MAXWELL'S EQUATIONS WITH A POLARIZATION INDEPENDENT WAVE VELOCITY: DIRECT AND INVERSE PROBLEMS

YAROSLAV KURYLEV, MATTI LASSAS, AND ERKKI SOMERSALO

Abstract: We study Maxwell's equations in time domain for an anisotropic medium of a special type, characterized by the polarization independent velocity of the wave propagation. In particular, this property is satisfied by all isotropic media. The analysis is based on an invariant formulation of the system of electrodynamics as a Dirac type first order system on a Riemannian 3−manifold. We study the properties of this system in the first part of the paper. The second part is devoted to the inverse problem of the identification of the Riemannian manifold M and the corresponding system of equations from the dynamic boundary data. These data are the boundary ∂M and the admittance map $\mathcal{Z}^T$. Physically, this map corresponds to the measurements of the tangential components of the electric and magnetic fields on the boundary at a finite time interval $[0,T]$. It is shown that, for sufficiently large $T>0$, $\mathcal{Z}^T$ determines the Riemannian manifold and the underlying electromagnetic parameters. Similar results are proven in the case when the boundary data are given only on an open part of the boundary. In domains of $\mathbb{R}^3$, we describe the group of transformations which preserve the admittance map $\mathcal{Z}^T$, providing a complete characterization of the non-uniqueness of the underlying physical problem. In the isotropic case with $M \subset \mathbb{R}^3$, we prove that the boundary data given on an open part of the boundary determine the domain M, the permittivity ϵ and the permeability μ uniquely.
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Introduction

In this paper we study direct and inverse boundary value problems for Maxwell's equations in Euclidean domains in $\mathbb{R}^3$ and on compact manifolds. In a bounded smooth domain $M^{\text{int}} \subset \mathbb{R}^3$, Maxwell's equations for the electric and magnetic fields E and H and the associated electric displacement D and magnetic flux density B are

$$\operatorname{curl} E(x,t) = -B_t(x,t), \ \text{(Maxwell–Faraday)}, \tag{1}$$

$$\operatorname{curl} H(x,t) = \ \ D_t(x,t), \ \text{(Maxwell–Ampère)}. \tag{2}$$

Under the assumption of a non-conducting, linear and non-chiral medium, these are augmented with the constitutive relations

$$D(x,t) = \epsilon(x)E(x,t), \quad B(x,t) = \mu(x)H(x,t). \tag{3}$$

Here electric permittivity ϵ and magnetic permeability μ are smooth 3×3 time-independent positive matrices.

The initial boundary value problem we mainly deal with in this paper is (1)-(3) with the homogeneous initial data and a prescribed tangential component of the electric field E, that is,

$$E(x,t)|_{t=0} = 0, \quad H(x,t)|_{t=0} = 0, \tag{4}$$

$$n \times E|_{\partial M\times\mathbb{R}_+} = f, \tag{5}$$

where n is the unit exterior normal vector to ∂M. The inverse problem associated with (1)-(5), which we are looking at, is the problem of describing all possible electromagnetic parameters $\epsilon(x)$ and $\mu(x)$ having the same impedance map

$$\mathcal{Z} : n \times E|_{\partial M\times\mathbb{R}_+} \mapsto n \times H|_{\partial M\times\mathbb{R}_+}.$$

In this connection, the present work consists of two parts. In the first part, we pursue further the invariant formulation of Maxwell's equations (1)–(3). In the invariant approach to Maxwell's equations, the domain M is considered as a 3–manifold and the vector fields E, H, D, and B as differential forms. This alternative formulation has several advantages both from the theoretical and practical points of view. First, the invariance of the system and boundary measurements with respect to the diffeomorphisms of M that preserve the part of the boundary where these measurements are done is essential for the inverse problem. Second, the formulation of electrodynamics in terms of differential forms reflects the way in which these fields are actually observed. For instance, flux quantities are expressed as 2–forms while field quantities that correspond to forces are naturally written as 1–forms. Therefore, the electromagnetic material paremeters $\epsilon(x)$ and $\mu(x)$ should be interpreted as Hodge-type operators from 1-forms to 2-forms thus defining two underlying Riemannian metrics g_ϵ and g_μ on M. This point of view has been adopted in modern physics, see e.g. [18], as well as in applications where the numerical treatment of the equations is done using the Whitney elements. An extensive treatment of this topic can be found in [11, 12]. For the original reference concerning the Whitney elements, see [71].

The Hodge-type operators $*_\epsilon$ and $*_\mu$ generated by $\epsilon(x)$ and $\mu(x)$ may or may not be proportional. The former case, where they are equal, up to some multiplicative scalar function, is the one addressed in this paper, see discussion in Section 1.1. Wave velocity is then independent of polarization, contrary to what happens in the latter case. Wavefronts may look ellipsoidal from the point of view of the outside observer who is using the vacuum natural metric. However, they are actually spherical with respect to the metric that makes both $*_\epsilon$ and $*_\mu$ scalar multiples of the associated Hodge operator defined by the underlying travel time metric. This metric is responsible for the velocity of the electromagnetic wave propagation in the medium. In other words, anisotropy is only apparent, genuine anisotropy only occurring when Hodge operators $*_\epsilon$ and $*_\mu$ are not proportional. On the level of the material parameters, ϵ and μ are 3×3 matrices that represent the action of the Hodge-operations in a given coordinate frame. In these coordinate frame proportionality of the Hodge operators $*_\epsilon$ and $*_\mu$ means that

$$\mu = \alpha^2 \epsilon, \tag{6}$$

where α is a positive scalar function. In literature, the parameter α is called the *wave impedance*, see e.g. [30, 31, 52]. When ϵ and μ are constant matrices, i.e. indendent of x, with α being a constant scalar parameter, the corresponding material is known as *affine isotropic* [50, 51]. Clearly, for a general anisotropic ϵ and μ such a scalar function α does not exist. In this paper we say that the material corresponding to ϵ and μ has *scalar wave impedance* $\alpha(x) > 0$ if (6) is satisfied.

Summarizing the above, we consider direct and inverse problems for the most general subclass of Maxwell's equations which is distinguished by the fact that electromagnetic fields with different polarization propagate with the same velocity which, of cause, may depend on the propagation direction. This case is encountered in many physical situations. For instance, in a curved spacetime with coordinates $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ and a "time-independent" metric $ds^2 = g_{jk}(x)dx^j dx^k - dt^2$, Maxwell's equations with scalar permittivity and permeability correspond, in the coordinate invariant form to Maxwell's equations (1)–(3) with scalar wave impedance, see [18, Sec. 14.1.c] or [42, Sec. 90]. Clearly, all isotropic media, i.e., with scalar ϵ and μ, have a scalar wave impedance.

The invariant approach leads us to formulate Maxwell's equations on 3–manifolds as a first order Dirac type system. From the operator theoretic point of view, this formulation is based on an elliptization procedure by extending Maxwell's equations to the bundle of the exterior differential forms over the manifold. This is a generalization of the elliptization of Birman and Solomyak and Picard (see [2, 60]).

In the second part of the work, we consider an inverse boundary value problem for Maxwell's equations with scalar wave impedance. In physical terms, the goal is to determine the material parameter tensors ϵ and μ in a bounded domain from field observations at the boundary or a part of the boundary of that domain. It is possible to prove unique identifiability in the invariant formulation and then use this result to completely characterize the groups of transformations between indistinguishable parameters ϵ and μ in the case $M \subset \mathbb{R}^3$. In particular, when ϵ and μ are scalar functions, this result implies the uniqueness of the determination of ϵ, μ, and M from data on a part of boundary not assuming an *a priori* knowledge of M.

As inverse problems of electrodynamics have a great significance in physics and applications, they have been studied starting from the 30's, see e.g. [43, 63], where the one-dimensional case was considered. However, results concerning the multidimensional inverse problems in electrodynamics are relatively recent. The first breakthrough achieved in [65, 14, 57, 58] was based on the use of the complex geometrical optics. These papers were devoted to the identifiability of isotropic material parameters ϵ and μ from the fixed-frequency data collected on ∂M, namely, the stationary admittance map. Under some mild geometric assumptions it was shown there that these data determine isotropic ϵ and μ and also isotropic conductivity, σ, uniquely. These works were based on the ideas previously developed in [68, 53, 54] to tackle the scalar Calderón problem, introduced in [13]. Other approaches to the isotropic inverse problem for Maxwell's equations work directly in the time domain, see [8, 61]. Regarding the case $\sigma = 0$ which is considered in this paper, the result obtained in [61] proves the identifiability of ϵ and μ from the time-dependent data collected on the whole ∂M in the case when M, considered as a Riemannian manifold with metric $dl^2 = \epsilon\mu|dx|^2$, is simple geodesic. We remind the reader that a Riemannian manifold with boundary is called simple geodesic if any two points $x, y \in M$ can be connected by a unique geodesic. Constructions of [8] make it possible to find the product, $\epsilon\mu$ of unknown parameters ϵ, μ. Moreover, the results of [8] are of a local nature making it possible to find this product only in some collar neighbourhood of ∂M. The time-dependent inverse problem for isotropic Maxwell's equations was also considered in [10] which used the time Fourier transform to reduce the problem to the one in the frequency domain so that to apply the results of [57, 58].

Much less is known in the anisotropic case, where the material parameters are matrix valued functions. The case of anisotropic $\epsilon = \mu$ was considered in [6] where it was shown that the time-dependent admittance map known on ∂M makes it possible to recover $\epsilon = \mu$ locally, i.e., in some collar neighbourhood of ∂M. In spite of a very little knowledge, it is, however, clear from the study of the scalar anisotropic problems that, instead of uniqueness, one obtains uniqueness only up to a group of transformations, involving proper coordinate changes, see e.g. [49, 66, 7, 33, 26, 48, 47]. A similar result for Maxwell's equations was conjectured in [67], based on the analysis of the linearized inverse problem. Therefore, it is natural to split the study of this problem into two steps. First, to formulate and solve the corresponding coordinate-invariant inverse problem, i.e., an inverse problem on a manifold. Second, to analyse the properties resulting from an embedding of the manifold into $\mathbb{R}^3$. For a systematic development of this approach, see [27].

In recent years, inverse problems with data on a part of the boundary have attracted much interest, see [20, 26, 32, 22, 48]. Part of the motivation come from the physical setting when only a part of the boundary is accessible. However, as far as we know, there are currently no results on identifiability of the shape of the domain M and/or the material parameters ϵ, μ on it from inverse data collected on an arbitrary open subset, $\Gamma \subset \partial M$.

A fruitful approach to the scalar inverse problems, including those with data on a part of the boundary, turned out to be the boundary control method, originated in [5] for the isotropic acoustic wave equation. In the anisotropic context, it has been developed for the Laplacian on Riemannian manifolds [7] and for general anisotropic self-adjoint [34, 35] and certain non-selfadjoint inverse problems [37].

The current article pursues the study of inverse problems for Maxwell's equations significantly further dealing with the global reconstruction of the shape of the domain or, more general, $3-$manifold M, metric tensor g and scalar wave impedance α, the latter two being equivalent to the reconstruction of ϵ and μ. Being based on the boundary control, the method developed here combines ideas of the articles [38] and [39] with those of [57] and [58]. What is more, to be able to study anisotropic Maxwell's equations, we introduce two essential new ideas. First, we characterize the subspaces controlled from the boundary *by duality*, thus avoiding the difficulties arising from the complicated topology of the domains of influence but still providing necessary information about the structure of achievable sets, see e.g. Theorem 1.16 in section 1.5. This makes our approach much different from that in [6, 8] also based on the boundary control method. Indeed, the method of [6, 8] requires local controllability in the domains of influence which is no more valid for large times, see e.g. [9] thus making the constructions of [6, 8] inappropriate outside a collar neighbourhood of ∂M. Second, we develop a *method of focused waves* which enables us to recover the pointwise values of electromagnetic waves on the manifold and, therefore, reconstruct not only the metric g, as in [8], but also the impedance α.

The main results of this paper can be summarized as follows.

(1) The knowledge of the complete dynamical boundary data over a sufficiently large finite period of time determines uniquely the compact manifold endowed with the travel time metric as well as the scalar wave impedance (Theorem 2.1). This is valid also when measurements are made on a part of the boundary (Theorem 2.15). The necessary time of observation is double of

the time required to fill the manifold from the observed part of the boundary.

(2) For the corresponding anisotropic inverse boundary value problem with scalar wave impedance for bounded domains in $\mathbb{R}^3$, the non-uniqueness is completely characterized by describing the class of possible transformations between material tensors that are indistinguishable from the observed part of the boundary (Theorem 2.19).

(3) For the corresponding isotropic inverse boundary value problem for bounded domains in $\mathbb{R}^3$, the shape of the domain and the material parameters inside it are uniquely determined from the measurements done on a part of the boundary (Theorem 2.21).

Some of the results of the paper have been announced in [40, 41].

1. Maxwell's equations on a manifold

This chapter is devoted to Maxwell's equations on a compact oriented 3–manifold with boundary. We concentrate on the properties of these equations important for the inverse problem considered in Chapter 2.

We start with the formulation of Maxwell's equations for the 1– and 2–forms. These equations are augmented to a complete Maxwell system on the full bundle of exterior differential forms over a 3–dimensional Riemannian manifold. This allows us to define and analyze properties of an elliptic operator related to Maxwell's equations and to study the corresponding initial boundary value problem. Crucial results of Sections 1.3 and 1.4 are the Blagovestchenskii formula, Theorem 1.10, enabling us to evaluate the inner products of electromagnetic waves in terms of the admittance map $\mathcal{Z}$, also defined in Section 1.3, and the unique continuation result for Maxwell's equations with Cauchy data on the lateral boundary. Building on these results, we obtain local and global controllability for electromagnetic waves generated by boundary sources and define, in a usual manner, spaces of generalized sources.

1.1. **Invariant definition of Maxwell's equations.**

To define Maxwell's equations invariantly, consider a smooth compact oriented connected Riemannian 3-manifold M, $\partial M \neq \emptyset$, with a metric g_0, that we call the background metric. Clearly, in physical applications we take $M \subset \mathbb{R}^3$ with g_0 being the Euclidean metric. Analogously to (1) and (2), Maxwell's equations on the manifold M are equations of the form

$$\operatorname{curl} E(x,t) = -B_t(x,t), \tag{7}$$

$$\operatorname{curl} H(x,t) = \quad D_t(x,t). \tag{8}$$

Here E, H, D, $B \in \Gamma M$, the space of C^∞-smooth vector fields on M. They are related by the constitutive relations,

$$D(x,t) = \epsilon(x)E(x,t), \quad B(x,t) = \mu(x)H(x,t), \tag{9}$$

where ϵ and μ are C^∞-smooth positive-definite $(1,1)$–tensor fields on M. We remind that, for $X \in \Gamma M$,

$$(10) \qquad (\mathrm{curl}\, X)^\flat = *_0 dX^\flat, \quad \mathrm{div}\, X = - *_0 d *_0 X^\flat.$$

Here, d is the exterior differential, $^\flat$ is the fiberwise duality between 1–forms and vector fields

$$X \in \Gamma M \to X^\flat \in \Omega^1 M, \quad X^\flat(Y) = g_0(X,Y),$$

with $\Omega^1 M$ and, generally, $\Omega^k M$ standing for the bundle of the differential $k-$forms on M. We define the 1-forms $\mathcal{E} = E^\flat$ and $\mathcal{H} = H^\flat$ and the 2-forms $\mathcal{B} = *_0 B^\flat$ and $\mathcal{D} = *_0 D^\flat$, where $*_0$ is the Hodge operator with respect to the metric g_0, acting fiberwise,

$$*_0 : \Omega^k M \to \Omega^{3-k} M.$$

Then we can write Maxwell's equations (7)–(8) in terms of differential forms as

$$(11) \qquad d\mathcal{E} = -\mathcal{B}_t, \quad d\mathcal{H} = \mathcal{D}_t,$$

where we used the identity $*_0 *_0 = \mathrm{id}$ valid in the 3–dimensional case.

Consider now the constitutive relations (9). Starting with equation $D = \epsilon E$, we will next construct a *metric* g_ϵ such that the Hodge-operator with respect to this metric, denoted by $*_\epsilon$, would satisfy the identity

$$(12) \qquad \mathcal{D} = *_0 (\epsilon E)^\flat = *_\epsilon \mathcal{E}.$$

For such metric, in local coordinates (x^1, x^2, x^3), the middle term of (12) yields

$$*_0(\epsilon E)^\flat = \sqrt{\mathrm{g}_0} g_0^{ij} e_{jpq} g_{0,ij} \epsilon^j_k E^k dx^p \wedge dx^q = \sqrt{\mathrm{g}_0} e_{jpq} \epsilon^j_k E^k dx^p \wedge dx^q,$$

where e_{jpq} is the totally antisymmetric permutation index and $\mathrm{g}_0 = \det(g_{0,ij})$. Likewise, the right-hand side reads

$$*_\epsilon \mathcal{E} = \sqrt{\mathrm{g}_\epsilon} g_\epsilon^{ij} e_{jpq} g_{0,ik} E^k dx^p \wedge dx^q,$$

so evidently equality (12) is valid if we set

$$\sqrt{\mathrm{g}_\epsilon} g_\epsilon^{ij} g_{0,ik} = \sqrt{\mathrm{g}_0} \epsilon^k_j.$$

By taking the determinants of the both sides we find that $\sqrt{\mathrm{g}_\epsilon} = \sqrt{\mathrm{g}_0} \det(\epsilon)$. Thus we see that for the metric tensor

$$(13) \qquad g_\epsilon^{ij} = \frac{1}{\det(\epsilon)} g_0^{ik} \epsilon^j_k$$

identity (12) is valid. In the same fashion, we see that for the metric $g_\mu^{ij} = \det(\mu)^{-1} g_0^{ik} \mu^j_k$ we have

$$\mathcal{B} = *_0 (\mu H)^\flat = *_\mu \mathcal{H}.$$

Thus the constitutive relations take the form

$$(14) \qquad \mathcal{D}(x,t) = *_\epsilon \mathcal{E}(x,t), \quad \mathcal{B}(x,t) = *_\mu \mathcal{H}(x,t).$$

In general, the metrics g_μ and g_ϵ can be very different from each other. In this article, we consider a particular case where the metrics g_μ and g_ϵ are equal up to a scalar factor.

Definition 1.1. *We say that a material has a* scalar wave impedance*, if the metrics corresponding to the tensors ϵ and μ satisfy*

$$g_\epsilon^{ij} = \alpha^4 g_\mu^{ij}, \quad \textit{or equivalently } *_\epsilon = \alpha^{-2} *_\mu : \Omega^1 M \to \Omega^2 M, \tag{15}$$

where the wave impedance $\alpha = \alpha(x)$ is a smooth positive function on M.

Note that, in terms of (1,1)-tensors ϵ and μ, (15) is equivalent to $\mu = \alpha^2 \epsilon$.

This allows us to introduce a new metric, g on M by

$$g^{ij} = \frac{1}{\alpha^2} g_\epsilon^{ij} = \alpha^2 g_\mu^{ij}. \tag{16}$$

As we see later, this metric defines the velocity of the electromagnetic wave propagation and we call it the *travel-time metric*. In other words, in this case of the scalar wave impedance the electromagnetic wave probagation has only one wave velocity to each direction, i.e., the vave velocity does not depend on polarization.

Next we consider the waves that satisfy the initial conditions

$$B(x,t)|_{t=0} = 0, \quad D(x,t)|_{t=0} = 0. \tag{17}$$

Operating with divergence to the Maxwell equations, this implies that

$$\operatorname{div} B(x,t) = 0, \quad \operatorname{div} D(x,t) = 0, \quad \text{for } t > 0,\ x \in M.$$

In terms of differential forms these read as

$$d\mathcal{D} = 0, \quad d\mathcal{E} = 0. \tag{18}$$

In the further considerations, we will use only the pair $(\mathcal{E}, \mathcal{B})$ and, as an auxiliary tool, we will consider a more general system of equations than the physical Maxwell equations. To reflect this we will denote $(\mathcal{E}, \mathcal{B})$ by a pair (ω^1, ω^2), where

$$\omega^1 = \mathcal{E} \in \Omega^1 M, \quad \omega^2 = \mathcal{B} \in \Omega^2 M. \tag{19}$$

Then equations (11), (14), and (18) imply

$$\omega_t^1 = \delta_\alpha \omega^2, \quad \delta_\alpha \omega^1 = 0, \tag{20}$$

$$\omega_t^2 = -d\omega^1, \quad d\omega^2 = 0. \tag{21}$$

where $\delta_\alpha : \Omega^k M \to \Omega^{3-k} M$ is the α–codifferential, given by

$$\delta_\alpha \omega^k = (-1)^k * \alpha\, d \frac{1}{\alpha} * \omega^k, \tag{22}$$

and $*$ is the Hodge operator with respect to the travel-time metric, g. These equations are called *Maxwell's equations for forms in the divergence free case* on a Riemannian manifold with a scalar wave impedance, (M, g, α).

To extend the above equations to the full bundle of exterior differential forms $\mathbf{\Omega} M = \Omega^0 M \times \Omega^1 M \times \Omega^2 M \times \Omega^3 M$, we introduce auxiliary forms, $\omega^0 \in \Omega^0 M$ and $\omega^3 \in \Omega^3 M$, which vanish in the electromagnetic theory, by

$$\omega_t^0 = \delta_\alpha \omega^1, \quad \omega_t^3 = -d\omega^2.$$

Since $\omega^0 = 0$ and $\omega^3 = 0$ in electromagnetics, we can modify equations (20) and (21) to read

$$\omega_t^1 = -d\omega_0 + \delta_\alpha \omega^2, \quad \omega_t^3 = -d\omega^2, \tag{23}$$

$$\omega_t^2 = -d\omega^1 + \delta_\alpha \omega^3, \quad \omega_t^0 = \delta_\alpha \omega^1, \tag{24}$$

or, in the matrix form,

$$\omega_t + \mathcal{M}\omega = 0, \tag{25}$$

where $\omega = (\omega^0, \omega^1, \omega^2, \omega^3) \in \mathbf{\Omega} M$, and the operator $\mathcal{M}$ (without prescribing its domain at this point, i.e., defined as a differential expression) is given as

$$\mathcal{M} = \begin{pmatrix} 0 & -\delta_\alpha & 0 & 0 \\ d & 0 & -\delta_\alpha & 0 \\ 0 & d & 0 & -\delta_\alpha \\ 0 & 0 & d & 0 \end{pmatrix}. \tag{26}$$

Equations (25), (26) are called *the complete Maxwell system.* We note that not all the solutions of (25) correspond to the physical electromagnetic waves in the absence of internal sources but only those with $\omega^0 = 0$ and $\omega^3 = 0$ (cf. Lemma 1.6). For similar extensions, see [2, 60].

Note that, identifying $\mathbf{\Omega} M$ with $\Omega^0 M \oplus \Omega^1 M \oplus \Omega^2 M \oplus \Omega^3 M$, the complete Maxwell operator can be written as

$$\mathcal{M} = d - \delta_\alpha, \tag{27}$$

i.e., it becomes a Dirac-type operator on $\mathbf{\Omega} M$.

An important property of $\mathcal{M}$ is that

$$\mathcal{M}^2 = -\mathrm{diag}(\Delta_\alpha^0, \Delta_\alpha^1, \Delta_\alpha^2, \Delta_\alpha^3) = -\mathbf{\Delta}_\alpha,$$

where the operator Δ_α^k acts on the k–forms as

$$\Delta_\alpha^k = d\delta_\alpha + \delta_\alpha d = \Delta_g^k + Q^k(x, D). \tag{28}$$

Here Δ_g^k is the Laplace-Beltrami operator in the metric g and $Q^k(x, D)$ is a first order perturbation. Hence, if ω satisfies equation (25), it satisfies also the wave equation

$$(\partial_t^2 + \mathbf{\Delta}_\alpha)\omega = (\partial_t - \mathcal{M})(\partial_t + \mathcal{M})\omega = 0. \tag{29}$$

This formula legitimates the notion of the travel time metric and makes it clear that in the Maxwell system with scalar wave impedance, the electromagnetic waves of different polarization propagate with the same speed determined by the metric g. On the other hand, as follows from

[24, 25] (see also [16]), when ε and μ are not proportional, waves with different polarization propagate with different velocity.

We end this section with a representation of the energy of electric and magnetic fields in terms of the corresponding differential forms, setting

$$\begin{aligned}\mathcal{E}(E) &= \frac{1}{2}\int_M g_0(\epsilon E, E)\, dV_0 = \frac{1}{2}\int_M \frac{1}{\alpha}\omega^1 \wedge *\omega^1,\\ \mathcal{E}(B) &= \frac{1}{2}\int_M g_0(\mu H, H)\, dV_0 = \frac{1}{2}\int_M \frac{1}{\alpha}\omega^2 \wedge *\omega^2,\end{aligned}$$

where dV_0 is the volume form of (M, g_0). These formulae serve as a motivation for our definition of the inner products in the following section.

1.2. **The Maxwell operator.** In this section we establish a number of notational conventions and definitions concerning differential forms used in this paper.

We define the L^2–inner products for the k–forms in $\Omega^k M$ as

$$(\omega^k, \eta^k)_{L^2} = \int_M \frac{1}{\alpha}\omega^k \wedge *\eta^k, \quad \omega^k,\ \eta^k \in \Omega^k M,$$

and denote by $L^2(\Omega^k M)$ the completion of $\Omega^k M$ in the corresponding norm. We also define

$$\mathbf{L}^2(M) = L^2(\Omega^0 M) \times L^2(\Omega^1 M) \times L^2(\Omega^2 M) \times L^2(\Omega^3 M),$$

with the Sobolev spaces $\mathbf{H}^s(M)$, $\mathbf{H}^s_0(M)$, $s \in \mathbb{R}$, given as

$$\begin{aligned}\mathbf{H}^s(M) &= H^s(\Omega^0 M) \times H^s(\Omega^1 M) \times H^s(\Omega^2 M) \times H^s(\Omega^3 M),\\ \mathbf{H}^s_0(M) &= H^s_0(\Omega^0 M) \times H^s_0(\Omega^1 M) \times H^s_0(\Omega^2 M) \times H^s_0(\Omega^3 M).\end{aligned}$$

Here, $H^s(\Omega^k M)$ is the Sobolev space of the k–forms and $H^s_0(\Omega^k M)$ is the closure in $H^s(\Omega^k M)$ of the set of the k–forms in $\Omega^k M$, which vanish near ∂M.

Clearly, a natural domain of the exterior derivative, d in $L^2(\Omega^k M)$ is

$$H(d, \Omega^k M) = \left\{\omega^k \in L^2(\Omega^k M) \mid d\omega^k \in L^2(\Omega^{k+1} M)\right\},$$

and a natural domain of δ_α is

$$H(\delta_\alpha, \Omega^k M) = \left\{\omega^k \in L^2(\Omega^k M) \mid \delta_\alpha\omega^k \in L^2(\Omega^{k-1} M)\right\}.$$

In the sequel, we drop the sub-index α from the codifferential.

The operators d and δ are adjoint, i.e., for the C^∞_0–forms ω^k, η^{k+1},

$$(d\omega^k, \eta^{k+1})_{L^2} = (\omega^k, \delta\eta^{k+1})_{L^2}.$$

To extend this formula to less regular forms, let us fix some notations. For $\omega^k \in \Omega^k M$, we define its *tangential* and *normal* boundary components on ∂M as

$$\mathbf{t}\omega^k = i^*\omega^k, \quad \mathbf{n}\omega^k = i^*(\alpha^{-1} * \omega^k),$$

respectively, where $i^*: \Omega^k M \to \Omega^k \partial M$ is the pull-back of the natural imbedding $i: \partial M \to M$. With these notations, Stokes' formula for forms can be written as

$$(d\omega^k, \eta^{k+1})_{L^2} - (\omega^k, \delta\eta^{k+1})_{L^2} = \langle \mathbf{t}\omega^k, \mathbf{n}\eta^{k+1}\rangle, \tag{30}$$

where, for $\omega^k \in \Omega^k M$ and $\eta^{k+1} \in \Omega^{k+1} M$,

$$\langle \mathbf{t}\omega^k, \mathbf{n}\eta^{k+1}\rangle = \int_{\partial M} \mathbf{t}\omega^k \wedge \mathbf{n}\eta^{k+1}.$$

There are well defined extensions of the boundary trace operators $\mathbf{t}$ and $\mathbf{n}$ to $H(d, \Omega^k M)$ and $H(\delta, \Omega^k M)$. The following result is due to Paquet [59]:

Proposition 1.2. *The operators* $\mathbf{t}$ *and* $\mathbf{n}$ *can be extended to continuous surjective maps*

$$\mathbf{t}: H(d, \Omega^k M) \to H^{-1/2}(d, \Omega^k \partial M),$$
$$\mathbf{n}: H(\delta, \Omega^{k+1} M) \to H^{-1/2}(d, \Omega^{2-k} \partial M),$$

where the space $H^{-1/2}(d, \Omega^k \partial M)$ *is the space of the* k*-forms* ω^k *on* ∂M *satisfying*

$$\omega^k \in H^{-1/2}(\Omega^k \partial M), \quad d\omega^k \in H^{-1/2}(\Omega^{k+1} \partial M).$$

Formula (30) is instrumental for characterizing the spaces of forms with vanishing boundary data. Introducing $\overset{\circ}{H}(d, \Omega^k M) = \operatorname{Ker}(\mathbf{t})$ and $\overset{\circ}{H}(\delta, \Omega^{k+1} M) = \operatorname{Ker}(\mathbf{n})$ and applying Stokes' formula, one can prove in standard way the following lemma.

Lemma 1.3. *The adjoint of the operator*

$$d: L^2(\Omega^k M) \supset H(d, \Omega^k M) \to L^2(\Omega^{k+1} M)$$

is the operator $\delta: L^2(\Omega^{k+1} M) \supset \overset{\circ}{H}(\delta, \Omega^{k+1} M) \to L^2(\Omega^k M)$ *and* vice versa. *Similarly, the adjoint of*

$$\delta: L^2(\Omega^{k+1} M) \supset H(\delta, \Omega^{k+1} M) \to L^2(\Omega^k M)$$

is the operator $d: L^2(\Omega^k M) \supset \overset{\circ}{H}(d, \Omega^k M) \to L^2(\Omega^{k+1} M)$.

When there is no risk of confusion we will write for brevity $H(d) = H(d, \Omega^k M)$ and similarly, *mutatis mutandis* for the other spaces.

For later references, we point out that Stokes' formula for the complete Maxwell system can be written as

$$(\eta, \mathcal{M}\omega)_{\mathbf{L}^2} + (\mathcal{M}\eta, \omega)_{\mathbf{L}^2} = \langle \mathbf{t}\omega, \mathbf{n}\eta\rangle + \langle \mathbf{t}\eta, \mathbf{n}\omega\rangle, \tag{31}$$

where $\omega \in \mathbf{H}$ with

$$\mathbf{H} = H(d) \times [H(d) \cap H(\delta)] \times [H(d) \cap H(\delta)] \times H(\delta), \tag{32}$$

$\eta \in \mathbf{H}^1(M)$, $\mathbf{t}\omega = (\mathbf{t}\omega^0, \mathbf{t}\omega^1, \mathbf{t}\omega^2)$, $\mathbf{n}\omega = (\mathbf{n}\omega^3, \mathbf{n}\omega^2, \mathbf{n}\omega^1)$, and

$$\langle \mathbf{t}\omega, \mathbf{n}\eta\rangle = \langle \mathbf{t}\omega^0, \mathbf{n}\eta^1\rangle + \langle \mathbf{t}\omega^1, \mathbf{n}\eta^2\rangle + \langle \mathbf{t}\omega^2, \mathbf{n}\eta^3\rangle.$$

With these notations, we give the following definition of the Maxwell operators with electric boundary condition.

Definition 1.4. *The Maxwell operator with electric boundary condition, $\mathcal{M}_{\rm e}$, is an operator in $\mathbf{L}^2(M)$, with*

$$\mathcal{D}(\mathcal{M}_{\rm e}) = \overset{\circ}{\mathbf{H}}_{\mathbf{t}} := \overset{\circ}{H}(d) \times [\overset{\circ}{H}(d) \cap H(\delta)] \times [\overset{\circ}{H}(d) \cap H(\delta)] \times H(\delta),$$

and $\mathcal{M}_{\rm e}\omega$, $\omega \in \mathcal{D}(\mathcal{M}_{\rm e})$ is given by the differential expression (26).

In terms of physics, the electric boundary condition is associated with electrically perfectly conducting boundaries, i.e., $n \times E = 0$, $n \cdot B = 0$, where n is the exterior normal vector at the boundary. In terms of differential forms, this means simply that $\mathbf{t}E^\flat = \mathbf{t}\omega^1 = 0$ and $\mathbf{t} *_0 B^\flat = \mathbf{t}\omega^2 = 0$. Although not used in the sequel, the Maxwell operator with magnetic boundary condition, $\mathcal{M}_{\rm m}$, is given by (26) with the domain

$$\mathcal{D}(\mathcal{M}_{\rm m}) = \overset{\circ}{\mathbf{H}}_{\mathbf{n}} := H(d) \times [H(d) \cap \overset{\circ}{H}(\delta)] \times [H(d) \cap \overset{\circ}{H}(\delta)] \times \overset{\circ}{H}(\delta).$$

Consider the intersections of spaces in the definition of $\mathcal{D}(\mathcal{M}_{\rm e})$ and $\mathcal{D}(\mathcal{M}_{\rm m})$. Let

$$\begin{aligned}
\overset{\circ}{H}{}^1_{\mathbf{t}}(\Omega^k M) &= \{\omega^k \in H^1(\Omega^k M) \mid \mathbf{t}\omega^k = 0\},\\
\overset{\circ}{H}{}^1_{\mathbf{n}}(\Omega^k M) &= \{\omega^k \in H^1(\Omega^k M) \mid \mathbf{n}\omega^k = 0\}.
\end{aligned}$$

It is a direct consequence of Gaffney's inequality (see [64]) that

$$\begin{aligned}
\overset{\circ}{H}(d, \Omega^k M) \cap H(\delta, \Omega^k M) &= \overset{\circ}{H}{}^1_{\mathbf{t}}(\Omega^k M),\\
H(d, \Omega^k M) \cap \overset{\circ}{H}(\delta, \Omega^k M) &= \overset{\circ}{H}{}^1_{\mathbf{n}}(\Omega^k M).
\end{aligned}$$

The following lemma is a straightforward application of Lemma 1.3 and the classical Hodge-Weyl decomposition [64].

Lemma 1.5. *The electric Maxwell operator has the following properties:*

(i) *The operator $\mathcal{M}_{\rm e}$ is skew-adjoint.*
(ii) *The operator $\mathcal{M}_{\rm e}$ defines an elliptic differential operator in $M^{\rm int}$.*
(iii) $\mathrm{Ker}\,(\mathcal{M}_{\rm e}) = \big\{(0, \omega^1, \omega^2, \omega^3) \in \overset{\circ}{\mathbf{H}}_{\mathbf{t}} \mid \ d\omega^1 = 0,\ \delta\omega^1 = 0,\ d\omega^2 = 0,\ \delta\omega^2 = 0,\ \delta\omega^3 = 0\big\}$.
(iv) $$\begin{aligned}\mathrm{Ran}\,(\mathcal{M}_{\rm e}) = L^2(\Omega^0 M) &\times \big(\delta H(\delta, \Omega^2 M) + d\overset{\circ}{H}(d, \Omega^0 M)\big)\times\\ &\times \big(\delta H(\delta, \Omega^3 M) + d\overset{\circ}{H}(d, \Omega^1 M)\big) \times d\overset{\circ}{H}(d, \Omega^2 M).\end{aligned}$$

By the skew-adjointness, it is possible to define weak solutions to initial boundary-value problems needed later.

1.3. **Initial–boundary value problem.** In the sequel, we denote the forms $\omega(x,t)$ by $\omega(t)$ or ω when there is no danger of misunderstanding.

By a *weak solution* to the initial boundary value problem

$$\begin{aligned}(33)\qquad &\partial_t\omega + \mathcal{M}\omega = \rho \in L^1_{\rm loc}(\mathbb{R}, \mathbf{L}^2(M)),\\ &\mathbf{t}\omega\big|_{\partial M\times\mathbb{R}} = 0, \quad \omega(0) = \omega_0 \in \mathbf{L}^2,\end{aligned}$$

we mean the form $\omega(t) \in C(\mathbb{R}, \mathbf{L}^2(M))$ defined as

$$(34)\qquad \omega(t) = \mathcal{U}(t)\omega_0 + \int_0^t \mathcal{U}(t-s)\rho(s)ds,$$

where $\mathcal{U}(t) = \exp(-t\mathcal{M}_{\rm e})$ is the unitary operator in $\mathbf{L}^2$ generated by $\mathcal{M}_{\rm e}$. Similarly, we define weak solutions with initial data at $t = T$, $T \in \mathbb{R}$. Assuming $\rho \in C(\mathbb{R}, \mathbf{L}^2)$, the solution has more regularity, $\omega \in C(\mathbb{R}, \mathbf{L}^2) \cap C^1(\mathbb{R}, \mathbf{H}')$, where $\mathbf{H}'$ denotes the dual of $\mathbf{H}$.

The following result gives a sufficient condition for a weak solution of the complete system to be also a solution of Maxwell's equations.

Lemma 1.6. *Assume that the initial data ω_0 is of the form $\omega_0 = (0, \omega_0^1, \omega_0^2, 0)$, where $\delta\omega_0^1 = 0$, $d\omega_0^2 = 0$, and $\rho = 0$. Then the weak solution $\omega(t)$ of form (34) satisfies also Maxwell's equations (20), (21), i.e., $\omega^0 = 0$ and $\omega^3 = 0$.*

Proof: As seen from (29), $\omega^0(t)$ satisfies the wave equation

$$\Delta^0_\alpha\omega^0 + \omega^0_{tt} = 0,$$

with the Dirichlet boundary condition $\mathbf{t}\omega^0 = 0$. The initial data for ω^0 is

$$\omega^0(0) = \omega^0_0 = 0, \quad \omega^0_t(0) = \delta\omega^1\big|_{t=0} = \delta\omega^1_0 = 0.$$

Hence, $\omega^0(t) = 0$ for all $t \in \mathbb{R}$.

Similarly, $\omega^3(t)$ satisfies the wave equation with the initial data

$$\omega^3(0) = \omega^0_3 = 0, \quad \omega^3_t(0) = -d\omega^2\big|_{t=0} = -d\omega^2_0 = 0.$$

As for the boundary condition, we observe that

$$\mathbf{t}\delta\omega^3 = \mathbf{t}\omega^2_t - \mathbf{t}d\omega^1 = \partial_t\mathbf{t}\omega^2 - d\mathbf{t}\omega^1 = 0,$$

i.e., the Neumann data for the function $*\omega^3$ vanish at ∂M. Thus, $\omega^3(t) = 0$. □

Assume that $\omega(t)$ is a smooth solution of the complete system (33). The *complete Cauchy data* of $\omega(t)$ consist of

$$\big(\mathbf{t}\omega(x,t), \mathbf{n}\omega(x,t)\big), \quad (x,t) \in \partial M \times \mathbb{R}.$$

The Cauchy data for the solutions $\omega(t)$ of Maxwell's equations have a particular structure. Indeed, by taking the tangential trace of equation (24), we obtain $\mathbf{t}\omega^2_t = -d\mathbf{t}\omega^1$. Further, by integrating,

$$(35)\qquad \mathbf{t}\omega^2(x,t) = \mathbf{t}\omega^2(x,0) - \int_0^t d(\mathbf{t}\omega^1(x,t'))\,dt', \quad x \in \partial M.$$

Similarly, by taking the normal trace of equation (23), we find that $\mathbf{n}\omega^1_t = d\mathbf{n}\omega^2$, so by integrating

$$\mathbf{n}\omega^1(x,t) = \mathbf{n}\omega^1(x,0) + \int_0^t d(\mathbf{n}\omega^2(x,t'))\,dt', \quad x \in \partial M. \tag{36}$$

In this work, we consider mainly the case $\omega(0) = 0$, when the lateral Cauchy data for the original problem of electrodynamics is simply

$$\mathbf{t}\omega = (0, f, -\int_0^t df(t')dt'), \tag{37}$$

$$\mathbf{n}\omega = (0, g, \int_0^t dg(t')dt'), \tag{38}$$

where f and g are functions of t with values in $\Omega^1\partial M$. The following theorem implies that solutions of Maxwell's equations are solutions of the complete Maxwell system and gives sufficient conditions for the converse result.

Theorem 1.7. *If $\omega(t) \in C(\mathbb{R}, \mathbf{H}) \cap C^1(\mathbb{R}, \mathbf{L}^2)$ satisfies the equation*

$$\omega_t + \mathcal{M}\omega = 0, \quad t > 0, \tag{39}$$

with $\omega(0) = 0$, and $\omega^0(t) = 0$, $\omega^3(t) = 0$, then $\mathbf{t}\omega$, $\mathbf{n}\omega$ are of the form (37)–(38).

Conversely, if $\mathbf{t}\omega$, $\mathbf{n}\omega$ are of the form (37)–(38) for $0 \le t \le T$, and $\omega(t)$ satisfies (39), with $\omega(0) = 0$, then $\omega(t)$ is a solution of Maxwell's equations (20), (21), , i.e., $\omega^0(t) = 0$, $\omega^3(t) = 0$.

Proof: The first part of the theorem follows from the above considerations if we show that $\omega(t)$ is sufficiently regular. For $\omega^2 \in C(\mathbb{R}, H(\delta, \Omega^2 M))$, by Proposition 1.2, $\mathbf{n}\omega^2 \in C(\mathbb{R}, H^{-1/2}(\Omega^1\partial M))$ with $d\mathbf{n}\omega^2 \in C(\mathbb{R}, H^{-1/2}(\Omega^2\partial M))$. As $\delta\omega^1_t(t) = \delta\delta\omega^2(t) = 0$, it holds also that

$$\mathbf{n}\omega^1_t \in C(\mathbb{R}, H^{-1/2}(\Omega^2\partial M)),$$

implying (38). To prove (37), we use the Maxwell duality: Consider the forms

$$\eta^{3-k} = (-1)^k * \alpha^{-1}\omega^k.$$

Then $\eta = (\eta^0, \eta^1, \eta^2, \eta^3)$ satisfies the complete system dual to the Maxwell system, $\eta_t + \widetilde{\mathcal{M}}\eta = 0$, where $\widetilde{\mathcal{M}}$ is the Maxwell operator with metric g and scalar impedance α^{-1}. Then formula (38) for the solution η implies (37) for ω.

To prove the converse, we observe that the equations,

$$\partial_t\omega^0(t) - \delta\omega^1(t) = 0, \quad \partial_t\omega^1(t) + d\omega^0(t) - \delta\omega^2(t) = 0 \tag{40}$$

imply that

$$\omega^0_{tt}(t) + \delta d\omega^0(t) = 0.$$

In addition, $\omega^0(0) = 0$, $\omega^0_t(0) = 0$, and from (37), $\mathbf{t}\omega^0(t) = 0$. Thus, $\omega^0 = 0$ for $0 \le t \le T$. By the Maxwell duality described earlier, this implies also that $\omega^3(t) = 0$. □

The following definition, where R is a right inverse to the mapping $\mathbf{t}$, fixes the solutions of the forward problem used in this work.

Definition 1.8. *Let $h = (h^0, h^1, h^2) \in C^\infty([0,T], \mathbf{\Omega}\partial M)$. The solution $\omega(t)$ of the initial boundary value problem*

$$\omega_t + \mathcal{M}\omega = 0, \quad t > 0,$$
$$\omega(0) = \omega_0 \in \mathbf{L}^2(M), \quad \mathbf{t}\omega = h,$$

is given by

$$\omega(t) = Rh(t) + \mathcal{U}(t)\omega_0 - \int_0^t \mathcal{U}(t-s)(\mathcal{M}Rh(s) + Rh_s(s))ds.$$

When $\omega_0 = 0$ and h is a smooth boundary source of form (37),

$$h = (0, f, -\int_0^t df(t')dt'), \quad f \in C_0^\infty(]0,T[, \Omega^1\partial M),$$

$\omega(t)$ is called the solution of Maxwell's equations in $M \times [0,T]$ with the boundary condition $\mathbf{t}\omega^1 = f$ and the initial condition $\omega(0) = 0$.

To emphasize the dependence of $\omega(t)$ on f above, we write occasionally

(41) $$\omega(t) = \omega^f(t) = (0, (\omega^f)^1, (\omega^f)^2, 0).$$

We note that f could be chosen from a wider class, e.g. from $H^{1/2}(\partial M \times [0,T])$.

We use the notation $\overset{\bullet}{C}{}^\infty([0,T], \Omega^1\partial M)$ for the space of C^∞ functions $[0,T] \to \Omega^1\partial M$ vanishing near $t = 0$. Theorem 1.7 motivates the following definition.

Definition 1.9. *The admittance map, $\mathcal{Z}^T$ is defined as*

$$\mathcal{Z}^T \;:\; \overset{\bullet}{C}{}^\infty([0,T], \Omega^1\partial M) \to \overset{\bullet}{C}{}^\infty([0,T], \Omega^1\partial M),$$
$$\mathbf{t}\omega^1|_{\partial M \times [0,T]} \mapsto \mathbf{n}\omega^2|_{\partial M \times [0,T]},$$

where $\omega(t)$ is the solution of Maxwell's equations (20), (21), in $M \times [0,T]$ with $\omega(0) = 0$.

Note that in the classical terminology of the electric and magnetic fields, $\mathcal{Z}^T$ maps the tangential electric field $n \times E|_{\partial M \times [0,T]}$ to the tangential magnetic field $n \times H|_{\partial M \times [0,T]}$.

The following result, which relates the boundary data and the energy of the electromagnetic field, is crucial for boundary control. It is a version of the Blagovestchenskii formula (see [4] for the scalar case).

Theorem 1.10. (1) *For any $T > 0$ and $f,\, h \in \overset{\bullet}{C}^\infty([0,2T],\Omega^1\partial M)$, the knowledge of the admittance map $\mathcal{Z}^{2T}$ allows us to evaluate the inner products*

$$((\omega^f)^j(t),(\omega^h)^j(s))_{L^2},\quad j=1,2,\quad 0\le s,t\le T.$$

(2) *For any $T > 0$ and $f \in \overset{\bullet}{C}^\infty([0,T],\Omega^1(\partial M))$, $\mathcal{Z}^T$ determines the energy of the field ω^f at $t = T$, defined as*

$$\mathcal{E}^T(\omega^f) = \frac{1}{2}\|(\omega^f)^1(T)\|^2_{L^2} + \frac{1}{2}\|(\omega^f)^2(T)\|^2_{L^2}.$$

Proof: 1. Let $\omega(t) = \omega^f(t)$ and $\eta(s) = \omega^h(s)$ with $F^j(s,t) = (\omega^j(s),\eta^j(t))_{L^2}$, $j = 1, 2$. By (25),

$$(42)\quad (\partial_s^2-\partial_t^2)F^j(s,t) = (\omega^j_{ss}(s),\eta^j(t))_{L^2} - (\omega^j(s),\eta^j_{tt}(t))_{L^2}$$
$$= -((d\delta+\delta d)\omega^j(s),\eta^j(t))_{L^2} + (\omega^j(s),(d\delta+\delta d)\eta^j(t))_{L^2} = b^j(s,t).$$

We apply Maxwell's equations (20), (21) and the commutation relations,

$$(43)\quad \mathbf{t}d\omega^j = d\mathbf{t}\omega^j,\quad \mathbf{n}\delta\omega^j = \mathbf{t}**d\frac{1}{\alpha}*\omega^j = d\mathbf{t}\frac{1}{\alpha}*\omega^j = d\mathbf{n}\omega^j,$$

where $j \in \{1, 2\}$, and d, in the right-hand side, is the exterior derivative on ∂M. A straightforward applications of Stokes' formula (30), yields

$$b^1(s,t) = \langle \mathbf{n}\omega^2_s(s),\mathbf{t}\eta^1(t)\rangle - \langle \mathbf{t}\omega^1(s),\mathbf{n}\eta^2_t(t)\rangle,$$
$$b^2(s,t) = \langle \mathbf{n}\omega^2(s),\mathbf{t}\eta^1_t(t)\rangle - \langle \mathbf{t}\omega^1_s(s),\mathbf{n}\eta^2(t)\rangle.$$

As $\mathcal{Z}^{2T}$ determines $b^1(s,t)$ and $b^2(s,t)$ for $t, s < 2T$ and

$$(44)\qquad F^j(0,t) = F^j(s,0) = 0,\quad F^j_s(0,t) = F^j_t(s,0) = 0,$$

the function $F^j(s,t)$ can be found from the wave equation (42) for $s + t < 2T$.

2. Again, by differentiating and using Maxwell's equations and Stokes' formula, we obtain

$$\partial_t\mathcal{E}^t(\omega^f) = -\langle \mathbf{t}\omega^1(t),\mathbf{n}\omega^2(t)\rangle = -\langle f(t),\mathcal{Z}^T f(t)\rangle.$$

As $\mathcal{E}^0(\omega^f) = 0$, the energy is readily obtained for $t \le T$. □

1.4. **Unique continuation results.** For further applications to inverse problems, in this section we consider the unique continuation of the Holmgren-John type for Maxwell's equations. We start with an extension of differential forms outside the manifold M. Let $\Gamma \subset \partial M$ be open and $\widetilde{M}$ be an extension of M across Γ, i.e., $M \subset \widetilde{M}$, $\Gamma \subset \widetilde{M}^{\rm int}$ and $\partial M \setminus \Gamma \subset \partial\widetilde{M}$. Let $\widetilde{g}$, $\widetilde{\alpha}$ be smooth continuations of g and α to $\widetilde{M}$. In this case, we say that the manifold $(\widetilde{M},\widetilde{g},\widetilde{\alpha})$ is an *extension of* (M,g,α) *across* Γ.

Let ω^k be a k-form on M and $\widetilde{\omega}^k$ its extension by zero to $\widetilde{M}$. It follows from Stokes' formula (30) that for $\omega^k \in H(d,\Omega^k M)$ with $\mathbf{t}\omega^k\big|_\Gamma = 0$, we have $\widetilde{\omega}^k \in H(d, \Omega^k \widetilde{M})$. Similarly, if $\omega^k \in H(\delta, \Omega^k M)$ and $\mathbf{n}\omega^k\big|_\Gamma = 0$, then $\widetilde{\omega}^k \in H(\delta, \Omega^k \widetilde{M})$. These yield the following result.

Proposition 1.11. *Let $\omega(t) \in C^1(\mathbb{R}, \mathbf{L}^2) \cap C(\mathbb{R}, \mathbf{H})$ with $\mathbf{t}\omega|_{\Gamma\times[0,T]} = 0$ and $\mathbf{n}\omega|_{\Gamma\times[0,T]} = 0$, be a solution of the complete Maxwell system (25) in $M \times [0,T]$. Let $\widetilde{\omega}$ be its extension by zero across $\Gamma \subset \partial M$. Then $\widetilde{\omega}(t)$ satisfies the complete Maxwell system (25) in $\widetilde{M} \times [0,T]$.*

We are particularly interested in the solutions of Maxwell's equations. The following result extends Proposition 1.11 to this case.

Lemma 1.12. *Let $\omega(t) \in C^1(\mathbb{R}, \mathbf{L}^2) \cap C(\mathbb{R}, \mathbf{H})$ be a solution of Maxwell's equations (20), (21) in $M \times [0,T]$, i.e., $\omega^0(t) = 0$, $\omega^3(t) = 0$. In addition, let $\mathbf{t}\omega^1|_{\Gamma\times[0,T]} = 0$, $\mathbf{n}\omega^2|_{\Gamma\times[0,T]} = 0$, and $\omega(0) = 0$. Then $\widetilde{\omega}(t) \in C^1(\mathbb{R}, \mathbf{L}^2(\widetilde{M})) \cap C(\mathbb{R}, \mathbf{H}(\widetilde{M}))$ is a solution of Maxwell's equations (20–21) in $\widetilde{M} \times [0,T]$.*

Proof: The above conditions together with Theorem 1.7 imply that

$$\mathbf{t}\omega = (0, \mathbf{t}\omega^1, -\int_0^t d\mathbf{t}\omega^1 dt') = 0, \quad \mathbf{n}\omega = (0, \mathbf{n}\omega^2, \int_0^t d\mathbf{n}\omega^2 dt') = 0$$

in $\Gamma \times [0,T]$. Therefore, by Proposition 1.11, $\widetilde{\omega}(t)$ satisfies (25) in $\widetilde{M} \times [0,T]$. Clearly, also $\widetilde{\omega}^0(t) = 0$, $\widetilde{\omega}^3(t) = 0$ in $\widetilde{M} \times [0,T]$, and $\widetilde{\omega}(0) = 0$.

□

When we deal with a general solution to Maxwell's equations (20)–(21), which may not satisfy zero initial conditions, and try to extend them by zero across Γ, the arguments of Lemma 1.12 fail. Indeed, if $\omega(0) \neq 0$, then (36) show that $\mathbf{n}\omega^2 = 0$ is not sufficient for $\mathbf{n}\omega^1 = 0$. However, by differentiating with respect to t, the parasite term $\mathbf{n}\omega^1(0)$ vanishes. This is the motivation why Theorem 1.13 below deals with the time derivatives of the weak solutions.

Let, again, $\Gamma \subset \partial M$ be open and $T > 0$. Denote by $K(\Gamma, T)$ the double cone of influence with the base on the slice $t = T$,

$$K(\Gamma, T) = \{(x,t) \in M \times [0, 2T] \mid \tau(x, \Gamma) < T - |T - t|\},$$

where $\tau(x,y)$ is the distance function on (M, g) (see Figure 1).

We prove the following unique continuation result for the time derivatives of the fields.

Theorem 1.13. *Let $\omega(t)$ be a weak solution (34) of the initial boundary value problem (33). Assume that $\omega_0 = (0, \omega_0^1, \omega_0^2, 0)$, $\delta\omega_0^1 = 0$, $d\omega_0^2 = 0$, and $\rho = 0$. If $\mathbf{n}\omega^2 = 0$ on $\Gamma \times]0, 2T[$, then $\omega_t = 0$ in $K(\Gamma, T)$.*

Proof: When $\omega(t) \in C^2(]0, 2T[, \mathbf{L}^2) \cap C^1(]0, 2T[, \mathbf{H})$, then $\eta(t) = \omega_t(t) \in C^1(]0, 2T[, \mathbf{L}^2) \cap C^0(]0, 2T[, \mathbf{H})$ also satisfies Maxwell's equations (20), (21). Let $\widetilde{M}$ be the extension of M across Γ and $\widetilde{\eta}$ be the extension of η by zero. In follows for (35) and (36) that $\mathbf{t}\eta^2 = -d\mathbf{t}\omega^1 = 0$

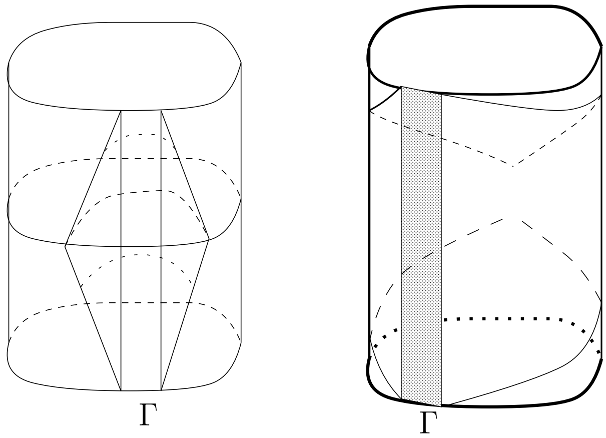

FIGURE 1. Left: The double cone of influence. Right: For T large enough, the double cone contains a slice $\{T/2\} \times M$.

and $\mathbf{n}\eta^1 = d\mathbf{n}\omega^2 = 0$ in $\Gamma\times]0, 2T[$. Therefore, by Proposition 1.11,

$$\widetilde{\eta}(t) \in C^1(]0, 2T[, \mathbf{L}^2(\widetilde{M})) \cap C^0(]0, 2T[, \mathbf{H}(\widetilde{M})),$$

is a solution of the complete system and obviously, also a solution of Maxwell's equations (20), (21) in $\widetilde{M}\times]0, 2T[$.

By the unique continuation for sufficiently smooth solutions (see [17] and Remark 1.14 below), we have that, for any $\sigma > 0$, $\widetilde{\eta} = 0$ in the double cone

$$\{(x, t),\, x \in \widetilde{M}, t \in \mathbb{R}|\, \widetilde{\tau}(x, \widetilde{M} \setminus M) < T - \sigma - |T - t|\}.$$

Thus, $\eta = 0$ in $K(\Gamma, T)$.

When $\omega(t) \in C^0(]0, 2T[, \mathbf{L}^2)$ is a weak solution, we use Friedrich's mollifier in t,

$$\omega_\sigma = \psi_\sigma * \omega, \quad \psi_\sigma(t) = (1/\sigma)\psi(t/\sigma) \quad \text{for } \sigma > 0,$$

where $\psi \in C_0^\infty([-1, 1])$, $\int \psi(s)ds = 1$. Then $\omega_\sigma(t)$ satisfies the conditions of the theorem with $\Gamma\times]0, 2T[$ replaced by $\Gamma\times]\sigma, 2T - \sigma[$. As

$$\mathcal{M}^j\omega_\sigma = (-\partial_t)^j\omega_\sigma \in C^\infty(]\sigma, 2T - \sigma[, \mathbf{L}^2(M)), \quad \text{for any } j > 0,$$

we have, in particular, that $\omega_\sigma(t) \in C^2(]\sigma, 2T - \sigma[, \mathbf{L}^2) \cap C^1(]\sigma, 2T - \sigma[, \mathbf{H})$. By the above, $\partial_t\omega_\sigma(t) = 0$ in $K_\sigma(\Gamma, T)$, where

$$K_\sigma(\Gamma, T) = \{(x, t) \mid \widetilde{\tau}(x, \widetilde{M} \setminus M) < T - \sigma - |T - t|\}.$$

As $\partial_t\omega_\sigma(t) \to \partial_t\omega(t)$ in the distribution sense, the result follows. □

Remark 1.14. The article [17], based on results of Tataru [69, 70] deals with scalar ϵ and μ. However, due to the polarization independence of the wave velocity, it is, in principle, possible to generalize it to the scalar impedance case. Another way to prove the desired unique continuation for the sufficiently smooth solutions of equation (29) is to use the simplified version of Tataru's construction, given in [27, sec. 2.5]. There, the unique continuation result is based on local Carleman estimates for the solutions of the scalar wave equation, $u_{tt} - a_{ij}(x)\partial_i\partial_j u + A_1(x, D)u = 0$, where $A_1(x, D)$ is a first-order differential operator. These estimates utilized a function $\phi(x, t)$ that is pseudoconvex with respect to the metric a_{ij}, and absorbed the perturbation due to $A_1(x, D)$ into the main terms of the Carleman estimates. By (28) and (29), the operator $\mathcal{M}^2$ is, in local coordinates, a principally diagonal operator with the same second order differential operator, $g^{ij}\partial_i\partial_j$, acting on all components of $\omega(t)$. As in [27], one can treat the first-order terms as a perturbation and obtain a desired Carleman estimate. In this manner, the constructions in [27] can be word-by-word generalized to the considered case of Maxwell's equations with scalar wave impedance.

Remark 1.15. It is clear from the above arguments that if $\omega(t)$ is a weak solution of the initial boundary value problem (33) and $\omega(t) \in C^\infty(]0, 2T[, \mathbf{L}^2(M))$, then

$$\omega(t) \in C^\infty(]0, 2T[, \mathcal{D}^\infty(\mathcal{M}_{\rm e})), \quad \omega(t) \in C^\infty(M^{\rm int} \times]0, 2T[),$$

where we used the notation $\mathcal{D}^\infty(\mathcal{M}_{\rm e}) = \bigcap_{N>0} \mathcal{D}(\mathcal{M}_{\rm e}^N)$.

1.5. **Controllability results.** In this section we derive controllability results for Maxwell's equations. We divide these results into *local results*, i.e., controllability at short times and *global results*, where the time of control is long enough so that the controlled electromagnetic waves fill the whole manifold. Both types of results are based on the unique continuation of Theorem 1.13.

Let $\omega^f(t)$, $f \in C_0^\infty(\mathbb{R}_+, \Omega^1\partial M)$ be a solution of Maxwell's equations in the sense of Definition 1.8 with the initial condition $\omega^f(0) = 0$. Let $\widetilde{\omega}$ be the weak solution of (33) given by (34) with $\rho = 0$ and $\widetilde{\omega}(T) = \omega_0 = (0, \omega_0^1, \omega_0^2, 0)$. Similar considerations to those in the proof of Theorem 1.10, show that

$$(\omega^f(T), \omega_0)_{\mathbf{L}^2} = -\int_0^T \langle \mathbf{t}\omega^f(t), \mathbf{n}\widetilde{\omega}(t)\rangle dt, \tag{45}$$

which we will refer to as the *control identity.*

1.5.1. *Local controllability.* In this section, we study the differential 1−forms in M generated by boundary sources active for short periods of time. Instead of a complete characterization of these forms, we show that they form a sufficiently large subspace in $L^2(\Omega^1 M)$. The difficulty that prevents a complete characterization of this subspace lies in the topology of the domains of influence, which can be very complicated.

Let $\Gamma \subset \partial M$ be open and $T > 0$. The *domain of influence*, $M(\Gamma, T)$, is defined as

$$M(\Gamma,T) = \{x \in M | \tau(x,\Gamma) < T\},$$
$$M(\Gamma,T)\times\{T\} = K(\Gamma,T) \cap \{t = T\}. \tag{46}$$

Let $C_0^\infty(]0,T[,\Omega^1\Gamma) \subset C_0^\infty(]0,T[,\Omega^1\partial M)$ consists of the forms supported in $\overline{\Gamma} \times [0,T]$ and

$$X(\Gamma,T) = \mathrm{cl}_{L^2}\{(\omega^f)^1(T) \mid f \in C_0^\infty(]0,T[,\Omega^1\Gamma)\}. \tag{47}$$

Furthermore, let

$$H(\delta, M(\Gamma,T)) = \{\omega^2 \in H(\delta,\Omega^2 M) \mid \mathrm{supp}\,(\omega^2) \in \overline{M(\Gamma,T)}\}.$$

For $S \subset M$, we define $H_0^1(\Omega^k S) \subset H_0^1(\Omega^k M)$ consisting of the k–forms with support in $\overline{S}$.

Theorem 1.16. *For any open $\Gamma \subset \partial M$ and $T > 0$,*

$$\delta H_0^1(\Omega^2 M(\Gamma,T)) \subset X(\Gamma,T) \subset \mathrm{cl}_{L^2}\left(\delta H(\delta, M(\Gamma,T))\right). \tag{48}$$

Proof: The rightmost inclusion being an immediate corollary of (20), we concentrate on the leftmost one.

Let $\omega_0^1 \in L^2(\Omega^1 M)$ satisfy

$$\left(\omega_0^1, (\omega^f)^1(T)\right)_{L^2} = 0, \quad \text{for all } f \in C_0^\infty(]0,T[,\Omega^1\Gamma). \tag{49}$$

By the Hodge decomposition (see [64]) in $L^2(\Omega^1 M)$, we have

$$\omega_0^1 = \widehat{\omega}_0^1 + \delta\eta_0^2, \tag{50}$$

where $d\widehat{\omega}_0^1 = 0$, $\mathbf{t}\widehat{\omega}_0^1 = 0$ and $\eta_0^2 \in H(\delta,\Omega^2 M)$. Thus, (49) is equivalent to

$$\left(\delta\eta_0^2, (\omega^f)^1(T)\right)_{L^2} = 0. \tag{51}$$

Let $\widetilde{\omega}(t)$ be the weak solution to (33) with $\rho = 0$ and the initial data at $t = T$ given by $\widetilde{\omega}(T) = (0, \delta\eta_0^2, 0, 0)$. By the control identity (45), the orthogonality (49) and the particular form of the boundary data for solutions of Maxwell's equations (37), (38), we see that

$$0 = \int_0^T \langle \mathbf{t}\omega^f(t), \mathbf{n}\widetilde{\omega}(t)\rangle = \int_0^T \langle \mathbf{t}(\omega^f)^1(t), \mathbf{n}\widetilde{\omega}^2(t)\rangle = \int_0^T \langle f, \mathbf{n}\widetilde{\omega}^2(t)\rangle,$$

i.e., $\mathbf{n}\widetilde{\omega}^2 = 0$ on $\Gamma\times]0,T[$. Since

$$\widetilde{\omega}(T+t) = (0, \widetilde{\omega}^1(T-t), -\widetilde{\omega}^2(T-t), 0),$$

also $\mathbf{n}\widetilde{\omega}^2 = 0$ on $\Gamma\times]T,2T[$. Since $\delta\widetilde{\omega}^2(t) = 0$, we see by using Proposition 1.2 that $\mathbf{n}\widetilde{\omega}^2 \in C^0(\mathbb{R}, H^{-1/2}(\Omega^1\partial M))$. Hence

$$\mathbf{n}\widetilde{\omega}^2 = 0 \quad \text{on } \Gamma\times]0,2T[.$$

Therefore, by Theorem 1.13, $\partial_t\widetilde{\omega}^2 = 0$ in $K(\Gamma,T)$, In particular, $d\delta\eta_0^2 = -\partial_t\widetilde{\omega}^2(T) = 0$ in $M(\Gamma,T)$. In other words, if $\omega_0^1 \in X(\Gamma,T)^\perp$, then the term η_0^2 in the decomposition (50) satisfies $d\delta\eta_0^2 = 0$ in $M(\Gamma, T)$. For any $\nu^2 \in H_0^1(\Omega^2 M(\Gamma, T))$, we have therefore

$$\left(\delta\nu^2, \omega_0^1\right)_{L^2} = \left(\nu^2, d\delta\eta_0^2\right)_{L^2} = 0,$$

and thus $\delta\nu^2 \in (X(\Gamma, T)^\perp)^\perp$. This is equivalent to the leftmost inclusion in (48). □

Remark 1.17. Later in this work, we deal mainly with the time derivatives of the electromagnetic fields. Since $\omega_t^f(t) = \omega^{\partial_t f}(t)$, we see by using

$$X(\Gamma, T) \subset \mathrm{cl}_{L^2}\{(\omega_t^f(T))^1 \mid f \in \overset{\bullet}{C}^\infty([0, T], \Omega^1\partial M)\} \tag{52}$$

and (20), that the inclusions (48) remain valid when $X(\Gamma, T)$ is replaced with the right-hand side of (52).

1.5.2. *Global controllability.* This section is devoted to the study of controllability results when the control times are large enough so that the waves fill the whole manifold.

For $\Gamma \subset \partial M$ and $T > 0$, we define

$$Y(\Gamma, T) = \{\omega_t^f(T) \mid f \in C_0^\infty(]0, T[, \Omega^1\Gamma)\}, \tag{53}$$

where $\Omega^1\Gamma$ is the set of the 1-forms in $\Omega^1\partial M$ supported on Γ and abbreviate $Y(\partial M, T) = Y(T)$. Our objective is to characterize $Y(\Gamma, T)$ for T large enough. In the following theorem, we use the notation

$$\mathrm{rad}_\Gamma(M) = \max_{x \in M} \tau(x, \Gamma), \quad \mathrm{rad}(M) = \mathrm{rad}_{\partial M}(M). \tag{54}$$

Theorem 1.18. *For an open non-empty $\Gamma \subset \partial M$ and $T \geq T_0 > 2\,\mathrm{rad}_\Gamma(M)$, we have $\mathrm{cl}_{\mathbf{L}^2(M)} Y(\Gamma, T) = Y$, where*

$$Y = \{0\} \times \delta H(\delta) \times d\overset{\circ}{H}(d) \times \{0\}. \tag{55}$$

Proof: Let $\omega(t) = \omega^f(t)$ be the solution, in the sense of Definition 1.8, of the initial boundary-value problem with $f \in C_0^\infty([0, T_0], \Omega^1\Gamma)$. Since $f = 0$ for $T \geq T_0$, we have $\mathbf{t}\omega^1(T) = 0$, and consequently, for $\omega_t(T) = -\mathcal{M}\omega(T)$,

$$\omega_t(T) = (0, \delta\omega^2(T), -d\omega^1(T), 0) \in \{0\} \times \delta H(\delta) \times d\overset{\circ}{H}(d) \times \{0\}.$$

To prove the converse, we will show that $Y(\Gamma, T)$ is dense in $\{0\} \times \delta H(\delta) \times d\overset{\circ}{H}(d) \times \{0\}$. To this end, let $\omega_0 \in \{0\} \times \delta H(\delta) \times d\overset{\circ}{H}(d) \times \{0\}$ and $\omega_0 \perp Y(\Gamma, T)$, i.e.,

$$(\omega_0, \omega_t(T))_{\mathbf{L}^2} = (\omega_0^1, \omega_t^1(T))_{L^2} + (\omega_0^2, \omega_t^2(T))_{L^2} = 0, \tag{56}$$

for any $f \in C_0^\infty([0, T_0], \Omega^1\Gamma)$.

Let $\widetilde{\omega}$ be the weak solution of the problem

$$\widetilde{\omega}_t + \mathcal{M}\widetilde{\omega} = 0, \quad \mathbf{t}\widetilde{\omega} = 0, \quad \widetilde{\omega}(T) = \omega_0.$$

Observe that ω_0 satisfies $\delta\omega_0^1 = 0$ and $d\omega_0^2 = 0$, so that $\widetilde{\omega}$ satisfies Maxwell's equations (20), (21). Consider the function $F : \mathbb{R} \to \mathbb{R}$, $F(t) = (\widetilde{\omega}(t), \omega_t(t))_{\mathbf{L}^2}$. By Maxwell's equations,

$$\begin{aligned} F_t(t) &= (\widetilde{\omega}(t), \omega_{tt}(t))_{\mathbf{L}^2} + (\widetilde{\omega}_t(t), \omega_t(t))_{\mathbf{L}^2} \\ &= -(\widetilde{\omega}^1, \delta d\omega^1)_{L^2} - (\widetilde{\omega}^2, d\delta\omega^2)_{L^2} + (d\widetilde{\omega}^1, d\omega^1)_{L^2} + (\delta\widetilde{\omega}^2, \delta\omega^2)_{L^2}, \end{aligned}$$

and further, by Stokes' formula (30),

$$F_t(t) = -\langle \mathbf{t}\widetilde{\omega}^1(t), \mathbf{n}d\omega^1(t)\rangle - \langle \mathbf{n}\widetilde{\omega}^2(t), \mathbf{t}\delta\omega^2(t)\rangle.$$

However, $\mathbf{t}\widetilde{\omega} = 0$ and $\delta\omega^2 = \omega_t^1$. Thus,

$$F_t(t) = -\langle \mathbf{n}\widetilde{\omega}^2(t), \mathbf{t}\omega_t^1(t)\rangle = -\langle \mathbf{n}\widetilde{\omega}^2(t), f_t(t)\rangle.$$

On the other hand, since $\omega(0) = 0$, the orthogonality condition (56) implies that $F(0) = F(T) = 0$, i.e.,

$$\int_0^T \langle \mathbf{n}\widetilde{\omega}^2(t), f_t(t)\rangle dt = -\int_0^T F_t(t)dt = 0.$$

Since $f \in C_0^\infty(]0,T[, \Omega^1\Gamma)$ is arbitrary, this implies that $\mathbf{n}\widetilde{\omega}_t^2 = 0$ in $\Gamma\times]0,T[$. Thus, by Theorem 1.13, $\widetilde{\omega}_{tt} = 0$ in the double cone $K(\Gamma, T/2)$. Since $T_0 > 2\,\mathrm{rad}_\Gamma(M)$, this double cone contains a cylinder of the form $C = M\times]T/2 - s, T/2 + s[$ with some $s > 0$. (See Figure 1).

As $\widetilde{\omega}_{tt}$ satisfies Maxwell's equations with the homogeneous boundary condition $\mathbf{t}\widetilde{\omega}_{tt} = 0$, this implies that $\widetilde{\omega}_{tt} = 0$ in $M \times \mathbb{R}$. Therefore, $\widetilde{\omega}(t) = \omega_1 + t\omega_2$, where ω_1 and ω_2 do not depend on t. Again, by Maxwell's equations,

$$\omega_2 = \widetilde{\omega}_t = \mathcal{M}\omega_1 + t\mathcal{M}\omega_2.$$

Therefore, $\omega_2 = \mathcal{M}\omega_1$ and $\mathcal{M}\omega_2 = 0$. But then Stokes' formula implies that

$$(\omega_2, \omega_2)_{\mathbf{L}^2} = (\omega_2, \mathcal{M}\omega_1)_{\mathbf{L}^2} = -(\mathcal{M}\omega_2, \omega_1)_{\mathbf{L}^2} = 0,$$

i.e., $\omega_2 = 0$ and $\mathcal{M}\omega_1 = 0$. Furthermore, by the choice of ω_0,

$$\omega_1 = \widetilde{\omega}(T) = \omega_0 = (0, -\delta\nu^2, d\nu^1, 0) = \mathcal{M}\nu,$$

with $\nu \in \{0\} \times \overset{\circ}{H}(d) \times H(\delta) \times \{0\}$. By a further application of Stokes' formula,

$$(\omega_1, \omega_1)_{\mathbf{L}^2} = (\omega_1, \mathcal{M}\nu)_{\mathbf{L}^2} = -(\mathcal{M}\omega_1, \nu)_{\mathbf{L}^2} = 0,$$

i.e., $\omega_1 = 0$ and, therefore, $\omega_0 = 0$. $\square$

1.6. **Generalized sources.** So far, we dealt only with smooth boundary sources and the corresponding fields. Later, we need more general sources which are described in this section.

Let W^T be the wave operator,

$$W^T : C_0^\infty(]0, T_0[, \Omega^1 \partial M) \to Y, \quad f \mapsto \omega_t^f(T),$$

where $T \geq T_0 > 2\,\mathrm{rad}\,(M)$. Let $\| \cdot \|_{\mathcal{F}}$ be a quasinorm on the space of boundary sources defined via W^T,

$$\|f\|_{\mathcal{F}} = \|W^T f\|_{\mathbf{L}^2}. \tag{57}$$

By the energy conservation, this norm is independent of $T \geq T_0$ and by Theorem 1.10, if the admittance map $\mathcal{Z}^T$ is given, we can evaluate $\|f\|_{\mathcal{F}}$ for $f \in C_0^\infty(]0, T_0[, \Omega^1 \partial M)$.

Using the standard procedure in PDE-control, e.g. [62, 44], there is a Hilbert space of *generalized boundary sources* with the norm defined by (57). Indeed, we first introduce the space $\mathcal{F}([0, T_0])$,

$$\mathcal{F}([0, T_0]) = C_0^\infty(]0, T_0[, \Omega^1 \partial M)/ \sim,$$

where $f \sim g$ iff $W^T f = W^T g$, and then complete it with respect to the norm (57) to obtain $\overline{\mathcal{F}}([0, T_0])$. By Theorem 1.18, W^T is an isometry between $\overline{\mathcal{F}}([0, T_0])$ and Y for any $T \geq T_0 > 2\,\mathrm{rad}\,(M)$. The elements of $\overline{\mathcal{F}}([0, T_0])$ are equivalence classes of Cauchy sequences $(f_j)_{j=1}^\infty$ and we denote them by $\widehat{f} = (f_j)_{j=1}^\infty$. (This is a slight abuse of notations, as $(f_j)_{j=1}^\infty$ is a representative of the equivalence class $\widehat{f}$.) To put it in another way, for any $\omega_0 \in Y$, there is a sequence $(f_j)_{j=0}^\infty$ with $f_j \in C_0^\infty(]0, T_0[, \Omega^1 \partial M)$, defining a generalized source $\widehat{f} \in \overline{\mathcal{F}}([0, T_0])$, and for the corresponding wave

$$\omega_t^{\widehat{f}}(t) := \lim_{j \to \infty} \omega_t^{f_j}(t), \quad \text{for } t \geq T_0, \tag{58}$$

we have $\omega_t^{\widehat{f}}(T) = \omega_0$. Since in this work T_0 is considered as a fixed parameter, we denote $\overline{\mathcal{F}}([0, T_0])$ for brevity as $\overline{\mathcal{F}}$.

We say that $\widehat{h} \in \overline{\mathcal{F}}$ is a *generalized time derivative* of $\widehat{f} \in \overline{\mathcal{F}}$, if for $T = T_0$,

$$\lim_{\sigma \to 0+} \left\| \frac{\widehat{f}(\cdot + \sigma) - \widehat{f}(\cdot\,)}{\sigma} - \widehat{h} \right\|_{\overline{\mathcal{F}}} = 0, \tag{59}$$

and write $\widehat{h} = \mathbb{D}\widehat{f}$, or simply $\widehat{h} = \partial_t \widehat{f}$. We also need spaces with s generalized derivatives, $\mathcal{F}^s = \mathcal{D}(\mathbb{D}^s)$, with $s \in \mathbb{Z}_+$, and $\mathcal{F}^\infty = \bigcap_{s \in \mathbb{Z}_+} \mathcal{F}^s$. As in Remark 1.15, if $\widehat{f} \in \mathcal{F}^s$,

$$\omega_t^{\widehat{f}} \in \bigcap_{j=0}^{s} \big(C^{s-j}([T_0, \infty[, \mathcal{D}(\mathcal{M}_e^j)) \cap \mathrm{Ran}\,(\mathcal{M}_e) \big), \tag{60}$$

so that $\omega_t^{\widehat{f}}(T) \in \mathbf{H}^s_{\mathrm{loc}}(M^{\mathrm{int}})$ for $T \geq T_0$.

We need also the dual of the space $\mathcal{D}(\mathcal{M}_e^s)$. Since $\mathbf{H}_0^s \subset \mathcal{D}(\mathcal{M}_\mathrm{e}^s)$, we have $(\mathcal{D}(\mathcal{M}_\mathrm{e}^s))' \subset \mathbf{H}^{-s}$. Similarly, $\mathbf{H}_0^{-s} \subset (\mathcal{D}(\mathcal{M}_\mathrm{e}^s))'$. These facts will be used later to construct focusing sequences.

1.7. **Continuation of the boundary data.** Theorems 1.10 and 1.18 make it possible to continue boundary data, originally given for $t \leq T$ to larger times $t > T$, when T is large enough, by using essentially the same ideas as in the scalar case, [27, 38] (see also [10] for another continuation method).

Lemma 1.19. *The admittance map $\mathcal{Z}^T$, given for $T > 2\,\mathrm{rad}(M)$, uniquely determines $\mathcal{Z}^t$ for any $t > 0$.*

Proof: Let $2\varepsilon = T - 2\,\mathrm{rad}(M)$. For $f \in C_0^\infty([0,T],\, \Omega^1\partial M)$, Theorem 1.18 guarantees that there is a sequence $f_n \in C_0^\infty([\varepsilon,T],\, \Omega^1\partial M)$ with

$$\lim_{n\to\infty} \omega_t^{f_n}(T) = \omega_t^f(T) \quad \text{in } L^2(\Omega^1 M) \times L^2(\Omega^2 M), \tag{61}$$

or, equivalently, in terms of the energy of a field,

$$\lim_{n\to\infty} \mathcal{E}^T(\omega^{g_n}) = 0, \quad g_n = \partial_t(f - f_n). \tag{62}$$

By using Theorem 1.10 one can verify, for an arbitrary sequence $(f_n)_{n=1}^\infty$, whether the convergence condition (62) is valid or not. Moreover, condition (62) is valid for some sequence $(f_n)_{n=1}^\infty$. Thus, when the map $\mathcal{Z}^T$ is given, one can find a sequence $(f_n)_{n=1}^\infty$ that satisfies condition (62).

From the definition (34) of a weak solution, (61) implies that

$$\lim_{n\to\infty} \mathbf{n}\partial_t(\omega^{f_n})^2|_{\partial M\times]T,\infty[} = \mathbf{n}\partial_t(\omega^f)^2|_{\partial M\times]T,\infty[}. \tag{63}$$

Let $h_n(x,t) = f_n(x, t+\varepsilon) \in C_0^\infty([0, T-\varepsilon])$. Since the function $\mathcal{Z}^T h_n$ determines $\mathbf{n}\partial_t(\omega^{f_n})^2|_{\partial M\times]T,T+\varepsilon[}$, we see that $\mathcal{Z}^T$ determines the form $\mathbf{n}(\omega^f)^2|_{\partial M\times]T,T+\varepsilon[}$. Iterating this procedure, we construct $\mathcal{Z}^t$ for any $t > 0$. ☐

In the sequel, we need $\mathcal{Z}^T$ with various values $T > 2\,\mathrm{rad}(M)$. Taking into account Lemma 1.19, we denote simply by $\mathcal{Z}$ the admittance map known for all t.

Remark 1.20. The controllability results, Theorem 1.16 together with the Blachovestchenskii formula, Theorem 1.10, make it is possible to verify from the knowledge of $\mathcal{Z}^T$ whether the condition $T > 2\,\mathrm{rad}(M)$ holds or not. Indeed, $T \leq 2\,\mathrm{rad}(M)$ if and only if, for any $\varepsilon > 0$,

$$M(\partial M, (T-\epsilon)/2) \neq M(\partial M, T/2).$$

This is equivalent to the fact that there are $f_n \in C_0^\infty(\partial M\times]0,T/2[)$, $n = 1, 2, \ldots$ such that $(\omega^{f_n})^1(T/2)$ form a Cauchy sequence in $L^2(M)$, have norm one and converge to a function that is orthogonal to all $(\omega^h)^1((T-\epsilon)/2)$, $h \in C_0^\infty(\partial M\times]0,(T-\epsilon)/2[)$. When $\mathcal{Z}^T$ is given, this can be verified for all f_n and h.

2. Inverse problem

This chapter is devoted to the inverse problem of electrodynamics. Building on the properties of Maxwell's equations obtained in Chapter 1, we prove the following uniqueness result.

Theorem 2.1. *The boundary ∂M and the admittance map $\mathcal{Z}^T$, $T > 2\,\mathrm{rad}(M)$, uniquely determine the manifold M, the travel-time metric g, and the scalar wave impedance α.*

The proof of this theorem consists of several steps. The first is to reconstruct the Riemannian manifold (M, g). Having (M, g), we then identify those boundary sources which generate the electromagnetic waves focusing in a fixed point in M at time $T > 2\,\mathrm{rad}(M)$. These sources are instrumental in reconstructing the impedance α. What is more, in section 2.4 we prove a generalization of the main Theorem 2.1 for the case where the admittance map is given only on a part of the boundary.

At the end, we return to $\mathbb{R}^3$ to characterize group of transformations of the parameters ϵ and μ leaving the boundary data intact.

2.1. Reconstruction of the manifold.

In this section we determine the manifold M and the travel time metric g from the admittance map $\mathcal{Z}$. The idea is to use a slicing procedure to control the supports of the waves from the boundary in order to determine the set of the *boundary distance functions.*

We start by fixing certain notations. Let $T_0 < T_1 < T_2$ satisfy

$$T_0 > 2\,\mathrm{rad}(M), \quad T_1 \geq T_0 + \mathrm{diam}(M), \quad T_2 \geq 2\,T_1.$$

Let $\Gamma_j \subset \partial M$ be arbitrary open disjoint sets and $0 < \tau_j^- < \tau_j^+ < \mathrm{diam}(M)$ be arbitrary times, $1 \leq j \leq J$. We define a set $S = S(\{\Gamma_j, \tau_j^-, \tau_j^+\}_{j=1}^J) \subset M$ as an intersection of slices,

$$S = \bigcap_{j=1}^{J} \left(M(\Gamma_j, \tau_j^+) \setminus M(\Gamma_j, \tau_j^-) \right). \tag{64}$$

(See Figure 2.) Our first goal is to find, in terms of $\mathcal{Z}$, whether the set S contains an open ball or not. To this end, we use the following definition.

Definition 2.2. *The set $Q = Q(\{\Gamma_j, \tau_j^-, \tau_j^+\}_{j=1}^J) \subset \mathcal{F}^\infty$ consists of the generalized sources $\widehat{f}$ such that the waves, $\omega_t = \omega_t^{\widehat{f}}$, satisfy*

(i) $\omega_t^1(T_1) \in X(\Gamma_j, \tau_j^+)$, *for all j, $1 \leq j \leq J$,*
(ii) $\omega_t^2(T_1) = 0$,
(iii) $\omega_{tt}(T_1) = 0$ *in $M(\Gamma_j, \tau_j^-)$, for all j, $1 \leq j \leq J$.*

Observe that Maxwell's equations for $\omega_t = \omega_t^{\widehat{f}}$, $\widehat{f} \in Q$ imply that $\omega_{tt} = (0, \delta\omega_t^2, -d\omega_t^1, 0)$, so, in particular, at $t = T_1$, we have $\omega_{tt}(T_1) =$

$(0, 0, d\eta^1, 0)$, where $\eta^1 = -\omega_t^1(T_1)$ has the support property $\mathrm{supp}(d\eta^1) \subset S$.

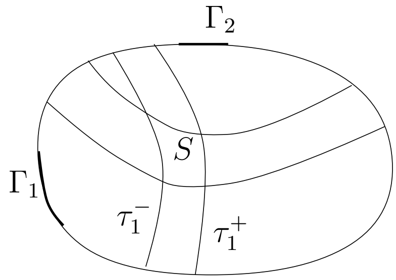

FIGURE 2. The set S in the case when $J = 2$.

The central tool for the reconstruction of the manifold is the following theorem.

Theorem 2.3. *Let S and Q be defined as above. Then:*

(1) *If S contains an open ball, then* $\dim(Q) = \infty$,
(2) *If S does not contain an open ball, then* $Q = \{0\}$.

The proof of Theorem 2.3 is given later.

Theorem 2.4. *For any $\widehat{f} \in \mathcal{F}^\infty$ it is possible, given $\mathcal{Z}$, to determine whether $\widehat{f} \in Q$ or not.*

Proof: Let $\widehat{f} = (f_k)_{k=0}^\infty$ be a generalized source. By Remark 1.17, Condition *(i)* of the definition of Q is equivalent to the existence of a sequence $\widehat{h} = (h_\ell)_{\ell=0}^\infty$, $h_\ell \in \overset{\bullet}{C}{}^\infty([0, \tau_j^+], \Omega^1\Gamma_j)$, such that

$$\lim_{k,\ell \to \infty} \|(\omega_t^{f_k})^1(T_1) - (\omega_t^{h_\ell})^1(\tau_j^+)\| = 0. \tag{65}$$

By the linearity of the initial boundary value problem, we have

$$\|(\omega_t^{f_k})^1(T_1) - (\omega_t^{h_\ell})^1(\tau_j^+)\| = \|(\omega^{g_{k,\ell}})^1(T_1)\|,$$

where the source $g_{k,\ell}$ is

$$g_{k,\ell}(t) = \partial_t\big(f_k(t) - h_\ell(t + \tau_j^+ - T_1)\big) \in C_0^\infty(]0, T_1[, \Omega^1\partial M).$$

Using Lemma 1.10, we can evaluate the norm $\|(\omega^{g_{k,\ell}})^1(T_1)\|$ for various (h_ℓ) and thus verify condition (65).

In a similar fashion, Condition *(ii)* is valid for $\widehat{f}$, if

$$\lim_{k \to \infty} \|(\omega_t^{f_k})^2(T_1))\| = 0,$$

which can also be verified via $\mathcal{Z}$ by Lemma 1.10.

Finally, consider Condition *(iii)* for $\widehat{f}$ satisfying Conditions *(i)* and *(ii)*. Observe that

$$(\partial_t + \mathcal{M})\omega_{tt} = 0 \text{ in } M \times \mathbb{R}_+,$$

where $\omega_{tt} = \omega_{tt}^{\widehat{f}}$, and $\mathbf{t}\omega_{tt} = 0$ in $\partial M \times [T_0, \infty[$. If Condition *(iii)* holds, then, by the finite propagation speed,

$$\omega_{tt} = 0 \text{ in } K_j = \{(x,t) \in M \times \mathbb{R}_+ \mid \tau(x, \Gamma_j) + |t - T_1| < \tau_j^-\},$$

i.e., ω_{tt} vanishes in the double cone of influence of $\Gamma_j \times]T_1 - \tau_j^-, T_1 + \tau_j^-[$, for all $j = 1, \dots, J$. Therefore, in each K_j, ω_t does not depend on time, and, by Condition *(ii)*, $\omega_t^2 = 0$ in K_j. Hence,

$$\mathbf{n}\omega_t^2 = \mathcal{Z}f = 0 \text{ on } \Gamma_j \times]T_1 - \tau_j^-, T_1 + \tau_j^-[. \tag{66}$$

Conversely, if condition (66) holds together with Conditions *(i)* and *(ii)*, then ω_t satisfies

$$(\partial_t + \mathcal{M})\omega_t = 0 \text{ in } M \times \mathbb{R}_+$$

with the boundary conditions

$$\mathbf{t}\omega_t^1 = 0, \quad \mathbf{n}\omega_t^2 = 0 \text{ in } \Gamma_j \times]T_1 - \tau_j^-, T_1 + \tau_j^-[,$$

because $T_1 - \tau_j^- > T_0$, so that $\widehat{f} = 0$ in $\Gamma_j \times]T_1 - \tau_j^-, T_1 + \tau_j^-[$. Thus, by Theorem 1.13, $\omega_{tt} = 0$ in K_j and, in particular, Condition *(iii)* is valid. As (66) is given in terms of $\mathcal{Z}$, this completes the proof. □

Proof of Theorem 2.3: Assume that there is an open ball $B \subset S$ and let $\varphi \in \Omega^2 M$ with $\text{supp}(\varphi) \subset\subset B$. By Theorem 1.18, there is $\widehat{f} \in \overline{\mathcal{F}}$ such that

$$\omega_t^{\widehat{f}}(T_1) = (0, \delta\varphi, 0, 0), \tag{67}$$

Clearly, $\varphi \in \mathcal{D}^\infty(\mathcal{M}_e)$, so that $\widehat{f} \in \mathcal{F}^\infty$.

Let us show that $\widehat{f} \in Q$. Conditions *(i)–(ii)* are immediate from (67) and Theorem 1.16. Finally, since

$$\omega_{tt}^{\widehat{f}}(T_1) = -\mathcal{M}\omega^{\widehat{f}}(T_1) = (0, 0, d\delta\varphi, 0), \quad \text{supp}(d\delta\varphi) \subset\subset B,$$

Condition *(iii)* is also valid. This proves the first part of the theorem.

To prove the second part, assume that S does not contain an open ball and, however, there is $\widehat{f} \in Q$, $\widehat{f} \neq 0$. Let $\omega(t) = \omega^{\widehat{f}}(t)$. Then, by Conditions *(i)–(ii)*,

$$\text{supp}(\omega_t(T_1)) \subset \bigcap_{j=1}^{J} M(\Gamma_j, \tau_j^+) = S^+, \tag{68}$$

implying, due to $\omega_{tt}(T_1) = -\mathcal{M}\omega_t(T_1)$, that $\text{supp}(\omega_{tt}(T_1)) \subset S^+$. On the other hand, by Condition *(iii)*,

$$\omega_{tt}(T_1) = 0 \quad \text{in } \bigcup_{j=1}^{J} M(\Gamma_j, \tau_j^-) = S^-.$$

Thus, $\text{supp}(\omega_{tt}(T_1)) \subset S^+ \setminus S^-$, which is nowhere dense in M. Since $\omega_{tt}(T_1)$ is smooth, $\omega_{tt}(T_1) = 0$, and, therefore,

$$d\omega_t^1(T_1) = -\omega_{tt}^2(T_1) = 0. \tag{69}$$

However, by Theorem 1.18,

$$\omega_t^1(T_1) = \delta\eta^2, \quad \text{with } \eta^2 \in H(\delta, \Omega^2 M).$$

Combining this equation with (69) and using $\mathbf{t}\omega_t^1(T_1) = 0$, we obtain, by Stokes' formula (30), that

$$(\omega_t^1(T_1), \omega_t^1(T_1))_{L^2} = (\delta\eta^2, \omega_t^1(T_1))_{L^2} = (\eta^2, d\omega_t^1(T_1))_{L^2} = 0,$$

i.e., $\omega_t^1(T_1) = 0$. Also, by Condition *(ii)*, $\omega_t^2(T_1) = 0$. These imply that $\widehat{f} = 0$.

□

We are now ready to construct the set of the *boundary distance functions*, r_x, which are defined, for any $x \in M$, as continuous functions on ∂M,

$$r_x : \partial M \to \mathbb{R}_+, \quad r_x(z) = \tau(x, z), \quad z \in \partial M.$$

They define *the boundary distance map* $\mathcal{R} : M \to C(\partial M)$, $\mathcal{R}(x) = r_x$, which is continuous and injective, [36, 27]. The set of all boundary distance functions, i.e., the image of $\mathcal{R}$,

$$\mathcal{R}(M) = \{r_x \in C(\partial M) \mid x \in M\},$$

can be endowed, in a natural way, with a differentiable structure and a metric tensor $\widetilde{g}$, so that $(\mathcal{R}(M), \widetilde{g})$ becomes isometric to (M, g), see e.g. [36, 27]. Hence, in order to reconstruct M (or more precisely, the isometry type of M), it suffices to determine the set $\mathcal{R}(M)$. The following result is therefore crucial.

Theorem 2.5. *For any $h \in C(\partial M)$, it is possible, given $\mathcal{Z}$, to determine whether $h \in \mathcal{R}(M)$ or not.*

Proof: The proof is based on a discrete approximation process. First, we observe that $h \in \mathcal{R}(M)$ if and only if, for any finite subset $\{z_1, \dots, z_J\}$ of ∂M, there is an $x \in M$ with

$$h(z_j) = \tau(x, z_j), \quad 1 \le j \le J.$$

Denote $\tau_j = h(z_j)$. By the continuity of the distance function, $\tau : M \times \partial M \to \mathbb{R}_+$, the above condition is equivalent to the following one: For any $\varepsilon > 0$, there are open sets $\Gamma_j \subset \partial M$, $z_j \in \Gamma_j$ with $\operatorname{diam}(\Gamma_j) < \varepsilon$, such that

$$\operatorname{int}\left(\bigcap_{j=1}^{J} M(\Gamma_j, \tau_j + \varepsilon) \setminus M(\Gamma_j, \tau_j - \varepsilon)\right) \neq \emptyset. \tag{70}$$

On the other hand, by Theorem 2.3, condition (70) is equivalent to

$$\dim\big(Q(\{\Gamma_j, \tau_j + \varepsilon, \tau_j - \varepsilon\}_{j=1}^J)\big) = \infty,$$

a condition that can be verified in terms of $\mathcal{Z}$ by means of Theorem 2.4. □

As a consequence, we obtain the main result of this section.

Corollary 2.6. *The boundary ∂M and the admittance map,*

$$\mathcal{Z} : \overset{\bullet}{C}^\infty(\partial M \times \mathbb{R}_+) \to \overset{\bullet}{C}^\infty(\partial M \times \mathbb{R}_+),$$

determine uniquely the manifold M and the travel time metric g.

Remark 2.7. *The considerations of this section can be, in principle, made constructive to provide a way to build the set of the boundary distance functions $R(M)$ and an isometric copy of (M, g) from ∂M and the admittance map $\mathcal{Z}$. Given a finite approximation of ∂M and the admittance map $\mathcal{Z}$ and using e.g. finite elements, it is possible to construct a finite metric space that is close to (M, g) in the Gromov-Hausdorff sense (for the construction in the scalar case, see* [29]*). For a numerical realization of a similar method for scalar equations, see* [23, Ch. 4].

Having found (M, g), we proceed in the next section to the reconstruction of the impedance α.

2.2. **Focusing sequence.** In this section, we construct sequences of sources, $(\widehat{f}_p)_{p=1}^\infty$ with the property that $(\omega_t^{\widehat{f}_p})^2(T_1) = 0$ and the sets $\mathrm{supp}\left((\omega_t^{\widehat{f}_p})^1(T_1)\right)$ converge, when $p \to \infty$, to a single point in M^{int}, i.e., the time derivative of the electric field focuses to a single point.

Let $y \in M^{\mathrm{int}}$ and $\underline{\delta}_y$ denote the Dirac delta at y in the sense that

$$\int_M \frac{1}{\alpha}\, \underline{\delta}_y(x) \wedge *\phi(x) = \phi(y), \quad \text{for any } \phi \in C_0^\infty(M).$$

Since the Riemannian manifold (M, g) is already found, we can choose $\Gamma_{jp} \subset \partial M,\ 0 < \tau_{jp}^- < \tau_{jp}^+ < \mathrm{diam}(M),\ j = 1, \dots, J(p)$, so that

$$S_{p+1} \subset S_p, \quad \bigcap_{p=1}^\infty S_p = \{y\}, \quad S_p = S\big(\{\Gamma_{jp}, \tau_{jp}^-, \tau_{jp}^+\}_{j=1}^{J(p)}\big). \tag{71}$$

Then, $Q_p = Q\big(\{\Gamma_{jp}, \tau_{jp}^-, \tau_{jp}^+\}_{j=1}^{J(p)}\}\big)$ are the spaces of the boundary sources, which correspond, by Definition 2.2, to the sets S_p.

Definition 2.8. *For $y \in M^{\mathrm{int}}$, let S_p, $p = 1, 2, \dots$, be given by (71). A sequence $(\widehat{f}_p)_{p=1}^\infty$ with $\widehat{f}_p \in Q_p$, is called a* focusing sequence *of boundary sources of the order s, $s \in \mathbb{Z}_+$, if there is a distribution form A_y on M, $A_y \neq 0$, such that*

$$\lim_{p\to\infty} (\omega_t^{\widehat{f}_p}(T_1), \eta)_{\mathbf{L}^2} = (A_y, \eta)_{\mathbf{L}^2}, \quad \textit{when } \eta \in \mathcal{D}(\mathcal{M}_e^s). \tag{72}$$

With a slight abuse of notations, we use the same notation for the inner product in $\mathbf{L}^2$ and for the distribution duality. We denote a focusing sequence converging to y by $\widetilde{f}_y = (\widehat{f}_p)_{p=1}^\infty$.

The following theorem characterizes a class of the limit distributions that can be produced by focusing sequences. This class is large enough for our further goal to solving the inverse problem. What is more, the sequences from this class can be constructed via the admittance map.

Theorem 2.9. *(1) Let $y \in M^{\rm int}$ and $(\widehat{f}_p)_{p=1}^\infty$ be a sequence of boundary sources, $\widehat{f}_p \in Q_p$. Given the admittance map, $\mathcal{Z}$, it is possible to determine, for any $s \in \mathbb{Z}_+$, whether $(\widehat{f}_p)$ is a focusing sequence of the order s or not.*
(2) Let $\widetilde{f}_y$ be a focusing sequence. Then $\mathrm{supp}(A_y) = \{y\}$.
(3) For $s = 3$, the limit distribution A_y has the form

$$A_y = (0, \delta(\lambda(y)\underline{\delta}_y), 0, 0), \tag{73}$$

where $\lambda(y) \in \Lambda^2 T_y^* M$.
(4) For any $y \in M^{\rm int}$ and $\lambda(y) \in \Lambda^2 T_y^* M$, there is a focusing sequence $\widetilde{f}_y$, of the order $s = 3$, with $(A_y)^1 = \delta(\lambda(y)\underline{\delta}_y)$.

Proof: 1. Take $\eta \in \mathcal{D}(\mathcal{M}_{\rm e}^s)$ and decompose it as $\eta = \eta_1 + \eta_2$, where

$$\eta_1 \in \mathcal{D}(\mathcal{M}_{\rm e}^s) \cap Y, \quad \eta_2 \in \mathcal{D}(\mathcal{M}_{\rm e}^s) \cap Y^\perp.$$

As $\mathcal{U}(t)$ in (34) is unitary in $\mathcal{D}(\mathcal{M}_{\rm e}^s)$, by Theorem 1.18 there is a boundary source $\widehat{h} \in \mathcal{F}^s$ such that $\eta_1 = \omega_t^{\widehat{h}}(T_1)$. Observe that $(\omega_t^{\widehat{f}_p}(T_1), \eta_2)_{\mathbf{L}^2} = 0$, so that $(\widehat{f}_p)$ is a focusing sequence if and only if there is a limit,

$$(A_y, \eta)_{\mathbf{L}^2} = \lim_{p\to\infty} (\omega_t^{\widehat{f}_p}(T_1), \omega_t^{\widehat{h}}(T_1))_{\mathbf{L}^2}, \quad \text{when } \widehat{h} \in \mathcal{F}^s. \tag{74}$$

By Theorem 1.10, the existence of this limit can be verified in terms of $\mathcal{Z}$.

Conversely, assume that the limit (74) does exist for all $\widehat{h} \in \mathcal{F}^s$. Then, by the Principle of Uniform Boundedness, the functionals

$$\eta \mapsto (\omega_t^{\widehat{f}_p}(T_1), \eta)_{\mathbf{L}^2}, \quad p \in \mathbb{Z}_+,$$

are uniformly bounded in the dual of $(\mathcal{D}(\mathcal{M}_{\rm e}^s))'$. By the Banach-Alaoglu theorem, there is a weak*-convergent subsequence,

$$\omega_t^{\widehat{f}_p}(T_1) \to A_y \in \Big(\mathcal{D}(\mathcal{M}_{\rm e}^s)\Big)',$$

where A_y is the sought after distribution for which (72) is valid.

2. Let $\widetilde{f}_y = (\widehat{f}_p)_{p=1}^\infty$ be a focusing sequence. Since $\widehat{f}_p \in Q_p$, Condition *(ii)* of Definition 2.2 implies that $A_y = (0, A_y^1, 0, 0)$ and Conditions *(i)–(iii)*, together with (71), yield

$$\mathrm{supp}(dA_y^1) \subset \liminf_{p\to\infty} \mathrm{supp}\big(d(\omega_t^{\widehat{f}_p})^1(T_1)\big) \subset \bigcap_{p=1}^\infty S_p = \{y\}. \tag{75}$$

As $\omega_t^{\widehat{f}_p}(T_1) \in Y$, $\delta A_y^1 = \lim_{p\to\infty} \delta(\omega_t^{\widehat{f}_p})^1(T_1) = 0$. Thus,

$$\mathrm{supp}(\Delta_\alpha A_y^1) \subset \{y\}. \tag{76}$$

On the other hand, by Condition *(i)* of Definition 2.2,

$$(\omega_t^{\widehat{f}_p})^1(T_1) = 0 \quad \text{in} \quad M \setminus S_p^+, \quad S_p^+ = \bigcap_{j=1}^{J(p)} M(\Gamma_{jp}, \tau_{jp}^+).$$

By the definition (72) of a focusing sequence, $A_y^1 = 0$ in $M \setminus S_p^+$. As $\mathrm{rad}(M) < \mathrm{diam}(M)$, we can always choose Γ_{jp}, τ_{jp}^+, so that $M \setminus S_p^+$ is non-empty. By the unique continuation principle for elliptic systems (see e.g. [21]), it then follows from the support property (76) that $\mathrm{supp}(A_y^1) \subset \{y\}$. Since A_y is non-zero by assumption, $\mathrm{supp}(A_y^1) = \{y\}$.

3. Let $s = 3$. By part *2.* of the theorem, in local coordinates the components of A_y are finite sums of the derivatives of the delta-distribution. Since $A_y \in (D(\mathcal{M}_e^s))' \subset \mathbf{H}^{-3}(M)$, it follows that

$$A_y^1 = \sum_{i,j=1}^{3} c_i^j \partial_j \underline{\delta}_y \, dx^i + \sum_{i=1}^{3} \widetilde{c}_i \underline{\delta}_y \, dx^i. \tag{77}$$

Substituting (77) into the identity $\delta A_y^1 = 0$, we obtain (73).

4. Let $\psi_p \in C_0^\infty(S_p)$, $p = 1, 2, \dots$, be 2–forms that converge to $\lambda \underline{\delta}_y$ in $H^{1-s}(\Omega^2 M)$. By the global control Theorem 1.18, there are boundary sources $\widehat{f}_p$ such that $\omega_t^{\widehat{f}_p}(T_1) = (0, \delta\psi_p, 0, 0)$. Then $\widetilde{f}_y = (\widehat{f}_p)_{p=1}^\infty$ is a desired focusing sequence. □

As y runs over M^{int}, we get a parameterized family of the focusing sequences $\{\widetilde{f}_y\}_{y \in M^{\mathrm{int}}}$ which defines a map $y \mapsto \lambda(y)$. However, the admittance map does not provide a direct access to the values $\lambda(y)$. Although this mapping is unknown, we have the following result.

Lemma 2.10. *Given the admittance map $\mathcal{Z}$, it is possible to determine whether the map $y \mapsto \lambda(y)$ is a 2–form valued C^∞–function in M^{int}.*

Proof: Let $\{A_y\}_{y \in M^{\mathrm{int}}}$ be a family of distributions of form (73) corresponding to a family $\{\widetilde{f}_y\}_{y \in M^{\mathrm{int}}}$ of the focusing sequences. Assume that $y \mapsto \lambda(y)$ is smooth, i.e., $\lambda \in \Omega^2 M^{\mathrm{int}}$. Then, for any generalized source $\widehat{h} = (h_j) \in \mathcal{F}^\infty$, we have

$$\begin{aligned}(A_y, \omega^{h_j}(T_1))_{\mathbf{L}^2} &= (\delta\lambda\underline{\delta}_y, (\omega^{h_j})^1(T_1))_{L^2} = (\lambda\underline{\delta}_y, d(\omega^{h_j})^1(T_1))_{L^2} \\ &= -(\lambda\underline{\delta}_y, (\omega_t^{h_j})^2(T_1))_{L^2}.\end{aligned}$$

By taking the limit $j \to \infty$ of the both sides and using notation $\langle \lambda, \eta \rangle_y = *(\lambda \wedge *\eta)$ for the inner product of $\lambda, \eta \in \Lambda^k T_y^* M$, we arrive at the identity

$$(A_y, \omega^{\widehat{h}}(T_1))_{\mathbf{L}^2} = -\langle \lambda(y), (\omega_t^{\widehat{h}})^2(y, T_1) \rangle_y. \tag{78}$$

As $(A_y, \omega^{\widehat{h}}(T_1))_{\mathbf{L}^2} = \lim_{j\to\infty} \lim_{p\to\infty} (\omega_t^{\widehat{f}_{p,y}}(T_1), \omega^{h_j}(T_1))_{\mathbf{L}^2}$, we can evaluate (78) in terms of $\mathcal{Z}$ by Theorem 1.10.

Conversely, if $(A_y, \omega^{\widehat{h}}(T_1))_{\mathbf{L}^2} \in C^\infty(M^{\rm int})$ for any $\widehat{h} \in \mathcal{F}^\infty$, then $\lambda(y) \in \Omega^2(M^{\rm int})$. Indeed, by Theorem 1.18, for any $\varphi \in \Omega^1 M$ with $\operatorname{supp}(\varphi) \subset\subset M^{\rm int}$, there is a generalized boundary source $\widehat{h} \in \mathcal{F}^\infty$ with $\omega_t^{\widehat{h}}(T_1) = (0, 0, -d\varphi, 0)$, and, by (78),

$$(A_y, \omega^{\widehat{h}}(T_1))_{\mathbf{L}^2} = \langle \lambda(y), d\varphi(y)\rangle_y \in C^\infty(M^{\rm int}). \tag{79}$$

As φ is arbitrary, condition (79) is equivalent to that $\lambda(y)$ is C^∞–smooth in $M^{\rm int}$. □

Returning to (78), we conclude that a focusing sequence $\{\widetilde{f}_y\}$ gives rise to a functional on $(\omega_t^{h_j})^2(T_1))$. It depends only on the value of $(\omega_t^{h_j})^2(T_1))$ at the point y and will be called the *point evaluation functional* in the sequel. By the above result this functional is determined up to an unknown factor $\lambda(y)$. Hence, by using three proper focusing sequences, we can evaluate the 2–form $(\omega_t^{\widehat{h}})^2$ at any point in $M^{\rm int}$, up to a linear transformation. The possibility to control the precise form of this transformation is discussed in the next section.

Lemma 2.11. *Let $t > T_1$ and $\widehat{h} \in \overline{\mathcal{F}}$. Given the admittance map $\mathcal{Z}$, it is possible to find the 2–forms*

$$K(y)(\omega_t^{\widehat{h}}(y,t))^2, \quad y \in M^{\rm int}. \tag{80}$$

Here $K(y) : \Lambda^2 T_y^* M \to \Lambda^2 T_y^* M$ is a smooth section of $\operatorname{End}(\Lambda^2 T^* M^{\rm int})$ *having the maximal rank.*

Proof: Let U be a relatively open coordinate patch in M with 2–forms $\xi_k \in \Omega^2 U$, $k = 1, 2, 3$, linearly independent at any $y \in U$. If $\{\widetilde{f}_k(y)\}_{y\in U}$, $k = 1, 2, 3$ are three families of focusing sequences with the corresponding limiting 2–forms $\lambda_k(y)$, we define the endomorphism $K_U(y)$ by

$$K_U(y)\omega^2(y) = \sum_{k=1}^{3} \langle \lambda_k(y),\, \omega^2(y)\rangle_y\, \xi_k(y), \quad y \in U. \tag{81}$$

As we can evaluate inner products (78) by using Theorem 1.10, it is possible, for any given three families of focusing sequences $\{\widetilde{f}_k(y)\}_{y\in U}$, $k = 1, 2, 3$ and $\widehat{h}$, to construct $K(y)(\omega_t^{\widehat{h}}(y,t))^2$ for $y \in U$, $t > T$. Further considerations are based on the result that we formulate separately for future references.

Proposition 2.12. *Let $U \subset M^{\rm int}$ be open and $\xi_k \in \Omega^2 U$, $k = 1, 2, 3$, linearly independent at each $y \in U$. There are focusing sequences $\{\widetilde{f}_k(y)\}_{y\in U}$ such that the corresponding endomorphism (81) is $K_U(y) = I_y$, $y \in U$, the identity in $\Lambda^2 T_y^* M$.*

Proof: Let $\lambda_k(y)$, $k = 1, 2, 3$ form the dual basis of $\xi_k(y)$, $k = 1, 2, 3$,

$$\langle \lambda_k(y), \xi_\ell(y)\rangle_y = \delta_{k\ell}.$$

It is a consequence of Theorem 2.9 that there are focusing sequences $\widetilde{f}_k(y)$ giving rise to the 2–forms $\lambda_k(y)$, which shows the claim. □

End of the proof of Lemma 2.11: By the above proposition, there are, for given linearly independent $\xi_k(y)$, focusing sequences $\widetilde{f}_k(y)$ so that $K_U(y)$ is of the maximal rank. Moreover, since $K_U\omega^2(y)$ can be evaluated for any $\omega^2 = (\omega_t^{\widehat{h}})^2(T_1)$, the maximality of the rank of $K_U(y)$ can be verified via $\mathcal{Z}$.

Let U_j, $j = 1,\dots,J$, be a finite covering of M by coordinate patches and K_j the corresponding local endomorphisms of form (81) in $U_j \cap M^{\rm int}$. As we can compute (81) for all $\widehat{h} \in \mathcal{F}^\infty$, $t > T$ and $x \in U$, it possible to verify that $K_j(y) = K_\ell(y)$ for $y \in U_j \cap U_\ell$ for all j and ℓ. As by Proposition 2.12 there are families of the focusing sequences for which this is true, we can construct the desired endomorphism. □

2.3. **Reconstruction of the wave impedance.** So far, we have found the waves $(\omega_t^{\widehat{h}})^2(t)$, $t > T_1$, up to a linear transformation K which, at this stage, is unknown. Since the choice of the focusing sequences is non-unique, we will choose them in such a manner that the endomorphism K becomes as simple as possible, i.e., $K = c_0 I$, an identity up to a constant multiplier. The first step in this direction is to consider the polarization of the *electric Green's function*, defined as the solution of the following initial boundary value problem,

$$
\begin{aligned}
(\partial_t + \mathcal{M})G_{\rm e}(x,y,t) &= 0 \text{ in } M \times \mathbb{R}_+, \\
\mathbf{t}G_{\rm e}(x,y,t) &= 0 \text{ in } (x,t) \in \partial M \times \mathbb{R}_+, \qquad (82)\\
G_{\rm e}(x,y,t)|_{t=0} &= (0, \delta(\lambda\underline{\delta}_y), 0, 0).
\end{aligned}
$$

Sometimes, we denote $G_{\rm e}(x,y,t) = G_{\rm e}(x,y,t;\lambda)$ to indicate the source $\lambda \in \Lambda^2 T_y^* M$. Assume that $\widetilde{h} = \widetilde{h}_y$ is a focusing sequence that produces a wave focusing at y, the corresponding 2–form being λ. Since the boundary sources are off when $t > T_1$, we must have

$$G_{\rm e}(x,y,t) = \omega_t^{\widetilde{h}}(x, t + T_1). \qquad (83)$$

On the other hand, by Lemma 2.11, we can calculate the 2–forms $K(x)(\omega_t^{\widetilde{h}})^2(x, t+T_1)$ for $x \in M^{\rm int}$ and $t > 0$. Hence, we know the electric Green's function up to a linear transformation.

Let us denote by $\Phi = \Phi^\lambda(x,y,t)$ the standard Green's 2–form, satisfying

$$
\begin{aligned}
(\partial_t^2 + \Delta_\alpha^2)\Phi(x,y,t) &= 0 \text{ in } M \times \mathbb{R}_+, \\
\Phi(x,y,t)|_{\partial M \times \mathbb{R}_+} &= 0, \qquad (84)
\end{aligned}
$$

$$\Phi(x,y,0) = 0, \qquad \Phi_t(x,y,0) = \lambda(y)\underline{\delta}_y(x), \qquad (85)$$

where $\lambda(y) \in \Lambda^2 T_y^* M$ and the boundary condition in (85) means that all three components of Φ vanish on $\partial M \times \mathbb{R}_+$.

Let $\widetilde{G}_{\rm e} = \widetilde{G}_{\rm e}(x, y, t)$ be defined as

$$\widetilde{G}_{\rm e} = (\partial_t - \mathcal{M})(0, \delta\Phi, 0, 0) = (0, \partial_t \delta\Phi, -d\delta\Phi, 0). \tag{86}$$

As $(\partial_t^2 + \mathbf{\Delta}_\alpha) = (\partial_t + \mathcal{M})(\partial_t - \mathcal{M})$, $\widetilde{G}_{\rm e}$ satisfies the complete Maxwell system and, by (85), the initial condition in (82). By the unit propagation speed, $\widetilde{G}_{\rm e} = 0$ near $\partial M \times]0, \tau(y, \partial M)[$, satisfying the boundary condition in (82). Thus, $\widetilde{G}_{\rm e}(x, y, t) = G_{\rm e}(x, y, t)$ for $t < \tau(y, \partial M)$.

To further the study of $G_{\rm e}$, we formulate the following result proved in the Appendix.

Lemma 2.13. *For every $y \in M^{\rm int}$ there is an open neighborhood $U \subset M^{\rm int}$ of y, a positive t_y and a mapping $Q_y(x)$ that is smooth with respect to $x \in U$, where $Q_y(x) : \Lambda^2 T_y^* M \to \Lambda^2 T_x^* M$ is bijective, such that*

$$\Phi(x, y, t) = Q_y(x) \lambda \underline{\delta}(t^2 - \tau^2(x, y)) + r(x, y, t) \tag{87}$$

for $(x, t) \in U \times]0, t_y[$. Moreover, with some smooth $Q_y^p(x) : \Lambda^2 T_y^* M \to \Lambda^2 T_x^* M$, $p = 1, 2$ and $C^{1,1}$–smooth 2–form $\widehat{r}(x, y, t)$, the remainder can be written as

$$r(x, y, t) = \sum_{p=1,2} Q_y^p(x) \lambda (t^2 - \tau^2(x, y))_+^{p-1} + \widehat{r}(x, y, t). \tag{88}$$

By (86), it follows from (87) that, for sufficiently small t,

$$G_{\rm e} = (0, G_{\rm e}^1, G_{\rm e}^2, 0) + r_1, \tag{89}$$

where

$$\begin{aligned} G_{\rm e}^1 &= -2t * (d\tau^2 \wedge *Q_y \lambda) \underline{\delta}^{(2)}(t^2 - \tau^2), \\ G_{\rm e}^2 &= d\tau^2 \wedge *(d\tau^2 \wedge *Q_y \lambda) \underline{\delta}^{(2)}(t^2 - \tau^2), \end{aligned}$$

and r_1 is a linear combination of a bounded function, the delta distribution on $\partial B_y(t)$ and its first derivative, $B_y(t)$ being the ball of radius t centered in y. Using Lemma 2.11 and (83) together with (89), we obtain the following result.

Lemma 2.14. *Given the admittance map $\mathcal{Z}$, it is possible to find the distribution 2–form*

$$K(x) G_{\rm e}^2(x, y, t) = K(x) (\omega_t^{\widetilde{h}})^2 (x, t + T_1),$$

where $K \in {\rm End}(\Omega^2 M^{\rm int})$ and $t > 0$. Moreover, the leading singularity of this form when $0 < t < t_y$ determines the 2–form

$$K(x) \big(d\tau^2(x, y) \wedge *(d\tau^2(x, y) \wedge *Q_y(x) \lambda) \big), \quad x \in \partial B_y(t). \tag{90}$$

The linear transformation $K(x)$ of Lemma 2.11 depends on $\widetilde{f}_k(x)$, $\xi_k(x)$, $k = 1, 2, 3$. Our next goal is to formulate conditions, in terms of $\mathcal{Z}$, on $\widetilde{f}_k, \xi_k$ to make K isotropic, i.e.,

$$K(x) = c(x) I, \quad c \in C^\infty(M^{\rm int}), \quad c(x) \neq 0. \tag{91}$$

To this end, observe that for $\lambda \in \Lambda^2 T_y^* M$,

$$\mathbf{t}_{B_y(t)}\big(d\tau^2 \wedge *(d\tau^2 \wedge *Q_y\lambda)\big) = 0, \tag{92}$$

where $\mathbf{t}_{B_y(t)}\omega^k$ is the tangential component of ω^k on $\partial B_y(t)$. Physically, condition (92) corresponds to the orthogonality of the polarization of the magnetic flux density and the direction of the wave propagation. (See Figure 3.) If K is isotropic, we have

$$\mathbf{t}_{B_y(t)}\Big(K\big(d\tau^2 \wedge *(d\tau^2 \wedge *Q_y\lambda)\big)\Big) = 0. \tag{93}$$

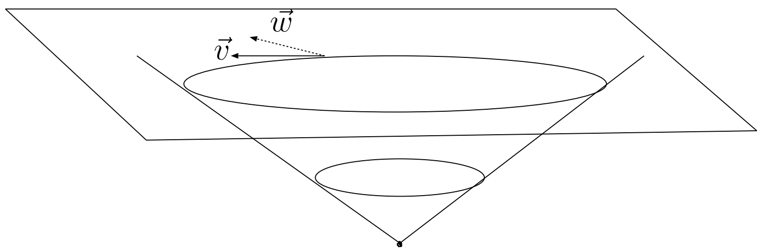

FIGURE 3. Vector $\vec{v}$ is the right polarization of the electromagnetic wave in the plane $M \times \{t\}$. The reconstructed polarization $\vec{w}$ has wrong direction, if the transformation matrix $K(x)$ is not isotropic.

Conversely, we show that condition (93) for all $y \in M^{\rm int}$ and $t < t_y$ guarantees that K is isotropic. What is more, condition (93) is verifiable from the knowledge of $\mathcal{Z}$. Indeed, for $\lambda(y) \in \Lambda^2 T_y^* M$ and $t = \tau(x,y)$, (93) means that $K\big(d\tau^2 \wedge *(d\tau^2 \wedge *Q_y\lambda)\big)$ is normal to $T_x\partial B_y(t) \subset T_x M$, i.e, for vectors X, $Y \in T_x\partial B_y(t)$, we have

$$K(x)\big(d\tau^2(x,y) \wedge *(d\tau^2(x,y) \wedge *Q_y(x)\lambda(y))\big)(X,Y) = 0.$$

Observe that, when $\lambda(y)$ runs through $\Lambda^2 T_y^* M$, then $*(Q_y(x)\lambda(y))$ runs through $T_x^* M$. Now we may vary y and t with a fixed x such that $\tau(x,y) = t$, making $T_x\partial B_y(t)$ run through the Grassmannian manifold $G_{3,2}(T_x M)$. The transformation $K(x)$ is kept invariant in this variation. Hence, we deduce that $K(x)$ keeps any 2–dimensional subspace of $\Lambda^2 T_x^* M$ invariant, so it must be isotropic as claimed.

Assume that the focusing sequences used for the point evaluation functionals are chosen so that $K(x) = c(x)I$. For any generalized source $\widehat{f} \in \overline{\mathcal{F}}$, we may thus evaluate

$$(\widetilde{\omega}_t^{\widehat{f}})^2(x,T_1) = c(x)(\omega_t^{\widehat{f}})^2(x,T_1),$$

with yet unknown $c(x)$. Since $\omega^{\widehat{f}}$ satisfies Maxwell's equations, we have

$$d(\widetilde{\omega}_t^{\widehat{f}})^2 = dc \wedge (\omega_t^{\widehat{f}})^2 + c d(\omega_t^{\widehat{f}})^2 = dc \wedge (\omega_t^{\widehat{f}})^2.$$

The global control Theorem 1.18 thus asserts that $c(x) = c_0$ is equivalent to

(94) $$d(\widetilde{\omega}_t^{\widehat{f}})^2(x, T_1) = 0, \quad \text{for all } \widehat{f} \in \overline{\mathcal{F}},$$

a condition that is verifiable from the knowledge of $\mathcal{Z}$. Hence, the focusing sequences used for point evaluation can be chosen such that $c(x) = c_0 \neq 0$.

To proceed with the reconstruction of α, consider the inner product,

(95)
$$\int_M (\widetilde{\omega}_t^{\widehat{f}})^2(x, T_1) \wedge *(\widetilde{\omega}_t^{\widehat{h}})^2(x, T_1) = c_0^2 \int_M (\omega_t^{\widehat{f}})^2(x, T_1) \wedge *(\omega_t^{\widehat{h}})^2(x, T_1),$$

which can be found via $\mathcal{Z}$. On the other hand, by Theorem 1.10, $\mathcal{Z}$ determines the energy inner product,

$$\frac{1}{2} \int_M \frac{1}{\alpha(x)} (\omega_t^{\widehat{f}})^2(x, T_1) \wedge *(\omega_t^{\widehat{h}})^2(x, T_1).$$

By choosing a boundary source $\widehat{h} = \widehat{h}_j$ such that $\widetilde{h} = (\widehat{h}_j)_{j=1}^{\infty}$ is a focusing sequence and by comparing the above inner products at the limit $j \to \infty$, we recover the value $c_0^2 \alpha(x)$ at any point $x \in M$.

Finally, we notice e.g. by considering the energy integrals that the admittance map has the scaling property $\mathcal{Z}_{(M,g,c_0^2\alpha)} = c_0^{-2}\, \mathcal{Z}_{(M,g,\alpha)}$, with evident notations. Therefore, given $\mathcal{Z}$ and $(g, c_0^2\alpha)$ already reconstructed, it is also possible to determine c_0 and hence α. This completes the proof of Theorem 2.1. □

2.4. **Data given on a part of the boundary.** In this section, we generalize the proof of Theorem 2.1 for the case when data is given on a non-empty open subset $\Gamma \subset \partial M$. In this case, instead of the complete admittance map $\mathcal{Z}^T$ we are given the local admittance map $\mathcal{Z}_\Gamma^T$, defined by

$$\mathcal{Z}_\Gamma^T f = \mathcal{Z}^T f|_{\Gamma \times]0,T[}, \quad f \in \overset{\bullet}{C}{}^\infty([0, T], \Omega^1\Gamma),$$

where $\Omega^1\Gamma$ is the space of the 1-forms $f \in \Omega^1 \partial M$ supported on Γ. Denote $\mathcal{Z} = \mathcal{Z}^T$ with $T = \infty$ and recall that $\mathrm{rad}_\Gamma(M)$ is the geodesic radius of M with respect to Γ, see (54).

Theorem 2.15. *Given Γ, the local admittance map $\mathcal{Z}_\Gamma^T$, $T > 2\,\mathrm{rad}_\Gamma(M)$, uniquely determines the manifold M, the metric g, and the scalar wave impedance α.*

Proof. Here we use notations of section 1.6. By Theorem 1.18 we have that the set $\mathcal{F}_\Gamma = C_0^\infty(]0, T_0[, \Omega^1\Gamma)/\!\sim$ with $T > T_0 > 2\,\mathrm{rad}_\Gamma(M)$ is a dense subset of $\overline{\mathcal{F}}$. Thus we can identify $\overline{\mathcal{F}}_\Gamma$ with $\overline{\mathcal{F}}$. This makes it possible to use, when the data is given on a part of the boundary, all the results about generalized sources obtained in section 1.6 for the whole boundary. In particular, we can define sets $\mathcal{F}_\Gamma^s \subset \overline{\mathcal{F}}_\Gamma$ that can be identified with $\mathcal{F}^s$.

Exactly as in section 1.4 we can show that the local admittance map $\mathcal{Z}_\Gamma^T$, $T > 2\,\mathrm{rad}_\Gamma(M)$, determine the map $\mathcal{Z}_\Gamma^t$ for all $t > 0$, i.e., the map $\mathcal{Z}_\Gamma$.

Our first aim is to reconstruct (M, g) near Γ. For this, let $\theta_{\Gamma} : \Gamma \to \mathbb{R}$ be

$$\theta_{\Gamma}(z) = \sup\{s > 0 : \tau(\gamma_{z,\nu}(s), \Gamma) = s\}$$

and

$$M_\Gamma = \{\gamma_{z,\nu}(s) \in M \ : \ z \in \Gamma,\ 0 \le s < \theta_{\Gamma}(z)\},$$

where $\gamma_{z,\nu}(s)$ is the geodesic starting from $z \in \partial M$ in the normal direction.

Lemma 2.16. *Given Γ, the local admittance map $\mathcal{Z}_\Gamma$ determines the function $\theta_{\Gamma} : \Gamma \to \mathbb{R}$, the Riemannian manifold (M_Γ, g), and the wave impedance α on M_Γ.*

Proof: Let $z \in \Gamma$. Using notations of section 2.1 we see that $s \le \theta_{\Gamma}(z)$ if and only if $S = M(\Gamma_z, s) \setminus M(\Gamma, s - \varepsilon)$ has a non-empty interior for all open $\Gamma_z \subset \Gamma$ containing z and $\varepsilon > 0$. In turn, $S = (M(\Gamma_z, s) \setminus M(\Gamma_z, 0)) \cap (M(\Gamma, s) \setminus M(\Gamma, s-\varepsilon))$ is of form (64). Thus, using Theorem 2.4 with $\mathcal{Z}_\Gamma$ instead of $\mathcal{Z}$, we can find out whether S has a non-empty interior or not. Thus we can find $\theta_{\Gamma}(z)$.

At this stage, we can proceed in a similar manner to the proof of Theorem 2.5 to find functions $R_\Gamma(M_\Gamma) = \{r_x|_\Gamma \ : \ x \in M_\Gamma\}$. To do this, we just use the procedure presented in the proof of Theorem 2.5 but consider only those functions $h \in C(\Gamma)$ that have a unique global minimum, say $z_0 \in \Gamma$, with $h(z_0) < \theta_{\Gamma}(z_0)$. After construction of this set, we see as in [27, Sect. 4.4] that the set $R_\Gamma(M_\Gamma)$ determines the Riemannian manifold (M_Γ, g).

Reconstruction of α in M_Γ follows the same route as with data given on the whole boundary by restricting our attention to the focusing sequences corresponding to points $y \in M_\Gamma$ and using an identification of $\mathcal{F}_\Gamma^s$ with $\mathcal{F}^s$. □

In the next step we will show that we can find the admittance map on the boundary of an arbitrary ball $B \subset M_\Gamma$. We will denote by $\mathcal{Z}_{\partial B}$ the local admittance map defined by using the manifold $M \setminus B$ instead of M and ∂B instead of Γ. For similar arguments in the scalar case, see [28].

Proposition 2.17. *Given (M_Γ, g, α) and the map $\mathcal{Z}_\Gamma$ for (M, g, α), we can find the local admittance map $\mathcal{Z}_{\partial B}$ for $(M \setminus B, g, \alpha)$.*

Proof. First we observe that (M_Γ, g, α) and $\mathcal{Z}_\Gamma$ determine the values of the electric Green's function $G_{\mathrm{e}}^2(x, y, t; \lambda)$ for any $x, y \in M_\Gamma$, $t > 0$, and $\lambda \in \Lambda^2 T_y^* M$. This result is proven in section 2.3 in the case when the admittance map is given on the whole boundary and the proof can be directly extended to the considered case.

Consider now the initial boundary value problem

$$\partial_t \eta + \mathcal{M}\eta = \kappa, \quad \text{in } M \times \mathbb{R}_+, \quad \mathbf{t}\eta\big|_{\partial M\times\mathbb{R}_+} = 0, \quad \eta(0) = 0, \tag{96}$$

where $\kappa = (0, \delta\beta, 0, 0)$ with $\beta \in \overset{\bullet}{C}{}^\infty(\mathbb{R}_+, \Omega^2 B)$ and denote its solution by $\eta = \eta_\beta = (0, \eta^1, \eta^2, 0)$. Writing η^2 in terms of the electric Green's function we obtain

$$\eta^2(x,t) = \int_{\mathbb{R}_+}\int_B G^2_{\rm e}(x,y,t-t'; \frac{\beta(y,t')}{\alpha(y)})\, dV_g(y) dt', \tag{97}$$

where dV_g is the Riemannian volume measure on (M, g). Using equation (96) we also find

$$\eta^1(x,t) = \int_0^t (\delta\eta^2(x,t') + \delta\beta(x,t'))dt'. \tag{98}$$

We continue the proof with the following lemma;

Lemma 2.18. *Let $\omega = (0, \omega^1, \omega^2, 0) \in \overset{\bullet}{C}{}^\infty(\mathbb{R}_+, \Omega M)$ be a solution of Maxwell's equations (23) and (24) in $(M\setminus B)\times\mathbb{R}_+$ which satisfies the electric boundary condition $\mathbf{t}\omega = 0$ on $\partial M\times\mathbb{R}_+$ and initial condition $\omega(0) = 0$. Then there is $\beta \in \overset{\bullet}{C}{}^\infty(\mathbb{R}_+, \Omega^2 B)$ such that the solution η_β of initial boundary value problem (96) coincides with ω_{tt} in $(M\setminus B)\times\mathbb{R}_+$.*

Proof. Let $\widetilde{\omega} = (0, \widetilde{\omega}^1, \widetilde{\omega}^2, 0) \in \overset{\bullet}{C}{}^\infty(\mathbb{R}_+, \Omega M)$ be an arbitrary smooth continuation of ω into $B\times\mathbb{R}_+$. Let

$$\rho = (0, \rho^1, \rho^2, 0), \quad \rho^1 = -\delta d\widetilde{\omega}^1, \quad \rho^2 = -d\delta\widetilde{\omega}^2.$$

Then $\rho = \omega_{tt}$ in $(M\setminus B)\times\mathbb{R}_+$, and ρ satisfies Maxwell's equations

$$\rho^1_t - \delta\rho^2 = \delta d a^1, \quad a^1 = -\widetilde{\omega}^1_t + \delta\widetilde{\omega}^2,$$
$$\rho^2_t + d\rho^1 = d\delta a^2, \quad a^2 = -\widetilde{\omega}^2_t - d\widetilde{\omega}^1,$$

in $M\times\mathbb{R}_+$. Then $\eta = (0, \rho^1 - \delta a^2, \rho^2, 0)$ satisfies the initial boundary value problem (96) with $\beta = da^1 - a^2_t$ supported in $B\times\mathbb{R}_+$. In particular, $\omega_{tt} = \eta_\beta$ in $(M\setminus B)\times\mathbb{R}_+$. □

To complete the proof of Proposition 2.17 we start with an arbitrary $\beta \in \overset{\bullet}{C}{}^\infty(\mathbb{R}_+, \Omega^2 B)$ and find, using formulae (97), (98), the wave $\eta_\beta(x,t)$ for $x \in M_\Gamma$. Let $\omega(x,t)$ be now defined as

$$\omega^0(x,t) = 0, \quad \omega^1(x,t) = \int_0^t\int_0^{t'} \eta^1_\beta(x,t'')\, dt'' dt', \tag{99}$$

$$\omega^2(x,t) = \int_0^t\int_0^{t'} \eta^2_\beta(x,t'')\, dt'' dt', \quad \omega^3(x,t) = 0.$$

Then $\omega(t)$ is the solution of the initial-boundary value problem,

$$\omega_t + \mathcal{M}\omega = 0 \quad \text{in } (M\setminus B)\times\mathbb{R}_+, \quad \omega(0) = 0,$$

$$\mathbf{t}\omega|_{\partial M\times\mathbb{R}_+} = 0, \quad \mathbf{t}\omega|_{\partial B\times\mathbb{R}_+} = (0, f_\beta, -\int_0^t df_\beta),$$

where

$$f_\beta = \int_0^t \int_0^{t'} \mathbf{t}\eta^1_\beta(x,t'')\,dt''dt' \in \overset{\bullet}{C}^\infty(\mathbb{R}_+, \Omega^1\partial B). \tag{100}$$

Using again formulae (97), (98), we see that (M_Γ, g, α) together with $\mathcal{Z}_\Gamma$ determine the map

$$f^1_\beta \longrightarrow \mathbf{n}\omega^2|_{\partial B\times\mathbb{R}_+} = \int_0^t \int_0^{t'} \mathbf{n}\eta^2_\beta(x,t'')\,dt''dt' \in \overset{\bullet}{C}^\infty(\mathbb{R}_+, \Omega^1\partial B) \tag{101}$$

for any f^1_β of form (100). As according to Lemma 2.18. the map $\beta \to f^1_\beta$ is a surjective map from $\overset{\bullet}{C}^\infty(\mathbb{R}_+, \Omega^2 B)$ onto $\overset{\bullet}{C}^\infty(\mathbb{R}_+, \Omega^1\partial B)$, the map (101) determines $\mathcal{Z}_{\partial B}$. This proves Proposition 2.17. □

Having found $\mathcal{Z}_{\partial B}$ we construct the Riemannian manifold $M_{\partial B} \subset M \setminus B$, metric g on $M_{\partial B}$ and impedance α on $M_{\partial B}$. Here $M_{\partial B}$ is defined in a similar way as M_Γ changing M to $M \setminus B$ and Γ to ∂B. Combining this with the previous results, we find the part $M_\Gamma \cup M_{\partial B}$ of M as well as the metric g and the wave impedance α on it. Iterating this procedure, we reconstruct, in a finite number of steps, the whole manifold (M, g, α). For details, see [27, Sect 4.4.9]. This proves Theorem 2.15. □

2.5. **Inverse problem for Maxwell's equations in $\mathbb{R}^3$.** In this section, the uniqueness results for Maxwell's equations on a manifold are used to characterize the non-uniqueness of inverse problems for Maxwell's equations (1)–(2) in a bounded domain of $M \subset \mathbb{R}^3$ with the Euclidean metric $(g_0)_{ij} = \delta_{ij}$.

Let $M_j \subset \mathbb{R}^3$, $j = 1, 2$, be two bounded smooth closed domains with a common part Γ of their boundaries, $\Gamma \subset \partial M_1 \cap \partial M_2$. Let ϵ_j and μ_j, $j = 1, 2$ be the permittivity and permeability matrices in M_j, respectively, with $\mu_j = \alpha_j^2 \epsilon_j$, $\alpha_j > 0$ being the corresponding scalar wave impedances. Assume that the local admittance maps $\mathcal{Z}_{\Gamma,j}$ for (M_j, ϵ_j, μ_j) coincide. By Theorem 2.15, both (M_1, ϵ_1, μ_1) and (M_2, ϵ_2, μ_2) correspond to the same abstract manifold $(\widetilde{M}, \widetilde{g}, \widetilde{\alpha})$ which is uniquely determined by $\mathcal{Z}_{\Gamma,j}$ with the part Γ corresponding to a part $\widetilde{\Gamma} \subset \partial\widetilde{M}$. This implies that there are embeddings $F_j : \widetilde{M} \to M_j \subset \mathbb{R}^3$ of the manifold $\widetilde{M}$ in the Euclidean space such that the metric tensors and the scalar wave impedances satisfy $\widetilde{g} = (F_j)^* g_j$ and $\widetilde{\alpha} = (F_j)^*\alpha_j$ and $F_1|_{\widetilde{\Gamma}} = F_2|_{\widetilde{\Gamma}}$. Recall that g_j are determined by expression (16) with ε_j and μ_j in place of ε, μ. The embeddings F_j induce a diffeomorphism

$$\Phi = F_2 \circ F_1^{-1} : M_1 \to M_2, \quad \Phi\big|_\Gamma = \mathrm{id}. \tag{102}$$

Consider two vector fields X_1 and Y_1 in M_1, and denote $X_2 = D\Phi X_1$, $Y_2 = D\Phi Y_1$. The electric energy inner product for the corresponding

1–forms ω^1, $\eta^1 \in \Omega^1 M$ is invariant, i.e., we have

$$\int_{M_1} g_0(X_1, \epsilon_1 Y_1) dV_0 = \int_{\widetilde M} \frac{1}{\widetilde{\alpha}} \omega^1 \wedge *\eta^1 = \int_{M_2} g_0(X_2, \epsilon_2 Y_2) dV_0.$$

On the other hand, as $X_2 = D\Phi\, X_1$ and $Y_2 = D\Phi Y_1$,

$$\int_{M_2} g_0(X_2, \epsilon_2 Y_2) dV_0 = \int_{M_1} g_0(X_1, \Phi^* \epsilon_2 Y_1) dV_0,$$

where

$$\Phi^* \epsilon_2 = \frac{1}{\det D\Phi} (D\Phi)^{\mathrm{T}} \left(\epsilon_2 \circ \Phi\right) D\Phi. \tag{103}$$

Since X_1 and Y_1 are arbitrary, we must have $\epsilon_1 = \Phi^* \epsilon_2$. Similar reasoning shows that $\mu_1 = \Phi^* \mu_2$.

Thus we have proven the following result.

Theorem 2.19. *Let $M_1, M_2 \subset \mathbb{R}^3$ be bounded smooth domains and $\Gamma \subset \partial M_1 \cap \partial M_2$ be open and non-empty. Let $\mathcal{Z}_{\Gamma,1}$ and $\mathcal{Z}_{\Gamma,2}$ be the local admittance maps corresponding to (M_1, ϵ_1, μ_1) and (M_2, ϵ_2, μ_2), respectively. Then $\mathcal{Z}_{\Gamma,1} = \mathcal{Z}_{\Gamma,2}$ if and only if there is a diffeomorphism $\Phi : M_1 \to M_2$, $\Phi\big|_\Gamma = \mathrm{id}$ and $\epsilon_1 = \Phi^* \epsilon_2$, $\mu_1 = \Phi^* \mu_2$.*

Remark 2.20. It follows from (103) that ϵ and μ do not transform like tensors of type (1,1). This is due to the special role played by the underlying Euclidean metric $g_0^{ij} = \delta^{ij}$, which is not changed by the diffeomorphisms Φ. These transformations were observed also in the study of the Calderón inverse conductivity problem. It is shown in [66] that, for $\Omega \subset \mathbb{R}^2$, boundary measurements determine the anisotropic conductivity up to the same group of transformations as described in Theorem 2.19. For $n \geq 3$, a similar result is conjectured, based on the analysis of the linearized inverse problem, see [67]. The Calderón problem is closely related to the inverse problem for Maxwell's equations, as the low-frequency limit of $\mathcal{Z}$ is related to the Dirichlet-to-Neumann map for the conductivity equation [46].

When ϵ and μ are isotropic, we obtain the following uniqueness result.

Theorem 2.21. *Let $M \subset \mathbb{R}^3$ be a bounded smooth domain, $\Gamma \subset \partial M$ be open and non-empty, ϵ and μ be smooth positive functions on $\overline{M}$ and $\mathcal{Z}_\Gamma$ be a local admittance map for (M, ε, μ). Then Γ and $\mathcal{Z}_\Gamma$ determine (M, ϵ, μ) uniquely.*

Note that the knowledge of M is not a priori assumed in the above theorem.

Proof. Assume that for (M_1, ϵ_1, μ_1) and (M_2, ϵ_2, μ_2) such that $\Gamma \subset \partial M_1 \cap \partial M_2$ we have $\mathcal{Z}_{\Gamma,1} = \mathcal{Z}_{\Gamma,2}$. Then there is a diffeomorphism $\Phi : M_1 \to M_2$ satisfying $\Phi\big|_\Gamma = \mathrm{id}$ and $\epsilon_1 = \Phi^* \epsilon_2$, $\mu_1 = \Phi^* \mu_2$. Since ϵ_1 and ϵ_2 are isotropic, it follows from the Liouville theorem that Φ is conformal. Since $\Phi\big|_\Gamma = \mathrm{id}$, it follows that Φ is identity. □

2.6. **Outlook.** There are several direction to which the present work can be extended.

1. A natural inverse problem is the inverse boundary spectral problem for the electric Maxwell operator $\mathcal{M}_{\rm e}$. The problem is to determine the metric g and wave impedance α, or, in other words, ε and μ from the non-zero eigenvalues λ_j of $\mathcal{M}_{\rm e}$ and the normal components of the corresponding eigenforms on ∂M. This problem was studied in, e.g. [45], for the scalar Maxwell's equations. For the considered anisotropic case, this requires significant modifications of the method developed in this paper and will be published elsewhere.

2. With the uniqueness of the inverse problem in hand, the next issue is to study stability of the inverse problem and develop stable reconstruction algorithms. A general approach to these questions, in the scalar case, is introduced in [29], in terms of certain geometrical a priori bounds on (M, g), with sharp results on conditional stability in [1]. Adding *a priori* analytical bounds on α, we intend to analyse these questions for Maxwell's equations in an anisotropic medium.

Appendix: The WKB approximation. Denote by $\Phi(x, y, t) = \Phi_\lambda(x, y, t)$ the Green's 2–form, i.e., the solution of

$$\begin{aligned}(104)\quad &(\partial_t^2 + \Delta_\alpha^2)\Phi_\lambda(x) = 0 \quad \text{in } M \times \mathbb{R}_+,\\ &\Phi_\lambda(x)|_{t=0} = 0, \quad \partial_t \Phi_\lambda(x)|_{t=0} = \lambda \underline{\delta}_y(x), \quad \Phi_\lambda(x)|_{\partial M \times \mathbb{R}_+} = 0,\end{aligned}$$

where $\lambda \in \Lambda^2 T_y^* M$. Let $B_y(\rho)$, $\rho < \tau(y, \partial M)$ be a domain of normal coordinates centered at y, so that

$$g^{ij}(0) = \delta^{ij},\ \partial_k g^{ij}(0) = 0. \tag{105}$$

Rewriting equations (104), componentwise, in these coordinates and using the unit propagation speed, we can, instead of (104), consider the fundamental solution, $\Phi(x, y, t)$, $t < \rho$,

$$\begin{aligned}(106)\quad &\left\{(\partial_t^2 - g^{ij}\partial_i\partial_j)I + B^i\partial_i + C\right\}\Phi = 0, \quad \text{in } M \times]0, \rho[,\\ &\Phi|_{t=0} = 0, \quad \partial_t \Phi|_{t=0} = I\underline{\delta}(x),\end{aligned}$$

where I is the 3×3 identity matrix and $B^i(x)$, $C(x)$ are smooth 3×3 matrices.

Following [15, 3], which deal with the scalar case, we search for the solution to (106) in the WKB form:

$$\Phi(x, t) \approx G_0(x)\, \underline{\delta}(t^2 - \tau^2) + \sum_{\ell \geq 1} G_\ell(x)\, (t^2 - \tau^2)_+^{\ell - 1}/(\ell - 1)!, \tag{107}$$

where $\tau(x, y) = |x|$. Substitution of (107) into equation (106) gives rise to a recurrent system of transport equations. The principal one is the equation for G_0,

$$4\tau \frac{dG_0}{d\tau}(\tau\widehat{x}) + \left\{(g^{ij}(\tau\widehat{x})\, \partial_i\partial_j\tau^2 - 6)\, I + B^i(\tau\widehat{x})\, \partial_i\tau^2\right\} G_0(\tau\widehat{x}) = 0,$$

where $\widehat{x} = x/\tau$. To satisfy the initial conditions in (106), we require that $G_0(0) = (2\pi)^{-1}I$. By (105), $g^{ij}\partial_i\partial_j\tau^2\big|_{x=0} - 6 = 0$. Also, $\partial_i\tau^2\big|_{x=0} = 0$. Therefore,

$$\frac{1}{4\tau}\left\{\left(g^{ij}(\tau\widehat{x})\,\partial_i\partial_j\tau^2 - 6\right) I + B^i(\tau\widehat{x})\,\partial_i\tau^2\right\}$$

is a smooth function of $(\tau, \widehat{x})$, so that $G_0(x)$ is a smooth 3×3 matrix-function of $(\tau, \widehat{x})$, for $\tau > 0$. Moreover, it can be shown that $G_0(x)$ is also smooth at $x = 0$.

For G_ℓ, $\ell \geq 1$, we obtain transport equations

$$\begin{aligned} &4\tau\frac{dG_\ell}{d\tau} + \left\{\left(4\ell - 6 + g^{ij}(x)\partial_i\partial_j\tau^2(x)\right) I + B^i(x)\partial_i\tau^2\right\} G_\ell \\ &= \left[g^{ij}\partial_i\partial_j I - B^i\partial_i - C\right] G_{\ell-1}, \end{aligned}$$

with $G_\ell(0) = 0$. If we write $G_\ell = G_0 F_\ell$, we obtain for F_ℓ the equations

$$4\tau\frac{dF_\ell}{d\tau} + 4\ell\, F_\ell = G_0^{-1}\,\left[g^{ij}\partial_i\partial_j I - B^i\partial_i - C\right] G_{\ell-1},$$

with $F_\ell(0) = 0$. Solving the above equations, we find

$$F_\ell(x) = \frac{1}{4}\tau^{-\ell}\int_0^\tau G_0^{-1}(s\widehat{x})\,\left\{\left[g^{ij}\partial_i\partial_j I - B^i\partial_i - C\right] G_{\ell-1}\right\}(s\widehat{x})\, s^{\ell-1}ds,$$

which are smooth functions of x. As (106) is a hyperbolic system, the right-hand side of (107) is the asymptotics, with respect to smoothness, of $\Phi(x, y, t)$, when $t < \rho$.

Clearly, the asymptotic expansion (107) implies decomposition (87), (88).

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Yaroslav Kurylev, Loughborough University, Department of Mathematical Sciences, Leicestershire LE11 3TU, UK

Matti Lassas and Erkki Somersalo, Helsinki University of Technology, Institute of Mathematics, PO Box 1100, FIN–02015 HUT, Finland