ENERGY MEASUREMENTS AND EQUIVALENCE OF BOUNDARY DATA FOR INVERSE PROBLEMS ON NON-COMPACT MANIFOLDS

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1. Introduction. Formulation of results.

1.1. The goal of this paper is to consider inverse problems with different types of boundary data given as boundary forms. In particular, the boundary measurements considered in this paper are related to the measurements of energy needed to force the boundary value of a physical field to a given one.

In various applications it is often not possible to measure the Cauchy data on the whole boundary, as one can not attach sources and measurement devices to the same locations. A perfect example is given by the seismic measurements when the boundary sources are often made of explosives. Similarly, in many cases it is difficult to measure both the amplitude and phase of a field. On the contrary, the total energy of a wave is often easily accessible and, in particular, it is possible to find the energy which is required to force the boundary value of the field to a given one, that is, to measure the energy needed to do a given measurement. In theoretical inverse problems this idea goes back to A. Calderón who in his 1980 seminal paper [Cl] considered the inverse problem for the conductivity equation from the point view of energy measurements. Similarly, the measurements based on energy or interference of waves have been used in many applications, e.g. in impedance tomography (see e.g. [CIN]) and near field optical tomography (see e.g. [SM]). The work of Calderón was extended by J. Sylvester and G. Uhlmann who developed a method of complex geometric optics to solve fully non-linear inverse problems [SU].

In the paper we consider also the question of the equivalence of different boundary data used in inverse problems. Our interest in the equivalence of inverse problems with various boundary data comes from the fact that there are numerous examples when the solution of a particular inverse problem is used to solve an inverse problem of another type. For example, A. Nachman, J. Sylvester, and G. Uhlmann [NSU] solved the inverse boundary spectral problem for a Schrödinger operator by reducing it to the inverse boundary value problem in fixed frequency and then using the method of the complex geometric optics. Similarly, inverse problems in the time-domain are often reduced to problems in the frequency domain (see e.g. [Is]). The equivalence is also useful for applications as it makes

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possible the use of reconstruction algorithms developed for some type of inverse problems to other inverse problems.

In this paper we concentrate mainly on the case of non-compact domains and manifolds and Robin boundary conditions. Our intention in doing so stems from the existence of many rather detailed expositions of these type of results for the compact case, e.g. [KaL], [KKLM] and, in particular, [KKL]. In addition, the non-compact case differs significantly from the compact one and we believe that it is worth a separate discussion. As [KKL] deals with the Dirichlet boundary condition we have decided to concentrate here on the Robin one.

1.2. We start with necessary definitions. Let M be a smooth, complete (possibly non-compact) m-dimensional Riemannian manifold with boundary ∂M and let $a(x,D)$ be an elliptic 2nd-order partial differential operator,

$$a(x,D)v(x) = -\Delta_g\, v(x) + \left(V(x), \nabla v(x)\right)_g + c(x)v(x), \tag{1}$$

where $\Delta_g = g^{-1/2}\partial_j g^{1/2} g^{jk}(x)\partial_k$ with $g = \det(g_{ij})$, $[g^{ij}] = [g_{ij}]^{-1}$ is the Laplace-Beltrami operator corresponding to the metric g_{ij}, V is a smooth real vector field (1st order operator) and c is a smooth real valued function. As usual we use Einstein's summation over repeated upper and lower indices.

Next we consider an operator A related to $a(x,D)$,

$$Av = a(x,D)v, \quad \mathcal{D}(A) = \{v \in H^2(M) :\ Bv|_{\partial M} = \partial_\nu v + \eta v|_{\partial M} = 0\}, \tag{2}$$

where ν is the exterior unit normal vector to ∂M and η is a smooth real valued function on ∂M. As usual $H^2(M)$ stands for the Sobolev space of functions having square integrable derivatives up to the second order. We assume that there is a smooth measure $d\mu$ on M,

$$d\mu = \rho\, dV_g, \quad dV_g = g^{1/2}\, dx^1 \cdots dx^m, \tag{3}$$

where dV_g is Riemannian volume on (M,g), so that A is self-adjoint with respect to $d\mu$. In particular,

$$\int_M vAu\, d\mu = \int_M uAv\, d\mu \tag{4}$$

for $u, v \in \mathcal{D}(A)$. Then

$$a(x,D)v(x) = -\rho^{-1} g^{-1/2}(\partial_i g^{1/2} g^{ij} \rho \partial_j v(x)) + q(x)v(x). \tag{5}$$

The fact that A of form (2), (5) is self-adjoint as well as just an invariant definition of the Sobolev spaces $H^s(M)$ puts some restrictions on $(M, g; \partial M)$, ρ, q, and η. For our purposes it is sufficient to assume that $(M, g; \partial M)$ is (possibly) a non-compact manifold of (finitely) bounded geometry and $\rho, \rho^{-1} \in C^\infty(M) \cap C^2_b(M)$, $q \in C^\infty(M) \cap C^0_b(M)$ and

$\eta \in C^\infty(\partial M) \cap C_b^1(\partial M)$. Here C_b^l is the class of functions having uniformly bounded derivatives upto order l. For the explanation of these facts as well the definition of a manifold of bounded geometry see Appendix.

As A is self-adjoint, it has a spectral resolution,

$$A = \int_{\mathbb{R}} \lambda dE(\lambda), \tag{6}$$

where $E(\lambda)$ is the spectral projector on the interval $(-\infty, \lambda]$.

Since $\mathcal{D}(A^n) \subset H^{2n}_{loc}(M)$ and $A \geq \lambda_0 I$ due to $q(x) \in C_b^0$, we see that for any $\lambda \in \mathbb{R}$ the projection $E(\lambda)$ is an infinitely smoothing operator, $E(\lambda) : H^s_{\text{comp}}(M) \to C^\infty(M)$ for any $s \in \mathbb{R}$. Thus $E(\lambda)$ has a smooth Schwartz kernel and there is a measure $dp(\cdot, \cdot; \lambda)$ on $M \times M$ such that

$$(u, E(\lambda)v)_{L^2(M,d\mu)} = \int_{M\times M} u(x)\overline{v(y)} d_{x,y}p(x, y; \lambda). \tag{7}$$

Here, we use the notation $d_{x,y}p(x, y; \lambda) = dp(x, y; \lambda)$ to indicate that integration variables are x and y and λ is a parameter. Clearly, the Radon-Nikodym derivative $dp_{x,y}(x, y; \lambda)/(d\mu(x)d\mu(y))$ is a real valued $C^\infty(M \times M)$-function for any λ. In particular, if M is compact and λ_j and φ_j are the eigenvalues and the normalized eigenfunctions of A, then

$$dp(x, y; \lambda) = \sum_{\lambda_j \leq \lambda} \varphi_j(x)\varphi_j(y) d\mu(x)d\mu(y).$$

Now we consider the measure $dP(x, y; \lambda)$ on $\partial M \times \partial M$,

$$dP_{x,y}(x, y; \lambda) = \frac{dp_{x,y}(x, y; \lambda)}{dV_g(x)dV_g(y)} dS_g(x)dS_g(y),$$

where dS_g is the Riemannian volume of the boundary.

Let $\delta_{\partial M}$ is the surface delta-measure with respect to dV_g. Then

$$\int_M h(x, y)\delta_{\partial M}(x)\delta_{\partial M}(y)dp(x, y; \lambda) = \int_{\partial M} h(x, y)dP(x, y; \lambda). \tag{8}$$

Remark 1. In the compact case, if the measure ρdS_g on the boundary is known, then λ_j and $\varphi_j|_{\partial M}$ determine $dP(x, y; \lambda)$. Operating as in [KKL], we also see that $dP(x, y; \lambda)$ determines λ_j and $\varphi_j|_{\partial M}$ upto a unitary transformation of the eigenfunctions corresponding to the same eigenvalue.

Therefore, it is natural to give the following generalization of the Gel'fand data to the case of a possibly non-compact manifold:

Definition 1. *We define the Gel'fand boundary spectral data of A to be the measure $dP(x, y; \lambda)$ given on $\partial M \times \partial M$ for all $\lambda \in \mathbb{R}$.*

Other objects on ∂M related to the spectral properties of A are the Robin-to-Dirichlet maps

$$\Lambda^z \phi = u_z^\phi|_{\partial M}, \tag{9}$$

where u_z^ϕ is the solution of the Robin problem

$$a(x,D)u_z^\phi = zu_z^\phi, \quad Bu_z^\phi|_{\partial M} = \phi \tag{10}$$

where $z \in \mathbb{C}$ is not an eigenvalue.

DEFINITION 2. *We define the Calderon-Gel'fand boundary form $\Lambda^z[\phi,\psi]$ related to problem (10) by the formula*

$$\Lambda^z[\phi,\psi] = \int_{\partial M} \Lambda^z\phi(x)\,\overline{\psi(x)}\,\rho(x)dS_g \tag{11}$$

which is equivalent to a Dirichlet-type form for u_z^ϕ and u_z^ψ,

$$\Lambda^z[\phi,\psi] = \int_{\partial M} \eta u_z^\phi \overline{u_z^\psi}\rho dS_g + \int_M \left((\nabla u_z^\phi, \nabla u_z^\psi)_g + (q-\overline{z})u_z^\phi\overline{u_z^\psi}\right)\rho dV_g, \tag{12}$$

where $(\nabla u, \nabla v)_g = g^{ij}\partial_i u\overline{\partial_j v}$ (compare with e.g. with [Cl], [SU]).

1.3. We consider also the hyperbolic initial boundary value problem corresponding to the elliptic operator A,

$$\begin{gathered} p_t^2 + a(x,D))u^f(x,t) = 0 \quad \text{in} \quad M\times\mathbb{R}_+, \\ Bu^f|_{\partial M\times\mathbb{R}_+} = f \in C_0^\infty(\partial M\times\mathbb{R}_+), \quad u^f|_{t=0}=0, \quad \partial_t u^f|_{t=0} = 0. \end{gathered} \tag{13}$$

For initial boundary value problem (13) we define the non-stationary Robin-to-Dirichlet map (response operator) Λ,

$$\Lambda f = u^f|_{\partial M\times\mathbb{R}_+}. \tag{14}$$

The operator Λ gives rise to the hyperbolic form $\mathcal{B}[f,h]$,

$$\begin{aligned} \mathcal{B}[f,h] &= \int_0^\infty \int_{\partial M} \left(\partial_\nu u^f\,\overline{u^h} - u^f\,\overline{\partial_\nu u^h}\right)\rho dS_g\,dt \\ &= \int_0^\infty \int_{\partial M} \left(f\overline{\Lambda h} - \Lambda f\overline{h}\right)\rho dS_g\,dt. \end{aligned} \tag{15}$$

There is a natural concept of energy for the wave equation (13) given by

$$\begin{aligned} E(u,t) &= \frac{1}{2}\int_{\partial M} \eta(x)|u(x,t)|^2\rho(x)dS_g \\ &+ \frac{1}{2}\int_M \left(|\nabla u(x,t)|_g^2 + q(x)|u(x,t)|^2 + |\partial_t u(x,t)|^2\right)d\mu(x). \end{aligned} \tag{16}$$

When $f \in C_0^\infty(\partial M\times[0,T])$ we see that $E(u^f,t)$ is constant for $t>T$, that is, the energy is conserved. Therefore, the energy $E(u^f,T)$ is brought into M through the boundary $\partial M\times\mathbb{R}_+$. We define the total energy flux $\Pi(f)$ through the boundary as

$$\Pi(f) = \lim_{t\to\infty} E(u^f,t). \tag{17}$$

Actually, $\Pi(f)$ is given by a quadratic form of f,

$$\Pi(f) = \text{Re} \int_{\partial M} \int_0^\infty f(x,t) \partial_t \overline{\Lambda f(x,t)}\, \rho(x) dS_x dt. \tag{18}$$

Because differential equation (13) is translation invariant in time, we extend the map Λ, the form $\mathcal{B}$, and the energy flux Π to $f \in C_0^\infty(\partial M \times \mathbb{R})$ by setting e.g. $\Lambda f(x,t) = (\Lambda(f(\cdot\,, \cdot - T)))(x, t+T)$.

1.4. Before formulating inverse problems, we need to introduce the notion of a gauge-transformation. We will consider all operators and boundary data in such a way that our considerations do not depend on a particular choice of the scale of measurements. For instance, if the change of the scale of measurements is described by a function $\kappa(x)$, $\kappa|_{\partial M} = 1$, that is at a point $x \in M$ the physical quantity $u(x)$ is replaced with $\kappa(x)u(x)$, this change of the scale of measurements does not affect the physical model or the measurements of this quantity on ∂M. However, it does change its mathematical description, i.e., the differential equation which describes the process. For this reason we formulate all our statements so that they are invariant in gauge transformations $u(x) \to \kappa(x)u(x)$.

Definition 3. *Let $\kappa \in C^\infty(M)$, $\kappa(x) \geq \kappa_0 > 0$ for $x \in M$. The gauge transformation generated by the function κ is the transformation*

$$S_\kappa : L^2(M, d\mu) \to L^2(M, d\mu_\kappa), \quad S_\kappa u(x) = \kappa(x)u(x),$$

with $d\mu_\kappa = \kappa^{-2}(x)d\mu$. If $\kappa|_{\partial M} = 1$ the gauge transformation S_κ is normalized on ∂M. Each gauge transformation determines the corresponding gauge transformation A_κ of the operator A,

$$A_\kappa u = \kappa A(\kappa^{-1}u).$$

If A is an elliptic differential operator in $L^2(M, d\mu)$ of form (2), (5) with $\rho, \rho^{-1} \in C_b^2(M)$, $q \in C_b^0(M)$ and $\eta \in C_b^1(\partial M)$, then A_κ is also an elliptic differential operator in $L^2(M, d\mu_\kappa)$ of form (2), (5) with ρ_κ, η_κ and q_κ from the same classes as soon as $\kappa, \kappa^{-1} \in C_b^2(M)$. Furthermore,

$$\mathcal{D}(A_\kappa) = \{v \in H^2(M) : \; B_\kappa v|_{\partial M} = \partial_\nu v + \eta_\kappa v|_{\partial M} = 0\}, \quad \eta_\kappa = \eta + \kappa^{-1}\partial_\nu \kappa,$$

and $a_\kappa(x, D)u(x)$ is given by the formula

$$a_\kappa(x, D)u(x) = \kappa^{-1}(x) a(x, D)(\kappa(x)^{-1}u(x)).$$

The gauge transformations $S_\kappa : L^2(M) \to L^2(M)$ parametrized by κ : $\kappa \in C^\infty(M)$, $\kappa, \kappa^{-1} \in C_b^2(M)$ form an Abelian group $\mathcal{G}$ with respect to composition

$$S_{\kappa_1} \circ S_{\kappa_2} = S_{\kappa_1 \kappa_2}.$$

The action of this group on the set of the second order elliptic differential operators is given by $S_\kappa(A) = A_\kappa = \kappa A \kappa^{-1}$. For any A

$$\sigma A = \{S_\kappa(A) : \quad S_\kappa \in \mathcal{G}\}$$

is the orbit of the group $\mathcal{G}$ through A. The gauge transformations normalized on ∂M form a subgroup $\mathcal{G}_{\partial M}$ and the corresponding orbit is denoted by $\sigma_{\partial M} A$.

Although gauge transformations change $a(x, D)$, the metric tensor $g^{ij} = a^{ij}$ associated with the operator A remains invariant, $g^{ij}_\kappa = g^{ij}$.

An important fact related to the gauge transformations is that any orbit σA of a self-adjoint operator A of form (2), (5) contains a unique Schrödinger operator which is called the Schrödinger operator corresponding to A.

Lemma 1. *i. Let A be an elliptic differential operator of form (2), (5). There is a unique Schrödinger operator $-\Delta_g + \widetilde{q}$ in the orbit σA, that is, for a given A there is a unique κ such that $A = \kappa(-\Delta_g + \widetilde{q})\kappa^{-1}$ and $d\mu = \kappa^{-2} dV_g$.*
ii. A is a Schrödinger operator if and only if $d\mu = dV_g$.

Proof. The assertion is proven in [KKL], see also [K1], [KK]. The basic idea is to consider the corresponding Dirichlet quadratic form in M and observe that a gauge transformation is equivalent to changing the measure in this form. □

In gauge transformations, the hyperbolic form $\mathcal{B}$, energy flux Π and Calderon-Gel'fand form Λ^z are also changed. Indeed, if for example $u^f(x,t)$ is a solution of problem (13) for the operator $a(x, D)$, then $v(x,t) = \kappa(x) u^f(x,t)$ is the solution of the problem

$$(19)\qquad \begin{aligned} &(\partial_t^2 + a_\kappa(x, D)) v(x,t) = 0 \quad \text{in} \quad M \times \mathbb{R}_+, \\ &B_\kappa v|_{\partial M \times \mathbb{R}_+} = \kappa f, \quad v|_{t=0} = 0, \ \partial_t v|_{t=0} = 0. \end{aligned}$$

Thus, if $\mathcal{B}, \Pi$ correspond to the operator A and $\mathcal{B}_\kappa, \Pi_\kappa$ – to its gauge transformation A_κ then

$$(20)\qquad \Lambda_\kappa[f, h] = \Lambda[\kappa f, \kappa h], \quad \Pi_\kappa[f, h] = \Pi[\kappa f, \kappa h].$$

Similarly, if Λ^z and $dP(x, y, \lambda)$ are the Calderon-Gel'fand forms and boundary spectral data of A and Λ^z_κ, $dP_\kappa(x, y, \lambda)$ – of A_κ, then

$$(21)\quad dP_\kappa(x, y; \lambda) = \kappa(x)^{-1} \kappa(y)^{-1} dP(x, y; \lambda), \quad \Lambda^z_\kappa[\phi, \psi] = \Lambda^z[\kappa\phi, \kappa\psi].$$

The forms Λ^z and Λ^z_κ, $\mathcal{B}$ and $\mathcal{B}_\kappa$, etc. which satisfy (20), (21) are called gauge equivalent.

1.5. Now we are in the position to formulate various inverse problems on M related to the above concepts.

Inverse problems. Determine (M, g) and A upto a normalized gauge transformation, i.e., determine $\sigma_{\partial M} A$ when we are given one of the following data:

i. The Gel'fand boundary spectral data $dP(x, y; \lambda)$ on $\partial M \times \partial M$ for all $\lambda \in \mathbb{R}$.
ii. The Calderon-Gel'fand forms Λ^z for all $z \in \mathbb{C} \setminus \sigma(A)$, where $\sigma(A)$ is the spectrum of A.
iii. The hyperbolic form $\mathcal{B}$.
iv. The energy flux Π.

Our aim is to show that, in particular, the energy measurements can be used to obtain the other data. Moreover, we will show:

THEOREM 1. *Inverse problems i.-iv. are equivalent, i.e., any of the data i.-iv. determine all other data.*

Thus all inverse problems *i.-iv.* can be reduced to solving one of them. Instead of solving problem *iii.* we will solve a more general problem which also answers the above problems. To formulate this more general problem denote by $\mathcal{B}^{2T}$ the hyperbolic form $\mathcal{B}$ restricted to the set of sources $f, h \in C_0^\infty(\partial M \times (0, 2T))$. We will show that

THEOREM 2. *Assume that we are given the hyperbolic form $\mathcal{B}^{2T}$ of an operator A of form (2), (5). This data determines uniquely the manifold $M^T = \{x \in M : \ d(x, \partial M) < T\}$ and the metric tensor g on M^T. Moreover, we can find $a(x, D)$ on M^T upto a normalized gauge transformation, i.e., we can find the orbit*

$$\sigma_{\partial M} A|_{M^T} = \{a_\kappa(x, D)|_{M^T} : \ \kappa > 0, \ \kappa|_{\partial M} = 1\}.$$

In particular, if any of data $i. - iv.$ is given, it is possible to determine the whole manifold (M, g) and the orbit $\sigma_{\partial M} A$. If, in addition, we have a priori knowledge about the structure of the operator, we can in many cases solve the inverse problem uniquely. For instance, we have:

COROLLARY 1. *Let $M \subset \mathbb{R}^m$ is given. Assume the metric g to be conformally Euclidean, that is $g_{jk}(x) = \sigma(x)\delta_{jk}$ where $\sigma(x) > 0$. Moreover, assume that we know any of the data i.-iv. for a Schrödinger operator $A = -\Delta_g + q$. Then we can determine g, q and η uniquely.*

At last we consider the case when the data is given only on an open subset $S \subset \partial M$. In this case we can define the Gel'fand boundary spectral data dP_S

$$dP_S(x, y; \lambda) = dP(x, y; \lambda), \quad x, y \in S, \tag{22}$$

the Calderon-Gel'fand form Λ_S^z

$$\Lambda_S^z[f, h] = \Lambda^z[f, h], \quad f, h \in C_0^\infty(S), \tag{23}$$

the hyperbolic boundary forms $\mathcal{B}_S$ and even $\mathcal{B}_S^{2T}$

$$\mathcal{B}_S^{2T}[f, h] = \mathcal{B}[f, h], \quad f, h \in C_0^\infty(S \times (0, 2T)), \tag{24}$$

and the energy flux Π_S,

$$\Pi_S(f) = \Pi(f), \quad f \in C_0^\infty(S \times \mathbb{R}_+). \tag{25}$$

Then the analogs of Theorems 1 and 2 remain valid for dP_S, Λ_S^z, etc:

THEOREM 3. *i. Assume that we are given $S \subset \partial M$. Then any data (22)–(25) determine all others.*
ii. Assume that we are given $S \subset \partial M$ and $\mathcal{B}_S^{2T}$. Then this data determine uniquely the manifold $M(S,T) = \{x \in M : \ d(x,S) < T\}$, the metric tensor g and the operator $a(x,D)$ on $M(S,T)$ upto a gauge transformation normalized on S, i.e., we can find $\sigma_S A|_{M(S,T)}$.

1.6. This paper gives a concise review of some of the results obtained in the multidimensional inverse boundary value problems, especially those regarding the equivalence of various types of data, gauge equivalence and also reconstruction procedures and uniqueness for hyperbolic inverse problems. The used techniques are based on various variants of the BC-method (for the original paper see [B1]). There are currently several extended expositions of this method, e.g. [B2], [KK], [KaL]. The monograph [KKL] is particularly close to our exposition in this paper and we refer the interested reader to this monograph for further details. Having said so we should stress that the majority of works on the BC-method deal with the case of inverse boundary problems on compact manifolds. More precisely, due to the local in time nature of the method, the treatment of hyperbolic inverse problems is essentially the same for compact and non-compact cases. This makes possible to closely follow in our proof of Theorem 2 the method described in [KKL], Ch. 4.2. An alternative approach to hyperbolic inverse problems also based on the BC-method and technique of Gaussian beams is given in [BKa] which deals with the wave equation for the Laplace operator on a Riemannian manifold. However, when coming to inverse spectral problems, i.e., problems *i.* and *ii.*, non-compact manifolds differ rather significantly from the compact ones due to a more complicated nature of the spectral properties of elliptic operators on non-compact manifolds. To our knowledge the only paper where the BC-method is applied to an inverse boundary spectral problem on a non-compact domain is [BKu1] where $M = \mathbb{R}^m_+$. In particular, the definition of the boundary spectral data (Definition 1) differs from that for the compact case. Moreover, the proof of the equivalence, although ideologically close to that in [KKL], Ch. 4.1 and [KKLM], implies some technical ideas absent in [KKL]. There are some other differences in our exposition as compared to the previous ones. For example, in the proof of Lemma 4 dealing with the inner products of waves we use variational technique which, we believe, is more appropriate for the numerical realization of the method. Furthermore, the step by step reconstruction of the manifold from a part of the boundary is based on the direct continuation of Green's function for the wave equation which, to our knowledge, has been unknown. And, of course, we deal with boundary forms rather then the corresponding operators. We believe that the invariance properties possessed by the forms better reflect the nature of the problem. This approach is essentially similar to that in, e.g. [SU], [LU], [Sy] which use differential forms rather then functions.

The plan of the paper is as follows: In section 2 we prove Theorem 1 about the equivalence of data *i.-iv.* in the case of the whole boundary, $S = \partial M$. In section 3 we describe the procedure of the reconstruction of a Riemannian manifold (M^T, g) and the Schrödinger operator on it from the hyperbolic form $\mathcal{B}^{2T}$ and give the proof of Theorem (2). An alternative approach based on Gaussian beams is given in Section 4. Section 5 is devoted to the generalization of the above results to the case $S \neq \partial M$. At last, in Appendix we collect some necessary results about properties of manifolds of bounded geometry and elliptic operators on such manifolds. Our exposition is rather concise, especially in Sections 4 and 5 and when the construction used is similar to those for the compact case. Nevertheless, we provide (at least brief) proofs of the main ingredients of the method, namely the Blagovestchenskii identity and controllability results.

2. Proof of equivalence of the boundary data.

2.1. We start with the observation that since $a(x, D)$ of form (5) is real,

$$\overline{u_z^\phi} = u_{\overline{z}}^{\overline{\phi}}, \quad \overline{u^f(x,t)} = u^{\overline{f}}(x,t), \quad (A - \overline{z})^{-1}\overline{\Phi} = \overline{(A-z)^{-1}\Phi}, \tag{26}$$

where $z \notin \sigma(A)$ and u_z^ϕ and $u^f(x,t)$ are solutions of problems (10) and (13), correspondingly. This allows us to work not with the inner product in $L^2(M)$ but with the complex bilinear pairing $\langle \cdot, \cdot \rangle$, namely, $\langle \Phi, \Psi \rangle = \int_M \Phi(x)\Psi(x) d\mu$, or the corresponding distribution duality. Using (26) we see that the forms $\Lambda^z[\phi, \psi]$, $\mathcal{B}[f, h]$ and $\Pi[f]$ determine complex bilinear forms

$$\begin{aligned} \Lambda_{\mathbb{C}}^z[\phi,\psi] &= \int_{\partial M} (\Lambda^z\phi)(x)\psi(x)\,\rho(x) dS_g = \Lambda^z[\phi, \overline{\psi}] \\ &= \int_{\partial M} \eta u_z^\phi u_z^\psi \rho dS_g + \int_M \left(\langle \nabla u_z^\phi, \nabla u_z^\psi\rangle_g + (q-z) u_z^\phi u_z^\psi\right) \rho dV_g, \end{aligned} \tag{27}$$

where $\langle \nabla u, \nabla v\rangle_g = g^{jk}\partial_j u\, \partial_k v$,

$$\mathcal{B}_{\mathbb{C}}[f,h] = \int_0^\infty \int_{\partial M} (f\Lambda h - \Lambda f h)\, \rho dS_g\, dt = \mathcal{B}[f, \overline{h}], \tag{28}$$

$$\Pi_{\mathbb{C}}[f,h] = \frac{1}{2}\int_{\partial M}\int_0^\infty \left(f(t)\partial_t \Lambda h(t) + \partial_t \Lambda f(t) h(t)\right) \rho dS_x dt \tag{29}$$

with

$$\Pi_{\mathbb{C}}[f,h] = \lim_{t\to\infty} E_{\mathbb{C}}[u^f(t), v^h(t)],$$

where

$$\begin{aligned} E_{\mathbb{C}}[u(t), v(t)] &= \frac{1}{2}\int_{\partial M} \eta u(t) v(t)\, \rho dS_g \\ &+ \frac{1}{2}\int_M \left(\langle \nabla u(t), \nabla v(t)\rangle_g + q u(t) v(t) + \partial_t u(t) \partial_t v(t)\right) d\mu. \end{aligned} \tag{30}$$

In the following, we will prove the equivalence for the complex forms. Clearly, due to (27)–(30) this will prove Theorem 1.

2.2. Spectral data. We start with the equivalence of the Gel'fand boundary data $dP(x,y;\lambda)$ and Calderon-Gel'fand forms $\Lambda^z_{\mathbb{C}}$.

$\underline{i. \to ii.}$ Consider the complex bilinear form

$$\langle (A-z)^{-1}\Phi, \Psi\rangle_{L^2(M)}, \quad z \notin \sigma(A). \tag{31}$$

As $H^{-1}(M) = \mathcal{D}((A+c)^{1/2})$, this form can be continued onto $H^{-1}(M)$, in particular, onto Ψ, Φ of the form

$$\Psi = \psi(x)\delta_{\partial M}(x), \;\; \Phi = \phi(x)\delta_{\partial M}(x), \tag{32}$$

where ψ, $\phi \in C_0^\infty(\partial M)$ or, more generally, in $H^{-1/2}(\partial M)$. Next we use the fact that

$$u_z^\phi = (A-z)^{-1}\Phi = \int_{\mathbb{R}} \frac{d_\lambda E(\lambda)\Phi}{\lambda - z}. \tag{33}$$

Then

$$\Lambda^z_{\mathbb{C}}[\phi,\psi] = \langle (A-z)^{-1}\Phi, \Psi\rangle_{L^2(M)} = \int_{\sigma(A)} \frac{dm_{\phi,\psi}(\lambda)}{\lambda - z}, \tag{34}$$

where $m_{\phi,\psi}(\lambda)$ is the complexified spectral measure

$$m_{\phi,\psi}(\lambda) = \langle E(\lambda)\Phi,\, \Psi\rangle = \big(E(\lambda)\Phi,\, \overline{\Psi}\big) = \int_{\partial M} \phi(y)\psi(x)\, dP(x,y;\lambda). \tag{35}$$

Thus the boundary spectral data uniquely determine the Calderon-Gel'fand forms, i.e., $i.$ determines $ii.$

$\underline{ii. \to i.}$ To prove the opposite we use the Pleijel-Stone formula (e.g. [Ko], [RS]) which together with (33)–(35) shows that

$$\begin{aligned} &\lim_{\varepsilon\to+0} \frac{1}{2\pi i}\int_{-\infty}^{\lambda} \langle [(A-k-i\varepsilon)^{-1} - (A-k+i\varepsilon)^{-1}]\Phi, \Psi\rangle_{L^2(M)} dk \\ &\qquad = \frac{1}{2}\int_{\partial M} \phi(x)\psi(x)\,(dP(x,y;\lambda-0) + dP(x,y;\lambda)), \end{aligned} \tag{36}$$

where $dP(x,y;\lambda-0) = \lim_{\varepsilon\to 0-} dP(x,y;\lambda-\varepsilon)$ is considered as a distribution limit. Because $\phi, \psi \in C_0^\infty(\partial M)$ are arbitrary, equations (34), (36) imply that the forms $\Lambda^z_{\mathbb{C}}$ determine $dP(x,y;\lambda-0) + dP(x,y;\lambda)$. Since the spectral projectors $E(\lambda)$ are continuous from the right we see that, in the sense of distributions,

$$\begin{aligned} dP(x,y;\lambda) &= \lim_{\varepsilon\to+0} dP(x,y;\lambda+\varepsilon) \\ &= \frac{1}{2}\lim_{\varepsilon\to+0}(dP(x,y;\lambda+\varepsilon) + dP(x,y;\lambda+\varepsilon-0)), \end{aligned}$$

Thus we can also determine $dP(x, y; \lambda)$, i.e., *ii.* determines *i.*

2.3. Hyperbolic data. Next we return to the time-domain problem. For $f \in C_0^\infty(\partial M \times \mathbb{R})$, let $\widehat{f}$ be its Fourier transform with respect to time,

$$\widehat{f}(x, k) = \int_{\mathbb{R}} e^{-ikt} f(x, t)\, dt.$$

If $f = 0$ for $|t| \geq c_0$, the function $\widehat{f}(x, k)$ is analytic in $k \in \mathbb{C}$ and $C^\infty-$smooth in x. Moreover, by the Paley-Wiener theorem this functions satisfies

$$||\widehat{f}(\cdot\,, k)||_{C^m(\partial M)} \leq C_{m,N}(1 + |k|)^{-N} \exp(c_0|\operatorname{Im} k|), \quad m, N > 0. \tag{37}$$

The solution $u^f(x, t)$ of problem (13) lies in $C^\infty(M \times \mathbb{R})$ and, using spectral resolution (6),

$$u^f(t) = \int_{\mathbb{R}} \int_{-\infty}^{t} \frac{\sin \sqrt{\lambda}(t - t')}{\sqrt{\lambda}}\, d_\lambda(E(\lambda)F(t'))\, dt', \tag{38}$$

where $F(x, t) = f(x, t)\delta_{\partial M}(x)$. Using this representation,

$$\|u^f(t)\|_{L^2(M)} \leq C_f e^{\tau_0 t}, \quad t > 0, \tag{39}$$

where

$$\tau_0 = \sqrt{\max(0, -\lambda_0)}, \tag{40}$$

and λ_0 is the bottom of the spectrum $\sigma(A)$. Then the Fourier transform $\widehat{u^f}(x, k)$ of $u^f(x, t)$ is well-defined for $\operatorname{Im} k < -\tau_0$ and is there the solution of elliptic problem (10) with $z = k^2$ and $\phi(x) = \widehat{f}(x, k)$. Thus, when $\operatorname{Im} k < -\tau_0$

$$\widehat{\Lambda f}(k) = \widehat{u^f}(k)|_{\partial M} = \Lambda^{k^2} \widehat{f}(k). \tag{41}$$

However, the right-hand side of (41) is analytic when $k^2 \notin \sigma(A)$ which determines an analytic continuation of $\widehat{\Lambda f}(k)$ onto $k \in \mathbb{C}$, $k^2 \notin \sigma(A) \subset [\lambda_0, \infty) \subset \mathbb{R}$.

After these preparations we can show that the hyperbolic data *iii.* and *iv.* are equivalent to each other and to *i.*

<u>*i.* $\to$ *iii.*</u> We will show that $\int_0^\infty \int_{\partial M} \Lambda f\, h\, \rho dS_g\, dt$, where $f, h \in C_0^\infty(\partial M \times \mathbb{R})$, may be represented in terms of the Gel'fand boundary spectral data $dP(x, y; \lambda)$. Due to (15) and (28) this will prove that *i.* determines *iii.* Indeed, by the Parseval identity and formula (41)

$$\begin{aligned} \int_0^\infty \int_{\partial M} \Lambda f\, h\, \rho dS_g\, dt &= \langle e^{-\tau t} \Lambda f, e^{\tau t} h \rangle_{L^2(\partial M \times \mathbb{R}_+)} \\ &= \int_{\mathbb{R}} \Lambda_{\mathbb{C}}^{(k - i\tau)^2} [\widehat{f}(k - i\tau), \widehat{h}(-k + i\tau)]\, dk. \end{aligned} \tag{42}$$

Using formulae (34), (35) we obtain from (42) that

$$\begin{aligned}(43)\qquad &\int_0^\infty \int_{\partial M} \Lambda f h\, \rho dS_g\, dt \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \int_{\partial M} \int_{\partial M} \frac{\widehat{f}(x, k-i\tau)\widehat{h}(y, -k+i\tau)}{\lambda - (k-i\tau)^2}\, d_{x,y,\lambda} P(x, y; \lambda) \right] dk.\end{aligned}$$

Hence $dP(x, y; \lambda)$ determines the forms Λ and $\mathcal{B}$.

<u>iii. → iv.</u> Let $f \in C_0^\infty(\partial M \times \mathbb{R}_+)$. It follows from (30) that

$$\partial_t E_{\mathbb{C}}(u^f, t) = \int_{\partial M} f(t) \partial_t \Lambda f(t)\, \rho dS.$$

As $\partial_t \Lambda f = \Lambda \partial_t f$, integrating by parts and using definitions (15), (17) we obtain

$$(44)\qquad \Pi_{\mathbb{C}}(f) = \int_0^\infty \int_{\partial M} f(t) \partial_t \Lambda f(t)\, \rho dS dt = \frac{1}{2} \mathcal{B}_{\mathbb{C}}[f, \partial_t f],$$

which implies that the hyperbolic form determines the energy flux.

<u>iv. → i.</u> Let us consider the form $\Pi_{\mathbb{C}}[f, h]$ when f, h are of the form

$$(45)\qquad f(x, t) = f_0(t)\phi(x), \quad h(x, t) = h_0(t)\psi(x),$$

with $f_0, h_0 \in C_0^\infty(\mathbb{R})$ and $\phi, \psi \in C_0^\infty(\partial M)$.

We intend to use the Parseval formula as in (42) to represent the integral in the rhs of (29) in terms of the Fourier transforms $\widehat{f}$, $\widehat{h}$. When f, h are of form (45), then $\widehat{f}(x, k) = \widehat{f}_0(k)\phi(x)$, $\widehat{h}(x, k) = \widehat{h}_0(k)\psi(x)$ and satisfy (37). Thus using partial integration we see that

$$\Pi_{\mathbb{C}}[f, h] = -\frac{1}{2} \int_0^\infty \int_{\partial M} \left[e^{\tau t} \partial_t f(t) (e^{-\tau t} \Lambda h(t)) + (e^{-\tau t} \Lambda f(t)) e^{\tau t} \partial_t h(t) \right] \rho dS_x dt.$$

As Λ^{k^2} is analytic for $k^2 \notin \sigma(A)$, applying the Parseval formula we see that

$$\begin{aligned}(46)\qquad \Pi_{\mathbb{C}}[f, h] &= \frac{1}{4\pi i} \int_{\mathbb{R}} \int_{\partial M} (k+i\tau) \langle \widehat{f}(k+i\tau), \widehat{\Lambda h}(-k-i\tau) \rangle_{L^2(\partial M)} dk \\ &\quad - \frac{1}{4\pi i} \int_{\mathbb{R}} \int_{\partial M} (k-i\tau) \langle \widehat{\Lambda f}(k-i\tau), \widehat{h}(-k+i\tau) \rangle_{L^2(\partial M)} dk\end{aligned}$$

Using the above formulae together with (34) and (41) we obtain that

$$\begin{aligned}(47)\qquad \Pi_{\mathbb{C}}[f, h] &= \frac{1}{4\pi i} \int_{\Gamma_\tau} \Lambda_{\mathbb{C}}^{k^2}[\phi, \psi] \widehat{f}_0(k)\, \widehat{h_0}(-k) k dk \\ &= \frac{1}{4\pi i} \int_{\Gamma_\tau} \langle (A - k^2)^{-1} \Phi,\, \Psi \rangle_{L^2(M)} \widehat{f}_0(k) \widehat{h_0}(-k)\, k dk,\end{aligned}$$

where Γ_τ is the boundary of the strip $\{k \in \mathbb{C} : \ |\operatorname{Im} k| \le \tau\}$ and Φ, Ψ are given by formula (32). Because $|\langle (A-k^2)^{-1}\Phi,\ \Psi\rangle| \le c_{\phi,\psi}$ for $k \in \Gamma_\tau$ when $\tau > \tau_0$ (40) and $\Phi, \Psi \in H^{-1}(M)$ while $\widehat{f_0}(k), \widehat{h_0}(-k) \in \mathcal{S}(\mathbb{R})$, integral (47) converges absolutely. Using again (34) we see that

$$\Pi_{\mathbb{C}}[f,h] = \frac{1}{4\pi i}\int_{\sigma(A)} \left[\int_{\Gamma_\tau} \widehat{f_0}(k)\widehat{h_0}(-k)\,\frac{kdk}{\lambda - k^2}\right] dm_{\phi,\psi}(\lambda), \tag{48}$$

Further considerations are based on the following lemma.

Lemma 2. *For any z, $Im\, z > \tau_0$ there is a sequence $f_0^n \in C_0^\infty(\mathbb{R})$ such that*

$$\lim_{n\to\infty} \Pi_{\mathbb{C}}[f^n] = \frac{1}{2}\int_{\sigma(A)} \frac{dm_\phi(\lambda)}{\lambda - z^2}, \tag{49}$$

where $f^n = f_0^n(t)\phi(x)$ and $dm_\phi(\lambda) = dm_{\phi,\phi}(\lambda)$.

Proof. i. We note that if $f_0(t) = h_0(t) = \exp{(izt)}H(t)$, where $H(t)$ is the Heaviside function, then $\widehat{f_0}(k)\widehat{h_0}(-k) = (k^2 - z^2)^{-1}$. Therefore, using formally (48) we obtain by means of the residue theorem that

$$\Pi_{\mathbb{C}}[f] = \frac{1}{2}\int_{\sigma(A)} \frac{dm_\phi(\lambda)}{\lambda - z^2}.$$

Next, let $u^f(t)$, $t > 0$ solve (13) with $f = H(t)\exp(izt)\phi(x)$. Then,

$$u^f(x,t) = \exp{(izt)} u^\phi_{z^2}(x) + w(x,t), \tag{50}$$

where $u^\phi_{z^2} = (A - z^2)^{-1}\Phi$ exists since $\operatorname{Im} z > \tau_0$ and w satisfies $(\partial_t^2 + A)w = 0$, $w|_{t=0} = -u^\phi_{z^2}$, $w_t|_{t=0} = -izu^\phi_{z^2}$. Using spectral resolution (6) and formula (33), we obtain the representation

$$\begin{aligned} w(t) &= -\int_{\mathbb{R}} W(\lambda,z,t)d_\lambda(E(\lambda)u^\phi_{z^2}), \\ W(\lambda,z,t) &= \cos(\sqrt{\lambda}t) + iz\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} \end{aligned} \tag{51}$$

Since by (34),

$$d_\lambda\langle E(\lambda)u^\phi_{z^2}, u^\phi_{z^2}\rangle = \frac{d_\lambda\langle E(\lambda)\Phi,\Phi\rangle}{(\lambda - z^2)^2} = \frac{dm_\phi(\lambda)}{(\lambda - z^2)^2},$$

formula (51) implies that

$$\begin{aligned} E_{\mathbb{C}}(w,t) &= \frac{1}{2}\int_{\sigma(A)} \left[\lambda W^2 + (\partial_t W)^2\right] d_\lambda\langle E(\lambda)u^\phi_{z^2}, u^\phi_{z^2}\rangle \\ &= \frac{1}{2}\int_{\sigma(A)} \frac{dm_\phi(\lambda)}{\lambda - z^2}. \end{aligned} \tag{52}$$

In particular, $E_{\mathbb{C}}(w,t) = E_{\mathbb{C}}(w,0)$. Moreover, it follows from (51) that

$$\|w(t)\|^2_{H^1(M)} + \|\partial_t w(t)\|^2_{L^2(M)} \le c_\phi \exp(2\tau_0 t), \quad \text{when } t > 0, \tag{53}$$

while

$$\|\exp(izt)u^\phi_{z^2}\|^2_{H^1(M)} + \|\partial_t(\exp(izt)u^\phi_{z^2})\|^2_{L^2(M)} \le c_\phi \exp[-2(\operatorname{Im} z)t]. \tag{54}$$

Because $\operatorname{Im} z > \tau_0$, representation (50) together with (52)–(54) imply that $E_{\mathbb{C}}(u^f,t)$ is defined for $t \ge 0$ and

$$\Pi_{\mathbb{C}}[f] = \lim_{t\to\infty} E_{\mathbb{C}}(u^f,t) = E_{\mathbb{C}}(w,0). \tag{55}$$

Comparing formulae (52), (55) and (34) we see that

$$\Pi_{\mathbb{C}}[f] = \lim_{t\to\infty} E_{\mathbb{C}}(u^f,t) = \frac{1}{2}\Lambda^{z^2}_{\mathbb{C}}[\phi,\phi].$$

Next we consider $u_{\varepsilon,N}(x,t)$ that is the solution of (13) with $f = f_{\varepsilon,N}(t)\phi(x)$, where $f_{\varepsilon,N}(t)$ is given by

$$f_{\varepsilon,N}(t) = f_\varepsilon(t)\widetilde{\chi}_N(t), \;\; f_\varepsilon(t) = (H(\cdot\,)\exp(iz\cdot\,) * \chi_\varepsilon)(t).$$

Here $*$ stands for the convolution in time, $\chi_\varepsilon(t)$ is a usual mollifier with $\operatorname{supp}(\chi_\varepsilon) \subset (-\varepsilon,\varepsilon)$ and $\widetilde{\chi}_N$ is a smooth cut-off function, $\widetilde{\chi}_N = 1$ for $t \le N$ and 0 for $t \ge N+1$. Then

$$u_{\varepsilon,N}(x,t) = f_{\varepsilon,N}(t)u^\phi_{z^2}(x) + w^1_\varepsilon(x,t) + w^2_{\varepsilon,N}(x,t), \tag{56}$$

where, for $t > 1$,

$$w^1_\varepsilon(t) = -\int_{\sigma(A)} (W(\lambda,z,\cdot) * \chi_\varepsilon)(t)\, d_\lambda(E(\lambda)u^\phi_{z^2}), \tag{57}$$

and, for $t > N+1$,

$$\begin{aligned} w^2_{\varepsilon,N}(t) &= -\int_{\sigma(A)} W^2_{\varepsilon,N}(\lambda,z,t) d_\lambda(E(\lambda)u^\phi_{z^2}), \\ W^2_{\varepsilon,N}(t) &= \int_0^t \frac{\sin(\sqrt{\lambda}(t-t'))}{\sqrt{\lambda}}\,[2\partial_t\widetilde{\chi}_N(t')\partial_t f_\varepsilon(t') + \partial_t^2\widetilde{\chi}_N(t')f_\varepsilon(t')]\,dt'. \end{aligned} \tag{58}$$

Clearly, $(W(\lambda,z,\cdot)*\chi_\varepsilon(\cdot))(t) \to W(\lambda,z,t)$ uniformly on any compact set of λ, t and are uniformly bounded when $\varepsilon \in (0,1)$, $\lambda \in \sigma(A)$ on any compact set of t. Because $\int_{\sigma(A)}(1+|\lambda|)d(E(\lambda)u^\phi_{z^2}, u^\phi_{z^2}) < \infty$ this implies that

$$\lim_{\varepsilon\to 0} E_{\mathbb{C}}(w^1_\varepsilon,t) = E_{\mathbb{C}}(w,t)$$

for bounded t. However, $E_{\mathbb{C}}(w^1_\varepsilon,t)$ does not depend on t for $t > 1$.

Furthermore, $E_{\mathbb{C}}(w^1_\varepsilon + w^2_{\varepsilon,N}, t)$ does not depend on t for $t > N+1$ and, as is seen from (58),

$$\lim_{N\to\infty} E_{\mathbb{C}}(w^1_\varepsilon + w^2_{\varepsilon,N}, N+1) = \lim_{N\to\infty} E_{\mathbb{C}}(w^1_\varepsilon, N+1) = E_{\mathbb{C}}(w^1_\varepsilon, 1). \tag{59}$$

In addition, $w^1_\varepsilon(t)$, $w^2_{\varepsilon,N}(t)$ satisfy estimate (53) uniformly for ε and N while $f_{\varepsilon,N}(t)u^\phi_{z^2}$ satisfies estimate (54). Thus,

$$\Pi_{\mathbb{C}}[f_{\varepsilon,N}] = \lim_{t\to\infty} E_{\mathbb{C}}(u_{\varepsilon,N}, t) = E_{\mathbb{C}}(w^1_\varepsilon + w^2_{\varepsilon,N}, N+1). \tag{60}$$

Combining (59) and (60) with (52), we can choose a sequence ε_n, N_n so that $f_n = f_{\varepsilon_n,N_n}$ satisfy

$$\lim_{n\to\infty} \Pi_{\mathbb{C}}(f_n) = \lim_{\varepsilon\to 0} E_{\mathbb{C}}(w^1_\varepsilon, 1) = \frac{1}{2}\int_{\sigma(A)} \frac{dm_\phi(\lambda)}{\lambda - z^2}.$$

□

Lemma 2 shows that $\Pi_{\mathbb{C}}[f]$ determines $F(\omega) = \int_{\sigma(A)} \frac{dm_\phi(\lambda)}{\lambda-\omega}$ for any $\omega = z^2,\ \mathrm{Im}\, z > \tau_0$. Because $F(\omega)$ is analytic outside $\sigma(A)$ we can continue it to $\mathbb{C}\setminus\sigma(A)$. Then the Pleijel formula (compare with (36)) may be used to find $m_\phi(\lambda)$ and, henceforth, using polarization, $m_{\phi,\psi}(\lambda)$. Clearly, this also determines $dP(x,y,\lambda)$.

Remark 2. When $\sigma(A) \subset \mathbb{R}_+,\ F(\omega),\ \omega \notin \sigma(A)$ can be directly found from $\Pi_{\mathbb{C}}$ without analytic continuation. Thus, the step $iv. \to i.$ does not require analytic continuation.

3. Reconstructions.

3.1. Blagovestchenskii identity. In this section we will describe a procedure to reconstruct the manifold and the Schrödinger operator on it. Here the given data is the hyperbolic form $\mathcal{B}^{2T}$ that is gauge equivalent to the form $\widetilde{\mathcal{B}}^{2T}$ of the Schrödinger operator. In Section 5 we will generalize our results to the case when data is given both on a finite part $S \subset \partial M$ and finite time-interval. To this end we present our constructions so that they can be easily extended to this general case. Remarks in the text often give generalizations of results which are used later in Section 5. As in [KKL], we will actually construct an isometric copy of (M,g) and an operator on it.

For $x \in M$ and $\xi \in T_x(M)$, $|\xi|_g = 1$, we denote by $\gamma_{x,\xi}(s)$ the geodesic parametrized by its path length which starts at x in the direction ξ.

By Lemma 1 there is a gauge transformation S_κ which makes A into a Schrödinger operator $A_\kappa = -\Delta_g + q$. We denote by $\widetilde{\Lambda}^{2T}$, $\widetilde{\mathcal{B}}^{2T}$ the Robin-to-Dirichlet and hyperbolic forms for $-\Delta_g + q$, and by Λ^{2T}, $\mathcal{B}^{2T}$ – those forms for A. By (20), $\mathcal{B}^{2T}[f,h] = \widetilde{\mathcal{B}}^{2T}[\kappa|_{\partial M} f, \kappa|_{\partial M} h]$, i.e., we are given the form

$$f, h \to \widetilde{\mathcal{B}}^{2T}[\kappa|_{\partial M} f, \kappa|_{\partial M} h],$$

where $\kappa|_{\partial M}$ is unknown. (In the future, when it does not cause confusion we will write κ instead of $\kappa|_{\partial M}$).

Consider the initial boundary value problem for the Schrödinger operator,

$$\begin{aligned} &\partial_t^2 u^f - \Delta_g u^f + q u^f = 0 \quad \text{in} \quad M\times\mathbb{R}, \\ &Bu^f\big|_{\partial M\times\mathbb{R}_+} = f, \quad u^f\big|_{t=0} = 0, \quad \partial_t u^f\big|_{t=0} = 0. \end{aligned} \tag{61}$$

The reconstruction is based on two main ingredients, namely the computation of the inner products of solutions to problem (61) and controllability results.

We start with the inner products, that is the Blagovestchenskii identity. Denote by $\mathring{C}^\infty(\Gamma\times\mathbb{R}_+), \Gamma\subset\partial M$ the class of functions

$$f\in C_0^\infty(\Gamma\times\mathbb{R}), \quad f=0 \text{ when } t<0.$$

Lemma 3. *Let $f,h\in\mathring{C}^\infty(\partial M\times\mathbb{R}_+)$. Then*

$$\int_M u^{\kappa f}(T)\overline{u^{\kappa h}(T)}\,dV_g = \frac{1}{2}\int_{-T}^{T} sign(\tau)\mathcal{B}^{2T}[Y_{T+\tau}(f), Y_{T-\tau}(h)]\,d\tau, \tag{62}$$

where Y_τ is the delay operator, $(Y_\tau f)(x,t) = f(x,t-\tau)$.

Proof. Let $w(t,s) = \int_M u^f(t)\overline{u^h(s)}\,dV_g$. Integrating by parts and using (61) we see that

$$\begin{aligned} (\partial_t^2-\partial_s^2)w(t,s) &= -\int_M [(-\Delta_g+q)u^f(t)\overline{u^h(s)} - u^f(t)\overline{(-\Delta_g+q)u^h(s)}]dV_g \\ &= -\int_{\partial M}[\partial_\nu u^f(t)\overline{u^h(s)} - u^f(t)\overline{\partial_\nu u^h(s)}]\,dS_g \\ &= \int_{\partial M}[f(t)\overline{\widetilde{\Lambda}^{2T}h(s)} - \widetilde{\Lambda}^{2T}f(t)\overline{h(s)}]\,dS_g. \end{aligned} \tag{63}$$

Moreover,

$$w|_{t=0} = w|_{s=0} = 0, \quad \partial_t w|_{t=0} = \partial_s w|_{s=0} = 0.$$

Thus, for $s\ge t$,

$$w(t,s) = \frac{1}{2}\int_{L(t,s)}\left[\int_{\partial M}[f(t')\overline{\widetilde{\Lambda}^{2T}h(s')} - \widetilde{\Lambda}^{2T}f(t')\overline{h(s')}]\,dS_g\right]dt'ds',$$

where $L(t,s)$ is the triangle bounded by $s'+t'=s+t$, $s'-t'=s-t$ and $t'=0$. Introducing $\tau=\frac{1}{2}(s'-t')$, $\theta=\frac{1}{2}(s'+t')$, we have

$$\begin{aligned} w(t,s) &= \int_{(s-t)/2}^{(s+t)/2}\int_\tau^{(s+t)/2}\Big[Y_\tau f(\theta)\,\overline{\widetilde{\Lambda}^{2T}Y_{-\tau}h(\theta)} - \widetilde{\Lambda}^{2T}Y_\tau f(\theta)\,\overline{Y_{-\tau}h(\theta)}\Big]d\theta d\tau \\ &= \int_{(s-t)/2}^{(s+t)/2}\widetilde{\mathcal{B}}^{2T}\,[Y_{\delta+\tau}f,\,Y_{\delta-\tau}h]\,d\tau, \quad \text{where } \delta = 2T-\frac{1}{2}(s+t). \end{aligned} \tag{64}$$

Similar formula can be written for $s \le t$. When $s = t$ we can use both to obtain a symmetrized one. Taking κf, κh instead of f, h and using relation (20), this symmetrized formula with $s = t = T$ takes the form (62). $\square$

Remark 3. It is clear from the proof, e.g. formula (64) that $\mathcal{B}^{2T}$ determines $\big(u^{\kappa f}(t), u^{\kappa h}(s)\big)$ for $s + t \le 2T$.

3.2. Approximate controllability. Let $t > 0$ and $\Gamma \subset \partial M$ be open and

$$M(\Gamma, t) = \{x \in M \ : \ \mathrm{dist}(x, \Gamma) < t\}, \tag{65}$$

be the domain of influence of Γ at time t. When $f \in C_0^\infty(\Gamma \times \mathbb{R}_+)$,

$$u^{\kappa f}(t) \in L^2(M(\Gamma, t)) = \{u \in L^2(M) : \ supp(u) \subset M(\Gamma, t)\}.$$

The controllability result we need is based on the celebrated Tataru's Holmgren-John unique continuation theorem [Ta1] (see also [Ta2], [Ho]).

THEOREM 4. *Let u be a solution of wave equation (61). Assume that*

$$u|_{\Gamma \times (0, 2\tau)} = 0, \ \ \partial_\nu u|_{\Gamma \times (0, 2\tau)} = 0 \tag{66}$$

where $\Gamma \subset \partial M$, $\Gamma \ne \emptyset$ is open. Then,

$$u(x, \tau) = 0, \ \ \partial_t u(x, \tau) = 0 \textit{ for } x \in M(\Gamma, \tau).$$

This result yields the following controllability result.

THEOREM 5. *Let $\Gamma \subset \partial M$ be open and $\tau > 0$. Then the linear subspace,*

$$\{u^f(\tau) \in L^2(M(\Gamma, \tau)) : \ f \in C_0^\infty(\Gamma \times [0, \tau])\},$$

is dense in $L^2(M(\Gamma, \tau))$.

Sketch of the proof. Let $\psi \in L^2(M(\Gamma, \tau))$ be such that

$$\langle u^f(\,\cdot\,, \tau), \psi\rangle = 0 \tag{67}$$

for all $f \in C_0^\infty(\Gamma \times [0, \tau])$. We need to show that $\psi = 0$. To this end, consider the following initial boundary value problem,

$$(\partial_t^2 - \Delta_g + q)e = 0, \quad Be|_{\partial M \times \mathbb{R}} = 0, \quad e|_{t=\tau} = 0, \quad \partial_t e|_{t=\tau} = \psi. \tag{68}$$

Integrating by parts and using equations (67) and (68), we obtain that

$$\begin{aligned} 0 &= \int_{M \times [0,\tau]} [u^f \overline{(\partial_t^2 - \Delta_g + q)e} - (\partial_t^2 - \Delta_g + q)u^f \, \overline{e}] \, dV_g \, dt \\ &= \int_{\partial M \times [0,\tau]} f \, \overline{e} \, dS_g \, dt. \end{aligned}$$

Since $f \in C_0^\infty(\Gamma \times [0,\tau])$ is arbitrary, $e|_{\Gamma\times[0,\tau]} = 0$. Together with boundary conditions in (68), this yields that the Cauchy data of e vanish on $\Gamma \times [0,\tau]$. Moreover, since $e(x,t) = -e(x, 2\tau - t)$ due to $e|_{t=\tau} = 0$, the Cauchy data of e vanish on $\Gamma \times [0, 2\tau]$. Therefore, $\psi = 0$ due to Theorem 4. □

3.3. Inner products of waves. Now our main ingredients are obtained and there have been various ways to proceed. In this section we will consider a method based on a minimization algorithm. In the next section we will briefly present a method based on Gaussian beams (for more detail see e.g. [BKa], [KK], [KKL]). Alternatively, one can use methods based on propagation of singularities, e.g. [B2] or methods based on a wave approximation of delta-distributions [KL2].

We start with projections onto domains of influences. Actually, there are several ways to obtain these projections. Here we describe an approach based on minimization because we want to use in the reconstruction as few unstable procedures as possible. A more explicit construction based on the Gram-Schmidt orthgonalization procedure may be found in e.g. [B1] or [KKL].

Let $P_{\Gamma,\tau} : L^2(M) \to L^2(M(\Gamma,\tau))$ be the orthoprojection,

$$P_{\Gamma,\tau} u(x) = \chi_{M(\Gamma,\tau)}(x) u(x),$$

where $\chi_{M(\Gamma,\tau)}$ is the characteristic function of the set $M(\Gamma,\tau)$.

Lemma 4. *Let $f, h \in C_0^\infty(\partial M \times \mathbb{R}_+)$, $T > 0$, $t, s, \tau_1, \tau_2 \in [0,T]$. Let also $\Gamma_1, \Gamma_2 \subset \partial M$ be open sets. Assume that we are given the form $\mathcal{B}^{2T}$. Then it is possible to find the inner products*

$$\begin{aligned} &\left(P_{\Gamma_1,\tau_1} u^{\kappa f}(t), P_{\Gamma_2,\tau_2} u^{\kappa h}(s)\right)_{L^2(M)} \\ &\qquad = \int_{M(\Gamma_1,\tau_1)\cap M(\Gamma_2,\tau_2)} u^{\kappa f}(x,t)\, \overline{u^{\kappa h}(x,s)}\, dV_g. \end{aligned} \tag{69}$$

Proof. Using Theorem 5 we see that

$$\begin{aligned} &||u^{\kappa f}(t)||^2 - ||P_{\Gamma_1,\tau_1} u^{\kappa f}(t)||^2 \\ &\qquad = ||(1 - P_{\Gamma_1,\tau_1}) u^{\kappa f}(t)||^2 \\ &\qquad = \inf\{\|u^{\kappa f}(t) - u^{\kappa\eta}(\tau_1)\|^2 :\ \eta \in C_0^\infty(\Gamma_1 \times [0,\tau_1])\}. \end{aligned} \tag{70}$$

Since

$$\begin{aligned} \|u^{\kappa f}(t) - u^{\kappa\eta}(\tau_1)\|^2 = \|u^{\kappa f}(t)\|^2 - 2\,\mathrm{Re}\,\left(u^{\kappa f}(t), u^{\kappa\eta}(\tau_1)\right) \\ + \|u^{\kappa\eta}(\tau_1)\|^2, \end{aligned} \tag{71}$$

the rhs of (70) can be computed by Lemma 3 and Remark 3. Therefore, we can choose a sequence $\eta_j \in C_0^\infty(\Gamma_1 \times [0,\tau_1])$ such that $\lim u^{\kappa\eta_j}(\tau_1) = P_{\Gamma_1,\tau_1} u^{\kappa f}(t)$. Similarly, we find $\widetilde{\eta}_k \in C_0^\infty(\Gamma_2 \times [0,\tau_2])$ with $\lim u^{\kappa\widetilde{\eta}_k}(\tau_2) = P_{\Gamma_2,\tau_2} u^{\kappa h}(s)$. As $\left(u^{\kappa\eta_j}(\tau_1), u^{\kappa\widetilde{\eta}_k}(\tau_2)\right)$ can be found by Lemma 3 and Remark 3 this proves the result. □

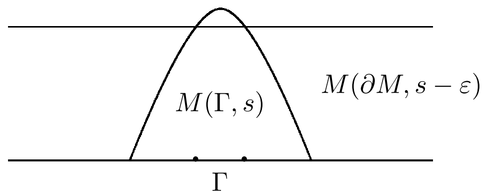

FIG. 1. *The set* $M(\Gamma, s) \setminus M(\partial M, s-\varepsilon)$.

Remark 4. Lemma 4 remains valid if

$$t+\tau_1 \leq 2T, \quad s+\tau_2 \leq 2T$$

This is true due to the remark after Lemma 3.

Let $z_1, z_2 \in \partial M$ and denote $M(z_1, \tau) = \{x \in M : \ d(x, z_1) < \tau\}$. If sequences of open sets $\Gamma_i^j \to z_i$, $i = 1, 2$, then $M(\Gamma_i^j, \tau_i) \to M(z_i, \tau_i)$. Hence we obtain

COROLLARY 2. *Let* $\mathcal{B}^{2T}$ *be given. Then for* $f, h \in C_0^\infty(\partial M \times \mathbb{R}_+)$, $t, s, \tau_1, \tau_2 \in [0, T]$ *we can find*

$$\left(P_{z_1,\tau_1} u^{\kappa f}(t), u^{\kappa h}(s)\right), \quad \left(P_{z_1,\tau_1} u^{\kappa f}(t), P_{z_2,\tau_2} u^{\kappa h}(s)\right),$$

where $P_{z,\tau}$ *is the projection to* $M(z, \tau)$.

3.4. Reconstruction of the manifold.

LEMMA 5. *Let* $y \in \partial M$, $s \in (0, T)$, *and* $\gamma_{y,\nu}([0, s])$ *be a normal geodesic. Then given* $\mathcal{B}^{2T}$ *we can determine whether* $\gamma_{y,\nu}$ *is the shortest geodesic between* $\gamma_{y,\nu}(s)$ *and the boundary* ∂M *or not. Moreover, when this geodesic is minimal,* $\mathcal{B}^{2T}$ *determines* $\min(d(\gamma_{y,\nu}(s), z), T)$ *for any* $z \in \partial M$.

Proof. i. The geodesic $\gamma_{y,\nu}$ is the shortest geodesic between $\gamma_{y,\nu}(s)$ and ∂M if and only if for any $\varepsilon > 0$ and any neighborhood Γ of y (see Fig. 1.),

$$M(\Gamma, s) \setminus M(\partial M, s-\varepsilon) \neq \emptyset. \tag{72}$$

By Theorem 5 property (72) is true if and only if for some $h \in C^\infty(\Gamma \times [0, s])$

$$\|u^{\kappa h}(s)\| > \|P_{M(\partial M, s-\varepsilon)} u^{\kappa h}(s)\|$$

However, this inequality can be checked when $\mathcal{B}^{2T}$ is given.

ii. Let now $\gamma_{y,\nu}$ be the shortest geodesic between $\gamma_{y,\nu}(s)$ and ∂M, $z \in \partial M$ and $t \in (0, T)$. Then $t \geq d(\gamma_{y,\nu}(s), z)$ if and only if there is a neighborhood $\Gamma \subset \partial M$ of y such that for sufficiently small $\varepsilon > 0$,

$$M(\Gamma, s) \subset M(\partial M, s-\varepsilon) \cup M(z, t). \tag{73}$$

Now, property (73) is true if and only if for any $h \in C_0^\infty(\Gamma \times [0, s])$,

$$\begin{aligned} \|u^{\kappa h}(s)\|^2 = {} & \|P_{M(\partial M, s-\varepsilon)} u^{\kappa h}(s)\|^2 + \|P_{M(z,t)} u^{\kappa h}(s)\|^2 \\ & - (P_{M(\partial M, s-\varepsilon)} u^{\kappa h}(s), P_{M(z,t)} u^{\kappa h}(s))_{L^2(M)}. \end{aligned}$$

This again can be checked using $\mathcal{B}^{2T}$. Thus taking infimum of all t for which property (73) is satisfied we find $\min(d(\gamma_{y,\nu}(s),z),T)$. $\square$

Next we introduce the truncated boundary distance functions

$$r_x^T(z) = \min(d(x,z),T), \quad z \in \partial M, \ \ x \in M,$$

and the truncated boundary distance map

$$R^T : M \mapsto C(\partial M), \ x \mapsto r_x^T.$$

Since each x has a closest boundary point z_x and the shortest geodesic from z_x to x is normal, Lemma 5 implies

Theorem 6. *The hyperbolic form $\mathcal{B}^{2T}$ determines the set $R^T(M(\partial M,T))$.*

One can show (e.g. [K2], [KKL]) that the map $R : M(\partial M,T) \to R^T(M(\partial M,T))$ is a homeomorphism.

Our further constructions use the evaluation functions, E_z, $z \in \partial M$,

$$E_z : R^T(M(\partial M,T)) \to \mathbb{R}, \qquad E_z(r_x^T) = r_x^T(z) = \min(d(x,z),T). \tag{74}$$

It can be shown that for any $r_{\tilde{x}}^T \in R^T(M(\partial M,T))$ we can choose $z_1, \cdots, z_m \in \partial M$ so that E_{z_j}, $j = 1, \cdots, m$, form a system of coordinates near $r_{\tilde{x}}^T$ which makes $R^T(M(\partial M,T))$ diffeomorphic to $M(\partial M,T)$. Let $\widetilde{g}$ be the metric on $R^T(M(\partial M,T))$ which makes it isometric to $(M(\partial M,T),g)$, i.e., $\widetilde{g} = ((R^T)^{-1})^{\cdot} g$. Then the differentials of E_z are covectors of length 1 on $(R^T(M(\partial M,T)),\widetilde{g})$. Using this observation it is possible to find infinitely many covectors of length 1 at any point and reconstruct the metric $\widetilde{g}$ (for details see e.g. [KKL]).

Identifying $(M(\partial M,T),g)$ with its isometric copy $(R^T(M(\partial M,T)),\widetilde{g})$ and using definition , we come to the following result:

Lemma 6. *Let the form $\mathcal{B}^{2T}$ be given. Then it possible to construct the Riemannian manifold $(M(\partial M,T),g) = (M^T,g)$.*

3.5. Construction of the gauge class of the operator. So far we have constructed an isometric copy of $(M(\partial M,T),g)$, namely, the manifold $(R^T(M(\partial M,T)),\widetilde{g})$. Our next goal is to find the potential q and impedance η on this manifold.

Lemma 7. *The form $\mathcal{B}^{2T}$ determines $\eta|_{\partial M}$ and $q|_{M(\partial M,T)}$, where η, q is the impedance and potential of the corresponding Schrödinger operator. Moreover, it determines also $\kappa|_{\partial M}$.*

Proof. We recall that any point $x \in M(\partial M,T)$ is represented as an endpoint of a shortest normal geodesic to ∂M.

Let $y \in \partial M$ and s be such that normal geodesic $\gamma_{y,\nu}([0,s])$ is shortest geodesic between its endpoints and let Γ be a neighborhood of y. Let $u^{\kappa f}(x,t)$ be the solution of (61) with $f \in C_0^\infty(\Gamma \times [0,T])$. By using Lemma 4, we can compute the inner products and find

$$\lim_{\varepsilon \to 0} \frac{||(P_{y,s} - P_{\Gamma,s-\varepsilon})u^{\kappa f}(t)||^2}{V_g(M(y,s) \setminus M(\partial M, s-\varepsilon))} = |u^{\kappa f}(x_0,t)|^2, \quad t \leq T, \tag{75}$$

where $x_0 = \gamma_{y,\nu}(s)$. Indeed, the sets $\overline{M(y,s) \setminus M(\partial M, s-\varepsilon)}$ converge to x_0 (see Fig. 1), the waves $u^{\kappa f}(x_0,t)$ are smooth while the denominator can be found since the metric g is already known. Thus we can find $|u^{\kappa f}(x_0,t)|$ at any t.

Next, let $x_0 \in \operatorname{int}(M(\partial M,T))$ be a point for which there is real-valued f with $u^{\kappa f}(x_0,T) \neq 0$. These points form an open dense subset of $\operatorname{int}(M(\partial M,T))$ as can be seen from Theorem 4 and openness of the sets $\{x : u^f(x,T) \neq 0\}$. Then,

$$\begin{aligned} q(x_0) &= \frac{-\partial_t^2 u^{\kappa f}(x_0,T) + \Delta_g u^{\kappa f}(x_0,T)}{u^{\kappa f}(x_0,T)} \\ &= \frac{-\partial_t^2 |u^{\kappa f}(x_0,T)| + \Delta_g |u^{\kappa f}(x_0,T)|}{|u^{\kappa f}(x_0,T)|} \end{aligned}$$

and the rhs can be found from (75). Thus q can be found in a dense subset, and by smoothness on the whole $M(\partial M,T)$.

Finally, varying f and t and using the fact that $\kappa > 0$ we can find $\kappa|_{\partial M}$ and $\eta|_{\partial M}$ from

$$Bu^f = [\partial_\nu u^{\kappa f}(t) + \eta u^{\kappa f}(t)]|_{\partial M} = \kappa|_{\partial M} f(t).$$

□

Remark 5. Formula (75) and Remark 4 show that given $\mathcal{B}^{2T}$ and f we can find $|u^{\kappa f}(x,t)|$ when $d(x,\partial M) \leq T$ and $t + d(x,\partial M) \leq 2T$. Moreover, as $u^{\kappa f}(x,t) = 0$ when $d(x,\partial M) > T$ and $t < T$ we thus can find $|u^{\kappa f}(x,t)|$ when $t + d(x,\partial M) \leq 2T$.

4. Alternative reconstruction via Gaussian beams.

4.1 Gaussian beams Exposition in this section is very concise. It is based on properties of Gaussian beams (see formula (76) below) which are essentially identical for compact and non-compact manifolds. We refer an interested reader to [KKL], Ch. 2.4 or [KK] where the necessary properties of Gaussian beams are discussed in detail.

Gaussian beams, called also "quasiphotons", are a special class of solutions of the wave equation depending on a parameter ε. They can be described as an asymptotic sum

$$(76)\quad U_\epsilon(x,t) = M_\epsilon \exp\{-(i\epsilon)^{-1}\theta(x,t)\} \sum_{n=0}^{N} u_n(x,t)(i\epsilon)^n, \quad x \in M,\ t \in [t_-,t_+],$$

where $M_\epsilon = (\pi\epsilon)^{-m/4}$ is the normalization constant. The function $\theta(x,t)$ is called the phase function and $u_n(x,t)$, $n = 0,1,\ldots,N$ – the amplitude functions. A phase function $\theta(x,t)$ is associated with a geodesic $t \mapsto \gamma(t) \in M$ so that

$$(77)\quad \operatorname{Im}\theta(\gamma(t),t) = 0,$$

$$(78)\quad \operatorname{Im}\theta(x,t) \geq C_0 d(x,\gamma(t))^2,$$

for $t \in [t_-, t_+]$. These conditions guarantee that the absolute value of $U_\epsilon(x,t)$ looks like a Gaussian distribution in x which moves in time along the geodesic $\gamma(t)$. The phase function satisfies the eikonal equation

$$(\partial_t \theta)^2 - g^{jl}(x)\partial_j\theta\partial_l\theta \asymp 0, \tag{79}$$

where $\asymp$ means the coincidence of the Taylor coefficients of both sides considered as functions of x depending on t as a parameter at the points $\gamma(t), t \in [t_-, t_+]$. The amplitude functions u_n, $n = 0, \dots, N$ can be constructed as solutions of the transport equations

$$\mathcal{L}_\theta u_n \asymp (\partial_t^2 - \Delta_g + q)u_{n-1}, \quad \text{with } u_{-1} = 0. \tag{80}$$

Here $\mathcal{L}_\theta$ is the transport operator

$$\mathcal{L}_\theta u = 2\partial_t\theta\partial_t u - 2\langle \nabla\theta, \nabla u\rangle_g + (\partial_t^2 - \Delta_g)\theta \cdot u. \tag{81}$$

The following existence result is proven e.g. in [KKL], [KK]:

Theorem 7. *Let $\gamma(t)$, $t \in [t_-, t_+]$ be a geodesic lying in int(M) when $t \in (t_-, t_+)$. Let $\theta_0(x)$, $u_n^0(x)$, $n = 0, 1, \dots$ be functions with θ_0 satisfying (78) for $t = t_0 \in [t_-, t_+]$. Assume that $\partial_t\gamma(t_0) = c\, Grad\, \theta_0(\gamma(t_0))$, $c > 0$.*

Then there are functions $\theta(x,t)$ and $u_n(x,t)$ satisfying (78)–(80) such that $\theta(x,t_0) = \theta_0(x)$, $u_n(x,t_0) = u_n^0(x,)$. Moreover, there is a solution $u_\varepsilon(x,t)$ of equation

$$(\partial_t^2 - \Delta_g + q)u_\epsilon(x,t) = 0, \quad (x,t) \in M \times [t_-, t_+], \tag{82}$$

such that

$$|u_\epsilon(x,t) - \chi(x,t)U_\epsilon(x,t)| \le C_{\widetilde{N}}\epsilon^{\widetilde{N}}, \tag{83}$$

where $\widetilde{N} \to \infty$ when $N \to \infty$. Here χ is the cut-off function, $\chi = 1$ near the trajectory $(\gamma(t), t)$, $t \in [t_-, t_+]$.

In the other words, for an arbitrary geodesic and matching initial data there is a Gaussian beam that propagates along this geodesic.

Next we consider a class of boundary sources in (61) which generate Gaussian beams. Let $z_0 \in \partial M$, $t_0 > 0$, and let $z = (z^1, \dots z^{m-1})$ be a local system of coordinates on ∂M near z_0. Consider a class of functions $f_\epsilon = f_{\epsilon, z_0, t_0}(z,t)$ on the boundary cylinder $\partial M \times \mathbb{R}$, where

$$f_\epsilon(z,t) = (\pi\epsilon)^{-m/4}\chi(z,t)\exp\{i\epsilon^{-1}\Theta(z,t)\}V(z,t). \tag{84}$$

Here χ is a smooth cut-off function near (z_0, t_0) and

$$\Theta(z,t) = -(t - t_0) + \frac{1}{2}\langle H_0(z - z_0), (z - z_0)\rangle + \frac{i}{2}(t - t_0)^2, \tag{85}$$

where $\langle \cdot, \cdot \rangle$ is the complexified Euclidean inner product, $\langle a, b\rangle = \sum a_j b_j$, and H_0 is a symmetric matrix with a positive definite imaginary part, i.e.,

$H_0 = H_0^t$, $\operatorname{Im} H_0 > 0$. At last $V(z,t)$ is a smooth function having non-zero value at (z_0, t_0).

The following result in valid (see e.g. [KKL]).

LEMMA 8. *For any function V and $t < t_0 + \tau(z_0)$ the solution $u_{\varepsilon,V}$ of problem (61) is a Gaussian beam propagating along the normal geodesic $\gamma_{z_0,\nu}$.* Here $\tau(z_0)$ is the first time when the geodesic $\gamma_{z_0,\nu}$ hits ∂M.

4.2 Reconstruction. Using the Gaussian beams described in Lemma 8 we can give an alternative method of the reconstruction of M, g, q and η of the Schrödinger operator in the orbit on an unknown operator A.

Indeed, by Corollary 2, the hyperbolic form $\mathcal{B}^{2T}$ uniquely determines $||P_{y,\tau} u^{\kappa f}(t)||$. Let $f = f_\varepsilon$ be of form (84)–(85) with $V = 1$. Then due to Lemma 8 $u^{\kappa f}(t)$, $f = f_\varepsilon$ is a Gaussian beam $u_{\varepsilon,\kappa}$ propagating along $\gamma_{z_0,\nu}$. The asymptotic expansion (76) of a Gaussian beam implies that for $s < \tau_{S,T}(z_0)$, $s + t_0 \le T$,

$$\lim_{\varepsilon \to 0} ||P_{y,\tau} u_{\varepsilon,\kappa}(s + t_0)|| = \begin{cases} h(t), & d(\gamma_{z_0,\nu}(s), y) < \tau, \\ 0, & d(\gamma_{z_0,\nu}(s), y) > \tau, \end{cases}$$

where $h(t)$ is a strictly positive function. Thus we can find $\min(d(\gamma_{z_0,\nu}(s), y), T)$. □

To find q, η and $\kappa|_{\partial M}$ we need to consider the Gaussian beam $u_{\varepsilon,\kappa}(x,t)$ more carefully. In particular, it follows from [KKL], Ch. 2.4 that

$$\lim_{\epsilon \to 0} ||u_{\varepsilon,\kappa}(t)||^2 = h^2(t) = \frac{|\kappa(z_0)|^2 \sqrt{g(z_0, 0))}}{\sqrt{\det(\operatorname{Im} H(t))}\, |\det(Y(t))|}. \tag{86}$$

Here the complex valued matrices $H(t)$ and $Y(t)$ explicitly depend only on the metric tensor g which is already in our disposal and the initial matrix $H_0(t_0)$. More precisely, these matrices are solutions of some Cauchy problems for the Hamilton system of equations and a linear system of ordinary differential equations along the geodesic $\gamma_{z_0,\nu}(s)$. Using (86) together with Corollary 2 this gives us $\kappa(z_0) > 0$.

To find potential q we can use the second term of the asymptotic expansion of $||P_{y,\tau} u_{\varepsilon,\kappa}(s + t_0)||^2$. Using results of section 2.4.19 of [KKL] we can easily obtain that for $d(\gamma_{z_0,\nu}(s), y) < \tau$

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} (||P_{y,\tau} u_{\varepsilon,\kappa}(s + t_0)||^2 - h^2(t)) = a(s) \int_0^s q(\gamma_{z_0,\nu}(t)) dt + b(s). \tag{87}$$

Here functions $a(s)$ and $b(s)$ again depend only on the metric g and initial data (84), (85). These functions can be found by solving some Cauchy problems for a system of ordinary differential equations along $\gamma_{z_0,\nu}$. Moreover, $a(s) \neq 0$ for any s. Therefore, by using (87) we can find the integral of the potential q along the normal geodesic $\gamma_{z_0,\nu}[0, s]$ and thus q itself. □

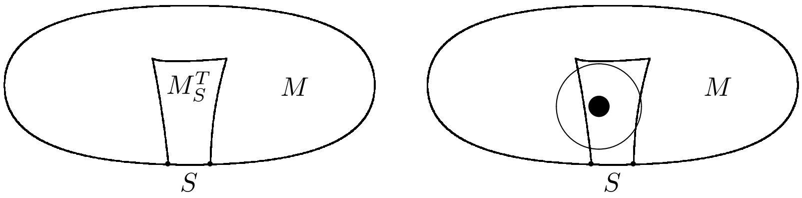

FIG. 2. *Left: Set M_S^T. Right: M^2 is union of domain of influences of balls $B_r(z) \subset M_S^T$.*

5. Reconstruction with data given on a part of the boundary.

5.1. Local constructions. In this section we will generalize constructions of Section 3 to the case when $\mathcal{B}_S^{2T}$ is given for open subset $S \subset \partial M$, i.e., the form $f \mapsto \widetilde{\mathcal{B}}^{2T}[\kappa f, \kappa f] = \mathcal{B}^{2T}[f, f]$, $f \in C_0^\infty(S \times \mathbb{R}_+)$ is given.

The construction of $M(S, T)$ will be given by iterating local constructions. First, we will construct a subset of M adjacent to S.

We use the function $\tau_{S,T} : S \to \mathbb{R}$,

$$\tau_{S,T}(z) = \sup\{s \in [0, T] : \, d(\gamma_{z,\nu}(s), S) = s\}.$$

We note that the function $\tau_{S,T}$ is, in general, not continuous but only upper semicontinuous. Let

$$\Omega_{S,T} = \{(z, s) \in S \times \mathbb{R}_+ : \quad s < \tau_{S,T}(z)\}$$

be the open set that lies under the graph of $\tau_{S,T}$. The exponential map,

$$\exp_{\partial M} : \Omega_{S,T} \to M, \quad (z, s) \mapsto \gamma_{z,\nu}(s), \tag{88}$$

is a diffeomorphism between $\Omega_{S,T}$ and M_S^T,

$$M_S^T = \exp_{\partial M}(\Omega_{S,T}) \subset M,$$

(see Fig 2).

Let $\widetilde{g} = (\exp_{\partial M})^* g$ be a metric on $\Omega_{S,T}$ which makes $\exp_{\partial M}$ an isometry. We will first construct the function $\tau_{S,T}$ and, therefore, $\Omega_{S,T}$ and then the metric tensor $\widetilde{g}$ in this set.

We want to apply the results of Section 3. We first observe that the main tools, Lemma 3 and Lemma 4 remain valid with $\mathcal{B}_S^{2T}$ instead of $\mathcal{B}^{2T}$ if we take $f, h \in C_0^\infty(S \times \mathbb{R}_+)$, $\Gamma_1, \Gamma_2 \subset S$.

To construct $\tau_{S,T}$ we observe that for $s < T$ we have $s \le \tau_{S,T}(y)$ if and only if for any $t < s$ and any neighborhood $\Gamma \subset S$ of y, $M(\Gamma, s) \not\subset M(S, t)$. On the other hand, by Theorem 5, $M(\Gamma, s) \subset M(S, t)$ if and only if

$$\|P_{S,t} u^{\kappa f}(s)\| = \|u^{\kappa f}(s)\| \tag{89}$$

for all $f \in C_0^\infty(\Gamma \times (0, s))$. As both sides of (89) can be computed when $\mathcal{B}_S^{2T}$ is given, we see that $\mathcal{B}_S^{2T}$ determines $\tau_{S,T}$.

Using some local coordinates $y = (y^1, \dots, y^{(m-1)})$ on S, we obtain local coordinates $(y^1, \dots, y^{(m-1)}, s)$ on $\Omega_{S,T}$.

Next we construct the metric $\widetilde{g}$ in these local coordinates. Using part *ii.* of the proof of Lemma 5 when $\Gamma \subset S$, and $z, y \in S$, we see that $\mathcal{B}_S^{2T}$ determines $\min(d(\gamma_{y,\nu}(s), z), T)$ for $z, y \in S$ and $s < T$.

Again, we use evaluation functions

$$E_z : \Omega_{S,T} \to \mathbb{R}, \quad E_z(y,s) = \min(d(z, \gamma_{(y,\nu)}(s)), T) = r^T_{(y,s)}(z),$$

where $z \in S$. Then $E_z(y,s)$, considered as a function of (y,s) with fixed z is a truncated distance function. Thus using them as in section 3.4 we find $\widetilde{g}$. Further steps to find $q|_{M_S^T}$ and η, $\kappa|_S$ are also the same.

5.2. Global constructions.

LEMMA 9. *Assume that we are given (M_S^T, g) and the gauge-equivalence class $\sigma_S a(x,D)|_{M_S^T}$ as well as the form $\mathcal{B}_S^{2T}$. Let $B_r(z) \subset M_S^T$ be a ball and consider the manifold $M \setminus B_r(z)$ with the boundary $\partial M \cup S_1$, $S_1 = \partial B_r(z)$. Then we can find the form $\mathcal{B}_{S_1}^{2T_1}$, $T_1 < T - (r + d(z,S))$ upto a gauge transformation.*

Proof. As we know the orbit $\sigma_S a(x,D)|_{M_S^T}$ we can work with the restriction to M_S^T of the corresponding Schrödinger operator $-\Delta_g + q$.
(i) We start with the reconstruction of the values of waves $u^f(x,t)$, $f \in \mathring{C}^\infty(S \times \mathbb{R}_+)$ in M_S^T. For at any point $x = \gamma_{y,\nu}(s) \in M_S^T$ it is possible to construct a wave u^{f_0} with $u^{f_0}(x,s) > 0$. Indeed, we can take, for instance, the real part of a Gaussian beam, $f_0 = f_{\varepsilon,y,0}$ (84) with sufficiently small ε. By remark after Lemma 4 we can find the inner products $\big((P_{y,s} - P_{S,s-\delta})u^f(t), u^{f_0}(s)\big)$ for $t + s < 2T$, $s < T$. As the metric in M_S^T is already found similar considerations to that in Lemma 5 make possible to find $u^f(\gamma_{y,\nu}(s), t)$. Thus we obtain $u^f(x,t)$ for $t < 2T - d(x,S)$ when $x \in M_S^T$, $f \in \mathring{C}^\infty(S \times \mathbb{R}_+)$.

Let now $G(x,y,t)$ be the Green function

$$\begin{aligned} &(\partial_t^2 - \Delta_g + q)G(x,y,t) = \delta_y(x)\delta(t) \text{ in } M \times \mathbb{R}, \quad y \in M \\ &BG(\cdot\,, y, \cdot)|_{\partial M \times \mathbb{R}} = 0; \quad G(x,y,t)|_{t<0} = 0. \end{aligned} \tag{90}$$

The distribution G has a limit when $y \to \partial M$, $G(\cdot\,, y, \cdot) \in \mathcal{D}'(M \times \mathbb{R})$. It is then the solution of problem (61) with $\kappa f = \delta_{\partial M, y}(x)\delta(t)$ where $\delta_{\partial M,y}$ is the delta-function on ∂M. Choosing a sequence $f_j \in C_0^\infty(S \times \mathbb{R}_+)$, $f_j \to \delta_{\partial M,y}(x)\delta(t)$ in $\mathcal{D}'(\partial M \times \mathbb{R})$ we find $G(x,y,t)$ as a limit of $u^{f_j}(x,t)$. Thus we can find $G(x,y,t)$ for $y \in S$, $x \in M_S^T$, $t + d(x,S) < 2T$.

Let us now fix $x \in M_S^T$. Because $G(x,y,t) = G(y,x,t)$ we know $G(x,y,t)$ when $x \in S$, $y \in M_S^T$, and $t + d(y,S) < 2T$.

Let $B_r(z) \subset M_S^T$ be a ball and consider the initial boundary value problem

$$\begin{aligned} &\partial_t^2 u - \Delta_g u^F + q u^F = F, \quad \text{in} \quad M \times \mathbb{R}, \\ &Bu^F\big|_{\partial M \times \mathbb{R}} = 0, \quad u^F\big|_{t=0} = 0, \quad \partial_t u^F\big|_{t=0} = 0, \end{aligned} \tag{91}$$

where $F(x,t) \in \mathring{C}^\infty(B_r(z)\times\mathbb{R}_+)$. Since

$$u^F(x,t) = \int_{\mathbb{R}}\int_{B_r(z)} G(x,y,t-s)F(y,s)\,dV_g(y)ds, \tag{92}$$

we can find $u^F(x,t)$ for $x \in S$, $t < 2T-(d(z,S)+r)$.

Next we consider the inner product $w(t,s) = (u^F(t), u^h(s))$ where $h \in \mathring{C}^\infty(S\times\mathbb{R}_+)$. The same considerations as in the proof of Lemma 3 show that

$$(\partial_t^2 - \partial_s^2)w(t,s) = -\int_{\partial M} u^F(t)|_S \cdot \overline{h(s)}\,dS_g + \int_M F(t)\overline{u^h(s)}dV_g. \tag{93}$$

Our previous results imply that we can evaluate the rhs of (93) for $t,s < 2T-(d(z,S)+r)$. Therefore we can find $(u^F(t), u^h(s))$ for any $F \in \mathring{C}^\infty(B_r(z)\times\mathbb{R}_+)$ and $h \in \mathring{C}^\infty(S\times\mathbb{R}_+)$ for $t+s < 2T-(d(z,S)+r)$.

Using an approximation to $(P_{y,s}-P_{S,s-\delta})u^h(s)$ by the waves u^{η_j}, $\eta_j \in C_0^\infty(S\times(0,s))$ as described in the proof of Lemma 4 we find the inner products $\big((P_{y,s}-P_{S,s-\delta})u^h(s), u^F(t)\big)$. Because $u^h(\gamma_{y,\nu}(s),s)$ is known this gives us $u^F(x,t)$ for any $F(x,t) \in \mathring{C}^\infty(B_r(z)\times\mathbb{R}_+)$, $t+d(x,S) < 2T-(d(z,S)+r)$, $x \in M_S^T$. Approximating $\delta_z(y)\delta(t)$ by smooth functions $F(t,y)$ it is thus possible to determine $G(x,z,t)$ in $X = \{(x,z,t) : d(x,S)+d(z,S)+t < 2T,\ x,z \in M_S^T\}$.

(ii) Consider the manifold $M\backslash B_r(z)$ and the initial boundary value problem

$$\begin{aligned} &\partial_t^2 e^f - \Delta_g e^f + q e^f = 0 \quad \text{in} \quad M\backslash B_r(z)\times\mathbb{R}_+ \\ &\qquad Be^f|_{\partial M\times\mathbb{R}_+} = 0,\ \partial_\nu e^f|_{S_1\times\mathbb{R}_+} = f; \quad e^f|_{t=0} = e^f_t|_{t=0} = 0, \end{aligned} \tag{94}$$

where $S_1 = \partial B_r(z)$ and $f \in C_0^\infty(S_1\times\mathbb{R}_+)$. If $\widetilde{e}^f$ is a smooth continuation of e^f inside $B_r(z)\times\mathbb{R}_+$ then $\widetilde{e}^f$ is the solution to (91) with $F = (\partial_t^2-\Delta_g+q)\widetilde{e}^f$.

Because we know $u^F(x,t)$ in $B_r(z)\times(0,2T(z,r))$, $T(z,r) = T - (d(z,S)+r)$ we use the above considerations to find $e^f|_{S_1\times(0,2T(z,r))}$ for any $f \in C_0^\infty(S_1\times\mathbb{R}_+)$, i.e., to construct the response operator $\Lambda_{S_1}^{2T(z,r)}$ for the manifold $M\setminus B_r(z)$. As the metric near $B_r(z)$ is known this provides us with the hyperbolic form $\mathcal{B}_{S_1}^{2T(z,r)}$ of the Schrödinger operator. □

After this it is possible to iterate the previous construction: We construct the manifold and the Schrödinger operator in the domain (see Fig. 2)

$$M^2 = \bigcup_{z\in M^1, r>0} M_{\partial B_r(z)}^{T(z,r)} \cup M^1, \quad \text{where } M^1 = M_S^T.$$

To glue together $M_{\partial B_r(z)}^{T(z,r)}$ with different z and r we use the fact that if $G(x,y,t) = G(\widetilde{x},y,t)$ for some $t>0$ and $y\in\Omega$ where $\Omega\subset M$ is open then $x = \widetilde{x}$.

In M^2 we can again take balls $B_r(z)$, and iterate the construction. Then it takes only a finite number of iterations to reconstruct any compact subset in $(M(S,T),g)$ and at most numerable many to construct the whole manifold $(M(S,T),g)$ and the Schrödinger operator on it and, therefore, the orbit $\sigma_S a(x,D)|_{M(S,T)}$.

We note that the extensive use of the Green functions for a step by step reconstruction of the manifold and the operator is somewhat similar to that in [LaU] which, however, deals with a fixed-frequency inverse problem.

6. Appendix: Elliptic operators on manifolds of bounded geometry. Here we consider a class of second-order elliptic self-adjoint differential operators on non-compact manifolds with boundary. Manifolds and coefficients of the operators are C^∞-smooth. Moreover, all Riemannian manifolds (M,g) are assumed to be complete as metric spaces, i.e., Cauchy sequences converge.

We denote by $B_r(x)$ the balls of (M,g) and by $B_{\partial,r}(y)$ the balls of $(\partial M, g_{\partial M})$, where $g_{\partial M}$ is the metric inherited from (M,g).

Assume that there is $r_0 > 0$ such that

a. For any $x_0 \in M$ and $r \le \min(r_0, d_g(x_0, \partial M))$ the ball $B_r(x_0)$ is the domain of Riemannian normal coordinates $x = (x^1, \cdots, x^m)$ centered in x_0;

b. For any $y_0 \in \partial M$ and $r \le r_0$ the ball $B_{\partial,r}(y_0)$ is the domain of Riemannian normal coordinates (on ∂M) $y = (y^1, \cdots, y^{m-1})$ centered in y_0;

c. For any $x_0 \in M$ with $d_g(x_0, \partial M) < r_0$ there exists a unique nearest boundary point $y_0 = y(x_0) \in \partial M$ and, when $r < r_0$, the cylinder $C_r(x_0) = \{x \in M : d_g(x, \partial M) < r,\, y(x) \in B_{\partial,r}(y_0)\}$, is the domain of boundary normal coordinates $x = (y^1, \cdots, y^{m-1}, n)$ where $(y^1, \cdots, y^{m-1})$ are Riemannian coordinates on $B_{\partial,r}(y(x_0))$ and $n = d_g(x, \partial M)$.

We note that the supremum of all such r_0 is called the injectivity radius of $(M, \partial M)$. In the future we denote by $\widehat{B_r}(x_0)$, $x_0 \in M$, $r \le r_0$ either a ball $B_r(x_0)$ if $r \ge d_g(x_0, \partial M)$ or a cylinder $C_r(x_0)$ if $r < d_g(x_0, \partial M)$. Moreover, by normal coordinates in $\widehat{B_r}(x_0)$ we mean Riemannian normal coordinates if $\widehat{B_r}(x_0) = B_r(x_0)$ and boundary normal coordinates if $\widehat{B_r}(x_0) = C_r(x_0)$.

Remark 6. By the Hopf-Rinow theorem any two points on a complete Riemannian manifold without boundary can be connected by a shortest geodesic. Moreover, for any x there is $r(x) > 0$ such that $B_{r(x)}(x)$ is a domain of Riemannian normal coordinates. Using the Hopf's double of M and a locally finite partition of unity, a manifold $(M, g, \partial M)$ can be considered as a subset of a manifold without boundary. Then the Hopf-Rinow theorem implies that any $x \in M$ can be connected with ∂M by a shortest geodesic which is normal to ∂M. Moreover, if $y \in \partial M$ there is $r(y) > 0$ such that $C_{r(y)}(y)$ is a domain of boundary normal coordinates (for results on geometry see e.g. [Cv]). What is essential in conditions a–c is that $r(x)$ is uniformly bounded from below.

Definition 4. *A complete connected smooth non-compact Riemannian manifold $(M, g, \partial M)$ is called a k-finite, $k \in \mathbb{Z}_+$, manifold of bounded geometry if there is r_0 such that the injectivity radius of Riemannian normal coordinates and injectivity radius of boundary normal coordinates are*

larger than r_0 and for some $C > 0$
i. In any generalized geodesic ball $\widehat{B_r}(x_0)$ of radius $r < r_0$ there exist normal coordinates $x = (x^1, \cdots, x^m)$ such that in these coordinates the metric tensor satisfies

$$C^{-1}I \leq g \leq CI, \quad |\partial^\alpha g_{ij}| \leq C \quad \text{when } |\alpha| \leq k. \tag{95}$$

ii. If $(\widehat{B_r}(x_0), x)$ and $(\widehat{B_r}(\widetilde{x}_0), \widetilde{x})$ are above coordinates, then transition functions satisfy $||\widetilde{x} \circ x^{-1}||_{C^{k+1}} \leq C$.

Remark 7. Conditions of this definition are satisfied if, for example, the injectivity radii of the Riemannian normal coordinates in M, the boundary normal coordinates in M and Riemannian normal coordinates in ∂M are uniformly bounded, and in addition the Riemannian curvature tensor R and the second fundamental form of the boundary S have bounded covariant derivatives upto order $k+1$. We note that above geometric requirements can be made weaker by using harmonic coordinates, see e.g., [HH], [dTK] and [KaKuLT].

Examples of manifolds of bounded geometry are given by manifolds which asymptotically at infinity look like as a finite number of cones and waveguides.

Manifolds of bounded geometry enjoy a number of important properties:
(i) For any $r < r_0$ the manifold M has a covering of a countable set of generalized balls $\widehat{B_{r/2}}(x_i)$ which have a finite intersection index I, i.e., if we take more than I balls $\widehat{B_r}(x_i)$ they will have empty intersection. In addition balls $\widehat{B_{r/4}}(x_i)$ do not intersect. This result is a direct generalization to the manifolds with boundary of the result by Gromov [Gr1] using uniform estimate of the volume of balls $B_R(x)$ (see e.g. [KaKuLa]).
(ii) There is a partition of unity ϕ_i, $\mathrm{supp}(\phi_i) \subset \widehat{B_{r/2}}(x_i)$ which satisfy $\|\phi_i\|_{C^{k+1}} \leq C_k$ in normal coordinates satisfying (95).
(iii) There is an invariant definition of spaces $C^q(M)$ for $q \leq k+1$. Also the Sobolev spaces $H^s(M)$ have an invariant meaning for $0 \leq s \leq k+1$. Furthermore, for such s

$$H^s(M) = \mathrm{cl}(C_c^\infty(M)), \tag{96}$$

where $C_c^\infty(M)$ consists of C^∞-functions which are equal to 0 outside some ball $B_R(x)$. The space of Sobolev functions having compact support is denoted by $H_c^s(M)$. For these results we refer to [Sh1].
(iv) Similar Sobolev spaces can be defined on ∂M and the usual embedding and extension theorems remain valid for $H^s(M)$, $s \leq k+1$.

For k-finite manifold of bounded geometry we can generalize the basic results for second order elliptic operators given in [Sh1] for non-compact manifolds without boundary.

Let us consider elliptic operators with smooth coefficients of the form

$$a(x,D)v(x) = -\Delta_g\, v(x) + (V(x), \nabla v(x))_g + c(x)v(x), \tag{97}$$

V is a real vector field (1st order operator) and c is a real valued function (cf. (1)). If $a(x,D)$ is symmetric on $C_0^\infty(\mathrm{int}(M))$ with respect to a smooth measure $d\mu = \rho dV_g$, $\rho > 0$, $dV_g = g^{1/2}\, dx^1 \cdots dx^m$, then

$$a(x,D)v(x) = -\rho^{-1} g^{-1/2} (\partial_i g^{1/2} g^{ij} \rho \partial_j v(x)) + q(x)v(x) \tag{98}$$

with a smooth real q (cf. (5)).

Lemma 10. *Let $(M, g, \partial M)$ be a k-finite manifold of bounded geometry and $a(x,D)$ be an operator of form (98) with $\rho, \rho^{-1} \in C^\infty(M) \cap C_b^2(M)$ $q \in C^\infty(M) \cap C_b^0(M)$.*
Then for any real valued $\eta \in C^\infty(\partial M) \cap C_b^1(\partial M)$ the operator A,

$$Av = a(x,D)v; \quad D(A) = \{v \in H^2(M) : \partial_\nu v + \eta v|_{\partial M} = 0\} \tag{99}$$

is self-adjoint and bounded from below, i.e. $A \geq -c_0$. Moreover, $D((A + c)^{1/2}) = H^1(M)$ when $c > c_0$.

We remind the readers that C_b^k stands for the class of functions having k uniformly bounded derivatives.

Lemma 10 can be proved by following arguments of [Sh1]. The idea of the proof is to consider a minimal operator A_0 defined by (98) on $C_\eta^\infty(M)$, i.e., the set of smooth functions satisfying the Robin boundary condition (99). Then $A = \mathrm{cl}(A_0)$ is a self-adjoint operator with $D(A) = \{v \in H^2(M) : \partial_\nu v + \eta v|_{\partial M} = 0\}$. This may be shown by the same technique of minimal-maximal operators and methods based on finite propagation speed as in [Sh1] (also [Ch]).

Remark 8. For readers convenience we note that Green's formula is valid on the manifolds of bounded geometry. Indeed, by using definition (96) we see for instance that for $u, v \in \mathcal{D}(A) \cap H_c^1(M)$

$$\int_M (v\, a(x,D)u - u\, a(x,D)v)\, \rho dV_g(x) = \int_{\partial M} (v\, Bu - u\, Bv))\, \rho dS_g(x).$$

This fact is used many times in our considerations.

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