satisfies the assumptions (4) and (9). Then for any wave number $k > 0$ there exists $\rho_0(k)$ such that for all $\rho > \rho_0(k)$, the truncated Bérenger system (19) has a unique solution $\tilde{u}_B = \tilde{u}_B(r, \theta, \rho)$. Furthermore, the solution has the approximation property*

$$\lim_{\rho \to \infty} \|u_{sc} - \tilde{u}_B\|_{H^1(B_1 \setminus \overline{\Omega})} = 0$$

the convergence being exponential.

Proof: The theorem follows immediately from Proposition 2.1 and Lemmas 2.1–2.4. □

3 Conclusions

This article is a contribution to the theoretical analysis of the Perfectly Matched Layer method of implementing nearly reflectionless boundary conditions for scattering problems. The work extends the previous theoretical analysis of [2]: With a certain extra assumption on the fictious absorption coefficient, it is possible to show that the PML equations are solvable for all frequencies, and an asymptotic error estimate is obtainable. Particularly, we have shown that the PML layer can be used to join the Helmholtz equations with real and complex wave numbers in a reflectionless manner. It remains open whether the extra assumption on the absorption coefficient that has been introduced here is really necessary theoretically or numerically.

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Matti Lassas
Rolf Nevanlinna Institute
University of Helsinki
P. O. Box 4, FIN-00014
Finland
e-mails:
Matti.Lassas@helsinki.fi

Erkki Somersalo
Department of Mathematics
Helsinki University of Technology
Otakaari 1, Espoo, FIN-02150
Finland
Erkki.Somersalo@hut.fi