Multidimensional Gel'fand Inverse Boundary Spectral Problem: Uniqueness and Stability

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1   Introduction

Here we consider the *Gel'fand Inverse Spectral Boundary Problem.* Let us start with a non-rigorous introduction to this class of problems. Assume we have a manifold with boundary $(M, \partial M)$, a vector bundle Λ over M, and a linear elliptic differential operator A acting on smooth sections of Λ which are denoted by $\mathcal{F}(M, \Lambda)$. The operator is defined with some boundary conditions $Bu = 0$ where B is a local operator making the boundary value problem $Au = F$, $Bu|_{\partial M} = 0$ elliptic, i.e., $\mathcal{D}(A) = \{u \in \Lambda : \ Bu|_{\partial M} = 0\}$. Note that all operators that we consider here are linear. Consider the boundary value problem

$$Au = \lambda u; \quad Bu|_{\partial M} = f. \tag{1}$$

If λ is not in spectrum of (A, B), the solution $u = u_\lambda^f$ to (1) exists and we define the "Dirichlet-to-Neumann" map as

$$R_\lambda : f \mapsto B^c u_\lambda^f|_{\partial M},$$

where B^c is the "complimentary" boundary operator for B such that the pair $(Bu|_{\partial M}, B^c u|_{\partial M})$ represents the whole Cauchy data of u with respect to the operator A.

The *Gel'fand's boundary spectral problem* (for the original form of the problem, see [9]) is the problem of finding M, Λ and (A, B) from the knowledge of the boundary ∂M, the bundle $\Lambda|_{\partial M}$ on the boundary and the map R_λ for all values of the spectral parameter $\lambda \notin \mathrm{spec}(A)$.

As we will see, Gel'fand's boundary spectral problem does not usually have a unique solution and the problem is to characterize (the group of) possible transformations which preserve the maps R_λ. It is also important to analyse subgroups of the transformation group due to various *a priori* restrictions on Λ and (A, B) and, in particular, to find when the subgroup becomes trivial,

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i.e. Gel'fand's boundary spectral problem possess a unique solution. Here we discuss mostly such a case with suitable a priori information.

The aim of this note is to concentrate on the following Gel'fand's boundary spectral problem:

Let M be an $m-$dimensional, $m \geq 2$, compact, connected $C^\infty-$smooth Riemannian manifold with non-empty boundary, ∂M. Let Δ_g be the Neumann Laplace operator on M acting on L^2 space of scalar functions. Thus, using Einstein summation convention we have in local coordinates,

$$\Delta_g u = -g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j u), \quad \mathcal{D}(\Delta_g) = \{u \in H^2(M) : \partial_\nu u|_{\partial M} = 0\}. \quad (2)$$

Here $g_{ij}(x)$ is the metric tensor, $g = \det[g_{ij}]$, $[g^{ij}]$ is the inverse matrix to $[g_{ij}]$, $\partial_i = \partial/\partial x^i$ and ∂_ν is the inward normal derivative. We note that the case of Dirichlet boundary condition can also be treated by the same method.

Denote by $0 = \lambda_1 < \lambda_2 \leq \ldots$ the eigenvalues and by $\phi_1(x) = V^{-1/2}$, $\phi_2, \ldots$ the orthonormal eigenfunctions of Δ_g, V being the volume of (M, g). Then, for each $\lambda \notin \text{spec}(\Delta_g)$, the data in Gel'fand's boundary spectral problem are given by the traditional Neumann-to-Dirichlet map,

$$R_\lambda f = u_\lambda^f|_{\partial M},$$

where u_λ^f solves the problem

$$\Delta_g u = \lambda u, \quad \partial_\nu u|_{\partial M} = f. \quad (3)$$

One can show, e.g. [15] that these data are equivalent to the *Gel'fand boundary spectral data (GBSD)*

$$\partial M, \quad \{(\lambda_k, \phi_k|_{\partial M})\}_{k=1}^\infty. \quad (4)$$

At this stage, let us make some remarks

i. It may seem that in the inverse problems occurring in real applications we deal only with domains in $\mathbb{R}^m$ and manifolds are introduced for the sake of maximal generality. However, when dealing with anisotropic operators similar to (2) in a domain $M \subset \mathbb{R}^m$, we need to take into account possible coordinate changes in M. If $y = \Phi(x)$ are other coordinates in M with $\Phi|_{\partial M} = \text{id}|_{\partial M}$, then operator (2) transforms into an operator of the same form with the metric tensor $\widetilde{g}$ given by

$$\widetilde{g}_{kl}(y)|_{y=\Phi(x)} = g_{ij}(x)\frac{\partial x^i}{\partial y^k}(y)|_{y=\Phi(x)} \frac{\partial x^j}{\partial y^l}(y)|_{y=\Phi(x)}. \quad (5)$$

Thus $\widetilde{\lambda}_k = \lambda_k$, $\widetilde{\phi}_k(\Phi(x)) = \phi_k(x)$ and we see that the boundary spectral data for g and $\widetilde{g}$ are the same. Therefore, in anisotropic inverse problems it is convenient to factorize out the non-uniqueness due to diffeomorphisms preserving ∂M, i.e., to work in the manifold formalism. Although this lies outside the scope of the current presentation, we note that in practical problems the domain M often

contains some *a priori* unknown "cavities" and measurement can be done only on the external part of the boundary. In invariant terms, the measurements can be done only part of a boundary of manifold which, in principle, may have non-trivial topology. This brings the problem even further into the realm of differential geometry.

ii. The method we will apply is applicable to a wider range of inverse problems to include general (scalar) 2nd order elliptic differential operators which are selfadjoint with respect to an appropriate (smooth) measure on M [13], some classes of non-selfadjoint operators [19] and Maxwell's system [20, 21]. When dealing with inverse problems for general operators, another source of non-uniqueness comes from *gauge transformations*. Indeed, by multiplying functions by a smooth (complex) factor, $\alpha(x) \neq 0$, $\alpha|_{\partial M} = 1$,

$$u \to \alpha u$$

and changing the measure accordingly, we obtain an operator with the same boundary data. We can factorize out this source of non-uniqueness by working with orbits of operators with respect to the action of the group of gauge transformations and choosing a *canonical representation* in each orbit. In the case of a general 2nd order selfadjoint elliptic operators with real coefficients, one can choose a canonical representation to be a (Riemannian) Schrödinger operator, $\Delta_g + q$ [14].

iii. Although we will discuss only the boundary spectral problem for the Laplace-Beltrami operator, the method is based essentially on properties of the wave equation

$$u_{tt} + \Delta_g u = 0. \tag{6}$$

Historically, it goes back to works of M. Krein at the end of 50th who used causality principle in dealing with the one-dimensional inverse problem for an inhomogeneous string, $u_{tt} - c^2(x)u_{xx} = 0$, see e.g. [18]. In his works, causality was transformed into analyticity (after Fourier transform). A more clear and straightforwardly hyperbolic version of the method was suggested by A. Blagovestchenskii at the end of 60th-70th [6]. In the multidimensional case the method was pioneered by M. Belishev [4] in late 80th who understood the role of the PDE-control for these problems (and gave it the name the *boundary control (BC)* method). Of crucial importance for the method was the result of D. Tataru [25] concerning a Holmgren-type uniqueness theorem for non-analytic coefficients. BC method was extended to anisotropic case (to deal exactly with the uniqueness problem of finding (M, g) from boundary spectral data of its Laplacian) by M. Belishev and Y. Kurylev [5]. The geometric version of the method, which we are going to present in this paper is developed by A. Katchalov, Y. Kurylev and M. Lassas in late 90th. It is summarized in [14], which will be the main reference for Section 2. In section 3 we will discuss some stability results for this problem based on [1] and [17]. In these notes, especially in Section 3, we often skip detailed proofs concentrating instead on basic ideas and referring to the literature, [1] and [14], for details.

3

2 Reconstruction with complete boundary spectral data

In order to reconstruct (M, g) we use a special representation, the *boundary distance representation*, $R(M)$ of M and later show that the boundary spectral data determine $R(M)$. Consider a map $R : M \to C(\partial M)$,

$$R(x) = r_x(\cdot); \quad r_x(z) = d(x, z), \ z \in \partial M, \tag{7}$$

i.e., $r_x(\cdot)$ is the *distance function* from x to various points on ∂M. The image $R(M) \subset C(\partial M)$ of R provides the boundary distance representation of M. The set $R(M)$ is a metric space with the distance inherited from $C(\partial M)$ which we denote by d_∞. The map R, due to the triangular inequality, is Lipschitz,

$$d_\infty(r_x, r_y) \leq d(x, y), \tag{8}$$

and, by compactness of (M, d), the metric space $(R(M), d_\infty)$ is also compact. Our first observation is

Lemma 1 *The map $R : (M, d) \to (R(M), d_\infty)$ is a homeomorphism. Moreover, given $R(M)$ as a subset of $C(\partial M)$ it is possible to construct a distance function d_R on $R(M)$ that makes the metric space $(R(M), d_R)$ isometric to (M, d).*

Remark. In the case when (M, g) is a simple manifold, that is all geodesics are the shortest curves between their endpoints and all geodesics can be continued to geodesics that hit the boundary, the claim is easy to prove. Indeed, then $||r_x - r_y||_{C(\partial M)} = d(x, y)$ for $x, y \in M$ and R is isometry of (M, d) and $(R(M), d_\infty)$. Next we prove this result for the general case.

Proof. We start by proving that R is a homeomorphism. Recall the following simple result from topology:

Assume that X and Y are Hausdorff spaces, X is compact and $F : X \to Y$ is a continuous, bijective map from X to Y. Then F is a homeomorphism.

By definition, R is surjective and, by (8), continuous. In order to prove injectivity, assume the contrary, i.e. $r_x(\cdot) = r_y(\cdot)$ but $x \neq y$. Denote by z_0 any point where

$$d(x, \partial M) = \min_{z \in \partial M} r_x(z) = r_x(z_0) = r_y(z_0) = \min_{z \in \partial M} r_y(z) = d(y, \partial M). \tag{9}$$

Then z_0 is a nearest boundary point to x implying that the shortest geodesic from z_0 to x is normal to ∂M. The same is true for y with the same point z_0. Both x and y lie on the geodesic $\gamma_{z_0}(s)$ to ∂M. It starts from z_0 normally to ∂M with s being the arclength. As the geodesics are unique solutions of a system of ordinary differential equations (the Hamilton-Jacobi equation), they are uniquely determined by their initial points and directions, that is the geodesics are non-branching. Thus we see that $x = \gamma_{z_0}(s_0) = y$, where $s_0 = r_x(z_0)$.

Notice that, although (M, d) is homeomorphic to $(R(M), d_\infty)$, they are not, in general, isometric. Imagine e.g. a unit sphere with a small circular hole

near the South pole of, say, diameter ε. Then, for any x, y on the equator and $z \in \partial M$, $\pi - \varepsilon \leq r_x(z) \leq \pi$ and $\pi - \varepsilon \leq r_y(z) \leq \pi$. Then $d_\infty(r_x, r_y) \leq \varepsilon$, while $d(x, y)$ may be equal to π.

Before going further, introduce the *boundary normal coordinates* on M which we have already used implicitly in the proof of the first part of this lemma. For a normal geodesic $\gamma_z(s)$ starting from z consider $d(\gamma_z(s), \partial M)$. For small s,

$$d(\gamma_z(s), \partial M) = s, \tag{10}$$

and z is the unique nearest point to $\gamma_z(s)$ on ∂M. Let $\tau(z)$ be the largest value for which (10) is valid for all $s \in [0, \tau(z)]$. Then for $s > \tau(z)$,

$$d(\gamma_z(s), \partial M) < s,$$

and z is no more the nearest boundary point. $\tau(z) \in C(\partial M)$ is called the cut locus distance function and the set

$$\omega = \{x_z : z \in \partial M, \ x_z = \gamma_z(\tau(z))\}, \tag{11}$$

is the *cut locus of M with respect to ∂M*. This is a zero-measure subset of M. In the remaining domain $M \setminus \omega$ we can use the coordinates

$$x \mapsto (z(x), t(x)), \tag{12}$$

where $z(x) \in \partial M$ is the unique nearest point to x and $t(x) = d(x, \partial M)$. (Strictly speaking, one also has to use some local coordinates of the boundary, $y : z \mapsto (y^1(z), \ldots, y^{(m-1)}(z))$ and define that

$$x \mapsto (y(z(x)), t(x)) = (y^1(z(x)), \ldots, y^{(m-1)}(z(x)), t(x)) \in \mathbb{R}^m, \tag{13}$$

are the boundary normal coordinates.) We will now use these coordinates to introduce a differential structure and metric tensor, g_R, on $R(M)$ to have an isometry

$$R : (M, g) \to (R(M), g_R).$$

We will concentrate mainly on doing so for $R(M) \setminus R(\omega)$, referring to [14] for details concerning vicinity of $R(\omega)$.

Observe first that we can identify those $r = r_x \in R(M)$ with $x \in M \setminus \omega$. Indeed, if $r = r_x$ with $x = \gamma_z(s)$, $s < \tau(z)$ then

i. $r(\cdot)$ has a unique global minimum at some point $z \in \partial M$;

ii. there is $\tilde{r} \in R(M)$ having a unique global minimum at the same z and $r(z) < \tilde{r}(z)$. This is equivalent to saying that there is y with $r_y(\cdot)$ having a unique global minimum at the same z and $r_x(z) < r_y(z)$.

A differential structure on $R(M \setminus \omega)$ can be defined by introducing coordinates near each $r^0 \in R(M \setminus \omega)$. In a sufficiently small neighborhood $V \subset R(M)$ of r^0 the coordinates $r \mapsto (Y(r), T(r)) = (y(\mathrm{argmin}_{z \in \partial M} r), \min_{z \in \partial M} r)$ are well defined. These coordinates have the property that the map $x \mapsto (Y(r_x), T(r_x))$ coincides with the boundary normal coordinates (12,13). When we choose the

differential structure on $R(M \setminus \omega)$ that corresponds to these coordinates, the map $R : M \setminus \omega \to R(M \setminus \omega)$ is a diffeomorphism.

Next we construct the metric g_R on $R(M)$. Let $r^0 \in R(M \setminus \omega)$. As above, in a sufficiently small neighborhood $V \subset R(M)$ of r^0 there are coordinates $r \mapsto X(r) := (Y(r), T(r))$ that correspond to the boundary normal coordinates. Let $(y^0, t^0) = X(r^0)$. We consider next the evaluation function $K_w : V \to \mathbb{R}$, $K_w(r) = r(w)$, where $w \in \partial M$. The inverse of $X : V \to \mathbb{R}^m$ is well defined in a neighborhood $U \subset \mathbb{R}^m$ of (y^0, t^0) and thus we can define the function $E_w = K_w \circ X^{-1} : U \to \mathbb{R}$ that satisfies

$$E_w(y, t) := d(w, \gamma_{z(y)}(t)), \quad (y, t) \in U, \tag{14}$$

where $\gamma_{z(y)}(t)$ is the normal geodesic starting from the boundary point $z(y)$ with coordinates $y = (y^1, \ldots, y^{m-1})$.

Let now $g_R = R_* g$ be the push-forward of g to $R(M \setminus \omega)$. We denote its representation in X-coordinates by g_{jk}. Since X corresponds to the boundary normal coordinates, the metric tensor satisfies $g_{mm} = 1$, $g_{\alpha m} = 0$, $\alpha = 1, \ldots, m - 1$.

Consider the function $E_w(y, t)$ as a function of (y, t) with a fixed w. Then its differential, dE_w at point (y^0, t^0) defines a covector in $T^*_{(y^0, t^0)}(U) = \mathbb{R}^m$. Since the gradient of a distance function is a unit vector field, we see from (14) that

$$||dE_w||^2_{[g_{jk}]} := (\frac{\partial}{\partial t} E_w)^2 + (g_R)^{\alpha\beta} \frac{\partial E_w}{\partial y^\alpha} \frac{\partial E_w}{\partial y^\beta} = 1, \quad \alpha, \beta = 1, \ldots, m - 1.$$

Varying $w \in \partial M$ we obtain a set of covectors $dE_w(y^0, t^0)$ in the unit ball of $(T^*_{(y^0, t^0)} U, g_{jk})$ which contains an open neighborhood of $(0, \ldots, 0, 1)$. This determines uniquely the tensor $g^{jk}(y^0, t^0)$. Thus we can construct the metric tensor in the boundary normal coordinates at arbitrary $r \in R(M \setminus \omega)$. This means that we can find the metric g_R on $R(M \setminus \omega)$ when $R(M)$ is given.

To complete the reconstruction, we need differential structure and metric tensor near $R(\omega)$. Observe that for any $x \in M^{\text{int}}$ there are points $z_1, \ldots, z_m$ on ∂M such that the distance functions $\tilde{x} \mapsto d(z_i, \tilde{x})$ form coordinates for $\tilde{x}$ near x. It is these coordinates we use near $R(\omega)$, and this determines on $R(M)$ a differential structure that makes $R : M \to R(M)$ a diffeomorphism. Since the metric g_R is a smooth tensor, and we have found it in a dense subset $R(M \setminus \omega)$ of $R(M)$, we can continue it in local coordinates. This gives us the metric on the whole $R(M)$ (for details, see [14]). Q.E.D.

Note that, if interested only in the uniqueness, rather than reconstruction, we could refer, at the last stage of proof, to the Myers-Steenrod theorem yielding that two Riemannian manifolds isometric as metric spaces are isometric as Riemannian manifolds.

To construct the set $R(M)$ from the boundary data we start with two auxiliary results related to the initial-boundary value problem for the wave equation. Let $u^f = u^f(x, t)$ be the solution of

$$u^f_{tt} + \Delta_g u^f = 0; \quad u^f|_{t<0} = 0; \quad \partial_\nu u^f|_{\partial M \times \mathbb{R}_+} = f \in \dot{C}^\infty(\partial M \times \mathbb{R}_+), \tag{15}$$

where $\dot{C}^\infty(\partial M \times \mathbb{R}_+)$ consists of smooth functions equal to 0 near $t = 0$. Denote by $\mathcal{F} : L^2(M) \to \ell^2$ the Fourier transform,

$$\mathcal{F}a = \{a_k\}_{k=1}^\infty, \quad \text{for } a(x) = \sum_{k=0}^\infty a_k \phi_k(x).$$

Let $u_k^f(t) = (u^f(\cdot, t), \phi_k)_{L^2(M)}$ be the Fourier coefficients of $u^f(\cdot, t)$ with $\mathcal{F}u^f(t) = \{u_k^f(t)\}_{k=1}^\infty$.

Lemma 2 *We have*

$$u_k^f(t) = \int_0^t \int_{\partial M} \frac{\sin\left[\sqrt{\lambda_k}(t-s)\right]}{\sqrt{\lambda_k}} \phi_k(x) f(x,s) dA_x \, ds \tag{16}$$

where dA_x is the Riemannian volume of $(\partial M, g)$.

The proof of this assertion is straightforward. If we time-differentiate twice the identity $u_k^f(t) = \left(u^f(\cdot, t), \phi_k(\cdot)\right)_{L^2(M)}$, use (15) and apply integration by parts, we obtain

$$u_{tt}^f(t) + \lambda_k u_k^f(t) = \int_{\partial M} f(x,t)\phi_k(x) \, dA_x.$$

Invoking the initial data, we get (16).

The other result is based on the following fundamental theorem by D. Tataru [25]

Theorem 1 *Let $u(x,t)$ solve the wave equation $u_{tt} + \Delta_g u = 0$ in $M \times \mathbb{R}$ and $u|_{\Gamma \times (0, 2t_0)} = \partial_\nu u|_{\Gamma \times (0, 2t_0)} = 0$, where $\emptyset \neq \Gamma \subset \partial M$ is open. Then*

$$u = 0 \text{ in } K_{\Gamma, t_0}; \quad \text{where } K_{\Gamma, t_0} = \{(x,t) \in M \times \mathbb{R} : d(x, \Gamma) < t_0 - |t - t_0|\} \tag{17}$$

is the double cone of influence (see Fig. 1).

(The proof of this theorem, in full generality, is in [25]. A simplified proof for the considered case is in [14].)

The observability Theorem 1 gives rise to the following approximate controllability:

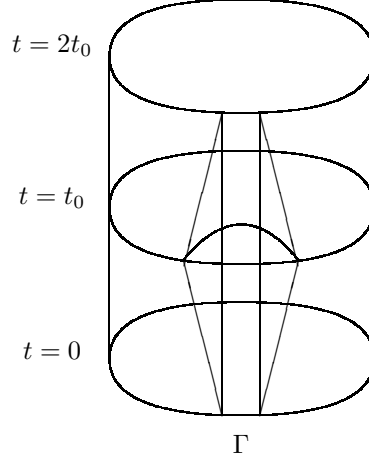**Corollary 1** *For any open $\Gamma \subset \partial M$ and $t_0 > 0$,*

$$cl_{L^2(M)}\{u^f(\cdot, t_0) : f \in C_0^\infty(\Gamma \times (0, t_0))\} = L^2(M(\Gamma, t_0)).$$

Here

$$M(\Gamma, t_0) = \{x : d(x, \Gamma) \le t_0\} = K_{\Gamma, t_0} \cap \{t = t_0\}$$

is the domain of influence of Γ at time t_0 and $L^2(M(\Gamma, t_0)) = \{a \in L^2(M) : supp\,(a) \subset M(\Gamma, t_0)\}$.

Figure 1: Double-cone of influence

This result lies within the realm of the well-known identity,

$$\overline{\mathrm{Ran}(A)} = N(A^*)^{\perp}.$$

Namely, let $a \in L^2(M(\Gamma, t_0))$ be orthogonal to all $u^f(\cdot, t_0)$, $f \in C_0^\infty(\Gamma \times (0, t_0))\}$. Denote by v the solution of the wave equation

$$(\partial_t^2 + \Delta_g)v = 0; \quad v|_{t=t_0} = 0, \ \partial_t v|_{t=t_0} = a; \quad \partial_\nu v|_{\partial M \times (0, t_0)} = 0.$$

Using integration by parts we obtain

$$\int_0^{t_0} \int_{\partial M} f(x, s)v(x, s) \, dA_x \, ds = \int_M a(x)u^f(x, t_0)dV = 0,$$

due to the orthogonality. Thus $v|_{\Gamma \times (0, t_0)} = 0$ and, as v is odd with respect to $t = t_0$, we conclude from Theorem 1 that $a = 0$. \hfill Q.E.D.

Note that if the surface measure dA_x which corresponds to the metric g in (16) is replaced by an arbitrary smooth positive surface measure $d\tilde{A}_x$, the collection $\{f(x, s)d\tilde{A}_x ds : \ f \in C_0^\infty(\Gamma \times [0, t_0])\}$ of measures do not change, that is,

$$\{f(x, s)d\tilde{A}_x ds : f \in C_0^\infty(\Gamma \times [0, t_0])\} = \{f(x, s)dA_x ds : f \in C_0^\infty(\Gamma \times [0, t_0])\}$$

Thus, by combining Lemma 2 and Corollary 1, we see that GBSD determine, for any open $\Gamma \subset \partial M$ and $t_0 > 0$, the subspace $\ell^2(\Gamma, t_0) \subset \ell^2$,

$$\ell^2(\Gamma, t_0) = \mathcal{F}L^2(M(\Gamma, t_0)).$$

Also, we define $\ell^2(z, \tau) = \mathcal{F}L^2(M(z, \tau))$, $z \in \partial M$. This set may be found using GBSD as limit of sets $\ell^2(\Gamma_p, \tau)$ when $\Gamma_p \to \{z\}$.

Theorem 2 *Let $\{z_n\}_{n=1}^{\infty}$ be a dense set on ∂M. Then $r(\cdot) \in C(\partial M)$ lies in $R(M)$ if and only if, for any $N > 0$,*

$$l^N := \bigcap_{n=1}^{N} \ell^2(z_n, r(z_n) + \frac{1}{N}) \cap \bigcap_{n=1}^{N} \ell^2(z_n, r(z_n) - \frac{1}{N})^{\perp} \neq \{0\}. \qquad (18)$$

Moreover, condition (18) can be verified using the Gel'fand boundary spectral data (4). Hence the Gel'fand boundary spectral data determines uniquely the boundary distance representation $(R(M), g_R)$ of (M, g) and therefore determines the isometry type of (M, g).

Proof *"If"–part.* Let $x \in M$ and denote for simplicity $r(\cdot) = r_x(\cdot)$. Consider a ball $B_{1/N}(x)$. Then,

$$B_{1/N}(x) \subset M(z, r(z) + 1/N) \setminus M(z, r(z) - 1/N).$$

Thus if $\operatorname{supp}(a) \subset B_{1/N}(x)$, then $\mathcal{F}a \in l^N$.
"Only if"–part. Let (18) be valid so that there is

$$x_N \in \bigcap_{n=1}^{N} \left(M(z_n, r(z_n) + \frac{1}{N}) \setminus M(z_n, r(z_n) - \frac{1}{N}) \right). \qquad (19)$$

By choosing a suitable subsequence of x_N (denoted also by x_N), there exists a limit $x = \lim_{n\to\infty} x_N$. By continuity of the distance function, it follows from (19) that

$$d(x, z_n) = r(z_n), \quad n = 1, 2, \ldots.$$

Since $\{z_n\}$ are dense in ∂M, we see that $r(z) = d(x, z)$ for all $z \in \partial M$, that is, $r = r_x$. \hfill Q.E.D.

Note that this proof provides an algorithm for construction of an isometric copy of (M, g) when the Gef'fand boundary spectral data are given.

3 Stability of the inverse problem

This section is based on joint results of authors with M. Anderson, A. Katsuda, and M. Taylor in [1].

It is well-known that inverse problems are ill-posed, i.e., arbitrary small variation of data can bring about arbitrary large change in the model (this is just a manifestation of the unboundedness of the inverse map). However, in order to deal with inverse problems in applications, we need to "stabilize" them, i.e., to find *a priori* conditions which render an *inverse problem* to become continuously dependent on data. The principal tool lies in the following basic topological lemma (already used in the proof that R is a homeomorphism).

Lemma 3 *Let X and Y be compact Hausdorff spaces with $F : X \to Y$ being continuous and bijective. Then F^{-1} is also continuous, i.e. F is a homeomorphism.*

Typically, in inverse problems in domains in $\mathbb{R}^m$, we assume that coefficients of the unknown operator, say the Schrödinger one, $-\Delta + q$, where now Δ is the Euclidean Laplacian, are bounded in some "strong" function space and then derive continuity in a "weaker" function space. As far as we know, the first result in this directions was obtained by G. Alessandrini [2] who proved that if q is *a priori* bounded in H^σ, $\sigma > m/2$, then inverse problem is continuous in L^∞. For the Laplace-Beltrami operator in a domain in $\mathbb{R}^m$, P. Stefanov and G. Uhlmann showed in [24] that, if the coefficients of the metric tensor, g_{ij} are close to δ_{ij} in $C^{\alpha(m)}(M)$ then the inverse problem is continuous in $C(M)$. Both results, and others obtained in this direction, fall in the framework of Lemma 3 since the embedding of H^σ into L^2 and $C^{\alpha(m)}$ into C are compact. It should be noted that the above works provided also some quantitative estimates for the corresponding moduli of continuity. For anisotropic inverse problems, due to possible changes of coordinates which can dramatically alter the metric tensor (cf. (5)) but do not affect the physical nature of the process, it is important to introduce *a priori* constraints in a coordinate-invariant form. Even more this is true for the inverse problems on manifolds when, in the beginning, even the topological type of the manifolds is not known. To be more rigorous, let $\Sigma_\mathcal{N}$ be the class of compact, connected Riemannian manifolds with the same, i.e. diffeomorphic boundaries $\mathcal{N}$. The boundary spectral data correspond to the *direct map*,

$$\mathcal{D}_S : (M, g) \mapsto ((\lambda_k)_{k=1}^\infty, (\phi_k|_\mathcal{N})_{k=1}^\infty) \in \mathcal{B}_\mathcal{N}, \tag{20}$$

where $\mathcal{B}_\mathcal{N}$ is the space of all pairs of sequences $(\underline{\mu}, \underline{\psi}) = ((\mu_k)_{k=1}^\infty, (\psi_k)_{k=1}^\infty)$ with $\mu_k \to \infty$ and $\psi_k \in L^2(\mathcal{N})$ (some other functions spaces are also appropriate as to be seen from further considerations).

For a sequence $(\psi_j)_{j=1}^N$ of functions we define the set $B((\psi_j)_{j=1}^N) \subset L^2(\mathcal{N})$,

$$B((\psi_j)_{j=1}^N) = \{v \in L^2(\mathcal{N}) : v = \sum_{k=1}^N \alpha_k \psi_k, \ \sum_{k=1}^N |\alpha_k|^2 \le 1\}.$$

Next we define a basis of open sets in the space $\mathcal{B}_\mathcal{N}$. This basis consists of sets $U_\varepsilon(\underline{\mu}, \underline{\psi})$ that are defined to be collection of sequences $((\widetilde{\mu}_j)_{j=1}^\infty, (\widetilde{\psi}_j)_{j=1}^\infty)$, such that the following is true:

There is a finite number of disjoint open intervals $I_p \subset [0, \varepsilon^{-1}]$, $p = 1, \ldots, P$ with lengths $|I_p| < \varepsilon$, such that

a. Each $\mu_k, \widetilde{\mu}_k < \varepsilon^{-1} - \varepsilon$ lies in some I_p;

b. $\sum_p d_H(B(\{\psi_j : \mu_j \in I_p\}, B(\{\tilde{\psi}_j : \tilde{\mu}_j \in I_p\}) < \varepsilon$ where d_H is the Hausdorff distance in $L^2(\partial M)$.

In layman terms the definition means that the first eigenvalues, $\mu_k, \widetilde{\mu}_k$ and restrictions to $\mathcal{N}$ of the corresponding eigenfunctions are close. However, we also need to take into account the possibility of multiple eigenvalues. Because of this, we group the close eigenvalues to clusters and require that for two operator having close spectral data have same number of eigenvalues is properly chosen

clusters. Note that the eigenvalues in interval $[\varepsilon^{-1} - \varepsilon, \varepsilon^{-1}]$ may or may not to belong to the considered clusters.

This definition may be given in a number of different forms. For example, instead of $\mathcal{D}_S$, direct map, $\mathcal{D}_h$, may correspond to the heat flow associated with the Laplacian,

$$\mathcal{D}_h : (M, g) \mapsto H(x, y, t), \quad x, y \in \mathcal{N}, \, t > 0. \tag{21}$$

Here H is the heat kernel for

$$(\partial_t + \Delta_g)H = 0 \text{ on } M \times \mathbb{R}_+, \quad \partial_\nu H|_{\partial M \times \mathbb{R}_+} = 0, \quad H(x, y, t)|_{t=0} = \delta_y(x).$$

Then $H(x, y, t)$, $x, y \in \mathcal{N}$, $t > 0$ is in $C(\mathcal{N} \times \mathcal{N} \times \mathbb{R}_+)$ which has topology of the uniform convergence on compact subsets of $\mathcal{N} \times \mathcal{N} \times \mathbb{R}_+$. Spaces $\mathcal{B}_\mathcal{N}$ and $C(\mathcal{N} \times \mathcal{N} \times \mathbb{R}_+)$ are not homeomorphic. However, on the classes of Riemannian manifolds which we intend to consider, convergence of GBSD in $\mathcal{B}_\mathcal{N}$ is equivalent to the convergence of te heat kernels in $C(\mathcal{N} \times \mathcal{N} \times \mathbb{R}_+)$. This is not surprising as

$$H(x, y, t) = \sum_{k=0}^{\infty} \exp(-\lambda_k t)\phi_k(x)\,\phi_k(y).$$

In the class $\Sigma_\mathcal{N}$ the convergence of the boundary data either in $\mathcal{B}_\mathcal{N}$ or $C(\mathcal{N} \times \mathcal{N} \times \mathbb{R}_+)$, by no means implies the convergence of the underlying Riemannian manifolds (although, of course, we should rigorously define what we mean by the convergence of Riemannian manifolds). Let us consider some examples of difficulties which may occur.

Example 1. Let S_+ be a two dimensional unit semisphere. Let us attach in a smooth manner a small handle near its north pole to obtain a Riemannian manifold M_δ, where δ characterizes the size of the handle. Then the first eigenfunctions almost do not feel the handle, namely, for any $\varepsilon > 0$, we can choose δ so that BSD of S_+ and M_δ are ε−close in $\mathcal{B}_\mathcal{N}$. Also the corresponding heat kernels can be made ε−close on an arbitrary given compact $K \subset \mathcal{N} \times \mathcal{N} \times \mathbb{R}_+$.

Example 2. Let again S_+ be a two dimensional unit semisphere and Ω be an arbitrary closed connected surface in $\mathbb{R}^3$. Connect S_+ with Ω by a thin long tube, close to the north pole of S_+. We obtain a manifold M_δ, with δ characterizing the size of the tube. Then, for any $\varepsilon > 0$, we can make tube so thin and long that the boundary values of the heat kernels of S_+ and M_δ are ε−close in $C(\mathcal{N} \times \mathcal{N} \times \mathbb{R}_+)$.

In both examples, S_+ and M_δ are, geometrically and even topologically, very different which we can not identify from our incomplete, imprecise boundary measurements. Clearly, such situation should be avoided. Observe that, in both cases, when $\delta \to 0$, then the curvature tends to ∞, and, in Example 2, diameter tends to ∞. Also, the second fundamental form of $\partial M = \mathcal{N}$ should be controlled. For technical reasons, namely, to prevent collapsing manifolds to those of smaller dimensions, we should control, in addition, the injectivity radii of manifolds in $\Sigma_\mathcal{N}$. This type of restrictions is typical, at least for manifolds

without boundary, in the Cheeger-Gromov theory of geometric convergence [11], [7]. Thus, it is natural to seek for conditions to guarantee stability of Gel'fand's boundary spectral problem in a properly modified to include manifolds with boundaries, framework of this theory.

Let us introduce the classes of manifolds we intend to consider. For a while, we go beyond the limits of inverse problems and intend to work both with manifolds with and without boundary.

Definition 1 *Let Λ, D, $i_0 > 0$. Denote by $\Sigma^m(\Lambda, D, i_0)$ the class of closed $m-$dimensional $C^\infty-$smooth Riemannian manifolds such that*

a. $||Ric_M||$, $||Ric_{\partial M}|| \leq \Lambda$,
b. $diam(M, g) \leq D$,
c. $||K||_{C^{0,1}(\partial M)} \leq \Lambda$,
d. $inj \geq i_0$.

Here Ric_M, $Ric_{\partial M}$ are the Ricci curvature of M and ∂M, respectively and K is the mean curvature of ∂M. (Clearly condition c. and the second condition in a. are void for manifolds without boundary). inj stands for the minimum of all three injectivity radii on a manifold with boundary, namely, injectivity radii of Riemann normal coordinates on M and ∂M and injectivity radius of boundary normal, i.e. $\min_{z \in \partial M} \tau(z)$.

The principal geometric result to get conditional stability of Gel'fand's boundary spectral problem in a class $\Sigma^m(\Lambda, D, i_0)$ is the following:

Theorem 3 *For any $m, \Lambda, D, i_0 > 0$, the class $\Sigma^m(\Lambda, D, i_0)$ is pre-compact in the $C^{1,\alpha}-$topology, for any $\alpha < 1$, and in the Gromov-Hausdorff topology [11]. Its closure, $\overline{\Sigma^m(\Lambda, D, i_0)}$ consists of Riemannian manifolds with the the metric tensor g which is, in proper coordinates, $C^{1,\alpha}-$smooth for any $\alpha < 1$. Conditions a. and c. of definition 1 are valid for the manifolds in the closure $\overline{\Sigma^m(\Lambda, D, i_0)}$.*

Let us explain what is meant by convergence in the above topologies. For $C^{1,\alpha}-$case, a sequence of Riemannian manifolds, $(M^{(n)}, g^{(n)})$ converge to (M, g) if, starting with some n_0, there are diffeomorphisms

$$\Phi_{(n)} : M \to M^{(n)}, \quad \Phi_{(n)} \in C^{2,\alpha}(M, M^{(n)}), \quad ||\Phi_{(n)}^*(g^{(n)}) - g||_{C^{1,\alpha}} \to 0. \quad (22)$$

Although in this paper we will speak predominantly about the $C^{1,\alpha}-$topology, let us explain briefly the Gromov-Hausdorff one and its relations to the inverse problems. The Gromov-Hausdorff distance is defined on the space of all compact metric spaces (X, d_X) with

$$d_{GH}((X, d_X), (Y, d_Y)) \leq \varepsilon,$$

if there are ε-nets $\{x_1, \ldots, x_N\} \subset X$ and $\{y_1, \ldots, y_N\} \subset Y$ such that

$$|d_X(x_i, x_j) - d_Y(y_i, y_j)| \leq \varepsilon.$$

Clearly, the Gromov-Hausdorff topology is weaker than $C^{1,\alpha}$. Its importance, in particular for the inverse problems, lies in its ability to compare objects of different dimensions. For example, let S_1 be a unit circle in $\mathbb{R}^3$ and T^ε be a two dimensional torus with its second radius equal to ε. Then $d_{GH}(S_1, T^\varepsilon) < 4\varepsilon$.

Theorem 3 has a direct analog for manifolds without boundary proven by M. Anderson [3]. However, the presence of boundary necessitates development of new techniques which is to be discussed later.

Let us discuss now the geometric properties of Σ which are needed for the analysis of Gel'fand's boundary spectral problem. According to the basic topological lemma, for the class Σ to be a proper candidate for the conditional stability of Gel'fand's boundary spectral problem, the map $\mathcal{D}_S$ should be well-defined and continuous from $\overline{\Sigma}$ to $\mathcal{B}_\mathcal{N}$ and injective. The first two statements follow easily from Kato's perturbation theory [16]. To analyze injectivity, recall two critical elements in the proof of uniqueness of Gel'fand's boundary spectral problem:

1. Tataru's unique continuation used in approximate controllability, Corollary 2. It requires Lipschitz continuity of the metric tensor and, evidently, is valid on $\overline{\Sigma}$.

2. Non-branching of geodesics used in the proof of the injectivity of the map $R : M \to C(\partial M)$. Theorem 3 fail to guarantee it, indeed, there are counterexamples going back, essentially, to Hartman [12] which show the existence of $g \in \bigcap_{0 < \alpha < 1} C^{1,\alpha}$ with branching geodesics. The remedy lies in the following

Theorem 4 *Let (M, g) be a complete Riemannian manifold with $C^{1,\alpha}$, $\alpha > 0$ metric. Assume, in addition, that conditions a., c. of Definition 1 are satisfied (in the case of non-compact manifolds we can assume only local boundedness of the Ricci curvatures and Lipschitz constant of the mean curvature). Then, in a proper coordinate system,*

$$g_{ij} \in C_*^2,$$

C_*^2 being the second Zygmund space.

Recall that a continuous function $f \in C_*^1$ if

$$\frac{|f(x) - 2f((x+y)/2) + f(y)|}{|x - y|} \le c_f < \infty,$$

and $f \in C_*^2$ if $f, \nabla f \in C_*^1$. For this and further properties of Zygmund spaces see e.g. H. Triebel [27] or M. Taylor [26].

Combining Theorems 3, 4, we see that $\overline{\Sigma^m(\Lambda, D, i_0)}$ consists of Riemannian manifolds with C_*^2 metric. It is compact with respect to $C^{1,\alpha}$−topology, for any $\alpha < 1$.

Corollary 2 *Let $(M, g) \in \overline{\Sigma^m(\Lambda, D, i_0)}$. Then the geodesics in M do not branch. Moreover, if $M^{(n)} \to M$ in $\overline{\Sigma^m(\Lambda, D, i_0)}$ with respect to $C^{1,\alpha}$ or Gromov-Hausdorff topology then the geodesic flow on $M^{(n)}$ converge to the geodesic flow on M.*

Summarizing Theorems 20, 21 and Corollary 2, topological lemma implies our principal result

Theorem 5 *The map $\mathcal{D}_S : \overline{\Sigma(\Lambda, D, i_0)} \to \mathcal{D}_S(\overline{\Sigma(\Lambda, D, i_0)}) \subset \mathcal{B}_\mathcal{N}$ is a homeomorphism, i.e., Gel'fand's boundary spectral problem depends continuously on the boundary spectral data.*

Similar result is valid for the heat kernels.

In the rest of these lectures we give principal ideas of proofs of Theorems 3, 4. They are related to the notion of proper coordinates which are the *boundary harmonic coordinates*. On manifolds without boundary, harmonic coordinates go back to Einstein and are widely used in differential geometry starting from DeTurck-Kazdan [8], with applications to geometric convergence by Peters [22], Greene-Wu [10], Anderson [3], etc. Coordinates $(x^1, \ldots, x^m)$ are harmonic if they satisfy the Laplace equation, $\Delta_g x^i = 0$ (e.g. Cartesian coordinates in $\mathbb{R}^m$). Harmonic coordinates enjoy a number of useful properties, in particular,

i. The metric tensor has maximal smoothness in these coordinates;

ii. Its components, g_{ij} satisfy, in harmonic coordinates, the Ricci equation,

$$\Delta_g g_{ij} = B_{ij}(g, \nabla g) - 2\mathrm{Ric}_{ij}. \tag{23}$$

Here B_{ij} is a quadratic function in ∇g and rational in g_{ij}.

When Ric_M is bounded and g is *a priori* sufficiently smooth, e.g. in $C^{0,1}$, interior elliptic regularity generalized to Zygmund spaces implies that $g \in C_*^2$ inside M. To deal with the boundary we use the following

Definition 2 *Coordinates $(x^1, \ldots, x^m)$ are boundary harmonic coordinates near ∂M if*

i. $(x^1, \ldots, x^m)$ are harmonic coordinates inside M;

ii. ∂M is defined by $x^m = 0$;

iii. Let $y^\gamma = x^\gamma|_{\partial M}$, $\gamma = 1, \ldots, m-1$. Then $(y^1, \ldots, y^{(m-1)})$ are harmonic coordinates on ∂M.

It may be shown that, in boundary harmonic coordinates, the components of the metric tensor, in addition to (23), satisfy the Dirichlet boundary conditions

$$g_{\beta\gamma} = h_{\beta\gamma}, \quad h_{\beta\gamma} \in C_*^2(\partial M), \ \beta, \gamma = 1, \ldots, m-1, \tag{24}$$

and third-type boundary conditions

$$\partial_\nu g^{mm} = -2(m-1)K\, g^{mm}, \tag{25}$$

$$\partial_\nu g^{\beta m} = -(m-1)K\, g^{\beta m} + \frac{1}{2\sqrt{g^{mm}}}\, g^{\beta k}\partial_k g^{mm},$$

where K is the mean curvature. A proper generalization of the boundary elliptic regularity to Zygmund spaces shows that $g_{\beta\gamma}, g^{im} \in C_*^2$. In turn these imply that $g_{ij} \in C_*^2$ proving Theorem 4.

The proof of Theorem 3 is based on regularity results of a geometric nature. They say, roughly, that any manifold from Σ can be covered by a uniformly finite number of domains of boundary harmonic coordinates which are uniformly large and smooth, namely

Lemma 4 *For any* $\Lambda, D, i_0 > 0$, *there are* $r > 0$ *(harmonic radius) and* $C > 0$ *so that there is a uniformly finite covering of any* $(M, g) \in \Sigma(\Lambda, D, i_0)$ *by coordinate patches,* U_p *of boundary harmonic coordinates,*

$$X_p : U_p \to \mathbb{R}^m, \quad \text{with } X_p(U_p) = B_r \text{ or } B_r^+,$$

B_r, B_r^+ *being, respectively, a ball and half-ball of radius* r *in* $\mathbb{R}^m$. *Moreover, the metric tensor, in these coordinates, satisfies*

$$\frac{1}{2} I \leq [g_{ij}] \leq 2I; \quad ||g_{ij}||_{C_*^2} \leq C.$$

The proof is based on blow-up arguments. Assuming the contrary, i.e. an existence of a sequence of manifolds $M^{(n)}$ with their harmonic radii $\varepsilon_{(n)} \to 0$, we rescale the metric tensors, $g^{(n)} \to \widetilde{g}^{(n)}$, $\widetilde{g}^{(n)} = \varepsilon_{(n)}^{-2} g^{(n)}$ to obtain a sequence of Riemannian manifolds, $\widetilde{M}^{(n)}$ with

$$\text{Ric}_{\widetilde{M}^{(n)}}, \text{Ric}_{\partial \widetilde{M}^{(n)}} \to 0; \quad \widetilde{K}^{(n)} \to 0; \quad \text{inj}_{\widetilde{M}^{(n)}} \to \infty, \tag{26}$$

with the radius of boundary harmonic coordinates, at some point, being equal to 1. Using the Riccati equation for the second fundamental form near the boundary or the Cheeger-Gromoll splitting theorem far from the boundary, we obtain a subsequence of $\widetilde{M}^{(n)}$ which tends either to $\mathbb{R}_+^m$ or $\mathbb{R}^m$. By the lower semicontinuity of the harmonic radius, we conclude that the harmonic radius of $\mathbb{R}_+^m$ and $\mathbb{R}^m$ are less than 1, which is a contradiction.

Below is a sketch of the proof of this statement for the case when the point $x^{(n)}$, where the geodesic radius $\widetilde{r}^n$ of $\widetilde{M}^{(n)}$ equals to 1, is near the boundary. We concentrate on this case since it reflects specific features of manifolds with boundary, otherwise the proof is essentially the same as for manifolds without boundary [3]. We use a rather standard construction of differential geometry, going back to J. Cheeger [7], see e.g. P. Petersen [23]. It says that if a pointed family of Riemannian manifolds, which have a locally finite coordinate covering by "uniformly large" charts with uniformly bounded, say in $C^{1,\alpha}$, metric tensors, then this family is pre-compact in $C^{1,\beta}$−topology for any $\beta < \alpha$. Note that, due to the blow-up procedure, we are exactly in this situation taking as coordinate charts those of the boundary harmonic coordinates, and by (26),

$$\text{Ric}_M, \text{Ric}_{\partial M} = 0, \quad K = 0, \quad \text{inj}_M = \infty, \tag{27}$$

where M is the limit manifold.

We use a consequence of the fundamental equations of Riemannian geometry (see e.g. [23])

$$\Delta_g \tau^{(n)} = \widetilde{K}^{(n)} \quad \text{on } \{x : \tau^{(n)}(x) = c\}, \tag{28}$$

where $\tau^{(n)}$ is the distance function to $\partial\widetilde{M}^{(n)}$, and

$$\partial_\tau \widetilde{K}^{(n)} \leq -\frac{1}{m-1}(\widetilde{K}^{(n)})^2 - \text{Ric}_{\widetilde{M}^{(n)}}(\partial_\tau, \partial_\tau), \qquad (29)$$

Conditions (26) imply that $\widetilde{K}^{(n)} \to 0$ in any layer $\tau^{(n)} < c$. Therefore, $K = 0$ everywhere on M, and by (28), τ is harmonic on M. In turn, this implies that $A(c) = 0$ for any $c \geq 0$, where $A(c)$ is the second fundamental form of the surface $\tau = c$. At last, we obtain from the above that M is isometric to the direct product $\partial M \times [0, \infty)$. Since the arguments adopted from the case of manifolds without boundary show that ∂M is isometric to $\mathbb{R}^{(m-1)}$, which completes the proof of Lemma 4.

The above arguments lead to Theorem 3 and, therefore, to Theorem 5. To this end, we appeal again to the above standard geometric construction using as coordinate charts those for the boundary harmonic coordinates and invoking Lemma 4.

Let us finish with few comments about reconstruction. We start with a finite approximation $\{\mu_k, \psi_k|_{\partial M}\}, k = 1, \ldots, \hat{k}$, to the boundary spectral data. We can then construct, using variational principle, a finite approximation $\widetilde{R} \subset C(\partial M)$ to the boundary distance representation $R(M)$. On $\widetilde{R}$ we can find approximate images of geodesics on M hitting ∂M and, using Alexandrov's lemma, equip a subset X of $\widetilde{R}$ with a metric, d_X so that (X, d_X) is an approximation to (M, d) in the Gromov-Hausdorff sense [17].

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