# Complex Riemannian metric and absorbing boundary conditions

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**Abstract**

In computational electromagnetic and acoustic scattering, the unbounded Euclidean space $\mathbb{R}^3$ is often modeled by a bounded domain with an absorbing boundary condition. One possible approach to create such absorbing boundary condition is to surround the computational domain by a non-reflecting artificial sponge layer that absorbs quickly the scattered waves. This approach is called the method of a Perfectly Matched Layer (PML). In this paper we prove that such absorbing boundary layers can be obtained by using complex Riemannian metric $g_{ij}$. We show that the boundary layer is non-reflecting when $g$ is flat, that is, the curvature tensor of the complex metric $g_{ij}$ is zero. This fact gives an invariant formulation for the absorbing boundary layers as well as give us new kind of absorbing boundary layers for Maxwell and Helmholtz equations. Moreover, we show that all Perfectly Matched Layers, that is, absorbing boundary layers obtained through a complexification of coordinates corresponds to flat complex manifolds. Finally, we discuss the relation of the absorbing boundary layers and the complex scaling technique, developed by Sjöstrand and Zworski for the study of scattering poles.

Keywords: Absorbing boundary conditions, Perfectly Matched Layer, Complex Riemannian metric, Scattering poles.

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# 1 Introduction

The initial motivation of this work comes from the problem of solving numerically scattering problems, e.g. to find out how a given electromagnetic wave scatters from a given obstacle. One difficulty to solve scattering problems numerically can be observed immediately: The scattering takes place in an unbounded space but the numerical simulations are usually made in a finite computational domain. Indeed, when finite element or finite difference methods to solve scattering problems are employed, a crucial question is how to terminate the mesh without creating excessive echoes from the artificial truncation surface that may spoil the quality of the solution completely. There is a wealth of articles suggesting different solutions, commonly known as Absorbing Boundary Conditions (ABC). In recent years, a large amount of work has been done to study a mesh truncation scheme known as the method of a Perfectly Matched Layer, or PML for short. The idea is to surround the scatterer and the near field region around it by a "sponge layer" that is reflectionless and absorbs strongly the scattered waves. Therefore, if the computational region is truncated within this sponge material, one expects that due to the strong attenuation, a homogeneous Dirichlet boundary condition, for example, is a good truncation condition. A large amount of numerical work (see e.g. [6], [8], [19]) and recently also theoretical work (see [7], [24], [23]) to study this scheme has been published.

It is known that the PML scheme can be understood as a method of complex stretching of the coordinates (see [6], [7], [23]). This was the starting point in the articles [15] and [16], where the scattering problem for the Helmholtz equation was studied in circular and general convex geometry, respectively. The idea in the cited works was to map $\mathbb{R}^n$ onto a surface in $\mathbb{C}^n$ and extend the Helmholtz equation analytically to this surface to get the PML equations. In the present work, we study the scattering problem for Maxwell's equations and the Helmholtz equation in $\mathbb{R}^3$. In this paper we use a different point of view from that in previously cited articles. Instead of stretching the coordinates, we change the metric defined on $\mathbb{R}^3$. There are several advantages of this point of view. First, when Maxwell's equations are written in terms of 1–forms, the differential operators take form of exterrior derivatives. In numerical literature this is expressed by saying that the differential operators are purely topological in nature, that is, the metric or the material parameters affect to the equation only through the Hodge–$*$ operator. This

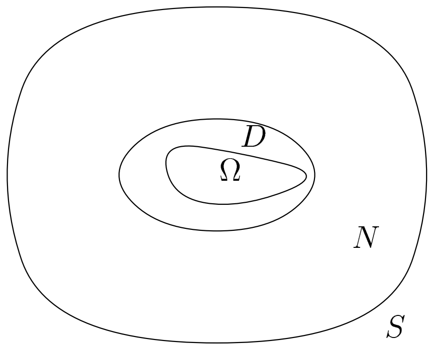


Figure 1: The scatterer $\Omega$ is in Euclidean domain $D \subset \mathbb{R}^3$. The domain $D$ is surrounded by the absorbing layer $N$. The Euclidean domain $D$ and the absorbing layer $N$ form together an absorbing manifold $M = D \cup N$. The computational domain is truncated to be finite by posing a boundary condition on a surface $S \subset M$.

is a clear advantage in numerics when e.g. the topological Whitney element bases are used in discretization ([4], [13]. Second, the stretching of the metric allows us to treat more general scattering geometries than before. We need not assume that the domain surrounded by the perfectly matched layer is convex. In fact, we may define a whole family of pseudo-Riemannian metrics for which the analysis can be carried out. Finally, the present formulation is completely invariant as it is done without a reference to specific coordinate systems. The main results of this paper on above topics are given in Section 4.

Let us mention that apart to the absorbing boundary conditions, the present work may turn out to be useful e.g. in analyzing scattering poles. In particular, we refer to the articles [20]-[22] for complex scaling method developed in connection to scattering poles.

## 2 Scattering problem and earlier results

We start this section by fixing some basic notations and concepts. The space $\mathbb{R}^3$ is considered as a manifold and it is equipped with the complex tangent bundle. The complex tangent spaces are denoted by $T_x\mathbb{R}^3 = \{u =$

$u^j(\partial/\partial x^j) \mid u^j \in \mathbb{C}\}$. For distinction, the real vector spaces are denoted by $T_x^{\mathbb{R}}\mathbb{R}^3$. The Euclidean space $\mathbb{R}^3$ is equipped with the standard metric tensor which is denoted by $g^{\mathcal{E}}$. The differential $r$–forms are denoted by $\Lambda^r(\mathbb{R}^3)$.

We adopt here the convention that vectors are denoted by lower case letters while forms are denoted by capital letters. We also use Einstein summation convention of summing over all repeated super- and sub-indeces.

In this work, we consider mainly time harmonic Maxwell's equations. Given a metric $g$, there is a well-known one-to-one correspondence between vector fields and 1–forms. In this work, we treat the electric and magnetic fields $E$ and $H$ exclusively as 1–forms. The time harmonic equations in vacuum corresponding to the time-harmonic time dependency $\exp(-i\omega t)$, $\omega > 0$, can be written as

$$dE = ik * H, \quad dH = -ik * E, \tag{1}$$

where $k = \omega/c = \omega/\sqrt{\epsilon_0\mu_0}$, and '$*$' denotes the Hodge–$*$ operator from 1–forms to 2–forms defined by the Euclidean metric. Here, we have used the scaling of the fields, $E \to \sqrt{\epsilon_0}E$ and $H \to \sqrt{\mu_0}H$ for reasons of notational symmetry. In the following, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary and a connected complement. By $B(x,R) \subset \mathbb{R}^3$ we denote the $x$-centered ball with radius $R$. For a submanifold $S \subset \mathbb{R}^3$ we denote by $i_S : S \to \mathbb{R}^3$ the natural embedding and by $i_S^* : \Lambda^r(\mathbb{R}^3) \to \Lambda^r(S)$ the corresponding pull-back.

Our main aim is to study the following scattering problem.

**Problem 2.1** *The exterior scattering problem for Maxwell's equation is to find 1-forms $E$ and $H$ in the exterior domain $\mathbb{R}^3 \setminus \overline{\Omega}$ which satisfy*

$$\begin{aligned} dE &= ik * H, & (2)\\ dH &= -ik * E, & (3)\\ i_{\partial\Omega}^* E &= \phi, & (4)\\ \lim_{R\to\infty}\int_{\partial B(0,R)} i_{\partial B(0,R)}^*(E-H) &\wedge\ i_{\partial B(0,R)}^*(\overline{E}-\overline{H}) = 0 & (5)\end{aligned}$$

*The solution of this problem is denoted by $(E_{\rm sc}, H_{\rm sc})$.*

Above the set $\Omega$ correspond to the scatter. At this point, we do not specify the smoothness properties of $\varphi$. The equations (2) and (3) say that the fields

$E$ and $H$ satisfy Maxwell's equations outside the scatterer. The equation (4) corresponds to the electric boundary condition on the boundary $\partial\Omega$ which says that the tangential component of the electric field on the boundary is given. The condition (5) is the weak type radiation condition for 1–forms corresponding to the standard Silver-Müller radiation condition (see e.g. [9]). Before defining the general absorbing manifolds, let us briefly summarize some of the earlier results obtained in the article [16] as a motivation for the discussion to ensue. In the cited article, it was assumed that the scatterer $\Omega$ is included in a strictly convex domain $D$ with a $C^3$-smooth boundary. In numerical approximation problems, $D \setminus \overline{\Omega}$ is the region where the fields are requested. In [16], a specific complex stretching of the space outside $D$ was defined as follows. Let $x \in \mathbb{R}^3 \setminus \overline{D}$. Since $D$ is strictly convex, there is a unique $p \in \partial D$ such that

$$h = \operatorname{dist}(x, \partial D) = |x - p|, \tag{6}$$

whence $x$ can be written as

$$x = p + hn, \tag{7}$$

where $n = n(p)$ is the exterior unit normal vector of $\partial D$ at $p$. Let $\tau : [0,\infty) \to [0,\infty)$ be a strictly increasing $C^3$–function with $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \tau'(t) = \infty$ and $\tau(0) = \tau'(0+) = 0$. We define the function $a : \mathbb{R}^3 \to \mathbb{R}^3$ as $a(x) = \tau(h)n(p)$, if $x \in \mathbb{R}^3 \setminus \overline{D}$, $h$ and $p$ given as in (6) and $a(x) = 0$ if $x \in D$. A complex stretching of $\mathbb{R}^3$ can be defined as

$$\tilde{x} : \mathbb{R}^3 \to \mathbb{C}^3, \quad \tilde{x}(x) = x + ia(x). \tag{8}$$

Let $\Gamma \subset \mathbb{C}^3$ be the manifold $\Gamma = \{z \in \mathbb{C}^3 \mid z = \tilde{x}(x),\ x \in \mathbb{R}^3\}$. In [16], the PML equation corresponding to the Helmholtz equation was defined via the analytic continuation of the Helmholtz equation to $\mathbb{C}^3$ and taking its restriction to $\Gamma$, that is,

$$(\partial_{z_1}^2 + \partial_{z_2}^2 + \partial_{z_3}^2 + k^2)u|_\Gamma = 0 \tag{9}$$

where $\partial_{z_j}$ are the complex derivatives in $\mathbb{C}^3$. The central features of the resulting equation are that

(a) the surface $\partial D$ is reflectionless

(b) outgoing waves are transformed to evanescent, exponentially decaying, waves outside $D$.

In particular, these properties imply that a truncation of the computational domain beyond the surface $\partial D$ by setting e.g. a Dirichlet condition, means effectively the introduction of an absorbing boundary condition. The main result of the cited article can be summarized as saying that

(c) the truncated problem is uniquely solvable, and the solution converges with exponential rate towards the physical scattering solution in $D$ as the absorbing layer gets thicker.

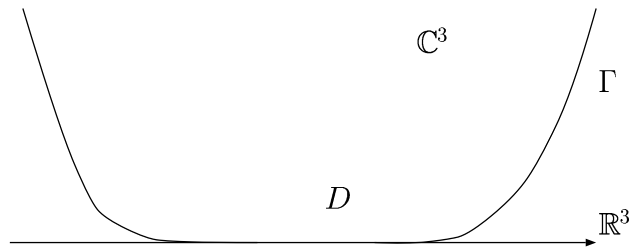


Figure 2: The PML-model can be represented as a stretching of the Euclidean space $\mathbb{R}^3$ to a surface $\Gamma \subset \mathbb{C}^3$.

# 3 Absorbing boundary conditions and manifolds with complex metric tensor

In the present work, we extend the previous results to Maxwell's equations and for more general geometric situations. Compared to the previous articles, we adopt a different a point of view: Rather than deforming the manifold $\mathbb{R}^3$ by complex coordinate stretching and considering the PML-equation via analytic continuation, we define a complex metric on the manifold.

For instance, the previous example (9) can be considered as a manifold with a complex metric in the following way. Let $u = (u^1, u^2, u^3)$ and $v = (v^1, v^2, v^3)$ be complex vectors in $\mathbb{C}^3$. Denote by $g^{\mathcal{C}}$ the complex Euclidean metric in $\mathbb{C}^3$, i.e., in Cartesian coordinates,

$$g^{\mathcal{C}}(u, v) = u \cdot v = \sum_{j=1}^{3} u^j v^j, \quad u^j,\ v^j \in \mathbb{C}.$$

Occasionally, we write $u^2 = g^{\mathcal{C}}(u,u)$, $u \in \mathbb{C}^3$. Then we define a complex metric on $\mathbb{R}^3$, denoted by $g_x : T_x\mathbb{R}^3 \times T_x\mathbb{R}^3 \to \mathbb{C}$, by setting

$$g_x(u,v) = g^{\mathcal{C}}(d\tilde{x}(u), d\tilde{x}(v)), \tag{10}$$

where $d\tilde{x}$ is the differential of the mapping $x \mapsto \tilde{x}(x)$ at $x$. It turns out that the metric $g$ thus defined has a number of properties that give rise to a more general class of what will be called *absorbing metrics.*

We start with some basic definitions. Given a manifold $(M,g)$ we say that $g$ is a complex pseudo-Riemannian metric, or simply a complex metric if $g$ is a symmetric complex valued 2-contravariant tensor which is non-degenerate, that is, the bilinear form $g(u,v) = g_x(u,v)$ in non-degenerate in the complex tangent space $T_xM$. The manifold $M$ with a complex metric is called a pseudo-Riemannian manifold. The Levi-Cività connection corresponding to $g$ is the connection $\nabla$ defined by the identity

$$\begin{aligned} 2g(\nabla_u v, w) &= ug(v,w) + vg(w,u) - wg(u,v) \\ &+ g([u,v],w) - g([u,w],v) - g([v,w],u), \end{aligned} \tag{11}$$

where $u$, $v$ and $w$ are vector fields on $M$. The connection coefficients $\Gamma^j_{ik}$ of $\nabla$ are complex. The connection $\nabla$ satisfies the identity

$$\nabla g = 0$$

and is torsion-free,

$$\nabla_u v - \nabla_v u = [u,v].$$

As usual, we use the notation $\nabla_k = \nabla_{\partial/\partial x^k}$, $x^k$ denoting the $k$th coordinate function. The Riemannian curvature tensor $R$ is defined by setting

$$R(u,v)w = \nabla_v \nabla_u w - \nabla_u \nabla_v w + \nabla_{[u,v]} w. \tag{12}$$

Next we are going to consider flat manifolds with complex metric, that is, manifolds for which $R = 0$.

**Lemma 3.1** *Assume that $(M,g)$ is a manifold with a complex metric which is diffeomorphic to $\mathbb{R}^3$ and flat, i.e., $R = 0$. Furthermore, assume that $g$ non-degenerate in the sense that for all tangent vectors $0 \neq u \in T_xM$, there is $v \in T_xM$ such that $g_x(u,v) \neq 0$. Then there are $g$-parallel orthonormal vector fields $e_j$ satisfying*

$$\nabla e_j = 0, \;\; g(e_j, e_k) = \delta_{jk}.$$

The proof of this result is postponed to Section 5.

For vector field $e_j$ there are the corresponding co-vector field $E_j$ such that at any $x \in M$ we have $\langle E_j, v\rangle = g(e_j, v)$ for $v \in T_xM$.

For the vector fields $e_j$ we can define the function $\tilde{x}^j : M \to \mathbb{C}$ which we call the integral functions of fields $e_j$. Let $x_0 \in M$ be a fixed point. For $x \in M$ we define the function

$$\tilde{x}^j(x) = \int_\gamma E_j = \int_\gamma g(e_j, \dot{\gamma}(t))\, dt$$

where $\gamma$ is an arbitrary path from $x_0$ to $x$. Later we show that these functions are well defined, independently of the path $\gamma$. The integral functions of $e_j$ define a function

$$\tilde{x} : M \to \mathbb{C}^3, \quad x \mapsto (\tilde{x}^1(x), \tilde{x}^2(x), \tilde{x}^3(x)).$$

Next we show that for flat pseudo-Riemannian 3-manifolds the mapping $\tilde{x}$ is an isometric immersion of $M$ to a totally real submanifold of $\mathbb{C}^3$ and conversely, any totally real embedding of $\mathbb{R}^3$ to $\mathbb{C}^3$ gives rise to a manifold with a flat complex metric.

To consider embeddings to $\mathbb{C}^3$ we identify $(\mathbb{C}^3, g^{\mathcal{C}})$ with $(\mathbb{R}^6, G)$ having the complex pseudo-metric as follows: If $(z^1, z^2, z^3) = (x^1 + iy^1, x^2 + iy^2, x^3 + iy^3) \in \mathbb{C}^3$ are the Euclidean coordinates, we define

$$G(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = \delta_{jk}, \quad G(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}) = -\delta_{jk}, \quad G(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}) = i\delta_{jk} \tag{13}$$

where $i$ is the imaginary unit. Furthermore, let us define the operator $J : T\mathbb{R}^6 \to T\mathbb{R}^6$ as

$$J : \frac{\partial}{\partial x^j} \mapsto \frac{\partial}{\partial y^j}, \quad J : \frac{\partial}{\partial y^j} \mapsto -\frac{\partial}{\partial x^j} \tag{14}$$

which defines an almost complex structure on $\mathbb{R}^6$. This almost complex structure corresponds the complex structure of $\mathbb{C}^3$ and $J$ corresponds the multiplication with the imaginary unit $i$ in $\mathbb{C}^3$. This identification is done to avoid the notion of analyticity related to complex manifolds.

We remind also that a submanifold $M$ of $\mathbb{C}^3 \cong \mathbb{R}^6$ is called totally real if for all $x \in M$, the real tangent spaces satisfy

$$T_x^{\mathbb{R}} M \cap J T_x^{\mathbb{R}} M = \{0\}.$$

Totally real manifolds play a crucial role in the theory of scattering poles (see [20]) and symplectic geometry (see [11]).

We can prove the following theorem for the embeddings to $\mathbb{C}^3$.

**Theorem 3.1** *Let $(M, g)$ be a flat pseudo-Riemannian manifold. Then the mapping (3) defines an immersion $\tilde{x} : M \to \mathbb{R}^6$ such that $g = \tilde{x}^* G$. In this case, the manifold $\tilde{x}(M) \subset \mathbb{R}^6$ is also totally real.*

*Conversely, if $M$ is a 3-submanifold such that there is a totally real immersion $\tilde{x} : M \to \mathbb{R}^6$, and $g = \tilde{x}^* G$, then $(M, g)$ is flat.*

The proof of the above Theorem is again postponed to Section 5.

We can now give the definition of an absorbing manifold.

**Definition 3.1** *A pseudo-Riemannian manifold $(M, g)$ is called an absorbing manifold, if*

1. *The manifold $(M, g)$ is flat and $M$ is diffeomorphic to $\mathbb{R}^3$. We denote this diffeomorphism by $\varphi : M \to \mathbb{R}^3$*

2. *There is relatively compact open set $D \subset M$ where the metric is Euclidean, that is, $g = \varphi^* g^{\mathcal{E}}$ in $D$.*

3. *For real tangent vectors $v \in T_x^{\mathbb{R}} M$, $v \neq 0$,*
$$g_x(v, v) \neq 0. \tag{15}$$

4. *The mapping $\tilde{x} : M \to \mathbb{C}^3$ given by formula (3) has the properties*
$$(\tilde{x}(x_1) - \tilde{x}(x_2))^2 = 0 \text{ if and only if } x_1 = x_2 \tag{16}$$
*and*
$$\mathrm{Re}\, \frac{\tilde{x}(x)^2}{|\varphi(x)|^2} < -2c_0 \text{ when } |\varphi(x)| > c_1 \tag{17}$$
*where $c_0, c_1 > 0$.*

For geometric interpretation of above conditions, let us consider the variety

$$\mathcal{L} = \{\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3 \mid \zeta^2 = \sum_{j=1}^{3} \zeta_j^2 = 0\}.$$

Then the condition (16) states that the fibres $z + \mathcal{L},\ z \in \tilde{x}(M)$ intersect the manifold $\tilde{x}(M)$ only once.

Furthermore, the condition (15) is equivalent that for real vectors $0 \neq v \in T_x^{\mathbb{R}} M$

$$G_{\tilde{x}(x)}(d\tilde{x}(v), d\tilde{x}(v)) = g_x(v, v) \neq 0, \tag{18}$$

which means that the real tangent vectors $T^{\mathbb{R}}_{\tilde{x}(x)}(\tilde{x}(M))$ are not tangent vectors of $\tilde{x}(x) + \mathcal{L}$. Hence the conditions (15) and (16) state that the manifold $\tilde{x}(M)$ intersects each fibre $z + \mathcal{L},\ z \in \tilde{x}(M)$ only once and transversally. In particular, by (16) the immersion $\tilde{x} : M \to \mathbb{C}^3$ is injective, i.e., an embedding. The condition (17) is related to the asymptotics of embedding $\tilde{x}$ at infinity.

The term *absorbing manifold* refers to the fact that if we place a source or a scatterer inside $D$, the electromagnetic fields within $D$ coincide with the scattering solution while outside $D$, the complex metric causes reflectionless attenuation of exponential type. Thus, $M \setminus \overline{D}$ is an absorbing layer. In Appendix 1 we show that the complex stretching introduced in Section 1 gives an example of an absorbing manifold.

To obtain absorbing manifolds without considering properties of the embedding $\tilde{x}$, we define the following class of manifolds with prescribed asymptotics at infinity.

**Definition 3.2** *Let $\eta \in \mathbb{C}$, $|\eta| = 1$, arg $\eta \in ]0, \pi]$. We say that the 3-manifold $(M, g)$ with a complex metric $g$ is asymptotically $\eta$-Euclidean if the following conditions hold:*

1. *There is a diffeomorphism $\varphi : M \to \mathbb{R}^3$ such that the metric $g$ satisfies*

$$\|g_x - \eta \mathring{g}_x\| \leq C_0 \frac{1}{(1 + |\varphi(x)|)^4}. \tag{19}$$

   *where $\mathring{g} = \varphi^* g^{\mathcal{E}}$ is the Euclidean metric on $M$. Above the norm is the tensor field norm in $(T_x^* M \otimes T_x^* M, \mathring{g})$.*

2. *The connection $\nabla$ corresponding to the metric $g$ approaches the metric $\mathring{\nabla}$ corresponding to the metric $\mathring{g}$ in the sense that*

$$\|\nabla - \mathring{\nabla}\| \leq C_1 \frac{1}{(1+|\varphi(x)|)^4}, \tag{20}$$

*the norm being the tensor field norm in $(T_x^*M \otimes T_x^*M \otimes T_xM, \mathring{g})$.*

In condition 2, observe that while the connections are not tensors, the difference of two connections is.

The above class is used to produce examples of absorbing manifolds. In Section 5, we prove the following result.

**Theorem 3.2** *Let $(M, g)$ be a flat, asymptotically $\eta$-Euclidean manifold containing a compact set $D$ where the metric is Euclidean, $g = \mathring{g}$. Then $(M, g)$ satisfies the condition (17). Moreover, if the constants $C_0$ and $C_1$ in the inequalities (19) and (20) are small enough, the conditions (15) and (16) are satisfied. In particular, then $(M, g)$ is an absorbing manifold.*

Summarizing, any totally real 3-manifold $\Gamma \subset \mathbb{C}^3$ which coincides with $\mathbb{R}^3$ near origin, intersect the fibres $z+\mathcal{L}$, $z \in \tilde{x}(M)$ only once transversally and is asymptotically the space $\eta^{1/2}\mathbb{R}^3 \subset \mathbb{C}^3$ is an example of absorbing manifolds.

It is our aim in the next section to develop a counterpart of the classical electromagnetic scattering theory on absorbing manifolds. What is more, we prove that the truncation of the computational domain in absorbing manifolds yield an exponentially converging approximation for the scattering solutions.

# 4 Scattering on absorbing manifolds

Let $(M, g)$ be an absorbing manifold. We define the Hodge–$*$ operator corresponding to the complex metric $g$ via the identity

$$U \wedge *V = g(U, V)d\mathrm{vol}_g, \quad d\mathrm{vol}_g = d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge d\tilde{x}^3.$$

Here the functions $\tilde{x}^j$ are the components of the embedding $\tilde{x} : M \to \mathbb{C}^3$ given in Theorem 3.1.

We start by considering the Helmholtz equation for $r$–forms,

$$(\Delta_g^r + k^2)U = 0,$$

where $U$ is an $r$-form and the $g$-Laplacian is defined in the usual manner as

$$\Delta_g^r = (-1)^r(*d*d - *d*d*), \tag{21}$$

where '$*$' is the Hodge–$*$ corresponding to the complex metric $g$. For Laplacian for 0-forms we use notation $\Delta_g = \Delta_g^0$. We remark that the scattering poles of the Euclidean Laplacian in $\mathbb{R}^3 \setminus \overline{\Omega}$ and the eigenvalues of the operator $-\Delta_g$ on the absorbing manifold $(M \setminus \overline{\Omega}, g)$ have a close connection which is discussed in Appendix 2.

Our first aim is to define a fundamental solution for the $g$–Helmholtz operator for 0–forms. For $x,\, y \in M$, denote $\tilde{x} = \tilde{x}(x)$, $\tilde{y} = \tilde{x}(y)$ and

$$\{\tilde{x} - \tilde{y}\} = (\sum_{j=1}^{3}(\tilde{x}^j - \tilde{y}^j)^2)^{1/2}. \tag{22}$$

Because of Condition 4 of Definition 3.1, we can choose the sign of the square root in a continuous way so that in a set $\{(x, y) \in M \times M \ : \ |\varphi(x)| > c(y)\}$ we have Im $\{\tilde{x} - \tilde{y}\} > 0$. Since the set $M \times M \setminus \mathrm{Diag}(M)$, $\mathrm{Diag}(M) = \{(x, x) : x \in M\}$ is simply connected, we can define the sign of $\{\tilde{x} - \tilde{y}\}$ in a unique continuous way in $M \times M$. In this way for all $x, y \in D$ we have either $\{\tilde{x} - \tilde{y}\} \geq 0$ or $\{\tilde{x} - \tilde{y}\} \leq 0$. The first condition corresponds absorbing manifolds for the outgoing radiation condition and the second one for the ingoing radiation condition. In the following we assume that the condition $\{\tilde{x} - \tilde{y}\} \geq 0$ is valid for $x, y \in D$, and call such manifolds *outgoing absorbing manifolds.*

With this notation, we can construct the fundamental solution of the Laplace-Beltrami equation.

**Theorem 4.1** *Let $(M, g)$ be an outgoing absorbing manifold. The function*

$$\Phi(x, y) = \frac{e^{ik\{\tilde{x}-\tilde{y}\}}}{4\pi\{\tilde{x} - \tilde{y}\}} \tag{23}$$

*is a fundamental solution of the operator $\Delta_g + k^2$ acting on 0-forms, i.e., it satisfies*

$$(\Delta_g + k^2)\Phi(\,\cdot\,, y) = -\delta_y. \tag{24}$$

*It satisfies the asymptotic estimate*

$$|\Phi(x,y)| \leq C_y e^{-c_0 k|\varphi(x)|}, \ \textit{when } |\varphi(x)| > c_2 \tag{25}$$

*where $c_0 > 0$ and diffeomorphism $\varphi : M \to \mathbb{R}^3$ are given in Definition 3.1 and $c_2 = c_2(y) > 0$.*

Above, the Dirac delta is to be interpreted with respect to the volume form defined by the metric $g$, e.g. if $\psi$ is a $C^\infty$ 0–form on $M$, we have

$$\int_M \psi(x)\delta_y(x) d\mathrm{vol}_g(x) = \psi(y).$$

The asymptotic condition (25) is the counterpart of the radiation condition on absorbing manifolds.

The proof of the above theorem as well as the other technical details in this section are again postponed to later sections.

Next we consider time harmonic Maxwell's equations on the absorbing manifold $(M, g)$, written for 1–forms as

$$dE = ik * H, \quad dH = -ik * E \tag{26}$$

with the Hodge–$*$ arising from the absorbing metric. Let $\Omega \subset D$ be a relatively compact open smooth subset of Euclidean part $D$ of the absorbing manifold $(M, g)$.

**Problem 4.1** *The exterior scattering problem for Maxwell's equation on the outgoing absorbing manifold in the set $M \setminus \Omega$ is to find 1-forms $E$ and $H$ which satisfy in $(M \setminus \Omega, g)$*

$$dE = ik * H, \tag{27}$$
$$dH = -ik * E, \tag{28}$$
$$i^*_{\partial\Omega} E = \phi \tag{29}$$

*and the radiation condition*

$$\|E(x)\|, \ \|H(x)\| \leq Ce^{-kc_0|\varphi(x)|}, \quad \textit{when } |\varphi(x)| > c_2, \ c_0, c_2 > 0. \tag{30}$$

In the above radiation condition the norms of the one–forms are the norms with respect to the $\overset{\circ}{g}$, i.e.

$$\|E(x)\|^2 = \|E(x)\|^2_{\overset{\circ}{g}} = \overset{\circ}{g}_x(\overline{E(x)}, E(x)),$$

We refer to this condition (30) as the *g-radiation condition.*

The existence and uniqueness proof of fields solving the above problem is based on a generalization of the Stratton-Chu representation theorems (see [9]). Indeed, similarly to the standard scattering theory, the fundamental solution for Maxwell's equations can be constructed by using the scalar fundamental solution. Let $A$ and $B$ be 1–forms given as

$$A = a_j d\tilde{x}^j, \quad B = b_j d\tilde{x}^j$$

with $\nabla A = \nabla B = 0$. We define $G(x,y) = G_{A,B}(x,y)$ as

$$G(x,y) = \begin{pmatrix} ik - (ik)^{-1} d * d* & *d \\ - * d & ik - (ik)^{-1} d * d* \end{pmatrix} \begin{pmatrix} \Phi(x,y)A \\ \Phi(x,y)B \end{pmatrix}. \tag{31}$$

We define the Maxwell operator $\mathcal{M}$ and the Hodge star operator for pairs $(E,H)$ as

$$\mathcal{M}\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix}, \quad *\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} *E \\ *H \end{pmatrix}.$$

As in the standard scattering theory, we have the following result.

**Theorem 4.2** *Let $(M,g)$ be an outgoing absorbing manifold, $y \in M$. The field $G = G_{A,B}$ satisfies*

$$(*\mathcal{M}^{\mathrm{T}} + ik)G(\,\cdot\,,y) = \begin{pmatrix} A\delta_y \\ B\delta_y \end{pmatrix} \ \text{in } M, \quad \mathcal{M}^{\mathrm{T}} = \begin{pmatrix} 0 & -d \\ d & 0 \end{pmatrix},$$

*and the asymptotic estimates*

$$\begin{aligned} |G(x,y)| &\leq C(y)e^{-kc_0|\varphi(x)|}, \\ |\nabla_x G(x,y)| &\leq C(y)e^{-kc_0|\varphi(x)|}, \end{aligned}$$

*when $|\varphi(x)| > c_2$, $c_2 = c_2(y)$.*

In the following, for pairs of 1-forms $P = (e, h) \in \Lambda^1(M) \times \Lambda^1(M)$ we use the notation $(P)_1 = e$, $(P)_2 = h$. The fundamental solution defined in the previous theorem gives following existence and uniqueness theorem for the scattering problem.

**Theorem 4.3** *Let $(M, g)$ be an outgoing absorbing manifold. The scattering problem 4.1 has a unique solution. Moreover, when $i^*_{\partial\Omega} E = \phi$ and $i^*_{\partial\Omega} H = \psi$ are boundary values of this solution, the field $E$ and $H$ can be given as*

$$\begin{aligned} E &= \Big\{ \int_{\partial\Omega} (-\phi \wedge (G_{d\tilde{x}^j,0})_2 + \psi \wedge (G_{d\tilde{x}^j,0})_1) \Big\} d\tilde{x}^j \\ H &= \Big\{ \int_{\partial\Omega} (-\phi \wedge (G_{0,d\tilde{x}^j})_2 + \psi \wedge (G_{0,d\tilde{x}^j})_1) \Big\} d\tilde{x}^j \end{aligned} \tag{32}$$

*Furthermore, in the neighborhood $D \setminus \overline{\Omega}$ where the metric is Euclidean, the solution $(E, H)$ equals in $D \setminus \overline{\Omega}$ to the solution $(E_{\rm sc}, H_{\rm sc})$ of the classical scattering problem (Problem 2.1).*

When the field computation is done in practice e.g. by the FEM, we need to replace the exterior domain problem by one that can be computed in a finite subdomain of the exterior domain. The scattering solution on an absorbing manifold serves as a useful tool for this purpose. First, it coincides with the physical exterior scattering solution near the scatterer. Secondly, it decays exponentially due to the $g$–radiation condition, so one could expect that if we force the solution to zero at a finite distance, the effect near the scatterer is minimal. To explain this in rigorous terms, we first formulate the truncated scattering problem.

**Problem 4.2** *The truncated scattering problem for Maxwell's equation on the outgoing absorbing manifold $(M, g)$ is to find 1–forms $\widetilde{E}$ and $\widetilde{H}$ which satisfy in a compact set $B \setminus \Omega$, $B \subset M$ the equation*

$$d\widetilde{E} = ik * \widetilde{H}, \tag{33}$$

$$d\widetilde{H} = -ik * \widetilde{E}, \tag{34}$$

$$i^*_{\partial\Omega} \widetilde{E} = \phi, \tag{35}$$

$$i^*_{\partial B} \widetilde{E} = 0. \tag{36}$$

Our aim is to show that the solution of this problem is close to the physical scattering problem (Problem 2.1) in the vicinity of the scatterer. To analyze the truncation of the manifold to a bounded domain, we need to specify the asymptotics of the manifold at infinity and pose a new local coercivity condition, as is explained below. Our main result for the truncated scattering problem is given in the following theorem.

**Theorem 4.4** *Let $(M,g)$ be an outgoing absorbing manifold which is asymptotically $\eta$-Euclidean. Moreover, assume that the metric $g$ satisfies the following local coercivity condition: For each bounded set $S \subset M$ there are constants $\alpha = \alpha(S) \in \mathbb{C}$ and $C = C(S) > 0$ such that*

$$\mathrm{Re}\,\left(\alpha \int_S \overline{U} \wedge *U\right) \geq C \int_S |U(x)|^2_{\overset{\circ}{g}} d\mathrm{vol}_{\overset{\circ}{g}}(x) \tag{37}$$

*for all 1–forms $U \in L^2(S,\Lambda^1)$. Furthermore, let $B \subset M$ be a bounded set so that*

$$R = \mathrm{dist}_{\overset{\circ}{g}}(\partial B, \Omega) > 0.$$

*When $R$ is large enough, the truncated scattering problem (4.2) for the 1–forms $(\widetilde{E},\widetilde{H})$ has a unique solution. Moreover, the solution $(\widetilde{E},\widetilde{H})$ is exponentially close in $D$ to the solution $(E_{\mathrm{sc}},H_{\mathrm{sc}})$ of Problem 2.1, that is, we have*

$$\begin{aligned} ||E_{\mathrm{sc}} - \widetilde{E}||_{L^2(D\setminus\Omega)} &\leq Ce^{-kc_0R}||\phi||_{\mathcal{H}^{-1/2}(\partial\Omega)}, \\ ||H_{\mathrm{sc}} - \widetilde{H}||_{L^2(D\setminus\Omega)} &\leq Ce^{-kc_0R}||\phi||_{\mathcal{H}^{-1/2}(\partial\Omega)}. \end{aligned} \tag{38}$$

Above $\mathcal{H}^{-1/2}(\partial\Omega)$ is a function space which defined later. At this point we note that for instance the space $C^1(\partial M,\Lambda^1)$ can be continuously be embedded to the space $\mathcal{H}^{-1/2}(\partial\Omega)$.

**Remark 1.** We note that the local coercivity condition (37) can be formulated for the metric tensor by requiring that for all $S$ there is $\alpha$ such that the matrix $\Re\alpha g^{ij}(x)\sqrt{\mathbf{g}(x)}$, $\mathbf{g}(x) = \det[g_{ij}(x)]$ is positive definite for $x \in S$.

# 5 Proofs concerning geometry and Green's functions

In this section, we have collected some detailed proofs that were omitted in the previous sections. We start by constructing a family of parallel vector fields that generalize the Euclidean coordinate basis in absorbing manifolds.

First we prove Lemma 3.1

*Proof.* (of Lemma 3.1) Let $(M,g)$ be a flat pseudo-Riemannian manifold, $\varphi : M \to \mathbb{R}^3$ a diffeomorphism such that $y = (y^1, y^2, y^3) = \varphi(x)$ are the Cartesian coordinates on $M$. Let $x_0 \in M$ be the point satisfying $\varphi(x_0) = 0$. First we construct linearly independent $g$–orthonormal vectors $e_j(x_0) \in T_{x_0}M$, $j = 1, 2, 3$. Since the matrix $G = [g_{ij}(x_0)]$ corresponding to the metric tensor in $y$-coordinates is non-degenerate, zero is not an eigenvalue of $G$. Thus we can define its power

$$G^{-1/2} = \frac{1}{2\pi i}\int_\gamma z^{-1/2}(G-z)^{-1}dz$$

where $z^{-1/2}$ is defined analytically in a set $\mathbb{C}\setminus z_0\overline{\mathbb{R}}_+,\ z_0 \neq 0$ and $\gamma \subset \mathbb{C}\setminus z_0\overline{\mathbb{R}}_+$ is a path having winding number one respect to the eigenvalues of $G$. The columns of the matrix $G^{-1/2}$ give us the $g$-orthonormal vectors $e_j(x_0)$. Let $\Gamma^j_{k\ell}$ denote the connection coefficients of $g$ in the coordinates $y^k$. By writing $e_j = \alpha^k_j \frac{\partial}{\partial y^k}$ the equation $\nabla_k e_j = \nabla_{\partial/(\partial y^k)} e_j = 0$ can be written as

$$(\frac{\partial}{\partial y^k} + A_k)\alpha_j = 0, \tag{39}$$

where $A_k = A_k(x) = (\Gamma^n_{k\ell})_{1\le n,\ell\le 3} \in \mathbb{C}^{3\times 3}$, $\alpha_j = \alpha_j(y) = (\alpha^\ell_j)_{1\le \ell\le 3}$ with $\alpha^\ell_j(0) = \delta^\ell_j$. Since the curvature vanishes, we have $[\nabla_k, \nabla_\ell] = 0$, or in the coordinate representation,

$$B_{k,\ell} = [\frac{\partial}{\partial y^k} + A_k, \frac{\partial}{\partial y^\ell} + A_\ell] = 0. \tag{40}$$

We can solve the equation (39) by a straightforward integration along paths originating from $y = 0$. To see that the solution is independent of the path, let $\gamma$ be a closed piecewise smooth path, and let $S$ be a smooth surface having

$\gamma$ as its boundary curve. By Stokes theorem and equations (39) and (40),

$$\oint_\gamma \frac{\partial \alpha_j}{\partial x^k} dx^k = -\oint_\gamma A_k \alpha_j dx^k$$

$$= -\int_S d(A_k\alpha_j) \wedge dx^k = \int_S \sum_{\ell<k} (B_{k,\ell}\alpha_j) dx^\ell \wedge dx^k = 0,$$

as a straightforward calculation shows.

Since $\nabla$ is a metric connection, it follows from the properties of the fields $e_j$ that

$$\nabla(g(e_j, e_k)) = 0.$$

This yields

$$g(e_j, e_k)|_x = g(e_j, e_k)|_{x_0} = \delta_{jk}, \quad x \in M.$$

The proof is thus complete. $\square$

**Lemma 5.1** *Let $(M, g)$ be as in Lemma 3.1. There exists an immersion $\tilde{x} : M \to \mathbb{C}^3$, $x \mapsto \tilde{x} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ such that for all vectors $v \in T_x M$ we have*

$$\langle d\tilde{x}^j, v\rangle = g(e_j, v).$$

*This mapping is called a* generalized coordinate stretching.

*Proof.* Let $E_j \in T^*M$ be the 1-form corresponding to $e_j$, i.e., $\langle E_j, v\rangle = g(e_j, v)$. Because the torsion of $\nabla$ vanishes,

$$\begin{aligned} \partial_\ell \langle E_j, \frac{\partial}{\partial x^k}\rangle - \partial_k \langle E_j, \frac{\partial}{\partial x^\ell}\rangle &= \nabla_\ell(g(e_j, \frac{\partial}{\partial x^k})) - \nabla_k(g(e_j, \frac{\partial}{\partial x^\ell})) \\ &= g(e_j, \left[\frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k}\right]) = 0 \end{aligned}$$

by the properties of the vectors $e_j$. But, this means just that $dE_j = 0$. Since $M$ is diffeomorphic to $\mathbb{R}^3$, by the Poincaré lemma, there is a mapping $\tilde{x}^j : M \to \mathbb{C}$ such that $E_j = d\tilde{x}^j$. Since the forms $E_j$ are linearly independent, $\tilde{x}$ is an immersion. This completes the proof. $\square$

To get further insight to the immersion $\tilde{x}$, let $\gamma : [0, 1] \to M$ be a given path from the point $x_0$ to the point $x \in M$, we have

$$\int_\gamma d\tilde{x}^j = \int_0^1 \langle E_j, \dot\gamma(t)\rangle dt = \int_0^1 g(e_j, \dot\gamma(t))dt.$$

By Stokes formula, the integrals depend only on the endpoints of the integration path. We may thus write

$$\tilde{x}^j = \int_{x_0}^{\varphi(x)} d\tilde{x}^j = \int_0^1 g(e_j, \dot{\gamma}(t))dt, \tag{41}$$

i.e., the functions $\tilde{x}^j$ are the integral functions of $e_j$. They are generalizations of the Cartesian coordinate functions on $M$.

Lemma 5.1 shows that there exists an immersion $\tilde{x} : M \to (\mathbb{R}^6, G)$ such that $g = \tilde{x}^*G$. Moreover, since the metric tensor $g$ in non-degenerate, the manifold $\tilde{x}(M)$ is totally real. Indeed, if $\tilde{x}(M)$ is not totally real at $x$, there are non-zero $a, b \in T_x^{\mathbb{R}}M$ such that $d\tilde{x}(a) + Jd\tilde{x}(b) = 0$. This would imply that $g(a + ib, c) = G(d\tilde{x}(a) + Jd\tilde{x}(b), d\tilde{x}(c)) = 0$ for all $c \in T_x^{\mathbb{R}}M$ and thus $g$ would not be non-degenerate. This gives us the first part of Theorem 3.1.

Consider the converse part of Theorem 3.1. We need to show that a totally real submanifold in $\mathbb{C}^3$ is flat.

**Lemma 5.2** *Let $M$ be a 3-manifold and assume that there is an embedding $\psi : M \to (\mathbb{R}^6, G)$ such that $\psi(M) \subset (\mathbb{R}^6, G)$ is a totally real submanifold. Let $g = \psi^*G$ be the pull-back of the complex pseudo-metric $G$. Then $(M, g)$ is flat.*

*Proof.* As before, we see that $g$ is non-degenerate since $\tilde{x}(M)$ is totally real. Denote the Levi-Cività connection of $(\mathbb{R}^6, G)$ by $\nabla^G$ and of $(M, g)$ by $\nabla$. By definition of the Levi-Cività connection (11), we get

$$\nabla_a b = \psi^*(\nabla^G_{\psi_* a} \psi_* b),$$

where $a$ and $b$ are real tangent vectors of $M$. We note that $\nabla^G_{\psi_* a}\psi_* b$ is well defined since $\psi_* a$ is a real tangent vector of $\psi(M)$ in $\mathbb{R}^6$. We want to generalize this for complex tangent vectors. We define a complexified push-forward map $\Psi$ as

$$\Psi : T_x M \to T_{\tilde{x}(x)}\mathbb{R}^6, \quad a + ib \mapsto \psi_* a + J\psi_* b,$$

where $a, b \in T_x^{\mathbb{R}}M$ are real tangent vectors. Since $M$ is totally real, we have

$$T^{\mathbb{R}}_{\psi(x)}\psi(M) \oplus JT^{\mathbb{R}}_{\psi(x)}\psi(M) = T^{\mathbb{R}}_{\psi(x)}\mathbb{R}^6,$$

implying that the mapping $\Psi$ is a bijection. In particular, there are smooth complex vector fields $e_j$ on $M$, $1 \leq j \leq 3$ such that

$$\Psi(e_j) = \frac{\partial}{\partial x^j}.$$

Since $G$ has the form (13), we see that $\nabla^G_A(JB) = J\nabla^G_A(B)$ for vector fields $A$ and $B$ in $\mathbb{R}^6$. Thus we have for real vector fields $a, b$ and $c$

$$\begin{aligned} &\Psi(\nabla_a(b+ic)) = \Psi(\nabla_a b + i\nabla_a c) = \psi_*(\nabla_a b) + J\psi_*(\nabla_a c) = \\ &= \nabla^G_{\psi_* a}\psi_* b + J\nabla^G_{\psi_* a}\psi_* c = \nabla^G_{\psi_* a}(\Psi(b+ic)). \end{aligned}$$

This together with $\nabla^G(\partial/\partial x^j) = 0$ implies that $\nabla e_j = 0$, i.e., the vector fields $e_j$ are parallel complex fields on $M$. By the definition of the curvature tensor $R$, we obtain

$$R(e_j, e_k)e_\ell = 0, \quad 1 \leq j, k, \ell \leq 3,$$

and since the vectors $e_j$ span the complex tangent space, we have $R = 0$. $\square$

Along with the above lemma, Theorem 3.1 is proved.

Next we consider the asymptotic behaviour of the immersion $\tilde{x}$ for $\eta$-Euclidean manifolds.

**Lemma 5.3** *For a flat, asymptotically $\eta$-Euclidean manifold $(M, g)$, the immersion $\tilde{x} : M \to \mathbb{C}^3$ satisfies*

$$\tilde{x}^1(x)^2 + \tilde{x}^2(x)^2 + \tilde{x}^3(x)^2 = \eta|\varphi(x)|^2 + \mathcal{O}(|\varphi(x)|),$$

*and*

$$|\tilde{x}^j(x)| \leq C|\varphi(x)| + \mathcal{O}(1),$$

*where $\varphi$ denotes the diffeomorphism appearing in the Definition 3.2 and $|\eta| = 1$, arg$(\eta) \in ]0, \pi]$.*

*Proof.* We can assume that the embedding $\varphi$ is given such a way that $\overset{\circ}{g}_{jk} = \delta_{jk}$. We denote the corresponding coordinates by $y = \varphi(x)$. Let $\Gamma^i_{jk}(y)$ be the Christoffel symbols of the connection $\nabla$ in $y$-coordinates. In the $y$-coordinates the Christoffel symbols of the connection $\nabla^\circ$ vanish. Thus by Condition 2 of Definition 3.2 implies

$$|\Gamma^i_{jk}(y)| \leq C_1(1 + |y|)^{-4}. \tag{42}$$

Let $e_j = \alpha_j^k \partial_k$ be the representations of the parallel vector fields $e_j$ in $y$-coordinates. The equation $\nabla e_j = 0$ takes the form

$$\frac{\partial \alpha_j^k}{\partial y^l} = -\Gamma_{lp}^k(y)\alpha_j^p(y). \tag{43}$$

Consider next $\alpha_j^k(y)$ at two points $y_1, y_2 \in \mathbb{R}^3$ and a path $\gamma_1$ connecting the points $y_1$ and $y_2$. We choose $\gamma_1$ to be a line segment between points $y_1$ and $y_2$. Then by using (42) and (43) along this line segment, we see that

$$|\alpha_j^k(y_2)| \leq |\alpha_j^k(y_1)| e^{C_1} \tag{44}$$

Thus, by considering a fixed $y_1$, we see that there is a uniform bound

$$|\alpha_j^k(y_2)| \leq C. \tag{45}$$

Observe that here the exact value of the constant $C$ has not yet estimated but we just know that $|\alpha_j^k(y)|$ is uniformly bounded by some constant $C$. To improve this estimate, let us connect the points $y_1, y_2 \in \mathbb{R}^3$ by a path $\gamma_2$ which the union of an arc from $y_1$ to $\frac{|y_1|}{|y_2|}y_2$ on the the ball $\{y : \ |y| = |y_1|\}$ and the line segment from the point $\frac{|y_1|}{|y_2|}y_2$ to the point $y_2$. By using differential equation (43) with estimate (42) and (44) on the path $\gamma_2$ together with the fact that the length of path $\gamma_2$ less than $(1+\pi)|y_1 - y_2|$, we obtain

$$|\alpha_j^k(y_2) - \alpha_j^k(y_1)| \leq \frac{C_1(1+\pi)|y_1 - y_2|}{\min((1+|y_1|)^4, (1+|y_2|)^4)} e^{C_1} |\alpha_j^k(y_1)|. \tag{46}$$

This relation is valid for any $y_1$ and $y_2$. Estimates (45) and (46) show for an arbitrary sequence $y_p \in \mathbb{R}^3$, $p \leq |y_p| < p+1$, $p = 0, 1, 2, \ldots$ that $(\alpha_j^k(y_p))_{p=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$. Indeed,

$$\sum_{q=p}^{\infty} |\alpha_j^k(y_{q+1}) - \alpha_j^k(y_q)| \leq \sum_{q=p}^{\infty} \frac{2CC_1e^{C_1}(1+\pi)}{(1+q)^3} \leq \frac{CC_1e^{C_1}(1+\pi)}{(1+p)^2}. \tag{47}$$

Thus there exists limits $d_j^k = \lim_{|\varphi(y)| \to \infty} \alpha_j^k(y)$ and

$$|\alpha_j^k(y) - d_j^k| \leq \frac{CC_1e^{C_1}(1+\pi)}{(1+|y|)^2}. \tag{48}$$

Since $e_j$ are $g$-orthonormal, i.e., $g(e_j, e_l) = \delta_{jl}$, Condition 1 of Definition 3.2 yields that

$$\sum_{k=1}^{3} \alpha_j^k(y)\alpha_l^k(y) = \eta^{-1}\delta_{jl} + \mathcal{O}((1+|y|)^{-4}). \tag{49}$$

Thus we see that

$$\sum_{k=1}^{3} d_j^k d_l^k = \eta^{-1}\delta_{jl}. \tag{50}$$

This means that the inverse of the matrix $[d_j^k]$ is its real transpose times $\eta$. Next, let us consider complex coordinates $\tilde{x}^j(y)$ defined in formula (41). By using as the path $\gamma$ a line segment from the origin to $y$ in formula (41), we obtain

$$\tilde{x}^j(y) = \int_\gamma g(e_j, \dot{\gamma})dt = \int_0^{|y|} g_{kl}(t\hat{y})\alpha_j^k(t\hat{y})\hat{y}^l\,dt, \quad \hat{y} = \frac{y}{|y|}. \tag{51}$$

Combining the above estimates, we get

$$\tilde{x}^j(y) = \int_0^{|y|} (\eta\delta_{kl}d_j^k\hat{y}^l + \mathcal{O}((1+|t|)^{-2}))dt = \eta\sum_{k=1}^{3} d_j^k y^k + \mathcal{O}(1). \tag{52}$$

By (50) and (52) we get

$$\sum_{j=1}^{3} \tilde{x}^j(y)^2 = \eta|y|^2 + \mathcal{O}(|y|).$$

This gives us the first assertion of Lemma 5.3. Moreover, formula (52) yield $|\tilde{x}^j(y)| \le C|y| + \mathcal{O}(1)$ which is the second assertion of Lemma 5.3. $\square$

By using previous considerations we can prove Theorem 3.2.

*Proof* (of Theorem 3.2) Let us consider a flat, $\eta$-Euclidean manifold $(M, g)$ which contain a compact Euclidean subset $D$. First we observe that condition (19) for the metric tensor with $C_0 < 1$ yields condition (15).

To obtain condition (16), we need to consider the proof of Lemma 5.3 in more detail. By replacing vector fields $e_j$ by their linear combinations, we can assume that $e_j$ are parallel $g$-orthonormal vector fields for which $d_j^k = \eta^{-1/2}\delta_j^k$ (see (50)). Thus when $\varphi(y_1)$ is large enough, we have $\sum_k |\alpha_j^k(y_1)|^2 \leq 2$. By using estimate (44) for this $y_1$ we obtain

$$|\alpha_j^k(y)| \leq C = 2e^{C_1},\ y \in M.$$

This gives now estimate (45) where $C = 2e^{C_1}$, that is, $C$ can be estimated by using $C_1$. Substituting this in to the formula (48) we see that

$$|\alpha_j^k(y) - d_j^k| \leq \frac{2C_1(1+\pi)e^{2C_1}}{(1+|y|)^2}. \tag{53}$$

Next, let $x_1, x_2 \in M$, $y_j = \varphi(x_j) \in \mathbb{R}^3$ and consider integral

$$\tilde{x}^j(x_1) - \tilde{x}^j(x_2) = \int_\gamma g(e_j, \dot\gamma(t))\, dt$$

(compare with 51) where the path $\gamma$ is a line segment between $y_1$ and $y_2$. Then inequalities (53) and (19) imply that

$$\begin{aligned} & |\tilde{x}^j(x_1) - \tilde{x}^j(x_2) - \sum_{l=1}^3 \eta(y_1^l - y_2^l)d_j^l| \\ \leq\ & \left|\int_\gamma g(e_j - d_j^k\partial_k, \dot\gamma)dt\right| + \left|\int_\gamma (g - \mathring{g})(d_j^k\partial_k, \dot\gamma)dt\right| \\ \leq\ & (1+C_0)\cdot 2C_1(1+\pi)e^{C_1}|y_1 - y_2| + C_0|y_1 - y_2|. \end{aligned}$$

Thus when $C_0 < \frac{1}{2}$ and $C_1$ is so small that $C_1(1+\pi)e^{2C_1} < \frac{1}{8}$, we see that conditions (16) and (17) are satisfied. $\square$

Next we give the proof of Theorem 4.1. We consider first the complex volume form

$$d\mathrm{vol}_g = \mathbf{g}^{1/2}(x)dx^1 \wedge dx^2 \wedge dx^3,\ \mathbf{g}(x) = \det(g_{ij}(x)).$$

This definition coincide with the definition given in Section 4. Indeed, $d\tilde{x}^j(e_k) = \delta_k^j$ which means that $d\tilde{x}^j$ are dual to $g$-ortonormal basis $e_k$. A direct computation gives

$$\begin{aligned} *d\tilde{x}^1 = d\tilde{x}^2 \wedge d\tilde{x}^3,\ *d\tilde{x}^2 &= -d\tilde{x}^1 \wedge d\tilde{x}^3,\ *d\tilde{x}^3 = d\tilde{x}^1 \wedge d\tilde{x}^2, \\ d\mathrm{vol}_g &= d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge d\tilde{x}^3. \end{aligned} \tag{54}$$

Next we prove Theorem 4.1.

*Proof of Theorem 4.1.* The asymptotics of $\Phi(x,y)$ follows from Lemma 5.3. We will show equation (24) in the case $k=0$. The proof for general case is analogous. At first stage, we keep $y$ fixed. We start by showing that for some $\beta \in \mathbb{C}$

$$\int_{\mathbb{R}^3} \Phi(x,y)\Delta_g \psi(x) d\mathrm{vol}_g(x) = \beta\psi(y) \tag{55}$$

for any $\psi \in C_0^\infty(\mathbb{R}^3)$. For this end, let us consider differential $d\tilde{x}|_y$ at point $y$. Next we fix some local coordinates near $y$ and thus we can identify the vectors $v \in T_y^{\mathbb{R}}M$ with $v \in \mathbb{R}^3$ and $d\tilde{x}|_y$ with the matrix $H \in \mathbb{C}^{3\times 3}$. By formula (18), the function

$$h(v) = g(v,v) = G(Hv,Hv) = H^t Hv \cdot v \neq 0$$

for $v \in \mathbb{R}^3 \setminus \{0\}$. Since $h(\mathbb{R}^3)$ is a convex set, and function $h(v)$ does not vanish for non-zero vectors $v$, we see there is $\alpha(y) \in \mathbb{C}$ such that

$$\alpha(y) g_y(v,v) > 0 \text{ for } v \in T_y^{\mathbb{R}}M \setminus \{0\}. \tag{56}$$

This implies that there is $\xi \in \mathbb{C},\ |\xi| = 1$ such that $g(v,v) \notin \xi\mathbb{R}_-$ for $v \in T_y^{\mathbb{R}}M$.

To analyze the Green's function, we define a regularized function

$$\Phi_\varepsilon(x,y) = \frac{1}{4\pi((\tilde{x}-\tilde{y})^2 + \xi\varepsilon^2)^{1/2}}$$

where $\xi \in \mathbb{C},\ |\xi| = 1$ is chosen as above. A simple calculation using (54) shows that

$$\Delta_g \Phi_\varepsilon(x,y) \quad = \quad *d*d\Phi_\varepsilon(x,y) = \frac{-3\xi\varepsilon^2}{4\pi((\tilde{x}-\tilde{y})^2 + \xi\varepsilon^2)^{5/2}}.$$

Since $\Delta_g$ is symmetric with respect to the inner product corresponding volume $d\mathrm{vol}_g$, we obtain in polar coordinates $(r,\theta) \in \mathbb{R}_+ \times S^2$, $r\theta = x - y$

$$\int_{\mathbb{R}^3} \Phi_\varepsilon(x,y)\Delta_g\psi(x) d\mathrm{vol}_g(x) = \int_{\mathbb{R}^3} \Delta_g\Phi_\varepsilon(x,y)\psi(x) d\mathrm{vol}_g(x)$$

$$= \int_{\mathbb{R}_+}\int_{S^2} \frac{-3\xi\varepsilon^2}{4\pi((\tilde{x}(r\theta)-\tilde{y})^2 + \xi\varepsilon^2)^{5/2}} \psi(r\theta) r^2 \mathbf{g}^{1/2}\, dr d\theta.$$

By using the variable $s = \varepsilon^{-1} r$, the matrix $H = d\tilde{x}|_y$ and $a = \mathbf{g}(y)^{1/2} = \det(H^t H)^{1/2}$ (where the sign of the root has been chosen continuously in $M$), we obtain

$$\int_{\mathbb{R}^3} \Phi_\varepsilon(x,y)\Delta_g \psi(x) d\mathrm{vol}_g(x) = \int_{\mathbb{R}_+}\int_{S^2} \frac{3a\xi s^2}{4\pi(s^2(H\theta)^2+\xi)^{5/2}}\psi(0)\, ds d\theta + \mathcal{O}(\varepsilon).$$

This shows that the equation (55) is valid for some $\beta$. To find out the value of $\beta$, we consider the last integral as an analytic function of the matrix $H$ defined in the set

$$W = \{H \in \mathbb{C}^{3\times 3}:\ (H\theta)^2 \not\in \xi\overline{\mathbb{R}}_-,\ v \in \mathbb{R}^3 \setminus \{0\}\}.$$

Next we use the fact that $W$ a path-connected and open. When $H$ is a real matrix, a change of variable shows that

$$\int_{\mathbb{R}_+}\int_{S^2} \frac{3a\xi s^2}{4\pi(s^2(H\theta)^2+\xi)^{5/2}}\, ds d\theta = \pm 1,$$

where the sign depends on the sign of $a$. Since this integral depends analytically on $H$ and is constant $\pm 1$ for real matrices $H \in W$, this integral has to be a constant in $W$. Hence we have shown for any point $y$ that $\beta = \pm 1$. As a function of $y$, $\beta = \beta(y)$ is continuous and get values $\pm 1$. Since metric is Euclidean in $D$, we see that $\beta(y) = 1$ in $D$. Since $M$ is connected and $\beta(y)$ depends continuously on $y$, we have to have $\beta(y) = 1$ in $M$. □

# 6 Proofs for Maxwell's equations on absorbing manifolds.

Having established the fundamental properties of absorbing manifolds, we discuss next the solutions of Maxwell's equations on these manifolds. For a rigorous discussion of Maxwell's equations, we fix first some notations and the function spaces needed in the sequel. Let us denote by $*^{\mathcal{E}}$ the Hodge-* operator defined by Euclidean metric $g^{\mathcal{E}}$ in $\mathbb{R}^3$. For for an open set $S \subset \mathbb{R}^3$ we use the space $L^2(S)$ of 1−forms with the norm

$$\|U\|_{L^2(S)} = (\int_S \overline{U} \wedge *^{\mathcal{E}} U)^{1/2}. \qquad (57)$$

Sometimes, we write simply $\|U\|_{L^2(S)} = \|U\|_2$. The corresponding space of square integrable $r$–forms is denoted as $L^2(S,\Lambda^r)$. We define an exterior Sobolev space

$$\mathcal{H}(S,\Lambda^r) = \{U \in L^2(S,\Lambda^r) \mid dU \in L^2(S,\Lambda^{r+1})\},$$

equipped with the norm $\|U\|_{\mathcal{H}} = \|dU\|_2 + k\|U\|_2$, where $k > 0$ is the wave number, $k = \omega/c$, included in this norm for dimensional reasons. To consider boundary value problems for forms, we define the space of $r$-forms, which tangential component vanish on the boundary:

$$\mathcal{H}_0(S,\Lambda^r) = \{U \in \mathcal{H}(S,\Lambda^r) \mid i^*_{\partial\Omega}U = 0\}$$

where $i^* : C^\infty(S,\Lambda^r) \to C^\infty(\partial S,\Lambda^r)$ is the pull-back corresponding embedding $i : \partial S \to \overline{S}$. Further, if $S \subset \mathbb{R}^3$ is an exterior domain, i.e., a neighborhood of infinity, we define

$$\mathcal{H}_{\rm rad}(S,\Lambda^r) = \{U \in \mathcal{H}(S,\Lambda^k) \mid \ U \text{ satisfies the radiation condition (5)}\}.$$

Finally, we denote

$$\mathcal{H}_{\rm rad,0}(S,\Lambda^r) = \mathcal{H}_0(S,\Lambda^r) \cap \mathcal{H}_{\rm rad}(S,\Lambda^r).$$

Next we generalize the definitions of the previous spaces on an absorbing manifold. When the Euclidean space $\mathbb{R}^3$ is replaced by the absorbing manifold $M$ we have two metric tensors $g$ and $\mathring{g}$. We denote by $*$ and $*^\circ$ the Hodge-* operators defined by metric $g$ and $\mathring{g}$, correspondingly. The space $L^2(S,\Lambda^r)$, $S \subset M$ is defined with the norm analogous to formula (57) by using the positive definite metric $\mathring{g}$. On the absorbing manifold $M$ the usual radiation condition needs to be replaced by an exponential decay condition (30). When $S$ is an exterrior domain in $M$, i.e., $M \setminus S$ is compact, we denote

$$\mathcal{H}^g_{\rm rad}(S,\Lambda^r) = \{U \in \mathcal{H}(S,\Lambda^k) \mid \ U \text{ satisfies the } g\text{–radiation condition (30)}\},$$

and, correspondingly we define $\mathcal{H}^g_{\rm rad,0}(S,\Lambda^r)$.

To solve boundary value problems, we need to specify the mapping properties of the boundary traces. We denote by $d_{\partial S} : C^\infty(\partial S,\Lambda^r) \to C^\infty(\partial S,\Lambda^{r+1})$ the exterior derivative of forms over the boundary. We denote by $\mathcal{H}^s(\partial S,\Lambda^r)$ the closure of $C^\infty(\partial S,\Lambda^r)$ with respect to the norm

$$\|\omega\|_{\mathcal{H}^s(\partial S)} = \|d_{\partial S}\omega\|_{H^s(\partial S,\Lambda^{r+1})} + k\|\omega\|_{H^s(\partial S,\Lambda^r)}.$$

By the theorem of Paquet ([18]), we have the following result: The mapping $i^*_{\partial S}$ extends to a continuous and surjective mapping

$$i^*_{\partial S}: \mathcal{H}(S,\Lambda^r) \to \mathcal{H}^{-1/2}(\partial S,\Lambda^r).$$

For later purposes, let us define the analog of scattering problem 2.1 on absorbing manifolds for non-smooth boundary values:

**Problem 6.1** *Find the fields $(E,H) \in \mathcal{H}^g_{\mathrm{rad}}(M\setminus\overline{\Omega},\Lambda^1)\times\mathcal{H}^g_{\mathrm{rad}}(M\setminus\overline{\Omega},\Lambda^1)$ satisfying Maxwell's equations on an absorbing manifold $(M,g)$ with the boundary condition*

$$i^*_{\partial\Omega}E = \varphi \in \mathcal{H}^{-1/2}(\partial\Omega).$$

We refer to this problem later as a *Full Space Problem.*

Next we consider the proof of Theorem 4.2. For this, we have to consider Green's functions of Maxwell's equations. Because of the form of Maxwell's Green's functions (31) is given in explicit terms, we can compute by using the formula (54) how the Maxwell's operator operates to the function $G_{A,B}(x,y)$. This computation together with Theorem 4.1 gives the formula

$$(*\mathcal{M}^{\mathrm{T}} + ik)G(\,\cdot\,,y) = \begin{pmatrix} A\delta_y \\ B\delta_y \end{pmatrix}$$

in $M$. Moreover, by formulas (54) and Lemma 5.3, we see that the function $G_{A,B}$ satisfies the $g$-radiation condition (30). This proves Theorem 4.2.

Next we apply Theorem 4.2 for the Problem 6.1.

**Theorem 6.1** *The Full Space Problem (Problem 6.1) is uniquely solvable.*

*Proof.* The existence and uniqueness of the solution is based on the representation theorems. Let $G = G_{A,B}$ be the fundamental solution of the Maxwell operator on $(M,g)$, and $G^{\mathcal{E}} = G^{\mathcal{E}}_{A,B}$ the corresponding solution on $(\mathbb{R}^3, g^{\mathcal{E}})$,

$$G^{\mathcal{E}}(x,y) = \begin{pmatrix} ik - (ik)^{-1} d *^{\mathcal{E}} d *^{\mathcal{E}} & *^{\mathcal{E}} d \\ - *^{\mathcal{E}} d & ik - (ik)^{-1} d *^{\mathcal{E}} d *^{\mathcal{E}} \end{pmatrix} \begin{pmatrix} \Phi^{\mathcal{E}}(x,y)A^{\mathcal{E}} \\ \Phi^{\mathcal{E}}(x,y)B^{\mathcal{E}} \end{pmatrix}$$

with

$$\Phi^{\mathcal{E}}(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|},$$

and $A^{\mathcal{E}} = a_j dx^j$ and $B^{\mathcal{E}} = b_j dx^j$. We have

$$(*^{\mathcal{E}}\mathcal{M}^{\mathcal{E}} + ik)G^{\mathcal{E}}(\,\cdot\,, y) = \left(\begin{array}{c} A^{\mathcal{E}}\delta_y \\ B^{\mathcal{E}}\delta_y \end{array}\right).$$

Now suppose that $(E_{\rm sc}, H_{\rm sc}) \in \mathcal{H}_{\rm rad}(\mathbb{R}^3 \setminus \overline{\Omega}, \Lambda^1) \times \mathcal{H}_{\rm rad}(\mathbb{R}^3 \setminus \overline{\Omega}, \Lambda^1)$ is the unique solution of the classical scattering problem of Maxwell's equations with the boundary data $i^*_{\partial\Omega}E^{\mathcal{E}} = \varphi \in \mathcal{H}^{-1/2}(\partial\Omega)$. Let us denote by brevity $X_{\rm sc} = (E_{\rm sc}, H_{\rm sc})$, $G^{\mathcal{E}}_{A^{\mathcal{E}},B^{\mathcal{E}}} = G^{\mathcal{E}} = (G^{\mathcal{E}}_1, G^{\mathcal{E}}_2)$. An application of Stokes formula in the exterior domain together with the classical radiation condition yields the representation formula

$$\begin{aligned} E_{\rm sc} \wedge *^{\mathcal{E}}A^{\mathcal{E}} + H_{\rm sc} \wedge *^{\mathcal{E}}B^{\mathcal{E}} &= \left\{\int_{\mathbb{R}^3\setminus\overline{\Omega}} (X_{\rm sc} \wedge \mathcal{M}^{\rm T}G^{\mathcal{E}} - \mathcal{M}X_{\rm sc} \wedge G^{\mathcal{E}})\right\} d{\rm vol}_{g^{\mathcal{E}}} \\ &= \left\{\int_{\partial\Omega} (H_{\rm sc} \wedge G^{\mathcal{E}}_1 - E_{\rm sc} \wedge G^{\mathcal{E}}_2)\right\} d{\rm vol}_{g^{\mathcal{E}}} \end{aligned}$$

By choosing first $A^{\mathcal{E}} = dx^j$, $B^{\mathcal{E}} = 0$ and then $A^{\mathcal{E}} = 0$, $B^{\mathcal{E}} = dx^j$, we obtain the classical Stratton-Chu representation formulas for 1–forms, corresponding to those in (32) with Euclidean metric. Now we simply define a solution of Maxwell's equation on the absorbing manifold $M$ by

$$\begin{aligned} E &= \left\{\int_{\partial\Omega} (-E_{\rm sc} \wedge (G_{d\tilde{x}^j,0})_2 + H_{\rm sc} \wedge (G_{d\tilde{x}^j,0})_1)\right\} d\tilde{x}^j \qquad (58) \\ H &= \left\{\int_{\partial\Omega} (-E_{\rm sc} \wedge (G_{0,d\tilde{x}^j})_2 + H_{\rm sc} \wedge (G_{0,d\tilde{x}^j})_1)\right\} d\tilde{x}^j. \end{aligned}$$

Obviously, these fields satisfy Maxwell's equations in $M \setminus \overline{\Omega}$. Furthermore, since $G(x, y) = G^{\mathcal{E}}(x, y)$, as $(x, y) \in D \times D$, we have $(E(x), H(x)) = (E_{\rm sc}(x), H_{\rm sc}(x))$ as $x \in D$. Thus $(E, H)$ satisfy the boundary condition (29). Thus we have proven the existence of a solution which satisfy formula (32).

Next we prove the uniqueness. First we observe that if $(E, H)$ is a solution of the Problem 6.1 with $f = 0$, then an application of the Stokes formula yields a representation formula (32). By setting

$$\begin{aligned} E'_{\rm sc} &= \left\{\int_{\partial\Omega} (-E \wedge (G^{\mathcal{E}}_{dx^j,0})_2 + H \wedge (G^{\mathcal{E}}_{dx^j,0})_1)\right\} dx^j \\ H'_{\rm sc} &= \left\{\{\int_{\partial\Omega} (-E \wedge (G^{\mathcal{E}}_{0,dx^j})_2 + H \wedge (G^{\mathcal{E}}_{0,dx^j})_1)\right\} dx^j, \end{aligned}$$

we have a solution of the classical scattering problem (Problem 2.1) with homogeneous electric boundary value. By the classical uniqueness theorem (see [9]), we deduce that this solution must be zero, so also $E = H = 0$ at the boundary. By using the Green's formula with $(E, H)$ and $G_{A,B}(x, y)$ on $M \setminus \Omega$, we see that $E = H = 0$ everywhere on $M \setminus \Omega$. $\square$

# 7 Approximation of scattering problem with bounded computational domain

Next, we discuss the truncation of the exterior domain of absorbing manifolds. Let us fix some notations by using local coordinates on $M$. If $U$ be a 1-form on $M$, $U = U_j dx^j$, we denote by $u = u^j(\partial/\partial x^j)$ the vector fields obtained as $u^j = g^{jk}U_k$.

The following definition has a counterpart in the previous work [16].

**Definition 7.1** *Let $(M, g)$ be an asymptotically $\eta$-Euclidean absorbing manifold and $C_0$ the constant appearing in Definition 3.2. The point $x \in M$ is said to be in the exponential range if $|\varphi(x)| > 4C_0^{1/4}(\mathrm{Im}\ \eta^{1/2})^{-1/4}$.*

The fact that $(M, g)$ is an outgoing absorbing manifold yields that $\sqrt{\mathbf{g}(x)} \to \eta^{3/2}\sqrt{\mathbf{g}^\circ}$, $\mathbf{g}^\circ = \det\,[\mathring{g}_{ij}]$ as $|\varphi(x)|$ grows, so the exponential range corresponds to a far field region.

Definition of the exponential range and the estimate (19) implies immediately the following estimate.

**Lemma 7.1** *Let $(M, g)$ be an asymptotically $\eta$-Euclidean manifold and let $x \in M$ be in the exponential range in the sense of Definition 7.1. Then*

$$\left| \sqrt{\frac{\mathbf{g}(x)}{\mathbf{g}^\circ(x)}} g(U, V) - \eta^{1/2}\mathring{g}(U, V) \right| \leq \frac{\mathrm{Im}\ \eta^{\frac{1}{2}}}{4} \|U\|\|V\|.$$

The previous lemma is used to establish an energy type estimate for fields in the exponential range.

**Lemma 7.2** *(a) Let $S \subset (M, g)$ be a bounded domain with a smooth boundary, and assume that $S$ is in the exponential range. Assume that $(E, H) \in \mathcal{H}_0(S, \Lambda^1) \times \mathcal{H}(S, \Lambda^1)$ satisfy Maxwell's equations*

$$dE = ik * H + K, \quad dH = -ik * E + J \tag{59}$$

*in $S$, where $(K, J) \in L^2(S, \Lambda^2) \times L^2(S, \Lambda^2)$. Then*

$$k(\|E\|_2 + \|H\|_2) \leq C(\|K\|_2 + \|J\|_2). \tag{60}$$

*(b) Let $S \subset (M, g)$ be the complement of a bounded domain with a smooth boundary, $S$ being in the exponential range. Assume that $(E, H) \in \mathcal{H}^g_{\mathrm{rad},0}(S, \Lambda^1) \times \mathcal{H}^g_{\mathrm{rad}}(S, \Lambda^1)$ satisfy Maxwell's equations (59) in $S$, where $(K, J) \in L^2(S, \Lambda^2) \times L^2(S, \Lambda^2)$. Then the estimate (60) holds.*

*Proof.* (a) Since $E \in \mathcal{H}_0(S, \Lambda^1)$, we have

$$\int_S dE \wedge \overline{H} - \int_S E \wedge d\overline{H} = 0. \tag{61}$$

By substituting Maxwell's equations it follows that

$$\begin{aligned} \left| -ik \int_S E \wedge \overline{*E} + ik \int_S \overline{H} \wedge *H \right| &= |\int_S E \wedge \overline{J} - \overline{H} \wedge K| \\ &\leq (\|E\|_2 + \|H\|_2)(\|J\|_2 + \|K\|_2). \end{aligned}$$

By definition of the Hodge-$*$, we have

$$E \wedge \overline{*E} = \overline{\sqrt{\frac{\mathbf{g}}{\mathbf{g}^\circ}} g(E, \overline{E})} \mathrm{dvol}_{\overset{\circ}{g}}, \quad \overline{H} \wedge *H = \sqrt{\frac{\mathbf{g}}{\mathbf{g}^\circ}} g(H, \overline{H}) \mathrm{dvol}_{\overset{\circ}{g}}.$$

From Lemma 7.1, we obtain the estimate

$$\begin{aligned} &\left| -ik \int_S \overline{\sqrt{\frac{\mathbf{g}}{\mathbf{g}^\circ}} g(E, \overline{E})} \mathrm{dvol}_{\overset{\circ}{g}} + ik \int_S \sqrt{\frac{\mathbf{g}}{\mathbf{g}^\circ}} g(H, \overline{H}) \mathrm{dvol}_{\overset{\circ}{g}} \right| \\ \geq\ & (\mathrm{Im}\ \eta^{\frac{1}{2}})(\|E\|_2^2 + \|H\|_2^2) - k \int_S \Big( \left| \sqrt{\frac{\mathbf{g}}{\mathbf{g}^\circ}} g(E, \overline{E}) - \eta^{1/2} \overset{\circ}{g}(E, \overline{E}) \right| \\ &+ \left| \sqrt{\frac{\mathbf{g}}{\mathbf{g}^\circ}} g(H, \overline{H}) - \eta^{1/2} \overset{\circ}{g}(H, \overline{H}) \right| \Big) \mathrm{dvol}_{\overset{\circ}{g}} \\ \geq\ & k(1 - \frac{2}{4})(\mathrm{Im}\ \eta^{\frac{1}{2}})(\|E\|_2^2 + \|H\|_2^2), \end{aligned}$$

and so the desired estimate follows.

(b) The proof of the claim is similar to the proof above. The only thing that needs to be observed is that the equation (61) follows from the exponential decay of the solutions. This in turn is a consequence of the representation formula (58). □

As a corollary, we get the following important result.

**Theorem 7.1** *Let $S \in (M, g)$ be as in Lemma 7.2. The Dirichlet problem*

$$dE = ik * H, \quad dH = -ik * E \text{ in } S,$$

$$i^*_{\partial S} E = \phi \in \mathcal{H}^{-1/2}(\partial S)$$

*has a unique solution $(E, H) \in \mathcal{H}(S, \Lambda^1) \times \mathcal{H}(S, \Lambda^1)$ if $S$ is bounded, or $(E, H) \in \mathcal{H}^g_{\mathrm{rad}}(S, \Lambda^1) \times \mathcal{H}^g_{\mathrm{rad}}(S, \Lambda^1)$ if $S$ is an exterior domain.*

*Proof.* We consider the case where $S$ is bounded. The case where $S$ is an exterior domain goes with obvious changes. Let $R : \mathcal{H}^{-1/2}(\partial S) \to \mathcal{H}(S, \Lambda^1)$ be a right inverse of $i^*_{\partial S}$. We seek to solve the Dirichlet problem in the form $(E, H) = (E_0 + R\phi, H)$, where $(E_0, H) \in \mathcal{H}_0 \times \mathcal{H}$ satisfies

$$dE_0 = ik * H - dR\phi, \quad dH = -ik * E_0 - ik * R\phi,$$

or in operator notation,

$$(\mathcal{M} + ik*)X = Y,$$

where $X = (E_0, H)$, $Y = (-ik * R\phi, dR\phi)$, and

$$\mathcal{M} = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} : \mathcal{D}(\mathcal{M}) \subset L^2(S, \Lambda^1) \times L^2(S, \Lambda^1) \to L^2(S, \Lambda^2) \times L^2(S, \Lambda^2),$$

the domain of $\mathcal{M}$ being $\mathcal{D}(\mathcal{M}) = \mathcal{H}_0(S, \Lambda^1) \times \mathcal{H}(S, \Lambda^1)$.

It follows from Lemma 7.2 that $(\mathcal{M} + ik*)$ is one-to-one. To show that it has a dense range, let us denote

$$\langle X, Y \rangle_S = \int_S X_1 \wedge Y_1 + \int_S X_2 \wedge Y_2, \tag{62}$$

where $X = (X_1, X_2) \in L^2(S, \Lambda^1) \times L^2(S, \Lambda^1)$, $Y = (Y_1, Y_2) \in L^2(S, \Lambda^2) \times L^2(S, \Lambda^2)$. By applying Stokes theorem, we can see that $\mathcal{M}^* = \mathcal{M}$, where

$\mathcal{M}^*$ is the adjoint of $\mathcal{M}$ respect of pairing $\langle\cdot\,,\cdot\rangle_S$. Since $(\mathcal{M}+ik*)^* = \mathcal{M}+ik*$ is one-to-one, the range of $\mathcal{M}+ik*$ is dense. Moreover, $\mathcal{M}$ is a closed operator in the sense that its graph is closed. Thus, by applying Lemma 7.2 we see that the range of $\mathcal{M}+ik*$ is closed, and hence $\mathcal{M}+ik*$ is surjective.

By open mapping theorem, $\mathcal{M}+ik*$ has a bounded inverse, proving the existence of the solution $(E,H)$ along with the norm estimate

$$k(\|E\|_2 + \|H\|_2) \leq C\|\phi\|_{\mathcal{H}^{-1/2}}.$$

□

Consider now the truncated scattering problem on an absorbing manifold:

**Problem 7.1** *Let $D \subset M$ be a neighborhood of the scatterer where $g$ is Euclidean, and let $B_R$ be a neighborhood of $D$, where distance in $\mathring{g}$-metric satisfies* $\mathrm{dist}(D, M\setminus B_R) = R > 0$. *The truncated scattering problem on an outgoing absorbing manifold problem is to find $(E,H) \in \mathcal{H}(B_R\setminus\overline{\Omega})\times\mathcal{H}(B_R\setminus\overline{\Omega})$ satisfying*

$$dE = ik * H, \quad H = -ik * E,$$

*with the boundary conditions*

$$i^*_{\partial\Omega}E = \phi \in \mathcal{H}^{-1/2}, \quad i^*_{\partial B_R}E = 0.$$

Unfortunately, Theorem 7.1 gives the existence of the Dirichlet problem only in domains away from the scatterer, so the solution of the above problem cannot be deduced to exist. However, we can reduce the existence of the solution to the existence in the far field region. To this end, we need to define appropriate equivalent problems corresponding to the scattering problem 6.1 and to the truncated problem 7.1. Therefore, for $r < R$ let $B_r \subset\subset B_R$ be a neighborhood of $D$ that is slightly smaller than $B_R$, $0 < \mathrm{dist}(D, M\setminus B_r) = r$. If $r$ is large enough, the exterior domain $M\setminus\overline{B}_r$ as well as the annulus $B_R\setminus\overline{B}_r$ lie entirely in the exponential range. Then, by Theorem 7.1 we may find a solution $(E,H) \in \mathcal{H}^g_{\mathrm{rad}}(M\setminus\overline{B}_r)\times\mathcal{H}^g_{\mathrm{rad}}(M\setminus\overline{B}_r)$ satisfying

$$i^*_{\partial B_r}E = \psi.$$

Let $\delta > 0$ be small enough such that $r+\delta < R$. We define a *double surface operator* $Z : \mathcal{H}^{-1/2}(\partial B_r) \to \mathcal{H}^{-1/2}(\partial B_{r+\delta})$ by setting

$$Z\psi = i^*_{\partial B_{r+\delta}}E,$$

where $E$ is the electric 1-form above. Similarly, Theorem 7.1 guarantees the existence of $(\widetilde{E}, \widetilde{H}) \in \mathcal{H}(B_R \backslash \overline{B}_r) \times \mathcal{H}(B_R \backslash \overline{B}_r)$ satisfying Maxwell's equations with the boundary conditions

$$i^*_{\partial D_r} \widetilde{E} = \psi \in \mathcal{H}^{-1/2}(\partial B_r), \quad i^*_{\partial D_R} \widetilde{E} = 0.$$

Thus, we may define the double surface operator in the truncated domain, $Z_R : \mathcal{H}^{-1/2}(\partial B_r) \to \mathcal{H}^{-1/2}(\partial B_{r+\delta})$ by setting

$$Z_R \psi = i^*_{\partial B_{r+\delta}} \widetilde{E}.$$

These definitions lead us to consider the following rather non-standard boundary value problems.

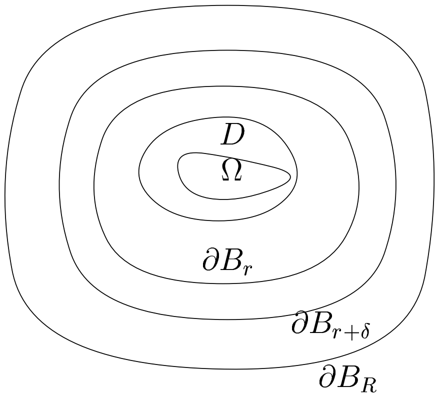


Figure 3: The absorbing layer is truncated by surface $\partial B_R$. To analyze the convergence when $R \to \infty$, the full space problem is transformed to problem in a bounded domain $B_{r+\delta}$ by using operator $Z_R$ which maps the value of the field on $\partial B_r$ to the value of the scattered field on $\partial B_{r+\delta}$.

**Problem 7.2** *(a) Find the fields* $(E, H) \in \mathcal{H}(B_{r+\delta} \setminus \overline{\Omega}) \times \mathcal{H}(B_{r+\delta} \setminus \overline{\Omega})$ *satisfying the Maxwell's equations with the boundary conditions*

$$i^*_{\partial\Omega} E = \phi \in \mathcal{H}^{-1/2}(\partial\Omega), \quad i^*_{\partial B_{r+\delta}} E = Z i^*_{\partial B_r} E.$$

*(b) Find the fields* $(E, H) \in \mathcal{H}(B_{r+\delta} \backslash \overline{\Omega}) \times \mathcal{H}(B_{r+\delta} \backslash \overline{\Omega})$ *satisfying the Maxwell's equations with the boundary conditions*

$$i^*_{\partial\Omega} E = \phi \in \mathcal{H}^{-1/2}(\partial\Omega), \quad i^*_{\partial B_{r+\delta}} E = Z_R i^*_{\partial B_r} E.$$

We have the following result.

**Theorem 7.2** *The problems 6.1 and 7.2 (a) are equivalent. More precisely, Problem 6.1 has a solution* $(E, H)$ *if and only if Problem 7.2 (a) has a solution* $(\widetilde{E}, \widetilde{H})$*, in which case* $(\widetilde{E}, \widetilde{H}) = (E, H)|_{B_{r+\delta}\setminus\overline{\Omega}}$.

*Similarly, the problems 7.1 and 7.2 (b) are equivalent in the same sense.*

*Proof.* If $(E, H)$ is a solution of the Problem 6.1, a solution for Problem 7.2 (a) is obtained by setting $(\widetilde{E}, \widetilde{H}) = (E, H)|_{B_r\setminus\overline{\Omega}}$. To prove the converse, let $(\widetilde{E}, \widetilde{H})$ be the solution of the Problem 7.2 (a). We define $(E', H') \in \mathcal{H}^g_{\mathrm{rad}}(M \setminus \overline{B}_r) \times \mathcal{H}^g_{\mathrm{rad}}(M \setminus \overline{B}_r)$ as the solution of Maxwell's equations with the Dirichlet data

$$i^*_{\partial B_r} E' = i^*_{\partial B_r} \widetilde{E},$$

which is possible by Theorem 7.1. Then also

$$i^*_{\partial B_{r+\delta}} E' = i^*_{\partial B_{r+\delta}} \widetilde{E},$$

and by Theorem 7.1, $(\widetilde{E}, \widetilde{H}) = (E', H')$ in the annulus $B_{r+\delta} \setminus \overline{B}_r$. Hence, $(\widetilde{E}, \widetilde{H})$ can be continued to a scattering solution by gluing it with $(E', H')$.

The proof of part (b) goes similarly. □

From now on, we consider solely the solvability of the Problems 7.2. From Section 6.1, we already know that the problem (a) is solvable. The solvability of problem (b) and thus the solvability of the Problem 7.1 is proved by showing that is in fact a small compact perturbation of the problem (a).

**Lemma 7.3** *The operators* $Z$*,* $Z_R : \mathcal{H}^{-1/2}(\partial B_r) \to \mathcal{H}^{-1/2}(\partial B_{r+\delta})$ *are compact operators with*

$$\|Z - Z_R\| < Ce^{-R/2}.$$

*Proof.* Let $(E, H) \in \mathcal{H}^g_{\mathrm{rad}}(M \setminus \overline{B}_r) \times \mathcal{H}^g_{\mathrm{rad}}(M \setminus \overline{B}_r)$ and $(\widetilde{E}, \widetilde{H}) \in \mathcal{H}(B_R \setminus \overline{B}_r) \times \mathcal{H}(B_R \setminus \overline{B}_r)$ be the solutions of Maxwell's equations in $M \setminus \overline{B}_r$ and $B_R \setminus \overline{B}_r$, respectively, with

$$i^*_{\partial B_r} E = i^*_{\partial B_r} \widetilde{E} = \psi, \quad i^*_{\partial B_R} \widetilde{E} = 0,$$

and write

$$E' = E|_{B_R\setminus\overline{B}_r} - \widetilde{E}, \quad H' = H|_{B_R\setminus\overline{B}_r} - \widetilde{H}.$$

Then $(E', H')$ satisfy the Maxwell's equations in $B_R \setminus \overline{B}_r$ with the boundary conditions

$$i^*_{\partial B_r} E' = 0, \quad i^*_{\partial B_R} E' = i^*_{\partial B_R} E.$$

From Theorem 4.2, Theorem 4.3, and Lemma 7.2, we obtain the estimate

$$k(\|E'\|_2 + \|H'\|_2) \leq C\|i^*_{\partial B_R} E\|_{\mathcal{H}^{-1/2}} \leq Ce^{-kc_0R/2}||\phi||_{\mathcal{H}^{-1/2}}.$$

Also, the same theorems imply that

$$\|dE'\|_2 + \|dH'\|_2 \leq C\|i^*_{\partial B_R} E\|_{\mathcal{H}^{-1/2}} \leq Ce^{-kc_0R/2}||\phi||_{\mathcal{H}^{-1/2}}.$$

Thus,

$$\|i^*_{\partial B_{r+\delta}} \widetilde{E}\|_{\mathcal{H}^{-1/2}(\partial B_{r+\delta})} \leq \|\widetilde{E}\|_{\mathcal{H}(B_{r+\delta}\setminus\overline{B}_r)} \leq Ce^{-kc_0R/2}||\phi||_{\mathcal{H}^{-1/2}},$$

as claimed. □

Our goal is to find Fredholm type equations that are equivalent to the problems (a) and (b) of Theorem 7.2 and that are close to each other. To this end, let

$$\rho : \mathcal{H}^{-1/2}(\partial\Omega) \to \mathcal{H}(B_{r+\delta} \setminus \overline{\Omega}), \quad \mathcal{R} : \mathcal{H}^{-1/2}(\partial B_{r+\delta}) \to \mathcal{H}(B_{r+\delta} \setminus \overline{\Omega})$$

be right inverse of the trace mappings $i^*_{\partial\Omega}$ and $i^*_{\partial B_{r+\delta}}$, respectively. We assume that $i^*_{\partial B_r}\rho = i^*_{\partial B_r}\mathcal{R} = 0$ and $i^*_{\partial B_R}\rho = i^*_{\partial\Omega}\mathcal{R} = 0$. Consider first the problem (a) of Theorem 7.2 that is known to be uniquely solvable. We write the component $E$ of the solution $(E, H)$ as

$$E = \rho\phi + \mathcal{R}i^*_{\partial B_{r+\delta}} E + \widetilde{E}, \quad \widetilde{E} \in \mathcal{H}_0(B_{r+\delta} \setminus \overline{\Omega}),$$

or, since $i^*_{\partial B_{r+\delta}} E = Zi^*_{\partial B_r} E$ and $i^*_{\partial B_r} E = i^*_{\partial B_r} \widetilde{E}$,

$$E = \rho\phi + \mathcal{R}Zi^*_{\partial B_r} \widetilde{E} + \widetilde{E}. \tag{63}$$

With these notations, we find that $X = (\widetilde{E}, H) \in \mathcal{H}_0(B_{r+\delta}\setminus\overline{\Omega}) \times \mathcal{H}(B_{r+\delta}\setminus\overline{\Omega})$ satisfies

$$(\mathcal{M} + ik*)X + \mathcal{B}X = Y, \tag{64}$$

where $\mathcal{M}$ is the Maxwell operator in $B_{r+\delta}\setminus\Omega$ with domain $\mathcal{D}(\mathcal{M}) = \mathcal{H}_0(B_{r+\delta}\setminus \overline{\Omega}) \times \mathcal{H}(B_{r+\delta}\setminus\overline{\Omega})$ and

$$\mathcal{B} = \begin{pmatrix} ik* & d \\ -d & ik* \end{pmatrix} \begin{pmatrix} \mathcal{R}Zi^*_{\partial B_r} & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} -ik*\rho\phi \\ d\rho\phi \end{pmatrix}. \tag{65}$$

To solve the problem (b) of theorem 7.2, we write a similar ansatz,

$$(\mathcal{M} + ik*)X + \mathcal{B}_R X = Y, \tag{66}$$

where the operator $\mathcal{B}_R$ is defined as the operator $\mathcal{B}$ in (65) but with $Z$ replaced by $Z_R$. We have

**Lemma 7.4** *Assume that there are constants $\alpha \in \mathbb{C}$ and $C > 0$ such that*

$$\mathrm{Re}\left(\alpha \int_{B_{r+\delta}} \overline{U} \wedge *U\right) \geq C\|U\|^2_{L^2(B_{r+\delta},\Lambda^1)}.$$

*Then the equations (64) and (66) are of Fredholm type*

*Proof.* Let us denote by $\mathrm{Ran}(\mathcal{M}) \subset L^2(B_{r+\delta}\setminus\overline{\Omega},\Lambda^2) \times L^2(B_{r+\delta}\setminus\overline{\Omega},\Lambda^2)$ the range of $\mathcal{M}$. As in the classical case (see [17]) one can show that $\mathrm{Ran}(\mathcal{M})$ and thus $*\mathrm{Ran}(\mathcal{M}) \subset L^2(B_{r+\delta}\setminus\overline{\Omega},\Lambda^1)\times L^2(B_{r+\delta}\setminus\overline{\Omega},\Lambda^1)$ is a closed subspace.

Let us next consider the operator $*\mathcal{M}$ in the space $\mathrm{Ran}(*\mathcal{M})$. For $X = (E,H) \in \mathrm{Ran}(*\mathcal{M})$ we observe that

$$(*\mathcal{M})^2 \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} \Delta^1_g E \\ \Delta^1_g H \end{pmatrix}$$

where $\Delta^1_g$ is the Laplace-Beltrami operator for 1-forms defined in (21). For these $(E,H)$ the boundary value problem

$$(*\mathcal{M})^2 \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}, \quad \begin{pmatrix} E \\ H \end{pmatrix} \in \mathcal{D}((*\mathcal{M})^2)\cap \mathrm{Ran}(*\mathcal{M})$$

can be written as

$$\begin{aligned} \begin{pmatrix} \Delta^1_g & 0 \\ 0 & \Delta^1_g \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} &= \begin{pmatrix} A \\ B \end{pmatrix} \\ i^*_{\partial(B_{r+\delta}\setminus\Omega)} E = 0, &\qquad i^*_{\partial(B_{r+\delta}\setminus\Omega)}(d*E) = 0, \\ i^*_{\partial(B_{r+\delta}\setminus\Omega)}(*H) = 0, &\qquad i^*_{\partial(B_{r+\delta}\setminus\Omega)}(*dH) = 0 \end{aligned}$$

The principal part of the differential operator $\Delta_g^1$ is diagonal matrix with diagonal elements $\Delta_g^0$. Thus by inequality (56) $\Delta_g^1$ is an elliptic differential operator. Moreover when $r+\delta$ is big enough, conditions (19) and (20) imply that the metric on $\partial B_{r+\delta}$ is $C^1$-close to $\eta\delta_{ij}$. Thus (see e.g. [2], Proposition 6.1.1 or [1]) we can see that the above boundary boundary conditions satisfy Shapiro-Lopatinskij condition. Thus, by e.g. [2], $(*\mathcal{M})^2 : \mathcal{D}((*\mathcal{M})^2) \cap \mathrm{Ran}(*\mathcal{M}) \to \mathrm{Ran}(*\mathcal{M})$ is a Fredholm operator. This in particular yields that the operator $*\mathcal{M}+ik$ in the space $\mathrm{Ran}(*\mathcal{M})$ has a parametrix $A$ for which $A(*\mathcal{M}+ik) \subset I+K$ in $\mathrm{Ran}(*\mathcal{M})$ where $K$ is a compact operator.

Consider the quadratic forms

$$\mathcal{F}(U,V) = \int_{B_{r+\delta}} \overline{U} \wedge *V, \quad \mathcal{F}^{\mathcal{E}}(U,V) = \int_{B_{r+\delta}} \overline{U} \wedge *^{\mathcal{E}} V.$$

The assumption of the lemma guarantees that $\mathcal{F}$ is coercive and thus the Lax-Milgram lemma gives that there exists a continuous projection

$$P : L^2(B_{r+\delta} \setminus \overline{\Omega}, \Lambda^1) \times L^2(B_{r+\delta} \setminus \overline{\Omega}, \Lambda^1) \to \mathrm{Ran}(*\mathcal{M}),$$

Let $Q = 1-P$. Next we show that $QX \in \mathrm{Ker}(\mathcal{M})$. First, we observe that for all $U \in L^2 \times L^2$, $V \in \mathcal{D}(\mathcal{M})$,

$$\mathcal{F}(U, i*\mathcal{M}V) = \int_{B_{r+\delta}} \overline{U} \wedge i\mathcal{M}V = \mathcal{F}^{\mathcal{E}}(U, i*^{\mathcal{E}} \mathcal{M}V).$$

On the other hand, we know that $i*^{\mathcal{E}}\mathcal{M}$ is self-adjoint with respect to $\mathcal{F}^{\mathcal{E}}$. Therefore, we have

$$0 = \mathcal{F}(QX, i*\mathcal{M}V) = \mathcal{F}^{\mathcal{E}}(QX, i*^{\mathcal{E}} \mathcal{M}V) = \mathcal{F}^{\mathcal{E}}(i*^{\mathcal{E}} \mathcal{M}QX, V)$$

for all $V \in \mathcal{D}(\mathcal{M})$. Since $\mathcal{D}\mathcal{M}$ is dense, we deduce that $*^{\mathcal{E}}\mathcal{M}QX = 0$ and hence $\mathrm{Ran}\,(Q) \subset \mathrm{Ker}\,(*\mathcal{M})$.

Consider now the equation (64). By writing $X = PX + QX$ the equation splits as

$$\begin{aligned} (*\mathcal{M}+ik)PX + P*\mathcal{B}(PX+QX) &= PY, \\ ikQX + Q*\mathcal{B}(PX+QX) &= QY. \end{aligned}$$

By operating with parametrix $A$ to the first equation we obtain

$$\begin{aligned}(I+K)PX + AP * \mathcal{B}(PX+QX) &= APY,\\ QX + \frac{1}{ik}Q * \mathcal{B}(PX+QX) &= \frac{1}{ik}QY,\end{aligned}$$

which is Fredholm by the compactness of $\mathcal{B}$ and $K$.

This equation (66) is treated similarly. $\square$

This result at hand, we are ready to give the proof of the Theorem 4.4.

*Proof* (of Theorem 4.4) From the previous lemma, the operator $(*\mathcal{M}+ik+*\mathcal{B})$ is invertible. Assume that

$$(*\mathcal{M} + ik + *\mathcal{B}_{\mathcal{R}})X_0 = 0.$$

Obviously, then $X_0$ satisfies the estimate

$$\|X\|_2 \leq \|(*\mathcal{M} + ik + *\mathcal{B}_{\mathcal{R}})\| \|\mathcal{B} - \mathcal{B}_R\| \|X_0\|_2. \tag{67}$$

When $R > 0$ is large enough, Lemma 7.3 implies that $\|\mathcal{B}-\mathcal{B}_R\|$ tends to zero, implying that $X_0 = 0$, and the claim follows from the Fredholm property. Finally, the norm estimate (38) follows from the formula (67) and Lemma 7.3. $\square$

# Appendix 1: Convex geometry

In this appendix we show that the previously discussed PML model around a strictly convex domain $D$ (see introduction) can be obtained as a special case of the equations for absorbing manifolds. More precisely, we shall prove the following result.

**Theorem 7.3** *Let $g$ be the complex metric on $M = \mathbb{R}^3$ defined in (10) and embedding $\tilde{x}$ given in formula (8). The manifold $(\mathbb{R}^3, g)$ is an absorbing manifold, i.e., it satisfies the properties listed in Definition 3.1.*

*Proof.* We have to check that the properties of Definition 3.1 are satisfied. For the diffeomorphism $\varphi : M \to \mathbb{R}^3$ we use just the identical mapping. Since

$$d\tilde{x}|_x = I + i\, da|_x$$

and $da|_x$ is real, symmetric and positive definite (see [16], Lemma 3.1) the metric $g$ is symmetric and non-degenerate. Moreover $g(v,v) \neq 0$ for real $v \in T_x^{\mathbb{R}}M$ and hence (15) is satisfied. In [16] one uses stretching function $\tilde{x}_s(x) = x + sa(x)$ which depends on a stretching parameter $s$, Re $s \geq 0$. For real $s$ the stretched manifold $\tilde{x}_s(M)$ is $\mathbb{R}^3$ and thus flat. Since the Riemannian curvature tensor of metric $g_s = \tilde{x}_s^* g^{\mathcal{C}}$ is an analytic function of $s$ the stretched manifold $(M, g_s)$ is also flat for complex $s$, particularly for $s = i$. The condition (16) is a straightforward consequence of Lemma 3.2 in [16] and (17) follows immediately from the definition of the stretching function and $\varphi$. This proves the claim. □

# Appendix 2: Scattering poles and absorbing manifolds

Next discuss shortly the relation of absorbing manifolds, the Sjöstrand-Zworski complex scaling, and scattering poles of obstacles.

Let $\Delta : H^2(\mathbb{R}^3 \setminus \Omega) \cap H_0^1(\mathbb{R}^3 \setminus \Omega) \to L^2(\mathbb{R}^3 \setminus \Omega)$ be the Laplacian in the exterior domain of the obstacle $\Omega \subset B(0,R)$ and consider the resolvent

$$(\Delta + k^2)^{-1} : L^2_{comp}(\mathbb{R}^3 \setminus \Omega) \to L^2_{loc}(\mathbb{R}^3 \setminus \Omega)$$

where Im $k > 0$. This operator has a meromorphic continuation through $\mathbb{R}_+$ to the lower half plane and the poles of the meromorphic continuation are called the scattering poles. For instance, when $\Omega$ is a sphere of radius $r$, the zeros $k$ of the Hankel function $H_0^{(1)}(rk)$ are scattering poles and the corresponding solutions of the Helmholtz equation, called resonances, are the solutions $H_0^{(1)}(k|x|)$. The canonical method to study the scattering poles is to use Sjöstrand-Zworski complex scaling, where one studies the equation

$$(\partial_{z_1}^2 + \partial_{z_2}^2 + \partial_{z_3}^2 + k^2) f(z_1, z_2, z_3) = 0 \tag{68}$$

on a totally real submanifold $\Gamma$ of $\mathbb{C}^3$, where complex derivatives $\partial_{z_j}$ are computed by using almost analytic continuation of $f$ to $\mathbb{C}^3$. Next, let $\Gamma = \tilde{x}(\mathbb{R}^3 \setminus \Omega)$, where $\tilde{x} : \mathbb{R}^3 \to \mathbb{C}^3$ , $\tilde{x}(x) = \alpha(|x|)x$, and $\alpha \in C^\infty(\mathbb{R}; \mathbb{C})$, $|\alpha(t)| = 1$, $\arg(\alpha(t)) \in [0, \pi/2]$ is a function for which $\alpha(t) = 1$ for $t < R$ and $\alpha(t) = i$ for $t > R + 1$. By using previous methods together with [16] the equation (68) can be written in the form

$$(\Delta_g + k^2) F = 0 \tag{69}$$

where $g = \tilde{x}^* g^{\mathcal{C}}$ and $F(x) = f(\tilde{x}(x))$. Thus we see that the eigenvalues of (68) on $\Gamma \setminus \Omega$ and (69) on $\mathbb{R}^3 \setminus \Omega$ coincide. By [20], the eigenvalues of the equation (68) and the scattering poles of the obstacle $\Omega$ coincide in the domain $S = \{\theta \in \mathbb{C} : \ -\pi < \arg\theta < 0\}$. Thus the scattering poles in $S$ can be considered as eigenvalues of Laplace-Beltrami operator on an absorbing manifold $(\mathbb{R}^3 \setminus \Omega, g)$. This fact will be discussed elsewhere in detail.

# References


[1] S. Agmon, A. Douglis, L. Nirenberg: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. Comm. Pure Appl. Math. **17**1964, 35–92.

[2] M. S. Agranovich: *Elliptic boundary problems.* Encyclopaedia Math. Sci., 79, Partial differential equations, IX, 1–144, 275–281, Springer, 1997.

[3] J. Bérenger: A perfectly matched layer for the absorption of electromagnetic waves. J. Comput. Phys **114** (1994) 185–200.

[4] A. Bossavit: *Computational Electromagnetics. Variational formulation, complementarity, edge elements.* Academic Press Inc. 1998.

[5] W. C. Chew and W. H. Weedon: A 3D perfectly matched medioum from modified Maxwell's equations with stretched coordinates. Michrowave Opt. Technol. Lett. **7** (1994) 599–604.

[6] W. Chew, J. Jin and E. Michielssen: Complex Coordinate System as a Generalized Absorbing Boundary Condition, IEEE Antennas and Propagation Society International Symposium in Montreal, 1997 Digest - Volume 3, pp., 2060-2063, 1997.

[7] F. Collino and P. Monk: The perfectly matched layer in curvilinear coordinates. Comput. Methods Appl. Mech. Engrg. **164** (1998), 157–171.

[8] F. Collino and P. Monk: Optimizing the Perfectly Matched Layer. SIAM J. Sci. Comput. **19** (1998), 2061–2090.

[9] D. Colton and R. Kress: *Integral Equation Methods in Scattering Theory.* Wiley&Sons, New York 1983.

[10] D. Colton and R. Kress: *Inverse Acoustic and Electromagnetic Scattering Theory.* Springer-Verlag, Berlin 1992.

[11] D. McDuff and D. Salamon: Introduction to symplectic topology. Oxford University Press, 1995. 425 pp.

[12] M. J. Grote and J. E. Keller: Exact non–reflecting boundary condition for the time dependent wave equation. SIAM J. Appl. Math. **55** (1995) 280–297.

[13] L. Kettunen, K. Forsman and A. Bossavit: Formulation of the eddy current problem in multiply connected regions in terms of $h$. Internat. J. Numer. Methods Engrg. 41 (1998), 935–954.

[14] A. Kirsch and P. Monk: An analysis of the coupling of finite-element and Nyström methods in acoustic scattering. IMA J. Numerical Anal. **14** (1994) 523–544.

[15] M. Lassas and E. Somersalo: On the existence and convergence of the solution of PML equations. Computing **60**(1998), 229–241.

[16] M. Lassas and E. Somersalo: Analysis of the PML equations in general convex geometry. To appear in Proc. Roy. Soc. Edinburgh Sect. A

[17] R. Leis: *Initial boundary value problems in mathematical physics.* B. G. Teubner, Stuttgart 1986.

[18] L. Paquet: Problèmes mixtes pour le système de Maxwell. Annales Facultè des Sciences Toulouse, Vol. IV (1982) 103–141.

[19] P. Petropoulos: Reflectionless sponge layers as absorbing boundary conditions for the numerical solution of Maxwell equations in rectangular, cylindrical, and spherical coordinates. SIAM J. Appl. Math. **60** (2000), 1037–1058.

[20] J. Sjöstrand and M. Zworski: Complex scaling and distribution of scattering poles J. Amer. Math. Soc. **4** (1991) 729–769.

[21] J. Sjstrand and M. Zworski: The complex scaling method for scattering by strictly convex obstacles. Ark. Mat. **33** (1995), 135–172.

[22] J. Sjöstrand and M. Zworski: Asymptotic distribution of resonances for convex obstacles. Acta Math. **183** (1999), 191–253.

[23] F. Teixera and W. Chew: Differential forms, metrics, and the reflectionless absorption of electromagetic waves. Journal of electromagnetic waves and applications, **13**(1999), 665-686.

[24] E. Turkel and A. Yefet: Absorbing PML boundary layers for wave-like equations. Absorbing boundary conditions. Appl. Numer. Math. **27** (1998), 533–557.