Gaussian beams and inverse boundary spectral problems

Alexander Katchalov * Matti Lassas †

December 21, 2000

1 Introduction.

In these lectures we consider inverse boundary spectral problems for elliptic operators on manifolds. This means the reconstruction of an unknown manifold and an elliptic operator on it from the knowledge of the boundary spectral data, i.e. the spectrum of the operator and normal derivatives of the normalized eigenfunctions on the boundary. Before we formulate and solve this problem in exact terms, we explain why the manifolds appear in the study of the inverse problems.

Let us consider an elliptic second order differential operator $a(x, D)$ in $\Omega \subset \mathbf{R}^n$ and $\varphi : \Omega \to \Omega$ be a diffeomorphism, that is, a change of coordinates, satisfying $\varphi(x) = x$ near $\partial\Omega$. Then the operator $a(x, D)$ in new coordinates $y = \varphi(x)$ is the operator $a(\varphi^{-1}(y), (D\varphi)D)$. Since the boundary spectral data of the operators $a(x, D)$ and $a(\varphi^{-1}(y), (D\varphi)D)$ coincide, we see that the boundary spectral data can not uniquely determine the operator $a(x, D)$. However, both operators can be considered as the same operator on a manifold represented in different coordinates. This example shows that it is natural to start from an operator on a manifold and ask if the boundary spectral data determine uniquely the manifold and the operator on it. A classical analog of reconstruction of the manifold structure is the reconstruction of material parameters in travel time coordinates.

*Steklov Mathematical Institute, RAN, Fontanka 27, 191011, St. Petersburg, Russia.
The author was partly supported by Russian grant RFFI 99-01-00107.
†Department of Mathematics, P.O. Box 4, 00014 University of Helsinki, Finland.

2 Formulation of the problem.

Let M be a C^∞-smooth compact m–dimensional manifold with boundary ∂M. It means that M is a topological space which is covered finite number of local coordinates (U_l, X_l), $l = 1, \dots, L$, $M = \cup U_l$. The coordinate functions $X_l = (x_l^1, \dots x_l^m)$ are homeomorphic maps from U_l onto open subsets of R^m (or R_+^m for boundary coordinate sets), X_l, $X_l : U_l \to U_l' \subset R^m(R_+^m)$, such that $X_l \circ X_k^{-1}$ are C^∞ functions. Using local coordinates we define the space $L^2(M)$, the Sobolev space $H^n(M)$ and the Sobolev space $H_0^n(M)$ with vanishing traces in the standard way.

Now we can define an elliptic selfadjoint second order differential operator $\mathcal{A}$ on a smooth differentiable manifold M. The differential expression $a(x, D)$ of the operator is given in any local coordinates as

$$(a(x,D)f)(x^1,\dots,x^m) = -a^{jk}(x^1,\dots x^m)\partial_j\partial_k f(x^1,\dots,x^m) +$$
$$+b^j(x^1,\dots,x^m)\partial_j f(x^1,\dots,x^m) + c(x^1,\dots,x^m)f(x^1,\dots,x^m),$$

where $\partial_j f = \frac{\partial f}{\partial x^j}$. Here and later we use the Einstein summation rule. The local representations, which are called the local differential expressions, are defined in such a manner that the value of $a(x, D)f$ at any point x independent of the choice of local coordinates (U, X) near x. In particular, it means that coefficients $a^{jk}(x^1, \dots, x^m)$ are transformed as components of 2–contravariant tensor.

The differential expression $a(x, D)$ is elliptic if the matrices $[a^{jk}](x^1, \dots, x^m)$ in a local coordinates (and consequently in any local coordinates) are positive definite,

$$a^{jk}p_jp_k \geq c\sum_{j=1}^m p_j^2.$$

The fact that $[a_{jk}] = [a^{jk}]^{-1}$ is positive definite matrix and a 2–covariant tensor implies that we can consider it as a Riemannian metric tensor $g_{ik} = a_{ik}$ on M. We call corresponding metric as the metric associated with operator $\mathcal{A}$. To define an operator on M we have to add a boundary condition on ∂M. We define the operator A by

$$\mathcal{A}u(x) = a(x,D)u(x), \quad u \in \mathcal{D}(\mathcal{A}) = H^2(M) \cap H_0^1(M).$$

To define self–adjoint operators on M we need to fix a volume element dV on the manifold. In local coordinates (U, X) near x we have representation

$$dV = m dV_g = m(x)g^{1/2}(x)dx^1dx^2\cdots dx^m,$$

where $g = \det[g_{ik}]$ and m is a positive function on M.

In the space $L^2(M, dV)$ we use the inner product

$$\langle u, v\rangle = \int_M u(x)\overline{v(x)}dV$$

and say that $\mathcal{A}$ is selfadjoint, if $\mathcal{A} = \mathcal{A}^*$ where the adjoint $\mathcal{A}^*$ is defined with respect to the inner product of $L^2(M, dV)$.

As example of a second order elliptic selfadjoint differential operator we consider the Schrödinger operator $\mathcal{A}_q$ on the Riemannian manifold (M, g),

$$\mathcal{A}_q = -\Delta_g + q, \tag{1}$$

where Δ_g is the Beltrami–Laplace operator,

$$\Delta_g u = g^{-1/2}\partial_j g^{1/2} g^{ij}\partial_i u,$$

and q is a smooth real valued function on M. The Schrödinger operator $\mathcal{A}_q$ is a second order elliptic differential operator on M which is selfadjoint in $L^2(M, dV_g)$, $dV_g = g^{1/2}(x)dx^1 \cdots dx^m$.

Let us return to the general elliptic selfadjoint second order differential operators on the manifold M. An easy computation show that the selfadjoint differential operators have a special form (see [14]).

Lemma 1 *Let $L^2(M, dV)$ be a Hilbert space with volume element dV and $\mathcal{A}$ be an operator with domain $H^2(M) \cap H^1_0(M)$. The operator $\mathcal{A}$ is selfadjoint in $L^2(M, dV)$ if and only if the corresponding differential expression has the form*

$$a(x, D)f = -m^{-1}g^{-1/2}\partial_i m g^{1/2} g^{ij}\partial_j f + qf.$$

In this case, $dV = m dV_g$.

The spectral properties of the selfadjoint operator $\mathcal{A}$ on M are given in the following well known theorem.

Theorem 1 *Let $\mathcal{A}$ be a selfadjoint second order differential operator with the Dirichlet boundary condition. Then the eigenvalues of the operator form an increasing sequence $\lambda_1, \lambda_2 \ldots$, $\lambda_j \leq \lambda_{j+1}$, $\lambda_j \to \infty$, when $j \to \infty$, where the eigenvalues are counted according to their multiplicities. For each eigenvalue λ_j there is an eigenfunction φ_j so that the collection of these eigenfunctions $\{\varphi_j\}_{j=1}^\infty$ form an orthonormal basis in $L^2(M, dV)$.*

Any function $f \in L^2(M)$ has a representation

$$f = \sum_{j=1}^{\infty} f_j \varphi_j, \quad \{f_j\} \in \ell^2,$$

which Fourier coefficients of f are $f_j = \langle f, \varphi_j \rangle$ and $\|f\|_{L^2} = \|\{f_j\}\|_{l^2}$. By standard application of Garding inequality, we see that $\varphi_j \in \mathcal{D}(\mathcal{A}^s) \subset H^{2s}(M)$ for any s. Henceforth $\varphi_j \in C^\infty(M)$.

3 Gauge transformations.

In our investigation we consider all operators and the boundary data in such a way that our considerations does not depend on the particular choice of scale of measurements. For instance, if the change of the scale of measurements is described by function $\kappa(x)$, $\kappa|_{\partial M} = 1$, that is, at point $x \in M$ the physical quantity $u(x)$ is replaced with $\kappa(x)u(x)$, this change of scale of measurements should not change the observations done on the boundary. For this reason we formulate all our considerations in such a way that they are invariant in the gauge transformations $u(x) \to \kappa(x)u(x)$. As we will see later, these transformations play an important role in the study of the multidimensional inverse problems.

Definition 1 *Let $\kappa \in C^\infty(M)$, $\kappa(x) > 0$ for $x \in M$. The gauge transformation generated by the function κ is the transformation*

$$S_\kappa : L^2(M, dV) \to L^2(M, dV_\kappa)$$

where $dV_\kappa = \kappa^{-2}(x)dV$. It is defined by the formula

$$S_k u(x) = \kappa(x)u(x).$$

Each gauge transformation determines the corresponding gauge transformation $\mathcal{A}_\kappa$ of the operator $\mathcal{A}$,

$$\mathcal{A}_\kappa u = \kappa \mathcal{A}(\kappa^{-1} u).$$

If $\mathcal{A}$ is an elliptic second order operator on M then operator $\mathcal{A}_\kappa$ is also an elliptic second order differential operator defined in $L^2(M, dV_\kappa)$ with the domain

$$\mathcal{D}(\mathcal{A}_\kappa) = H^2(M) \cap H_0^1(M).$$

Its differential expression is given by the formula

$$a_\kappa(x,D)u(x) = \kappa^{-1}(x)a(x,D)(\kappa(x)^{-1}u(x)).$$

It is easy to verify the following properties of the gauge transformations.

Lemma 2 *i. Transformation S_κ is a unitary transformation, i.e.*

$$\|S_\kappa u\|_{L^2(dV_\kappa)} = \|u\|_{L^2(dV)}.$$

ii. Operator $\mathcal{A}_\kappa$ in $L^2(M,dV)$ is selfadjoint if and only if $\mathcal{A}$ is selfadjoint in $L^2(M,dV_\kappa)$.

Gauge transformations $S_\kappa : L^2(M) \to L^2(M)$ form an Abelian group

$$\mathcal{G} = \{S_\kappa \ : \ \kappa \in C^\infty(M), \ \kappa > 0\}$$

with respect of composition

$$S_{\kappa_1} \circ S_{\kappa_2} = S_{\kappa_1\kappa_2}.$$

This group is action on the set of the second order elliptic differential operators

$$S_\kappa(\mathcal{A}) = \mathcal{A}_\kappa = \kappa\mathcal{A}\kappa^{-1}.$$

For any $\mathcal{A}$

$$\sigma\mathcal{A} = \{S_\kappa(\mathcal{A}) : \quad S_\kappa \in \mathcal{G}\}$$

is the orbit of the group $\mathcal{G}$ generated by $\mathcal{A}$. If $\mathcal{A}$ is selfadjoint operator than all operators $\mathcal{A}_\kappa \in \sigma\mathcal{A}$ are selfadjoint.

Next we investigate what objects are invariant in the action of group $\mathcal{G}$.

Firstly, the metric tensor $g^{ij} = a^{ij}$ associated to the operator $\mathcal{A}$ is invariant with respect to the gauge transformations,

$$g^{ij}_\kappa = g^{ij}.$$

Secondly, since S_κ is unitary, the eigenvalues are invariant,

$$\lambda_j(\mathcal{A}) = \lambda_j(\mathcal{A}_\kappa). \tag{2}$$

On the other hand for the eigenvectors we have that

$$\varphi_j^\kappa = S_\kappa \varphi_j = \kappa \varphi_j \tag{3}$$

where φ_j and φ_j^κ are eigenfunctions of operators $\mathcal{A}$ and $\mathcal{A}_\kappa$, correspondingly.

An important fact related to the gauge transformations is that any orbit $\sigma\mathcal{A}$ of a selfadjoint operator $\mathcal{A}$ contains a unique Schrödinger operator. We call this operator the Schrödinger operator corresponding to $\mathcal{A}$.

Lemma 3 *i. Let $\mathcal{A}$ be a selfadjoint second order elliptic differential operator in $L^2(M, dV)$. Then in the orbit $\sigma\mathcal{A}$ there is a unique Schrödinger operator $-\Delta_g + q$, that is, for a given $\mathcal{A}$ there is a unique κ such that $\mathcal{A} = \kappa(-\Delta_g + q)\kappa^{-1}$ and $dV = \kappa^{-2} dV_g$.*

ii. $\mathcal{A}$ is a Schrödinger operator if and only if $dV = dV_g$

Proof. It is clear from the definition of the Schrödinger operator that a general elliptic selfadjoint second order operator $\mathcal{A}$ is a Schrödinger operator if and only if $m = 1$. Using formulae describing the change of $\mathcal{A}$ in the gauge transformation we see that the gauge transformation S_κ, $\kappa = m^{1/2}$ transforms $\mathcal{A}$ to the Schrödinger operator. $\diamondsuit$

4 Boundary spectral data and main results.

Next, let ν be the unit inward normal vector field on ∂M.

Definition 2 *Let $\{\lambda_n\}_{n=1}^\infty$ and $\{\varphi_n\}_{n=1}^\infty$ be the eigenvalues (counted according their multiplicities) and the orthonormal eigenfunctions of an elliptic second order differential operator selfadjoint in $L^2(M, dV)$. Then the boundary spectral data of the operator $\mathcal{A}$ is the set*

$$\partial M, \quad \{\lambda_n\}_{n=1}^\infty, \quad \{\partial_\nu \varphi_n|_{\partial M}\}_{n=1}^\infty,$$

where ∂_ν is the normal derivative defined via duality $\partial_\nu \varphi_n|_{\partial M} = (d\varphi_n, \nu)$.

Formulae (2) and (3) imply that the boundary spectral data of the $\mathcal{A}_\kappa$ are

$$\partial M, \quad \{\lambda_n\}_{n=1}^\infty, \quad \{\partial_\nu(\kappa\varphi_n)|_{\partial M}\}_{n=1}^\infty = \{\kappa_0 \partial_\nu \varphi_n|_{\partial M}\}_{n=1}^\infty,$$

where $\kappa_0 = \kappa|_{\partial M}$. We say that these data are gauge equivalent to the the boundary spectral data of the operator $\mathcal{A}$. Our aim is to prove the following theorem.

Theorem 2 *Let $\mathcal{A}$ be a general second order differential operator on the manifold M, selfadjoint in $L^2(M, dV)$. Then its gauge equivalent boundary spectral data ∂M, $\{\lambda_n\}_{n=1}^\infty$, $\{\kappa_0\, \partial_\nu \varphi_n|_{\partial M}\}_{n=1}^\infty$ determine Riemannian manifold (M, g) and the orbit $\sigma\mathcal{A}$ of the operator $\mathcal{A}$.*

It is obvious that operators $\mathcal{A}$ and $\mathcal{A}_\kappa$ with the function κ satisfying $\kappa|_{\partial M} = 1$ have the same boundary spectral data. Therefore there is no possibility to reconstruct the operator $\mathcal{A}$ uniquely. Thus the best that we can hope to construct by using the gauge equivalent boundary spectral data is the orbit $\sigma\mathcal{A}$ of the operator $\mathcal{A}$. To construct the orbit it is sufficient to construct one particular operator in the orbit, for instance the Schrödinger operator corresponding to $\mathcal{A}$. To reconstruct the Schrödinger operator we have to reconstruct first of all the manifold M and the Riemannian metric on it. Before this, we explain what we mean by the reconstruction of a Riemannian manifold (M, g). Since a manifold is by definition a collection of coordinate patches our objective is to construct a representative in a class of equivalence of Riemannian manifolds isometric to (M, g). In other words, we consider all isometric Riemannian manifolds as the same manifold. After reconstruction of the Riemannian manifold we reconstruct the potential q on the manifold.

It can be shown that by knowing gauge equivalent boundary spectral data we can reconstruct metric $g|_{\partial M}$ and $\kappa_0 = \kappa|_{\partial M}$, where S_κ transforms the operator $\mathcal{A}$ to the Schrödinger operator. For simplicity we do not present these constructions and restrict ourself to the case of reconstruction of the Schrödinger operator when we know the boundary spectral data and the boundary as a Riemannian manifold $(\partial M, g_{\partial M})$. For these omitted constructions, we refer to [14]. We will prove the following result:

Theorem 3 *Let $\mathcal{A} = \mathcal{A}_q$ be a Schrödinger operator on the manifold M. Its boundary spectral data $(\partial M, g_{\partial M})$, $\{\lambda_n\}_{n=1}^\infty$, $\{\partial_\nu \varphi_n|_{\partial M}\}_{n=1}^\infty$ determine the Riemannian manifold (M, g) and potential q uniquely.*

Before we start to prove the theorem we discuss shortly the history of the problem. The proof of the theorem is based a transformation of the problem to an inverse problem for the wave equation and applying control theoretical results for computing projections of waves. This approach is called usually the boundary control method. It has its origin in the study of one dimensional inverse problems considered by A. Blagovestchenskii (see e.g. [12])

who applied the fact that the inner products of waves can be computed from the boundary data. The boundary control method was first time developed to multidimensional inverse problems in domains of $\Omega \subset \mathbf{R}^n$ by M. Belishev (see [3]). He made the crucial observation that the control theoretical results can be used to compute the projections of waves to subdomains of Ω. For further development of these results, see e.g. [6], [9], [10]. The study of more general operators led to the observation of the gauge-invariant nature of the problem. This was done by Y. Kurylev in the study of the Schrödinger operator (see e.g. [15], [17], [18]). The further invariant nature of the inverse problem was discovered by M. Belishev and Y. Kurylev who introduced the problem on Riemannian manifold at 1992 (see [11], [16], [20]). At 1995 D. Tataru extended the Holmgren-John unique continuation result for non-analytic equations [26]. This breakthrough made the applications of boundary control results possible for equations with non-analytic coefficients. We mention also that the analogous inverse problems are studied for more general cases than which are considered here, e.g. for non-selfadjoint systems, [1], [21], [22], for Maxwell system [5], [7] and operator pencils [23] or when the data is given only on a part of the boundary [22] or with incomplete data [13].

In these notes we consider the inverse problem by using the Gaussian beam solutions. The Gaussian beams (or quasiphotons) are introduced by V. Babich, V. Ulin, and J. Ralston (see [2], [24]) and they were applied for the inverse problems first time in [8], [13]. An other main tool in these notes are the boundary distance functions. These functions were introduced at 1995 by Y. Kurylev who observed that the set of the boundary distance functions can in fact be identified with the original manifold [20]. The importance of this observation was related to its geometrical nature; Instead of doing constructions in boundary normal coordinates the manifold is constructed as a global object and thus all difficulties related to non-regular coordinates near cut locus were removed. In these notes we do our analysis in the same manner by using geometrically invariant point of view when it is possible. Some technical details of the constructions are omitted, and for these details we refer for [14].

5 Spectral representation of waves.

We start with the reconstruction procedure the Riemannian manifold (M, g). For this consider an initial boundary value problem for the wave equation associated to the Schrödinger operator. Let

$$\begin{aligned} \partial_t^2 u - \Delta_g u + qu &= 0, \quad (x,t) \in M \times [0,T], \\ u|_{\partial M \times [0,T]} = f, &\qquad u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0, \end{aligned}$$

where $f \in L^2(\partial M \times [0,T])$. This initial boundary value problem has the unique solution $u = u^f$, $u^f \in C([0,T], L^2(M)) \cap C^1([0,T], H^{-1}(M))$. For any $t \in [0,T]$ the solution $u^f(\cdot, t) \in L^2(M)$ and, consequently, it can be decomposed in the Fourier series

$$u^f(\cdot, t) = \sum_{i=1}^{\infty} u_i^f(t) \varphi_i(\cdot)$$

Lemma 4 *The Fourier coefficients $u_i^f(t)$, $i = 1, 2, \ldots$, of the solution $u^f(x,t)$ have the following representation*

$$u_i^f(t) = \int_0^t \int_{\partial M} s_i(t - t') f(s, t') \partial_\nu \varphi_i(s) dS_g dt', \tag{4}$$

where

$$s_i(t) = - \begin{cases} \frac{\sin\sqrt{\lambda_i} t}{\sqrt{\lambda_i}} & \lambda_i > 0 \\ \frac{\sinh\sqrt{-\lambda_i} t}{\sqrt{-\lambda_i}} & \lambda_i < 0 \\ t & \lambda_i = 0. \end{cases}$$

Proof. For a smooth function f

$$\begin{aligned} \partial_t^2 u_i^f(t) &= \int_M \partial_t^2 u^f(x,t) \varphi_i(x) dV_g = \int_M (\Delta_g - q) u^f(x,t) \varphi_i(x) dV_g \\ &= -\lambda_i u_i^f(t) - \int_{\partial M} f(s,t) \partial_\nu \varphi_i(s) dS_g. \end{aligned}$$

This ordinary differential equation for $u_i^f(t)$ can be solved with the initial conditions

$$u_i^f(0) = \partial_t u_i^f(0) = 0,$$

which are followed from initial conditions for u^f. Thus we obtain the formula for the Fourier coefficients $u_i^f(t)$. For $f \in L^2(\partial M \times [0,T])$ claim follows by approximation of f by smooth functions in $L^2(\partial M \times [0,T])$. $\diamondsuit$

We see from formula (4) that if the boundary function f, called **the boundary control**, and the boundary spectral data are given, we can find the Fourier coefficients of the solution $u^f(x,t)$. In particular, for any t and t' we can find the inner product of two different solutions generated by controls f and h.

Corollary 1 *Let $f, h \in H_0^1(\partial M \times [0,T])$. Then*

$$\langle u^f(\cdot,t), u^h(\cdot,t')\rangle = \sum_{i=1}^{\infty} u_i^f(t)\overline{u_i^h(t')} \tag{5}$$

where the Fourier coefficients $u_i^f(t)$ and $u_i^h(t)$ can be found by means of knowing f, h, and the inverse boundary spectral data by using formula (4).

6 Gaussian beams.

Here we describe a special class of solutions of the initial boundary value problem for the wave equation on the Riemannian manifold related to the Schrödinger operator,

$$\partial_t^2 u - \Delta_g u + qu = 0.$$

We construct solutions which are known as Gaussian beam solutions or quasiphotons. The name of quasiphoton reflects the fact that the solutions of this type are concentrated at any time t in a neighborhood of a point $x = x(t)$. The path $x = x(t)$ is, in fact, a geodesic in the corresponding Riemannian metric. The quasiphoton moves along this geodesic with unit speed. Moreover, many properties of such solutions (the energy conservation law, reflection from the boundary etc.) are analogous to the properties of particles. The name of Gaussian beam reflects the fact that at any time t the absolute value of the Gaussian beam coincide with a Gaussian function.

The construction of the Gaussian beams is divided into several steps. Firstly, we construct a special **asymptotic** solution of the wave equation which is called a formal Gaussian beam. Secondly, we chose a special boundary data, so that the solution of the corresponding initial boundary value problem has an asymptotic expansion which coincides with a a formal Gaussian beam. These solutions are called Gaussian beams.

Definition 3 *A formal Gaussian beam of order N is a function $U_\epsilon^N(x,t)$ of form*

$$U_\epsilon^N(x,t) = (\pi\epsilon)^{-m/4}\exp\{-(i\epsilon)^{-1}\theta(x,t)\}\sum_{n=0}^{N} u_n(x,t)(i\epsilon)^n$$

satisfying the following properties: The phase function $\theta(x,t)$ and the amplitude functions $u_n(x,t)$, $n=0,1,\ldots,N$, are smooth complex valued functions of the variables x and t. The phase function $\theta(x,t)$ satisfies conditions

$$Im\,\theta(x(t),t) = 0, \tag{6}$$

$$Im\,\theta(x,t) \geq C(t)d(x,x(t))^2, \tag{7}$$

where $d(\cdot\,,\cdot\,)$ is the distance function in the metric g, $x(t)\in M$ depends continuously on t, and $C(t)$ is a continuous positive function. Finally, inequality

$$|(\partial_t^2 - \Delta_g + q)U_\epsilon^N| \leq C\epsilon^{N-m/4}$$

is satisfied uniformly for (x,t) belonging to any compact set.

Our goal is to find the phase function θ and amplitude functions u^n, $n=0,\ldots,N$, such that $U_\epsilon^N(t,x)$ is a formal Gaussian beam. We note that it will turn out that the curve $x(t)$ has to be a geodesic. Because the Gaussian beam exponentially decreasing respect of ϵ outside an $O(\epsilon^{1/2-\sigma})$ neighborhood of the point $x=x(t)$, $0<\sigma<1/6$, it is enough to construct the Gaussian beam in any small neighborhood of the point $x=x(t)$. Because of this, we start the construction on a local coordinate chart (U,X) around $x(0)$. In U we identify x with its local coordinates in $\mathbf{R}^n$. Moreover, in $U\times\mathbf{R}$ we use variables (y,t) where

$$y = x - x(t).$$

Substitution of the asymptotic expansion of the Gaussian beam into the wave equation yields

$$\left(\partial_t^2 - \Delta_g + q\right)U_\epsilon^N(x,t) = (\pi\epsilon)^{-m/4}(i\epsilon)^{-2}\exp\{-(i\epsilon)^{-1}\theta(x,t)\}\sum_{n=0}^{N+2} v_n(x,t)(i\epsilon)^n$$

where v_n has the form

$$v_n(x,t) = [(\partial_t\theta)^2 - g^{jl}(x)\partial_j\theta\partial_l\theta]u_n(x,t) - \mathcal{L}_\theta u_{n-1}(x,t) + (\partial_t^2 - \Delta_g + q)u_{n-2}(x,t),$$

$n = 0,\dots,N+2$, with $u_{-2}(x,t) \equiv u_{-1}(x,t) \equiv 0$. Here $\mathcal{L}_\theta$ is a transport operator

$$\mathcal{L}_\theta u = 2\partial_t\theta\partial_t u - 2g^{jl}\partial_j\theta\partial_l u + ((\partial_t^2 - \Delta_g)\theta)u.$$

7 The Hamilton–Jacobi equation and transport equations.

Next we study the equations for u_n. We start with the following simple lemma.

Lemma 5 *Let $\theta(x,t)$ satisfy condition (6) and*

$$|v_n(x,t)| \le Cd(x,x(t))^{2(N+2-n)}, \quad n = 0,\dots,N+2.$$

Then there is a constant C', such that $|(\partial_t^2 - \Delta_g + q)U_\epsilon^N| \le C'\epsilon^{N-m/4}$ uniformly on any compact interval of t.

Proof. Clearly

$$|v_n(x,t)\exp\{-i\epsilon^{-1}\theta(x,t)\}| \le C|y|^{2(N+2-n)}e^{-C(t)|y|^2/\epsilon} \le C'\epsilon^{N+2-n}.$$

$\diamondsuit$

It follows from Lemma 5 that it is sufficient to construct U_ϵ^N, such that corresponding v_n, $n = 0,\dots,N+1$, vanish to order $2(N+2-n)$ at $y=0$. Later we consider v_n as function of t and y. At this stage we do not prescribe the value of N. For this reason, we construct θ and u_n so that θ satisfies (6) and (7) and Taylor coefficients of v_n vanishes to any arbitrary order at $y=0$. We use notation

$$v \asymp 0,$$

when $\partial_y^\alpha v(y,t)|_{y=0} = 0$ for any multi index $\alpha = (\alpha_1,\dots,\alpha_m)$ at any time t.

We start our construction of a formal Gaussian beam by analyzing the term v_0. To satisfy this equation it is sufficient to find a phase function $\theta(x,t)$, such that

$$(\partial_t\theta)^2 - g^{jl}(x)\partial_j\theta\partial_l\theta \asymp 0. \tag{8}$$

The equation is called the Hamilton–Jacobi equation for the phase $\theta(x,t)$. Given θ which satisfies the equation the other equations $v_n \asymp 0$, $n = 1,\ldots,N+1$ take the form of the transport equations

$$\mathcal{L}_\theta u_n \asymp (\partial_t^2 - \Delta_g + q)u_{n-1},$$

$n = 0,\ldots,N$. The equations are solved successively, with known right hand side in each step. In particular, the transport equation with $n=0$ has the form

$$\mathcal{L}_\theta u_0 = 0.$$

Since we are interested in the Taylor expansion of v_n it is natural to represent the phase function θ and amplitude functions u_n as their Taylor expansions with respect to y

$$\begin{aligned} \theta(x,t) &= \sum_{l\geq 0}\theta_l(t) = \sum_{|\gamma|\geq 0}\frac{\theta_\gamma(t)}{\gamma!}y^\gamma, \\ u_n(x,t) &= \sum_{\sigma\geq 0}u_{n,l}(t) = \sum_{|\gamma|\geq 0}\frac{u_{n,\gamma}(t)}{\gamma!}y^\gamma. \end{aligned}$$

Here we use the notation $\theta_l(t)$ and $u_{n,l}(t)$ for the homogeneous terms of order l with respect to y.

Next we consider the geometrical implications of Hamilton system. To simplify notations, we denote the coefficients of θ_1 by p_j, so that

$$\theta_1(t) = p_j(t)y^j.$$

The coefficients of θ_2 are denoted by H_{jl}, so that

$$\theta_2(t) = \frac{1}{2}H_{jl}(t)y^jy^l,$$

where $H(t)$ is a complex valued symmetric matrix, $H_{jl} = H_{lj}$. For the real transpose of H we use notation H^t, and for its adjoint the notation H^*.

Before we continue the construction, we consider the behaviour of $\theta_0(t)$, $p(t)$ and $H(t)$ in the change of variables in the vicinity of $x = x(t)$. Obviously $\theta_0(t)$ has the same value in this change of variables, which means that $\theta_0(t)$ is a scalar function along the path $x = x(t)$. The components $(p_1(t), \dots, p_m(t))$ form a covariant vector field $p(t)$ along the path $x = x(t)$. On the other hand, the complex valued matrix function $H(t)$ does not represent a tensor field along the path. Indeed, a transformation of coordinates from x to $\tilde{x}$ gives a transformation of $H(t)$ to $\widetilde{H}(t)$, where

$$H_{ij}(t) = \widetilde{H}_{lk}(t)\frac{\partial \tilde{x}^l(x(t))}{\partial x^i}\frac{\partial \tilde{x}^k(x(t))}{\partial x^j} + \frac{\partial^2 \tilde{x}^l(x(t))}{\partial x^i \partial x^j}\tilde{p}_l(t).$$

However, we see from the formula that $Im\, H(t)$ is a 2–covariant tensor along the path $x = x(t)$. Although $H(t)$ is not a tensor, nevertheless, there is a symmetric tensor $G(t)$ which is closely related to $H(t)$,

$$G_{ij}(t) = H_{ij}(t) - \Gamma^k_{ij}(x(t))p_k(t),$$

where Γ^k_{jl} are the Christoffel symbols of the metric tensor g. Clearly, $Im\, H = Im\, G$.

One of the most important properties of a Gaussian beam is that the corresponding path $x = x(t)$ form a geodesic. Next we show this.

Lemma 6 *Let $U^N_\epsilon(x,t)$ be a formal Gaussian beam of order N corresponding to wave equation. Then*

i. $\theta_0(t) = \theta_0$ i.e. $\theta(x(t),t)$ is constant.

ii. $(x(t), p(t))$ (or $(x(-t), p(-t))$) is a bicharacteristic corresponding to the Hamiltonian $h(x,p) = (g^{ij}(x)p_ip_j)^{1/2}$.

We note ii. means that $(x(t), p(t))$ satisfy

$$\frac{dx^i}{dt} = \frac{\partial h}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial h}{\partial x^i} \tag{9}$$

with some initial conditions $x|_{t=0} = x_0$, $p|_{t=0} = p_0$.

Proof. We proof of the lemma with an additional assumption that for any t the instant frequency $\omega(t)$,

$$\omega(t) = -\,\partial_t\theta(x,t)|_{x=x(t)} = p_j\frac{dx^j}{dt} - \frac{d\theta_0}{dt} \neq 0.$$

It is easy to prove that if the assumption is not satisfied then corresponding Gaussian beam does not exist. On the other hand if the assumption is valid for $t = t_0$ than it is valid for any t.

We analyze the terms of (8) of homogeneity 0 and 1 with respect to y. These terms give rise to the following equations, which are considered together

$$\omega^2(t) - g^{jl} p_j p_l = 0, \tag{10}$$

$$\omega(t) \left[2\frac{dp_l}{dt} - (H_{lr} + H_{rl}) \frac{dx^r}{dt} \right] + \frac{\partial g^{jr}}{\partial x^l} p_j p_r + g^{jr} p_j (H_{lr} + H_{rl}) = 0. \tag{11}$$

Here the metric tensor g^{jl} and its derivatives $\frac{\partial g^{jr}}{\partial x^l}$ are evaluated in the point $x = x(t)$. Consider the imaginary part of equation (11). In view of conditions (6) and (7) for the phase function, we have

$$Im\, \theta_0(t) = 0, \quad Im\, p_j(t) = 0, \quad j = 1, \ldots, m.$$

Hence, we obtain the following equation

$$Im\, (H_{lr} + H_{rl}) \eta^r = 0, \quad l = 1, \ldots, m, \tag{12}$$

where $\eta = (\eta^1, \ldots, \eta^m)$ has the form

$$\eta^r = -\omega(t) \frac{dx^r}{dt} + g^{jr} p_j.$$

By (7), $Im\, H(t)$ must be positive defined. Hence equation (12) yields $\eta = 0$. Since η form a vector and p is a covector we can consider the scalar (η, p),

$$(\eta, p) = \eta^r p_r = -\omega \frac{dx^r}{dt} p_r + g^{jr} p_j p_r = 0.$$

Equation (10) implies that

$$g^{jr} p_j p_r = -\omega \left(\frac{d\theta_0}{dt} - p_r \frac{dx^r}{dt} \right) = -\omega \frac{d\theta_0}{dt} + \omega p_r \frac{dx^r}{dt} = -\omega \frac{d\theta_0}{dt} + g^{jr} p_j p_r.$$

Comparing the beginning and the end of the equation we see that

$$\omega \frac{d\theta_0}{dt} = 0$$

and, consequently, $d\theta_0/dt = 0$. Without lost of generality we can take $\theta_0(t) \equiv 0$. Thus statement i) is proven.

To analyze equation (10) we distinguish between the cases $\omega > 0$ and $\omega < 0$. If $\omega(t) > 0$, then

$$\omega(t) = (g^{jl}p_j p_l)^{1/2} = h(x(t), p(t)).$$

Then equations $\eta = 0$ imply that

$$\frac{dx^r}{dt} = [g^{il}p_i p_l]^{-1/2} g^{jr} p_j = \frac{\partial h}{\partial p_r}, \quad r = 1, \ldots, m,$$

which coincide with the first m equations of Hamilton system of equations (9). After this the last m equations of the Hamilton system are obtained from real part of equations (11). $\diamondsuit$

Next we consider the geometrical nature of $x(t)$ and $p(t)$. Since the Hamiltonian is a positive homogeneous function of order 1 with respect to p, we can assume that the initial data for p_0 satisfies $|p_0| = h(x_0, p_0) = 1$. The Hamilton system (9) yields that the Hamiltonian is constant along any bicharacteristic,

$$h(x(t), p(t)) = h(x_0, p_0) = 1,$$

and, consequently, $|p(t)| = |p_0| = 1$. In this case $p(t)$ is the covariant representation of the velocity vector $\frac{dx}{dt}$ of the path $x(t)$,

$$I_g p(t) = \frac{dx}{dt}, \quad or \quad \frac{dx^j}{dt} = g^{jl}(x(t))p_j(t).$$

Next, let $\gamma_{x,v}$ be a geodesic for which $\gamma_{x,v}(0) = x$ and $\dot{\gamma}_{x,v}(0) = v$. The following property of bicharacteristics is well known.

Lemma 7 *Let $(x(t), p(t))$ be a bicharacteristic of the Hamiltonian $h(x,p) = [g^{jl}(x)p_j p_l]^{1/2}$ with initial data (x_0, p_0), $|p_0| = 1$. Then the path $x(t)$ is a geodesic γ_{x_0,v_0}, where $v_0 = I_g p_0$, and t is the arclength along γ_{x_0,v_0}.*

We call the geodesic γ_{x_0,v_0} as the geodesic corresponding to the formal Gaussian beam $U_\varepsilon^N(x,t)$. Remark that the path $x(t)$ remains to be a geodesic with t being arclength even if $|p_0| = h_0 \neq 1$. Indeed, Hamiltonian h is a positive homogeneous function of the order 1 with respect to p

$$h(x, \lambda p) = \lambda h(x,p), \quad \lambda > 0.$$

Henceforth, $x(t)$ and $p(t)$ are homogeneous functions with respect to h_0 of the order 0 and 1, correspondingly.

In the following we always consider initial data (x_0, p_0), $\omega(0) = |p_0| = 1$.

8 Riccati equation.

To obtain an equation for $H(t)$ we analyze the term of homogeneity 2 with respect to y in $(\partial_t\theta)^2 - g^{jl}\partial_j\theta\partial_l\theta \asymp 0$. This gives us a matrix Riccati equation for $H(t)$ along the bicharacteristic $(x(t), p(t))$,

$$\frac{d}{dt}H + D + (BH + HB^t) + HCH = 0. \tag{13}$$

Matrices B, C, and D are $m \times m$ matrices with components given by the second derivatives of the Hamiltonian $h(x, p)$,

$$B = [B_l^j]_{j,l=1}^m, \quad C = [C^{jl}]_{j,l=1}^m, \quad D = [D_{jl}]_{j,l=1}^m$$

with

$$B_l^j = \frac{\partial^2 h}{\partial x^l \partial p_j}; \quad C^{jl} = \frac{\partial^2 h}{\partial p_j \partial p_l}; \quad D_{jl} = \frac{\partial^2 h}{\partial x^j \partial x^l},$$

where the derivatives are evaluated in the point $(x, p) = (x(t), p(t))$, i. e. on the bicharacteristic of Hamiltonian $h(x, p)$.

We supplement the Riccati equation with initial condition

$$H|_{t=0} = H_0, \tag{14}$$

where

$$H_0 = H_0^t, \quad Im\, H_0 > 0. \tag{15}$$

We remind that the last inequality is necessary to satisfy condition (7).

Lemma 8 . *i. The initial value problem for Riccati equation (13) with initial values (14-15) is uniquely solvable.*

ii. Its solution $H(t)$, $t \in R$, satisfies the following conditions

$$H(t) = H^t(t), \quad Im\, H(t) > 0.$$

iii. For any Y_0, Z_0, such that

$$H_0 = Z_0 Y_0^{-1},$$

the matrix $H(t)$ *is represented in the form*

$$H(t) = Z(t)Y(t)^{-1}.$$

The matrices $Z(t)$, $Y(t)$ *satisfy the linear initial value problem,*

$$\frac{d}{dt}Y(t) = B^t \cdot Y + C \cdot Z, \quad Y|_{t=0} = Y_0,$$

$$\frac{d}{dt}Z(t) = -D \cdot Y - B \cdot Z, \quad Z|_{t=0} = Z_0, \tag{16}$$

and $Y(t)$ *is non-degenerate for all* $t \in R$,

$$det Y(t) \neq 0.$$

The proof of the lemma is based upon the following conservation laws:

Lemma 9 *Let* $Z(t)$, $Y(t)$ *solve initial value problem (16). Then*

$$Z^t(t)Y(t) - Y^t(t)Z(t) = Z_0^t Y_0 - Y_0^t Z_0 = const, \tag{17}$$

$$Z^*(t)Y(t) - Y^*(t)Z(t) = Z_0^* Y_0 - Y_0^* Z_0 = const, \tag{18}$$

Proof. To proof the lemma it is enough to show that the derivatives on t of the left hand sides of the equations are equal to zero. It can be proven by direct differentiation of the left hand sides and using equations (16) and symmetry properties of matrices B, C, and D,

$$B^t = B^*, \quad C^t = C^* = C, \quad D^t = D^* = D.$$

$\diamondsuit$

Next we return to the proof of Lemma 8. We chose

$$Z_0 = H_0, \quad Y_0 = I,$$

where I is the identity matrix, so that $H_0 = Z_0 Y_0^{-1}$. Since system (16) is linear, it has a unique solution $Y(t)$, $Z(t)$.

Firstly, we show that $Y(t)$ is non-degenerate for all $t \in \mathbf{R}$. Assume on the contrary that there is $t_0 \in \mathbf{R}$ and a complex valued vector $\eta \in \mathbf{C}^m$, $\eta \neq 0$, such that

$$Y(t_0)\eta = 0.$$

Clearly, when $(\cdot\,,\cdot)$ is the inner product in $\mathbf{C}^n$,

$$(Y(t_0)\eta, Z(t_0)\eta) - (Z(t_0)\eta, Y(t_0)\eta) = 0. \tag{19}$$

Hence

$$0 = ((Z^*(t_0)Y(t_0) - Y^*(t_0)Z(t_0))\eta, \eta) = ((Z_0^*Y_0 - Y_0^*Z_0)\eta, \eta) =$$

$$= ((H_0^* - H_0)\eta, \eta) = -2i(Im\,H_0\eta, \eta).$$

Because $Im\,H_0 > 0$ this equation implies that $\eta = 0$. The contradiction shows that $Y(t)$ is non-degenerate.

Symmetry of H, $(H = H^t)$ follows from the first conservation law for Z and Y, and positive definiteness of H $(Im\,H > 0)$ follows from the second conservation law. ♢

Later we need the following result.

Lemma 10 *For any Gaussian beam*

$$det(Im\,H(t)) \cdot |det Y(t)|^2 = c_0 \tag{20}$$

where c_0 is independent of t.

Proof. Due to differential equation for matrix function $Y(t)$

$$\frac{d}{dt}(\ln(\det Y(t))) = tr\left(\frac{dY(t)}{dt}Y^{-1}(t)\right) = tr(B^t(t) + C(t)H(t)), \tag{21}$$

we see that

$$|\det Y(t)| = |\det Y(0)| \exp\{\int_0^t tr(B^t(\tau) + C(\tau)Re\,H(\tau))d\tau\}.$$

On the other hand,

$$\frac{d}{dt}(\ln(\det(Im\,H(t)))) = tr\left(\frac{dIm\,H(t)}{dt}(Im\,H(t))^{-1}\right) =$$

$$= -tr\{(B(t) + Re\,H(t)C(t)) - Im\,H(t)(B^t(t) + C(t)Re\,H(t))(Im\,H(t))^{-1}\} =$$

$$-tr(B(t) + Re\,H(t)C(t)) - tr(B^t(t) + C(t)Re\,H(t)) = -2tr(B(t) + C(t)Re\,H(t)),$$

where we use that $C^t = C$, $(Re\,H)^t = Re\,H$, and $trA = trA^t$, $tr(AB) = tr(BA)$. Hence,

$$\det(Im\,H(t)) = \det(Im\,H(0))\exp\{-2\int_0^t tr(B(\tau) + C(\tau)Re\,H(\tau))d\tau\}. \quad (22)$$

Combining formulae (21) and (22) we obtain formula (20). $\diamondsuit$

Next, we give a new invariant geometrical interpretation of the Riccati equation (13). Instead of the matrix function $H(t)$, which is not a tensor, let us consider the tensor field $G(t)$. We see that this tensor field is also symmetric and its imaginary part is positive definite,

$$G^t(t) = G(t), \quad Im\,G > 0.$$

To formulate Lemma 8 in invariant terms, we consider the $(1,1)$-tensor field

$$\widetilde{G} = I_g G, \quad \widetilde{G}^i_j = g^{ik}G_{kj}$$

Here we identify $(1,1)$-tensors with linear operators in the tangent space, so that $\widetilde{G}(t)$ is an operator in $T_{x(t)}N$. Next we introduce the operators $\widetilde{C}(t)$ and $\widetilde{R}_\gamma(t)$ in $T_{x(t)}N$. Let

$$\widetilde{C}(t) = I - P_\gamma(t),$$

where P_γ is a one–dimensional projector,

$$P_\gamma(t)w = \left(w, \frac{dx}{dt}\right)_g \frac{dx}{dt}, \quad w \in T_{x(t)}N,$$

where $\frac{dx}{dt}$ is the unit velocity vector of the geodesic γ.

The operator $\widetilde{R}_\gamma$ is obtained from the curvature operator R.

$$\widetilde{R}_\gamma(t)w = R\left(w, \frac{dx}{dt}\right)\frac{dx}{dt}, \quad w \in T_{x(t)}N.$$

Riccati equation (13) yields to the Riccati equation for $\widetilde{G}$.

Lemma 11 *The* $(1,1)$ *tensor* $\widetilde{G}(t)$ *satisfy the covariant Riccati equation*

$$\frac{D\widetilde{G}}{dt} + \widetilde{G}\widetilde{C}\widetilde{G} + \widetilde{R}_\gamma = 0.$$

This Riccati equation is complex analog of the Riccati equation for the shape operator in the distance coordinates [25].

After construction the first two Taylor coefficients of θ, we construct the higher order terms by solving linear equations. The equations for the homogeneous polynomials θ_l, $l \geq 3$, are obtained by considering the higher order homogeneous polynomials in the equation

$$(\partial\theta)^2 - g^{jl}\partial_j\theta\partial_l\theta = (\partial_t\theta) - h^2(x, \partial\theta) \asymp 0.$$

The resulting differential equations for the homogeneous polynomials θ_l, $l \geq 3$, are linear differential equations,

$$\frac{\partial\theta_l}{\partial t} + N^i_j \frac{\partial\theta_l}{\partial y^i} y^j = \mathcal{F}_l, \quad l = 3, 4, \ldots,$$

where the right–hand sides $\mathcal{F}_l$ depend upon θ_j, $j < l$. The matrix $N^i_j = N^i_j(t)$ is an $m \times m$ matrix of the form

$$N^i_j(t) = \frac{\partial^2 h}{\partial x^j \partial p_i} + \frac{\partial^2 h}{\partial p_i \partial p_k} H_{kj} = B^i_j(t) + C^{is}(t)H_{sj}(t) = [B^t + CH]^i_j.$$

Equations for $\theta_l(y,t) = \sum_{|\gamma|=l} \theta_\gamma(y) y^\gamma$ are ordinary differential equations with respect to t. By considering y as a parameter in these equations, we can define $\theta_l(y,t)$ also for $y \in \mathbf{C}^m$.

To simplify the analysis we introduce new coordinates $(\tilde{t}, \tilde{y})$ of the form

$$\tilde{t} = t, \quad \tilde{y} = Y^{-1}(t)y.$$

Let $\tilde{\theta}_l(\tilde{y}, \tilde{t})$ be the representation of the polynomial $\theta_l(y,t)$. Then equations for $\tilde{\theta}_l$ take the form

$$\frac{\partial}{\partial \tilde{t}} \tilde{\theta}_l = \widetilde{\mathcal{F}}_l, \quad l = 3, 4, \ldots. \tag{23}$$

These equations with initial data

$$\tilde{\theta}_l(\tilde{y}, \tilde{t})\Big|_{\tilde{t}=0} = \tilde{\theta}^0_l(\tilde{y}) = \theta^0_l(y) \tag{24}$$

determine $\tilde{\theta}_l(\tilde{y}, \tilde{t})$ for any $\tilde{t}$. Thus we find $\theta_l(y,t)$ for any t. Next we show that θ can be solved uniquely by giving an appropriate initial data at time $t = 0$.

Lemma 12 *Let $\Theta(x)$ be smooth function near x_0 having the Taylor expansion*

$$\Theta(x) \asymp \sum_{l\geq 1} \Theta_l = \sum_{|\gamma|\geq 1} \frac{1}{\gamma!}\Theta_\gamma y^\gamma, \quad y = x - x_0.$$

Let in addition $\Theta_1 = (p_0, y)$, $|p_0| = 1$, be real and $\Theta_2 = \frac{1}{2}(H_0 y, y)$, with $Im\, H_0 > 0$. Then for any integer $K > 1$ there exists a function $\theta(x,t) = \theta_K(x,t)$, satisfying conditions (6), (7), such that

$$|(\partial_t\theta)^2 - g^{ij}(x)\partial_i\theta\partial_j\theta| \leq C_K d(x, x(t))^K,$$

$$\theta(x, 0) \asymp \Theta(x),$$

where $x(t)$ is the geodesic $\gamma_{x_0,v_0}(t)$, $v_0 = I_g p_0$.

Proof. Using (x_0, p_0) as initial conditions in Lemma 6 we obtain the bicharacteristic $(x(t), p(t))$ and $\theta_1(t)$,

$$\theta_1(t) = (p(t), y).$$

Having constructing $(x(t), p(t))$ and using H_0 as initial data in Lemma 8 we construct $\theta_2(t)$,

$$\theta_2(t) = \frac{1}{2}(H(t)y, y).$$

Finally with arbitrary Θ_l, $3 \leq l \leq K-1$, as initial data we find $\theta_l(t)$ as solutions of initial problem (23), (24).

The desired function $\theta_K(x,t)$ is given by formula

$$\theta_K(x,t) = \sum_{l=1}^{K-1} \theta_l(t) = \sum_{|\gamma|=1}^{K-1} \frac{1}{\gamma!}\theta_\gamma(t)(x - x(t))^\gamma.$$

$\diamondsuit$

After construction θ, we find u_n by using the transport equation. The analysis of transport equations

$$\mathcal{L}_\theta u_n - (\partial_t^2 - \Delta_g + q)u_{n-1} \asymp v_n \asymp 0 \tag{25}$$

is also based upon the Taylor expansion for the amplitudes u_n near $x = x(t)$

$$u_n(x,t) \asymp \sum_{l\geq 0} u_{n,l}(y,t) \asymp \sum_{l\geq 0} \tilde{u}_{n,l}(\tilde{y},t),$$

where $u_{n,l}$ and $\tilde{u}_{n,l}$ are homogeneous polynomials of order l, $l = 0, 1, \ldots$, with respect to y and $\tilde{y}$, $\tilde{y} = Y^{-1}(t)y$. Here the operator $\mathcal{L}_\theta$,

$$\mathcal{L}_\theta u = 2\partial_t\theta\partial_t u - 2g^{jl}\partial_j\theta\partial_l u + (\partial_t^2 - \Delta_g)\theta \cdot u,$$

is a first order linear differential operator.

We consider $\tilde{u}_{n,l}$ as a function of t with values in the space of homogeneous polynomials of the order l with respect to $\tilde{y}$. By (25), we obtain differential equations for $\tilde{u}_{n,l}$,

$$\frac{d}{dt}\tilde{u}_{n,l}(t) + r(t)\tilde{u}_{n,l}(t) = \widetilde{\mathcal{F}}_{n,l}(t), \quad n, l = 0, 1, \ldots.$$

The right–hand sides $\widetilde{\mathcal{F}}_{n,l}(t)$ are homogeneous polynomials of order l depending on $\tilde{u}_{r,k}(t)$ and $\tilde{\theta}_k$ with $k \leq l+2$, $r < n$. The factor $r(t)$ in this equation has the form

$$r(t) = \frac{1}{2}tr(B^t + CH) + \frac{1}{4}\frac{d}{dt}\ln g(t).$$

Using factorization $H(t) = Z(t)Y^{-1}(t)$ and differential equation for $Y(t)$ we can see that

$$\frac{1}{2}tr(B^t + CH) = \frac{1}{2}tr[(B^tY + CZ)Y^{-1}] = \frac{1}{2}tr\left(\frac{dY}{dt}Y^{-1}\right).$$

Using well known formula

$$tr\left(\frac{dY}{dt}Y^{-1}\right) = \frac{d}{dt}\ln[\det Y(t)]$$

we obtain ordinary differential equations for $\tilde{u}_{n,l}$

$$\frac{d}{dt}\tilde{u}_{n,l}(t) + \left(\frac{1}{4}\frac{d}{dt}\ln[(det(Y(t))^2 g(t)]\right)\tilde{u}_{n,l}(t) = \widetilde{\mathcal{F}}_{n,l}(t).$$

Solutions to the equations are given by the formula

$$\tilde{u}_{n,l}(t) = \varrho(t)\{\tilde{u}_{n,l}(0) + \int_0^t \varrho^{-1}(t')\widetilde{\mathcal{F}}_{n,l}(t')\,dt'\}$$

with

$$\varrho(t) = \sqrt{\frac{\det Y(0)}{\det Y(t)}}\sqrt[4]{\frac{g(0)}{g(t)}}.$$

Simple investigations show, for example, that

$$\begin{aligned} u_{0,0}(t) &= u_{0,0}(0)\varrho(t), \\ \tilde{u}_{1,0}(t) &= -\frac{1}{2}\tilde{u}_{0,0}(t)\int_0^t q(x(t'))\,dt' + \tilde{u}^1_{1,0}(t). \end{aligned}$$

We remark that this formula is in fact integration along the geodesic γ_{x_0,v_0}, $v_0 = I_g p_0$. The function $\tilde{u}^1_{1,0}(t)$ depends upon $\tilde{u}_{0,0}$, $\tilde{u}_{0,1}$, $\tilde{u}_{0,2}$, p, and Y, but not q.

To this far we have constructed a formal Gaussian beam on one coordinate chart with a given initial data at time $t = 0$. By using terms of the asyptotical expansion at time $t = t_0$ as a new initial data and repeating previous considerations, we can construct a formal Gaussian beam on local coordinate patches which cover the geodesic $\gamma_{x_0,v_0}([0,T])$, $T < l(x_0,v_0)$, where $l(x_0,v_0) \in (0,\infty]$ is the first time when the geodesic γ_{x_0,v_0} hits the boundary. We summarize this in the following theorem.

Theorem 4 *Consider the wave equation on the compact manifold M. Let $\Theta(x)$, $U_n(x)$, $n = 0,\ldots,N$, be smooth complex valued functions given in an open neighborhood V of x_0, $x_0 \in M$. Assume that Θ, and U_0 satisfy the following conditions i) $\Theta(x_0) = 0$,*

ii) $Im\,\Theta(x) \geq cd^2(x,x_0)$, $c > 0$,

iii) $\partial_x\Theta(x_0) = p_0$, $|p_0|_g = 1$,

iv) $U_0(x_0) \neq 0$.

Let γ, $x = x(t)$ be the geodesic with initial data (x_0,v_0), $v_0 = I_g p_0$ and $T < l(x_0,v_0)$. Then there exist smooth complex valued functions $\theta(x,t)$, $u_n(x,t)$,

$n = 0, \dots, N$ *in an open neighborhood* W *of the path* $\mu : [0,T] \to M \times \mathbf{R}$, $\mu(t) = (x(t), t)$ *such that the Gaussian beam* U_ϵ^N *of the order* N,

$$U_\epsilon^N(x,t) = (\pi\epsilon)^{-m/4} \exp\{-(i\epsilon)^{-1}\theta(x,t)\} \sum_{n=0}^{N} u_n(x,t)(i\epsilon)^n,$$

satisfies he following conditions

i) $\theta(x(t),t) = 0$,

ii) $Im\,\theta(x,t) \geq cd^2(x, x(t))$,

iii) $\partial_x\theta(x(t),t) = p(t)$, $|p(t)|_g = 1$,

iv) $u_0(x(t),t) \neq 0$. *v) In a neighborhood* $U' \subset V$ *of* x_0

$$|U_\epsilon^N(x,0) - (\pi\epsilon)^{-m/4} \exp\{-(i\epsilon)^{-1}\Theta(x)\} \sum_{n=0}^{N} U_n(x)(i\epsilon)^n| \leq C\epsilon^{N+1-m/4}.$$

vi) $|(\partial_t^2 - \Delta_g + q)U_\epsilon^N| \leq C(\epsilon)^{N-m/4}$.

Corollary 2 *Let* $U_\epsilon^N(t,x)$ *be a formal Gaussian beam constructed in Theorem 4. Then for any* $j \geq 0$ *and multi index* α

$$|\partial_t^j \partial_x^\alpha (\partial_t^2 - \Delta_g + q)U_\epsilon^N| \leq C_{j\alpha} M_\epsilon \epsilon^{N-j-|\alpha|-m/4}.$$

Proof. By analyzing the construction of functions θ and u_n, we observe the corresponding v_n satisfy

$$|\partial_t^j \partial_x^\alpha v_n(t,x)| \leq Cd(x, x(t))^{2(N+2-n-j-\alpha)}.$$

From these inequalities the statement follows. $\diamondsuit$

Theorem 5 *Let* $\Theta(x)$ *and* $U_n(x)$, $n = 0, \dots, N$, *be defined as in Theorem 4. Then there is a solution* $u_\epsilon(x,t)$ *of the equation*

$$(\partial_t^2 - \Delta_g + q)u_\epsilon(x,t) = 0, \quad (x,t) \in M \times [0,T],$$

$0 < T < d(x_0, \partial M)$, *such that for any* $j > 0$ *and multi index* α

$$|\partial_t^j \partial_x^\alpha (u_\epsilon(t,x) - \chi(t,x)U_\epsilon^N(t,x))| \leq C\epsilon^{N-(j+|\alpha|)-m/4}.$$

Here $U_\epsilon^N(x,t)$ *is the formal Gaussian beam of Theorem 4 and* χ *is the cutting function,* $\chi = 1$ *near* $\mu([0,T])$.

Proof. Let $u_\epsilon(t,x) = \chi(t,x)U_\epsilon^N(t,x) + v^N(x,t)$, where v^N satisfies the initial boundary value problem

$$\begin{aligned}(\partial_t^2 - \Delta_g + q)v^N &= -(\partial_t^2 - \Delta_g + q)(\chi(t,x)U_\varepsilon^N),\\ v^N|_{t=0} = 0, &\qquad \partial_t v^N|_{t=0} = 0.\end{aligned}$$

Then the assertion follows from general hyperbolic estimates, Theorem 4 and Corollary 2. ◊

The functions $u_\varepsilon(x,t)$ introduced in Theorem 5 are called **Gaussian beams** of order N.

9 Gaussian beams from the boundary.

Here we construct Gaussian beams generated by boundary sources.

Let $z_0 \in \partial M$, $t_0 > 0$, and $z = (z^1, \dots z^{m-1})$ be a local system of coordinates on ∂M near z_0. Consider a class of functions $f_\epsilon = f_{\epsilon, z_0, t_0}(z,t)$ on the boundary cylinder $\partial M \times \mathbf{R}$ where

$$f_\epsilon(z,t) = (\pi\epsilon)^{-m/4} \exp\{i\epsilon^{-1}\Theta(z,t)\} V(z). \tag{26}$$

Here

$$\Theta(z,t) = -(t-t_0) + (H_0(z-z_0), (z-z_0)) + i(t-t_0)^2, \tag{27}$$

where H_0 is a given symmetric matrix with positive definite imaginary part, $H_0 = H_0^t$, $Im\, H_0 > 0$ and V is a given smooth function with Taylor expansion $V \asymp \sum_l V_l$ near $z = z_0$.

Consider the initial boundary value problem

$$\begin{aligned}&\partial_t^2 u - \Delta_g u + qu = 0,\\ &u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0, \quad u|_{\partial M \times \mathbf{R}_+} = f_\epsilon(z,t)\chi(z,t)\end{aligned} \tag{28}$$

where χ is smooth cut–off function near (z_0, t_0). In this section we prove that the solution of initial boundary value problem (28) is a Gaussian beam. The corresponding geodesic starts at z_0 and is normal to the boundary. To this end, we first construct a formal Gaussian beam which asymptotically has the right boundary value. Then the solution of problem (28) is close to the constructed formal Gaussian beam until the corresponding trajectory hits the boundary.

In the construction we use the boundary normal coordinates $(z, n) = (z^1, \ldots, z^{m-1}, n)$ where n is the distance to the boundary and $(z^1, \ldots, z^{m-1})$ are the coordinates of the nearest boundary point in the local coordinates of the boundary. The system of coordinates is smooth near the boundary and the length element has the form

$$ds^2 = g_{\alpha\beta}(z, n) dz^\alpha dz^\beta + dn^2, \tag{29}$$

$\alpha, \beta = 1, \ldots, m-1$. Here the metric tensor $g_{\alpha\beta}(z, 0)$ is the metric tensor on the boundary ∂M.

Theorem 6 *There is a unique formal Gaussian beam $U_\epsilon^N(z, n, t)$ such that near (z_0, t_0) its phase function θ and the amplitude functions u_k, $k = 0, 1, \ldots$, satisfy the following boundary conditions*

$$\begin{aligned} &\theta(z, 0, t) \asymp -(t - t_0) + (H_0(z - z_0), (z - z_0)) + i(t - t_0)^2, \qquad (30) \\ &u_0(z, 0, t) \asymp V(z), \quad u_k(z, 0, t) \asymp 0, \quad K = 1, \ldots. \end{aligned}$$

Moreover, corresponding geodesic $\gamma(t)$ is the normal geodesic, starting at point z_0 at time t_0, i.e.γ is given in the boundary normal coordinates by the curve

$$z(t) = z_0, \quad n(t) = t - t_0. \tag{31}$$

The proof consists of several lemmas.

Lemma 13 *The geodesic $\gamma(t)$ corresponding to the formal Gaussian beam U_ϵ^N which satisfy the boundary data is the normal geodesic γ_{z_0, ν_0}.*

Proof. Consider a formal Gaussian beam in the boundary normal coordinates (z, n, t). Previously it was shown that the phase function θ of the Gaussian beam has the form

$$\theta(z, n, t) = \sum_{l \geq 1} \theta_l(t),$$

where

$$\begin{aligned} \theta_1(t) &= p_\alpha(t)(z^\alpha - z^\alpha(t)) + p_m(t)(n - n(t)) && (32) \\ \theta_2(t) &= \frac{1}{2}[H_{\alpha\beta}(t)(z^\alpha - z^\alpha(t))(z^\beta - z^\beta(t)) + && (33) \\ &+ 2H_{\alpha m}(t)(z^\alpha - z^\alpha(t))(n - n(t)) + H_{mm}(t)(n - n(t))^2]. \end{aligned}$$

We remind that $p(t)$ is the canonical transformation of the unit velocity vector along the geodesic γ and $\theta_l(t)$ are homogeneous polynomials of order l with respect to $z - z(t)$ and $n - n(t)$.

Consider linear term of $\theta(z, 0, t)$ near (z_0, t_0) and compare it with the linear term of $\Theta(z, t)$ appearing in Lemma 12. We can see that

$$(z(t_0), n(t_0)) = (z_0, 0), \quad p(t_0) = (0, \cdots, 0, p_{0m}),$$

Moreover,

$$p_{0m} \frac{dn(t_0)}{dt} = 1.$$

As $|p(t)| = 1$ for all t we have, in particular, that $p_{0m} = 1$. By using the Hamilton system and the initial conditions

$$z|_{t=t_0} = z_0, \quad n|_{t=t_0} = 0, \quad p|_{t=t_0} = (0, \ldots, 0, 1),$$

it is easy to prove that the unique solution of the Cauchy problem has the form

$$z(t) = z_0, \quad n(t) = t - t_0, \quad p(t) = (0, \ldots, 0, 1). \tag{34}$$

To complete the proof we mention that it is necessary that $\gamma(t_0) = z_0$. Indeed, if $\gamma(t_0) = \tilde{z}_0 \neq z_0$ then Taylor's expansion for $\theta(z, 0, t)$ would not match Taylor's expansion (30). $\diamondsuit$

As our geodesic is normal to the boundary then θ_l is the homogeneous polynomials of order l with respect to $(z - z_0, n - (t - t_0))$.

Next we complete the proof of Theorem 6. Considering the quadratic term of $\theta_2(z, 0, t)$ and compare it with the quadratic form in expansion (30) we obtain that

$$\begin{aligned} H(t_0) &= \widehat{H}_0, \\ \widehat{H}_{0\alpha\beta} = H_{0\alpha\beta}, \quad \widehat{H}_{0\alpha m} = 0, &\quad \widehat{H}_{0mm} = i, \quad \alpha, \beta = 1, \ldots, m-1. \end{aligned} \tag{35}$$

Having found initial data for the quadratic form $H(t_0)$ we can find $H(t)$. In boundary normal coordinates $H(t)$ has the form

$$H(t) = Z(t) Y^{-1}(t) = \widehat{H}_0 (I + \int_{t_0}^{t} C(t') dt' \cdot \widehat{H}_0)^{-1}, \tag{36}$$

where

$$Z(t) = \widehat{H}_0, \quad Y(t) = I + \int_{t_0}^{t} C(t')dt' \cdot \widehat{H}_0, \tag{37}$$

$$C^{ik}(t) = g^{ik}(z_0, t - t_0) - \delta^{im}\delta^{km}. \tag{38}$$

In boundary normal coordinates matrices B and D in the Riccati equation are equal to zero and the matrix C has form (38). The fact can be used to prove formulae (36), (37).

Let us now return to the formula for $\theta(z, 0, t)$. The homogeneous terms $\tilde{\theta}_l$ involves $\theta_l(t_0)|_{n=0}$ and also derivatives of $\theta_j(t_0)|_{n=0}$ for $j = 2, 3, \dots l - 1$. Hence, if we know $\theta_1(t), \dots, \theta_{l-1}(t)$, comparing expansion for $\theta(z, 0, t)$ with expansion (30) we obtain initial data $\theta_l(t_0)$. Solving initial value problem for $\theta_l(t)$ we find the functions.

Similarly, we find Taylor's expansion for the amplitudes $u_l(z, n, t)|_{t=t_0}$, $l = 0, 1, \dots$, where

$$u_l(z, n, t) \asymp \sum_{k \geq 0} u_{lk}(t).$$

Here $u_{lk}(t)$ are homogeneous polynomials of order k of variables $(z - z_0, n - (t - t_0))$. In particular,

$$\begin{aligned} u_{00}(t_0) &= V(z_0), \\ u_{00}(t) &= (det\, Y(t))^{-1/2} \left(\frac{g(z_0, 0)}{g(z_0, t - t_0)} \right)^{1/4} V(z_0). \end{aligned}$$

$\diamondsuit$

Denote by $u_\epsilon^f(x, t)$ the solution of the initial boundary value problem (28). Let l_{z_0} be the time when normal geodesic $\gamma_{z_0,\nu}$ hits the boundary ∂M for the first time.

Theorem 7 *Let $T < l_{z_0}$. Then, for any $j \geq 0$ and multi index α,*

$$|\partial_t^j \partial_x^\alpha (u_\epsilon^f(x, t) - \chi(x, t) U_\epsilon^N(x, t))| \leq C\epsilon^{N-(j+|\alpha|)-m/4}. \tag{39}$$

Here $U_\epsilon^N(x, t)$ is a formal Gaussian beam of Theorem 6 and $\chi(x, t)$ is a cutting function having value 1 near the trajectory $(x(t), t)$, $t \in [0, T]$.

Proof. The proof follows from Theorem 6. $\diamondsuit$

10 Construction of manifold and boundary distance functions

To this far we have constructed special solutions, Gaussian beams, on the manifold M. Next, we study the inner products of the Gaussian beams to construct the Riemannian manifold (M, g). First we start with the domains of influences.

Definition 4 *Let $\Gamma \subset \partial M$ be a subset of boundary and $\tau > 0$. The domain of influence corresponding to the set Γ and $\tau > 0$ is*

$$\Sigma(\Gamma, \tau) = \{x \in M \ : \ d(x, \Gamma) \leq \tau\}. \tag{40}$$

Particularly, we consider the case where Γ consist of exactly on point y and use notation

$$\Sigma(y, \tau) = \{x \in M \ : \ d(x, y) \leq \tau\}. \tag{41}$$

For each domain of influence Σ we denote by $L^2(\Sigma)$ the subspace of $L^2(M)$ which contains all functions having support in Σ:

$$L^2(\Sigma) = \{u \in L^2(M) : \text{ supp } (u) \subset \Sigma\}.$$

Let P_Σ be the orthoprojector in $L^2(M)$ onto the space $L^2(\Sigma)$. It has a simple form

$$(P_\Sigma a)(x) = \chi_\Sigma(x) a(x) \tag{42}$$

where $\chi_\Sigma(x)$ is the characteristic function of the set Σ.

Our main plan below is to find out when a Gaussian beam is located in a given domain of influence. Before going to details, we explain how this can be done in principle. Let us consider the P_Σ-projections of Gaussian beams. If the Gaussian beam $u_\varepsilon^f(x, t)$ is at the time t at the point $x(t)$, then the norm $||P_\Sigma u_\varepsilon(\cdot\,, t)||_{L^2(M)}$ is approximately zero if $x(t) \notin \Sigma$. Vice versa, the norm is approximately one if $x(t) \in \Sigma^{int}$. In this way we can found out if the point $x(t)$ is in a given ball having center at boundary point or not. This gives us information about the global metric structure of M. To compute the norms of projections, we use the formula

$$||P_\Sigma u||^2_{L^2(M)} = \sum_{j,k=0}^{\infty} (m_\Sigma)_{jk} \langle u, \varphi_j \rangle \overline{\langle u, \varphi_k \rangle} \tag{43}$$

where m_Σ is the Gram–Schmidt matrix of projection P_Σ,

$$(m_\Sigma)_{jk} = \langle P_\Sigma \varphi_j, \varphi_k \rangle, \quad j, k = 1, 2, \dots \tag{44}$$

where φ_j are the eigenfunctions of $-\Delta_g + q$. Finally, to find the Gram–Schmidt matrix of an ortonormal projection we construct a special ortonormal basis for the image of the projection. This basis, called wave basis, consists of functions $u^{f_j} = u^{f_j}(\cdot\,, \tau)$ where u^{f_j} are solutions of wave equation (28). However, first we have to show that such a basis exists and for this we need so-called controllability results.

Let $\Gamma \subset \partial M$ be an open subset of the boundary and define a mapping

$$\mathcal{W}^\tau : H_0^1(\Gamma \times [0, \tau]) \to L^2(M), \; f \mapsto u^f(\cdot\,, \tau)$$

where u^f solves initial boundary value problem (28). By finite speed of wave propagation,

$$\text{supp } (u^f(\cdot\,, \tau)) \subset M(\Gamma, \tau) \tag{45}$$

for supp $(f) \subset \Gamma \times \overline{\mathbf{R}}_+$. This equation explains the term 'domain of influence'. Indeed, the waves sent from Γ can not propagate in time τ outside the set $\Sigma(\Gamma, \tau)$.

Next we show that the set of the possible final states $u^f(\cdot\,, \tau)$ are dense in $L^2(M(\Gamma, \tau))$. In other words, with the boundary source f supported in $\Gamma \times [0, \tau]$ one can control the wave in such a way that the final state $u^f(\cdot\,, \tau)$ is arbitrarily close to any state in $L^2(M(\Gamma, \tau))$.

Example. Let us consider one-dimensional wave equation

$$\begin{aligned} &u_{tt}(x, t) - u_{xx}(x, t) = 0, \quad (x, t) \in [0, 2] \times [0, 1] \\ &u(x, 0) = u_t(x, 0) = 0, \quad u(0, t) = f(t), \quad u(2, t) = 0. \end{aligned}$$

The solution of this equation is $u(x, t) = f(t - x)$. Particularly, at time $t = 1$ the solution $u(x, 1)$ vanishes outside the domain of influence, that is, for $x > 1$. We see that for any $v \in L^2([0, 1])$ there is $f(t) = v(1 - t)$ such that $u(x, 1) = v(x)$. This fact is called exact controllability, since by source with $f \in L^2([0, 1])$ we can control the final state $u(x, 1) \in L^2([0, 1])$.

Next we generalize this example in m-dimensional case.

Theorem 8 *Let $\tau > 0$ and $\Gamma \subset \partial M$ be an open subset of boundary. Then the set*

$$\mathcal{W}^\tau(C_0^\infty(\Gamma \times [0,\tau])) = \{u^f(\cdot\,,\tau) \in L^2(\Sigma(\Gamma,\tau)) :\ f \in C_0^\infty(\Gamma \times [0,\tau])\}$$

is dense in $L^2(\Sigma(\Gamma,\tau))$.

Proof. Let us consider a function $\psi \in L^2(\Sigma(\Gamma,\tau))$ in the orthocomplement of the range of $\mathcal{W}^\tau$, that is,

$$\langle u^f(\cdot\,,\tau), \psi\rangle = 0$$

for all $f \in C_0^\infty(\Gamma \times [0,\tau])$. To show that the range of $\mathcal{W}^\tau$ is dense we have to show that ψ vanishes. For this we consider the equation

$$\begin{aligned} e_{tt} - \Delta_g e + qe &= 0 \text{ in } M \times [0,\tau] \\ e|_{\partial M \times [0,\tau]} &= 0, \quad e|_{t=\tau} = 0,\ e_t|_{t=\tau} = \psi. \end{aligned}$$

By integration by parts

$$\begin{aligned} 0 &= \int_{M\times[0,\tau]} [(e_{tt} - \Delta_g e + qe)\overline{u^f} - e\overline{(u^f_{tt} - \Delta_g u^f + qu^f)}]\, dx\, dt \\ &= -\int_M u^f(\tau)\,\overline{\psi}\, dx + \int_{\partial M}\int_0^\tau f\,\overline{\partial_\nu e}\, dS_x\, dt \\ &= \int_{\partial M}\int_0^\tau f\,\overline{\partial_\nu e}\, dS_x\, dt \end{aligned}$$

for all $f \in C_0^\infty(\gamma \times [0,\tau])$. This yields

$$e|_{\Gamma\times[0,\tau]} = \partial_\nu e|_{\Gamma\times[0,\tau]} = 0$$

and hence the Cauchy data of e vanishes on $\Gamma \times [0,\tau]$. Since $e|_{t=\tau} = 0$, we can define a reflection of e over the surface $t = \tau$,

$$E(x,t) = \begin{cases} e(x,t), & for\ t \le \tau, \\ -e(x, 2\tau - t), & for\ t > \tau. \end{cases}$$

Since the traces of E and E_t coincide from both sides of the surface $t = \tau$, Green's formula shows that

$$E_{tt} + a(x,D)E = 0 \text{ in } M \times [0, 2\tau].$$

Moreover, the Cauchy data of E vanishes on $\Gamma \times [0, 2\tau]$,

$$E|_{\Gamma\times[0,2\tau]} = 0, \quad \partial_\nu E|_{\Gamma\times[0,2\tau]} = 0.$$

Thus we see by using Tataru's Holmgren-John theorem [26] that E vanishes in the set

$$K = \{(x,t) : \ d(x,\Gamma) < \tau - |\tau - t|\}.$$

Indeed, by deforming the surface $\Gamma \times [0, 2\tau]$ in continuous way so that the surface is non-characteristic, we can reach any point of the the double cone K. Thus by Tataru's Holmgren-John theorem the solution E vanish in the set K. Particularly

$$\psi(x) = E_t(x,t)|_{t=\tau} = 0, \text{ for } x \in \Sigma(\Gamma, \tau).$$

Hence the assertion is proven. $\diamondsuit$

Since $u^f(\tau)$ are dense in $L^2(\Sigma(\Gamma, \tau))$, there are f_j such that $u^{f_j}(\tau)$ form an ortonormal basis in $L^2(\Sigma(\Gamma, \tau))$. Next we construct this kind of functions f_j by using boundary spectral data.

Lemma 14 *Let $\tau > 0$. By using the boundary spectral data we can find boundary sources $\beta_j \in C_0^\infty(\Gamma \times [0, \tau])$ such that*

$$v_l(x) = u^{\beta_l}(x, \tau) \tag{46}$$

forms an ortonormal basis of $L^2(\Sigma(y, \tau))$.

Proof. Let (α_j) be a complete set in $L^2(\Gamma \times [0, \tau])$ and let us use Gram–Schmidt orthonormalisation procedure to the functions $u^{f_j}(\tau)$ with known inner products

$$c_{jk} = \langle u^{\alpha_j}(\tau), u^{\alpha_k}(\tau)\rangle.$$

More precisely, we define $\beta_j \in C_0^\infty(\Gamma \times [0, \tau])$ recursively as

$$\eta_j = \alpha_j - \sum_{k=1}^{j-1} \langle u^{\alpha_j}(\tau), u^{\beta_k}(\tau)\rangle, \quad \beta_j = \frac{\eta_j}{\langle u^{\eta_j}(\tau), u^{\eta_j}(\tau)\rangle}.$$

In the case when $\eta_j = 0$, we remove the corresponding α_j from the original sequence and continue the procedure with the next α_j. By Theorem 8, the sequence (β_j) obtained in this way is dense in $L^2(\Sigma(\Gamma, \tau))$ and ortonormal. $\diamondsuit$

After finding the basis for the space $L^2(\Sigma)$ we construct the Gram–Schmidt matrix of the corresponding projection.

Lemma 15 *Let $\Gamma \subset \partial M$ be open. Then for a domain of influence $\Sigma = \Sigma(\Gamma, \tau)$ the boundary spectral data determines the corresponding Gram–Schmidt matrix M_Σ,*

$$(M_\Sigma)_{jk} = \langle P_\Sigma \varphi_j, \varphi_k \rangle, \quad j, k = 1, 2, \dots. \tag{47}$$

Proof. By Lemma 14, we can construct a sequence $\beta_j \in C_0^\infty(\Gamma \times [0, \tau])$ such that the corresponding solutions of the wave equation form an ortonormal basis of $L^2(\Sigma)$. We denote this basis by

$$v_l(x) = u^{\beta_l}(x, \tau). \tag{48}$$

In this wave-basis all functions $u \in L^2(\Sigma(\Gamma, \tau))$ have representations

$$u(x) = \sum_{l=0}^{\infty} \langle u, v_l \rangle \, v_l(x).$$

By applying this for $u = P_\Sigma \varphi_j$ and using the fact that $\langle P_\Sigma \varphi_j, v_l \rangle = \langle \varphi_j, v_l \rangle$ we obtain

$$\langle P_{\Sigma(\Gamma,\tau)} \varphi_j, \varphi_k \rangle = \sum_{l=0}^{\infty} \langle \varphi_j, v_l \rangle \langle v_l, \varphi_k \rangle.$$

By Corollary 1 the inner products can be computed from the boundary spectral data imply that $\langle P_{\Sigma(\Gamma,\tau)} \varphi_j, \varphi_k \rangle$ can be found. ♢

Letting Γ tends to a point $\{y\}$ we obtain the following result.

Lemma 16 *The boundary spectral data determines the the Gram–Schmidt matrix M_Σ for a slice $\Sigma = \Sigma(y, \tau)$.*

Next we start to construct the manifold structure by using projections to domains of influences. We consider a normal geodesics $\gamma_{z_0,\nu}(t)$, $\gamma_{z_0,\nu}(t) = \exp_{z_0}(\nu t)$, where $z_0 \in \partial M$ and ν is the unit normal vector to the boundary at the point z_0. Let $u_\epsilon(x, t; z_0, t_0) = u_\epsilon^f(x, t)$ be the Gaussian beam sent from the point z_0 at the time t_0 in the normal direction, i.e. the solution of initial boundary value problem with the boundary source $f_\epsilon(z, t)$ given by formulae (26), (27).

By means of boundary spectral data we can find when the Gaussian beam is in a given domain of influence.

Lemma 17 *Let $y, z \in \partial M$, and $a > 0$. Consider the point $x = \gamma_{z,\nu}(t)$, where $t < l_z + a$. Then for the normal Gaussian beam $u_\epsilon(x, t; z, a)$ sent at the time a from the boundary point z we have*

$$\lim_{\varepsilon \to 0+} \langle P_{\Sigma(y,\tau)} u_\varepsilon(\,\cdot\,, t; z, a), u_\varepsilon(\,\cdot\,, t; z, a)\rangle = \begin{cases} 0, & for\ \tau < d(x, y), \\ \alpha, & for\ \tau > d(x, y) \end{cases}$$

where α is a positive constant $\alpha = (g(z_0, 0)/det(Im\, H_0))^{1/2} |V(z_0)|^2$.

Proof. When $\tau < d(x, y)$ we have $x \notin \Sigma(y, \tau)$. Then for sufficiently small r, the ball $B(x, r) \subset M$ satisfies $B(x, r) \cap \Sigma(y, \tau) = \emptyset$. For $\epsilon < r^3$ the formal Gaussian beam $U_\epsilon^N(x, t; z, a)$ exponentially small in $\Sigma(y, \tau)$. Using Theorem 7 we have for any $K < N - \frac{m}{4}$

$$\|P_{\Sigma(y,\tau)} u_\epsilon(\cdot, t; z, a)\|_{L^\infty} = O(\epsilon^K).$$

Assume that $\tau < d(x, y)$. Then for sufficiently small r, $B(x, r) \subset \Sigma(y, \tau)$. Using the same arguments as above we see that

$$\|P_{\Sigma(y,\tau)} u_\epsilon(\cdot, t; z, a)\|_{L^\infty} = \|u_\epsilon(\cdot, t; z, a)\|_{L^\infty} + O(\epsilon^K). \tag{49}$$

By Theorem 7 we can use in the right hand side of formula (49) formal Gaussian beam $U_\epsilon^N(\cdot, t; z, a)$ for sufficiently large N instead of Gaussian beam $u_\epsilon(\cdot, t; z, a)$ with the same estimate. Using the main terms of the formal Gaussian beam U_ϵ^N we obtain

$$\langle U_\epsilon^N(\cdot, t; z, a), U_\epsilon^N(\cdot, t; z, a)\rangle = \tag{50}$$

$$= (\pi\epsilon)^{-m/2} \alpha \int_{\mathbf{R}^m} e^{-\epsilon^{-1}(Im\, H(t) y, y)}\, |det\, Y(t)|^{-1} dy^1 \cdots dy^m + O(\epsilon) =$$

$$= \frac{\alpha \sqrt{det\,(Im\, H_0)}}{\sqrt{det\,(Im\, H(t)) |det Y(t)|^2}} + O(\epsilon) = \alpha + O(\epsilon)$$

where on the last step we used Lemma 10. ◇

By using Lemma 17 we can find the time l_z. Indeed, l_z is largest T such that for $\tau \in (0, T)$ and

$$\lim_{\rho \to 0} \lim_{\varepsilon \to 0+} \langle P_{\Sigma(\partial M, \rho)} u_\varepsilon(\,\cdot\,, \tau; z, a), u_\varepsilon(\,\cdot\,, \tau; z, a)\rangle = 0.$$

Using above observations we can now compute the distances of a Gaussian beam to the boundary at any time.

Lemma 18 *For any $z \in \partial M$ and $t < l_z$ we can find $d(\gamma_{z,\nu}(t), y)$ for any $y \in \partial M$.*

Proof. The claim follows from Corollary 2. $\diamondsuit$.

Now we are ready to define the set $R(M)$ of boundary distance functions. The boundary distance functions are the functions

$$r_x(y) = d(x,y), \quad y \in \partial M$$

where $x \in M$. Obviously r_x is continuous and bounded by diameter of M. Thus we can define a mapping

$$R : M \mapsto C(\partial M), \ x \mapsto r_x$$

which assign to any point x the continuous function which gives the distance of x to the boundary points. The set of all boundary distance functions is denoted by

$$R(M) = \{r_x \in C(\partial M) : \ x \in M\}.$$

It is well known that any manifold M of dimension m can be embedded to an Euclidean space $\mathbf{R}^n$ having sufficiently large dimension $n = 2m+1$. In our case we go even further and construct a map from M to an infinite dimensional space $C(\partial M)$. By triangle inequality,

$$||r_x - r_z||_C = \sup_{y \in \partial M} |d(x,y) - d(z,y)| \leq d(x,z).$$

Hence $R(M)$ is a continuous image of m-dimensional manifold M in $C(\partial M)$. We will show later that the set $R(M)$ can be consider as a smooth surface or a sub-manifold in the space $C(\partial M)$. First we prove that we can find $R(M)$.

Theorem 9 *The boundary spectral data determines the set $R(M)$.*

Proof. Each point $x \in M$ has a nearest boundary point z and by variational principle, the shortest geodesic from z to x is normal to the boundary. It means that every point lies on a normal geodesic starting from boundary. Thus

$$M = \{x = \gamma_{z,\nu}(t) \ : \ z \in \partial M, \quad t \in [0, l_z)\}. \tag{51}$$

By Lemma 18 we can find for any given $z \in \partial M$ and $t < l_z$ the boundary distance function

$$d(\gamma_{z,\nu}(t), y) = r_x(y),$$

where $x = \gamma_{z,\nu}(t)$. This shows that we can construct the set

$$R(M) = \{d(\gamma_{z,\nu}(t), \cdot\,) : \; z \in \partial M, \quad t \in [0, l_z)\}.$$

Note that each point $x \in M$ may lie on several normal geodesics but at least on one normal geodesic. Thus each function r_x has many representatives $d(\gamma_{z,\nu}(t), \cdot\,)$ but each at least one. $\diamondsuit$

Our next aim is to define a differentiable structure on $R(M)$ such that it becomes a differentiable manifold. After this we construct a metric which makes it a Riemannian manifold.

Example. Consider Riemannian manifold (M, g) which is geodesically simple. This means

1. For any $x, y \in M$ there is a unique geodesic γ joining these point.
2. Any geodesic $\gamma([a, b])$ can be continued to a geodesic $\gamma([a', b'])$ which endpoints are boundary points.

Let us consider the set of boundary distance functions $R(M)$. By triangular inequality,

$$||r_x - r_y||_{C(\partial M)} \le d(x, y), \; x, y \in M.$$

Moreover, let $\gamma([a, b])$ be the shortest geodesic from y to x. This geodesic can be continued to the shortest geodesic $\gamma([a', b])$ where $z = \gamma(a') \in \partial M$. Then

$$r_x(z) - r_y(z) = |\gamma([a', b])| - |\gamma([a', a])| = d(x, y).$$

Thus

$$||r_x - r_y||_{C(\partial M)} = d(x, y)$$

and the mapping $R : M \to R(M)$ is an isometry.

In the case of a general manifold we need more delicate constructions. At first we prove that $R(M)$ is a topological manifold. For this, we equip $R(M)$ with the relative topology as a subset $R(M) \subset C(\partial M)$.

Lemma 19 *The mapping $R : M \to R(M)$ is a homeomorphism.*

Proof. Let $x, y \in M$ and $z \in \partial M$. By triangular inequality

$$||r_x - r_y||_{C(\partial M)} \leq d(x,y),\ \ x, y \in M,$$

which shows that R is continuous. Next we show that it is one-to-one. Assume that $r_x = r_y$ for $x, y \in M$. Let us choose a point $z \in \partial M$ for which

$$r_x(z) = \min_{z' \in \partial M} r_x(z')$$

and denote by $h = r_x(z)$ this minimum. Then $z \in \partial M$ is one of the nearest boundary points to x, and $h = d(z,x)$. Since $r_x = r_y$, we see that z is also some of the nearest boundary points to y having the same distance h from y.

By variational principle, the shortest geodesic γ from x to z is normal to ∂M at the point z. Thus $x = \exp_z(h\nu_z)$. Since y has the same representation, we see that $x = y$. This shows that $R : M \to R(M)$ is one-to-one.

Since by definition the mapping R maps M onto $R(M)$, we see R is a bijective, continuous mapping defined on a compact set M. Since M is compact, elementary topological arguments show from this that R is open and thus a homeomorphism. $\diamondsuit$

Since $R : M \to R(M)$ is a homeomorphism and M is a differentiable manifold, also $R(M)$ has a differentiable manifold structure. However, $R(M)$ can have several differentiable structures, and we have to choose the one which coincides with the structure of M. Next we give a sketch of the construction which specifies the right differentiable structure on the set $R(M)$. So, consider the evaluation functions

$$E_z : R(M) \to \mathbf{R}, r \mapsto r(z)$$

where $z \in \partial M$. For any $r \in R(M) \setminus R(\partial M)$, consider a point $z_0 \in \partial M$ at which r gets its minimum. Thus, if $x \in M$ is such that $r = r_x$, then z_0 is the nearest point of ∂M to x. By using the fact that the normal geodesic from z_0 to x is the shortest geodesic from x boundary, it is possible to show that the distance function $(y, z) \mapsto d(x, z)$ is differentiable when y is near x and z is near z_0. Thus by choosing points $z_1, \dots, z_m$ near z_0 in a right way, the evaluation functions $E_{z_j},\ j = 1, \dots, m$ define coordinates near r. These points can be chosen for instance in the following way. Let $\eta_j \in T_{z_0}\partial M,\ j = 1, \dots, m-1$ be a basis of the tangent space of the boundary. Let $z_j = \mu_{z_0,\eta_j}(\varepsilon)$

where μ_{z_0,η_j} are geodesics of the boundary ∂M satisfying $\mu_{z_0,\eta_j}(0) = z_0$ and $\dot\mu_{z_0,\eta_j}(0) = \eta_j$. When ε is small enough, the points $z_1, \dots, z_{m-1}$ and $z_m = z_0$ are such that E_{z_j}, $j = 1, \dots, m$ define coordinates in a neighborhood of r in $R(M)$. Near boundary $R(\partial M)$ of $R(M)$ we have to define coordinates in different way by using boundary normal coordinates. These coordinates are the pair $(z(r), s(r))$ where $s(r)$ is the minimal value of $r \in R(M)$ and $z = z(r) \in \partial M$ is the point where this minimal value is achieved. These coordinates on $R(M)$ define for $R(M)$ a differentiable structure which makes the function $R : M \to R(M)$ diffeomorphism.

Next we construct the Riemannian metric on $R(M)$ such that $R : M \to R(M)$ becomes an an isometry.

Lemma 20 *The set $R(M)$ determines a Riemannian metric tensor G on $R(M)$ such that $(R(M), G)$ is isometric to (M, g).*

Proof. We give the sketch of the proof. Let us consider the metric tensor $G = R_* g$ which is the push forward of the metric g on M to the manifold $R(M)$. When we equip $R(M)$ with this metric, the mapping R is an isometry by definition, and thus an appropriate metric G exists of $R(M)$. Next we show that by knowing the set $R(M)$ we can find this metric tensor.

For each $r \in R(M)$, we define functions $s(r)$ and $z(r)$ where $s(r) = \min\{r(z) : z \in \partial M\}$ and $z(r) \in \partial M$ is one of the points where $r(z) = s(r)$. If $r = R(x)$, then $z(r)$ is a nearest boundary point to x and $s(r) = d(x, \partial M)$. Let us denote by $\tau(z)$ the maximum of the set $\{s(r) : r \in R(M), z(r) = z\}$. The geometrical meaning of $\tau(z)$ is related to the normal geodesic $\gamma_{z,\nu}$. Indeed, $\tau(z)$ is the maximal t for which z is the nearest boundary point of the point $\gamma_{z,\nu}(t)$.

Let $r_0 \in R(M)^{int}$, and let $x_0 \in M$ be such a point that $r_0 = R(x_0)$. Next we construct the metric tensor at the point r_0 in some local coordinates, e.g. the local coordinates defined with evaluation functions E_{z_j}. Let $z_0 = z(r_0)$ be a nearest boundary point to x_0. Next, for simplicity, we assume that $s(r_0) < \tau(z_0)$. From this one can show that if we perturb the geodesic $\gamma_{z_0,\nu}([0, s(r_0)])$ to a geodesic $\gamma_{z,\xi}$, where (z, ξ) is near to (x_0, ν), then the perturbed geodesic $\gamma_{z,\xi}$ is the shortest geodesic between its endpoints. Particularly, this and the inverse function theorem yield that the distance function

$$e : M \times \partial M \to \mathbf{R}, (x, z) \mapsto d(x, z) \tag{52}$$

is smooth function near the point (x_0, z_0).

Next we consider the evaluation functions on $R(M)$. If for $x = R^{-1}(r)$ then

$$E_z(r) = d(x, z).$$

Since mapping (52) is smooth, also $(r, z) \mapsto E_z(r)$ is a smooth function on $R(M) \times \partial M$ near (r_0, x_0). Let $z \in \partial M$ be a fixed point. Since $x \mapsto e(x, z)$ is a distance function, its differential respect of x, denoted by $d_x e(\cdot\,, z) : T_x M \to \mathbf{R}$, is a covector having length 1. Since $R : (M, g) \to (R(M), G)$ is an isometry, this yields that

$$||dE_z|_{r0}||_G = ||d_x e(\cdot\,, z)|_{x0}||_g = 1.$$

Hence we can construct unit covectors in the space $T^*_{r_0} R(M)$. Moreover, one can show that the mapping

$$z \mapsto dE_z|_{r_0} \in T^*_{r_0} R(M)$$

maps a neighborhood of z_0 to an open set. Thus we can construct an open set of the unit sphere

$$S^*_{r_0} R(M) = \{v \in T^*_{r_0} R(M) : \ ||v||_G = 1\}.$$

The G-unit sphere is an ellipsoid in local coordinates, and since an open subset of the surface of an ellipsoid determines the ellipsoid uniquely, we can find the whole G-unit sphere in $T^*_{r_0} R(M)$. This determines the metric tensor G at r_0.

Finally, the metric tensor G is a smooth 2-form. Since we have constructed G is a dense subset of $R(M)$, we can continue it on the whole $R(M)$. This proves the assertion. $\diamondsuit$

In the following we can identify the isometric Riemannian manifolds (M, g) and $(R(M), G)$. Since we have now reconstructed the manifold and the metric on it, it remains to show the following result.

Lemma 21 *The boundary spectral data determines the potential q uniquely.*

For this, we consider the projection of Gaussian beams and the inner product of the projection of the Gaussian beam and eigenfunctions.

Lemma 22 *Let $z_0 \in \partial M$ and $\Gamma \subset \partial M$ is an open neighborhood of z_0. Let $\tau < \tau_{\partial M}(z_0)$. Then for arbitrary eigenfunction φ_j, $j = 1, 2, \ldots$,*

$$\langle P_{\Sigma(z_0,\tau)}\varphi_j, u_\epsilon^f(t)\rangle = \tag{53}$$
$$-i\epsilon^{(m+2)/4}\pi^{(m-2)/2}[det(-iH(t))]^{-1/2}u_{00}(t)\varphi_j(z_0,\tau)g^{1/2}(z_0,\tau) + O(e^{(m+6)/4}),$$

where $t = \tau + t_0$. Here $u_{00}(t)$ is given by formula (39) and $H(t)$ is given by formula (36).

Proof. We can compute the inner product of the eigenfunction and the projection of the Gaussian beam by using the boundary normal coordinates analogously to the proof of Lemma 17. The formula (53) can be obtained by applying stationary phase method. ◊

Proof. (of Lemma 21) Let us consider result of Lemma 22. Since we have already determined the manifold M and the metric on it, we have determined also the coefficients $u_{00}(t)$ and $H(t)$ which depend only on the metric g (and not on q). Hence from formula (44) we can find the values of absolute values of the eigenfunctions $|\varphi_j|$ on all normal geodesics $\gamma_{z,\nu}([0, \tau(z)))$. In particular, we can find values of $|\varphi_j|$ in a dense set, and since the eigenfunctions are continuous, we can find them on the whole manifold M.

To finish the reconstruction of q, let $h(x) = |\varphi_j(x)|$. Since φ_j vanish in nowhere dense set X, we can find for any r_x such that $\varphi_j(x) \neq 0$ the value of

$$q(x) = \frac{(\Delta_g + \lambda_j)\varphi_j(x)}{\varphi_j(x)} = \frac{(\Delta_g + \lambda_j)h(x)}{h(x)}$$

Since q is continuous and we know it in a dense set, we can find $q(x)$ for any given r_x. ◊

References

[1] Avdonin, S., Belishev, M., Rozhkov, Y.: A dynamic inverse problem for the nonselfadjoint Sturm-Liouville operator. Zap. Nauchn. Sem. (POMI) 250 (1998),7–21, (in Russian).

[2] Babich V., Ulin V.: The comlex space–time ray method and quasiphotons, Zap. Nauch. Semin. LOMI, 117(1981), 5-12 (in Russian)

[3] Belishev M.: On an approach to multidimensional inverse problems for the wave equation. Dokl. Akad. Nauk SSSR., 297(1987), 524–527 (in Russian).

[4] Belishev, M.: Wave bases in multidimensional inverse problems. Mat. Sb. 180(1989), 584–602, 720. (in Russian)
[5] Belishev, M., Glasman, A.: A dynamic inverse problem for the Maxwell system: reconstruction of the velocity in the regular zone (the BC-method). Algebra i Analiz 12(2000), 131–187.
[6] Belishev, M., Gotlib, V.: Dynamical variant of the BC-method: theory and numerical testing. J. Inverse Ill-Posed Probl. 7(1999), 221–240.
[7] Belishev, M., Isakov, V., Pestov, L., Sharafutdinov, V.: On the reconstruction of a metric from external electromagnetic measurements. Dokl. Akad. Nauk 372(2000), 298–300, (in Russian)
[8] Belishev M., Katchalov A.: Boundary control and quasiphotons in the problem of a Riemannian manifold reconstruction via its dynamical data. Zap. Nauchn. Semin. POMI, (203)1992, 21–50 (in Russian).
[9] Belishev M. I. Boundary control in reconstruction of manifolds and metrics (the BC-method), Inverse Probl. 13(1997), R1-R45.
[10] Belishev M., Kurylev Y.: Nonstationary inverse problem for the wave equation (in large), Zap. Nauchn. Semin. LOMI 165(1987), 21-30.
[11] Belishev M., Kurylev Y. To the reconstruction of a Riemannian manifold via its spectral data (BC–method). Comm. PDE, 17(1992), 767–804.
[12] Blagovestchenskii A.: On an inverse problem of seimic waves propagation. Problem of Math. Physics. Leningrag University, 1966, 68–81 (in Russian).
[13] Katchalov, A., Kurylev, Y.: Multidimensional inverse problem with incomplete boundary spectral data. Comm. Partial Differential Equations 23 (1998), 55–95.
[14] Katchalov, A., Kurylev, Y., Lassas, M.: Inverse boundary spectral problems. To appear on CRC-press.
[15] Kurylev, Y.: Admissible groups of transformations that preserve the boundary spectral data in multidimensional inverse problems. Dokl. Akad. Nauk 327 (1992), 322–325 (in Russian).
[16] Kurylev, Y.: Inverse boundary problems on Riemannian manifolds, Geometry of the spectrum, Contemp. Math., 173(1993), 181–192.
[17] Kurylev, Y.: Multi-dimensional inverse boundary problems by BC-method: groups of transformations and uniqueness results. Math. Comput. Modelling 18 (1993), 33–45.
[18] Kurylev, Y.: An inverse boundary problem for the Schrdinger operator with magnetic field. J. Math. Phys. 36 (1995), 2761–2776.
[19] Kurylev Y.: A multidimensional Gel'fand-Levitan inverse boundary problem, Differential Equations and Mathematical Physics (ed. I.Knowles), Intern. Press (1995), 117-131.
[20] Kurylev Y.: Multidimensional Gel'fand inverse problem and boundary distance map, Inverse Probl. Rel. Geometry (ed. H.Soga) (1997), 1-15.

[21] Kurylev, Y., Lassas, M.: The multidimensional Gel'fand inverse problem for non-self-adjoint operators. Inverse Problems 13 (1997), 1495–1501.

[22] Kurylev, Y., Lassas, M.: Hyperbolic inverse problem with data on a part of the boundary. Differential eq. and math. physics (Birmingham), AMS/IP Stud. Adv. Math., 16(2000), 259–272.

[23] Kurylev y., Lassas, M.: Gelf'and Inverse Problem for a Quadratic Operator Pensil. J. Funct. Anal., 176(2000), 247-263.

[24] Ralston, J.: Gaussian beams and the propagation of singularities. Studies in partial differential equations, 206–248, MAA Stud. Math., 23, 1982.

[25] Petersen, P.: Riemannian geometry. Springer-Verlag, 1998.

[26] Tataru D.: Unique continuation for the solutions to PDE's; between Hormander's theorem and Holmgren's theorem, Comm. Part. Diff. Eq. 20(1995), 855-884.

[27] Tataru D.: Boundary controllability for conservative PDE's, Appl. Math. Optim. 31 (1995), 257-295.