THE CALDERÓN'S INVERSE PROBLEM - IMAGING AND INVISIBILITY

KARI ASTALA, MATTI LASSAS, LASSI PÄIVÄRINTA

1. Introduction

In electrical impedance tomography one aims to determine the internal structure of a body from electrical measurements on its surface. To consider the precise mathematical formulation of the electrical impedance tomography problem, suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with connected complement and let us start with the case when $\sigma : \Omega \to (0, \infty)$ be a measurable function that is bounded away from zero and infinity.

Then the Dirichlet problem

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega, \tag{1}$$

$$u\big|_{\partial\Omega} = \phi \in W^{1/2,2}(\partial\Omega) \tag{2}$$

admits a unique solution u in the Sobolev space $W^{1,2}(\Omega)$. Here

$$W^{1/2,2}(\partial\Omega) = H^{1/2}(\partial\Omega) = W^{1,2}(\Omega)/W_0^{1,2}(\Omega)$$

stands for the space of equivalent classes of functions $W_0^{1,2}(\Omega)$ that are same up to a function in $W_0^{1,2}(\Omega) = \mathrm{cl}_{W^{1,2}(\Omega)}(C_0^\infty(\Omega))$. This is the most general space of functions that can possibly arise as Dirichlet boundary values or traces of general $W^{1,2}(\Omega)$-functions in a bounded domain Ω.

In terms of physics, if the electric potential of a body Ω at point x is $u(x)$, having the boundary value $\phi = u|_{\partial\Omega}$, and there are no sources inside the body, u satisfies the equations (1)-(2). The electric current J in the body is equal to

$$J = -\sigma \nabla u.$$

In electrical impedance tomography, one measures only the normal component of the current, $\nu \cdot J|_{\partial\Omega} = -\nu \cdot \sigma \nabla u$, where ν is the unit outer normal to the boundary. For smooth σ this quantity is well

The authors were supported by the Academy of Finland, by the Finnish Center of Exellences Analysis and Dynamics and Inverse Problems and Mathematical Sciences Research Institute.

defined pointwise, while for general bounded measurable σ we need to use the (equivalent) definition of $\nu \cdot \sigma \nabla u|_{\partial\Omega}$,

(3)
$$\langle \nu \cdot \sigma \nabla u, \psi \rangle = \int_\Omega \sigma(x) \nabla u(x) \cdot \nabla \psi(x)\, dm(x)\,, \quad \text{for all } \psi \in W^{1,2}(\Omega)(\Omega),$$

as an element of $H^{-1/2}(\partial\Omega)$, the dual of space of $H^{1/2}(\partial\Omega) = W^{1/2,2}(\partial\Omega)$. Here, m is the Lebesgue measure.

Calderón's inverse problem is the question whether an unknown conductivity distribution inside a domain can be determined from the voltage and current measurements made on the boundary. The measurements correspond to the knowledge of the Dirichlet-to-Neumann map Λ_σ (or the quadratic form) associated to σ, i.e., the map taking the Dirichlet boundary values of the solution of the conductivity equation

$$\nabla \cdot \sigma(x) \nabla u(x) = 0 \tag{4}$$

to the corresponding Neumann boundary values,

$$\Lambda_\sigma :\ u|_{\partial\Omega} \mapsto \nu\ \cdot \sigma \nabla u|_{\partial\Omega}. \tag{5}$$

For sufficiently regular conductivities the Dirichlet-to-Neumann map Λ_σ can be considered as an operator from $W^{1/2,2}(\partial\Omega)$ to $W^{-1/2,2}(\partial\Omega)$. In the classical theory of the problem, the conductivity σ is bounded uniformly from above and below. The problem was originally proposed by Calderón [19] in 1980. Sylvester and Uhlmann [74] proved the unique identifiability of the conductivity in dimensions three and higher for isotropic conductivities which are C^∞-smooth, and Nachman gave a reconstruction method [62]. In three dimensions or higher unique identifiability of the conductivity is proven for conductivities with 3/2 derivatives [69], [17] and $C^{1,\alpha}$-smooth conductivities which are C^∞ smooth outside surfaces on which they have conormal singularities [34]. In a recent preprint of Haberman and Tataru, the uniqueness has been proven for the C^1-smooth conductivities [37]. The problems has also been solved with measurements only on a part of the boundary [43].

In two dimensions the first global solution of the inverse conductivity problem is due to Nachman [63] for conductivities with two derivatives. In this seminal paper the $\overline{\partial}$ technique was first time used in the study of Calderón's inverse problem. The smoothness requirements were reduced in [18] to Lipschitz conductivities. Finally, in [11] the uniqueness of the inverse problem was proven in the form that the problem was originally formulated in [19], i.e., for general isotropic conductivities in L^∞ which are bounded from below and above by positive constants.

The Calderón problem with an anisotropic, i.e., matrix-valued, conductivity that is uniformly bounded from above and below has been studied in two dimensions [73, 63, 51, 7, 38] and in dimensions $n \geq 3$ [53, 51, 70]. For example, for the anisotropic inverse conductivity problem in the two dimensional case it is known that the Dirichlet-to-Neumann map determines a regular conductivity tensor up to a diffeomorphism $F : \overline{\Omega} \to \overline{\Omega}$, i.e., one can obtain an image of the interior of Ω in deformed coordinates. This implies that the inverse problem is not uniquely solvable, but the non-uniqueness of the problem can be characterized. We note that the problem in higher dimensions is presently solved only in special cases, like when the conductivity is real analytic.

Electrical impedance tomography has a variety of different applications for instance in engineering and medical diagnostics. For a general expository presentations see [16, 22], for medical applications see [25].

In the last section we will study the inverse conductivity problem in the two dimensional case with degenerate conductivities. Such conductivities appear in physical models where the medium varies continuously from a perfect conductor to a perfect insulator. As an example, we may consider a case where the conductivity goes to zero or to infinity near ∂D where $D \subset \Omega$ is a smooth open set. We ask what kind of degeneracy prevents solving the inverse problem, that is, we study what is the border of visibility. We also ask what kind of degeneracy makes it even possible to coat of an arbitrary object so that it appears the same as a homogeneous body in all static measurements, that is, we study what is the border of the invisibility cloaking. Surprisingly, these borders are not the same; We identify these borderlines and show that between them there are the electric holograms, that is, the conductivities creating an illusion of a non-existing body (see Fig. 1). These conductivities are the counterexamples for the unique solvability of inverse problems for which even the topology of the domain can not be determined using boundary measurements.

In this presentation we concentrate on solving Calderón's inverse problem in two dimensions. The presentation is based on the works [11, 5, 7, 8] where the problem is considered using the quasiconformal techniques. In higher dimensions the usual method is to reduce, by substituting $v = \sigma^{1/2}u$, the conductivity equation (1) to the Schrödinger equation and then to apply the methods of scattering theory. Indeed, after such a substitution v satisfies

$$\Delta v - qv = 0,$$

where $q = \sigma^{-1/2}\Delta\sigma^{1/2}$. This substitution is possible only if σ has some smoothness. In the case $\sigma \in L^\infty$, relevant for practical applications the reduction to the Schrödinger equation fails. In two dimensional case we can overcome this by using methods of complex analysis. However, what we adopt from the scattering theory type approaches is the use of exponentially growing solutions, the so-called geometric optics solutions to the conductivity equation (1). These are specified by the condition

$$u(z,\xi) = e^{i\xi z}\left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right) \quad \text{as } |z| \to \infty, \tag{6}$$

where $\xi, z \in \mathbb{C}$ and ξz denotes the usual product of these complex numbers. Here we have set $\sigma \equiv 1$ outside Ω to get an equation defined globally. Studying the ξ-dependence of these solutions then gives rise to the basic concept of this presentation, the *nonlinear Fourier transform* $\tau_\sigma(\xi)$. The detailed definition will be given Section 2.6.

Thus to start the study of $\tau_\sigma(\xi)$ we need first to establish the existence of exponential solutions. Already here the quasiconformal techniques are essential. We note that the study of the inverse problems is closely related to the non-linear Fourier transform: It is not difficult to show that the Dirichlet-to-Neumann boundary operator Λ_σ determines the nonlinear Fourier transforms $\tau_\sigma(\xi)$ for all $\xi \in \mathbb{C}$. Therefore the main difficulty, and our main strategy, is to invert the nonlinear Fourier transform, show that $\tau_\sigma(\xi)$ determines $\sigma(z)$ almost everywhere.

The properties of the nonlinear Fourier transform depend on the underlying differential equation. In one dimension the basic properties of the transform are fairly well understood, while deeper results such as analogs of Carleson's L^2-converge theorem remain open. The reader should consult the excellent lecture notes of Tao and Thiele [75] for an introduction to the one-dimensional theory.

For (1) with nonsmooth σ, many basic questions concerning the nonlinear Fourier transform, even such as finding a right version of the Plancherel formula, remain open. What we are able to show is that for $\sigma^{\pm 1} \in L^\infty$, with $\sigma \equiv 1$ near ∞, we have a Riemann-Lebesgue type result,

$$\tau_\sigma \in C_0(\mathbb{C}).$$

Indeed, this requires the asymptotic estimates of the solutions (6), and these are the key point and main technical part of our argument. For results on related equations, see [66]. The nonlinear Fourier transform in two dimensional case is also closely related to the Novikov-Veselov

(NV) equation that is a (2+1)-dimensional nonlinearevolution equation that generalizes the (1+1)-dimensional Korteweg-deVries(KdV) equation, see [14, 50, 76, 78].

2. Calderón's inverse for isotropic L^∞-conductivity

To avoid some of the technical complications, below we assume that the domain $\Omega = \mathbb{D} = \mathbb{D}(1)$, the unit disk. In fact, see [11], the reduction of general Ω to this case is not difficult. Our main aim in this section is to consider the the following uniqueness result and its generalizations:

Theorem 2.1. [11] *Let $\sigma_j \in L^\infty(\mathbb{D})$, $j = 1, 2$. Suppose that there is a constant $c > 0$ such that $c^{-1} \leq \sigma_j \leq c$. If*

$$\Lambda_{\sigma_1} = \Lambda_{\sigma_2},$$

then $\sigma_1 = \sigma_2$ almost everywhere. Here Λ_{σ_i}, $i = 1, 2$, are defined by (5).

For the first steps in numerical implementation of the solution of the inverse problem based on quasiconformal methods see [9].

Our approach will be based on quasiconformal methods, which also enables the use of tools from complex analysis. These are not available in higher dimensions, at least to the same extent, and this is one of the reasons why the problem is still open for L^∞-coefficients in the three and higher dimensional case. The complex analytic connection comes as follows: From Theorem 2.3 below we see that if $u \in W^{1,2}(\mathbb{D})$ is a real-valued solution of (1), then it has the σ-harmonic conjugate $v \in W^{1,2}(\mathbb{D})$ such that

$$\partial_x v = -\sigma \partial_y u, \quad \partial_y v = \sigma \partial_x u. \tag{7}$$

Equivalently (see (26)), the function $f = u + iv$ satisfies the $\mathbb{R}$-linear Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \overline{\frac{\partial f}{\partial z}}, \tag{8}$$

where $\frac{\partial f}{\partial \bar{z}} = \partial_{\bar{z}} f = \frac{1}{2}(\partial_x f + i\partial_y f)$, $\frac{\partial f}{\partial z} = \partial_{\bar{z}} f = \frac{1}{2}(\partial_x f - i\partial_y f)$, and

$$\mu = \frac{1 - \sigma}{1 + \sigma}.$$

In particular, note that μ is real-valued and that the assumptions on σ in Theorem 2.1 imply $\|\mu\|_{L^\infty} \leq k < 1$. This reduction to the Beltrami equation and the complex analytic methods it provides will be the main tools in our analysis of the Dirichlet-to-Neumann map and the solutions to (1).

2.1. **Linear and non-linear Beltrami equations.** A powerful tool for finding the exponential growing solutions to the conductivity equation (including degenerate conductivities) are given by the non-linear Beltrami equation. We therefore first review few of the basic facts here. For more details and results see [5].

We start with general facts on the linear divergence-type equation

$$\operatorname{div} A(z)\nabla u = 0, \qquad z \in \Omega \subset \mathbb{R}^2 \tag{9}$$

where we assume that $u \in W^{1,2}_{loc}(\Omega)$ and that the coefficient matrix

$$A = A(z) = \left[\begin{array}{cc} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{array} \right], \qquad \alpha_{21} = \alpha_{12}, \tag{10}$$

is symmetric and elliptic,

$$\frac{1}{K(z)}|\xi|^2 \leq \langle A(z)\xi, \xi\rangle \leq K(z)|\xi|^2, \qquad \xi \in \mathbb{R}^2, \tag{11}$$

almost everywhere in Ω. Here, $\langle \eta, \xi \rangle = \eta_1\xi_1 + \eta_2\xi_2$ for $\eta, \xi \in \mathbb{R}^2$. We denote by $K_A(z)$ the smallest number for which (11) is valid. We start with the case when $A(z)$ is assumed to be isotropic, $A(z) = \sigma(z)\mathbf{I}$ with $\sigma(z) \in \mathbb{R}_+$. We also assume that there is $K \in \mathbb{R}_+$ such that $K_A(z) \leq K$ almost everywhere.

For many purposes it is convenient to express the above ellipticity condition (11) in terms of the following single inequality:

$$|\xi|^2 + |A(z)\xi|^2 \leq \left(K_A(z) + \frac{1}{K_A(z)} \right) \langle A(z)\xi, \xi\rangle \tag{12}$$

for almost every $z \in \Omega$ and all $\xi \in \mathbb{R}^2$. For the symmetric matrix $A(z)$ this is seen by representing the matrix as a diagonal matrix in the coordinates given by the eigenvectors.

Below we will study the divergence equation (9) by reducing it to the complex Beltrami system. For solutions to the divergence equation (9) a conjugate structure, similar to harmonic functions, is provided by the *Hodge star* operator $*$, which here really is nothing more than the (counterclockwise) rotation by 90 degrees,

$$* \;= \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right] : \mathbb{R}^2 \to \mathbb{R}^2, \qquad ** = -\mathbf{I}. \tag{13}$$

There are two vector fields associated with each solution to the homogeneous equation

$$\operatorname{div} A(z)\nabla u = 0, \qquad u \in W^{1,2}_{loc}(\Omega).$$

The first, $E = \nabla u$, has zero curl (in the sense of distributions, the curl of any gradient field is zero), while the second, $B = A(z)\nabla u$, is divergence-free as a solution to the equation.

It is the Hodge star $*$ operator that transforms curl-free fields into divergence-free fields, and vice versa. In particular, if

$$E = \nabla w = (w_x, w_y), \qquad w \in W^{1,1}_{loc}(\Omega),$$

then $*E = (-w_y, w_x)$ and hence

$$\operatorname{div}(*E) = \operatorname{div}(*\nabla w) = 0,$$

at least in the distributional sense. We recall here the following well-known fact from calculus (the Poincaré lemma).

Lemma 2.2. *Let $E \in L^p(\Omega, \mathbb{R}^2)$, $p \geq 1$, be a vector field defined on a simply connected domain Ω. If $\operatorname{Curl} E = 0$, then E is a gradient field; that is, there exists a real-valued function $u \in W^{1,p}(\Omega)$ such that $\nabla u = E$.*

When u is A-harmonic function in a simply connected domain Ω, that is, u solves the equation $\operatorname{div} A(z)\nabla u = 0$, then the field $*A\nabla u$ is curl-free and may be rewritten as

$$\nabla v = *A(z)\nabla u, \tag{14}$$

where $v \in W^{1,2}_{loc}(\Omega)$ is some Sobolev function unique up to an additive constant. This function v we call the *A-harmonic conjugate* of u. Sometimes in the literature one also finds the term *stream function* used for v.

The ellipticity conditions for A can be equivalently formulated for the induced complex function $f = u+iv$. We arrive, after a lengthy but quite routine purely algebraic manipulation, at the equivalent complex first-order equation for $f = u + iv$, which we record in the following theorem.

Theorem 2.3. *Let Ω be a simply connected domain and let $u \in W^{1,1}_{loc}(\Omega)$ be a solution to*

$$\operatorname{div} A\,\nabla u = 0. \tag{15}$$

If $v \in W^{1,1}(\Omega)$ is a solution to the conjugate A-harmonic equation (14), then the function $f = u + iv$ satisfies the homogeneous Beltrami equation

$$\frac{\partial f}{\partial \bar z} - \mu(z)\frac{\partial f}{\partial z} - \nu(z)\,\overline{\frac{\partial f}{\partial z}} = 0. \tag{16}$$

The coefficients are given by

$$\mu = \frac{\alpha_{22} - \alpha_{11} - 2i\alpha_{12}}{1 + \mathrm{Trace}(A) + \det A}, \qquad \nu = \frac{1 - \det A}{1 + \mathrm{Trace}(A) + \det A}. \tag{17}$$

Conversely, if $f \in W^{1,1}_{loc}(\Omega, \mathbb{C})$ is a mapping satisfying (16), then $u = Re(f)$ and $v = Im(f)$ satisfy (14) with A given by solving the complex equations in (17),

$$\alpha_{11}(z) = \frac{|1-\mu|^2 - |\nu|^2}{|1+\nu|^2 - |\mu|^2}, \tag{18}$$

$$\alpha_{22}(z) = \frac{|1+\mu|^2 - |\nu|^2}{|1+\nu|^2 - |\mu|^2}, \tag{19}$$

$$\alpha_{12}(z) = \alpha_{21}(z) = \frac{-2\, Im\,(\mu)}{|1+\nu|^2 - |\mu|^2}, \tag{20}$$

The ellipticity of A can be explicitly measured in terms of μ and ν. The optimal ellipticity bound in (11) is

$$K_A(z) = \max\{\lambda_1(z), 1/\lambda_2(z)\}, \tag{21}$$

where $0 < \lambda_2(z) \leq \lambda_1(z) < \infty$ are the eigenvalues of $A(z)$. With this choice we have pointwise

$$|\mu(z)| + |\nu(z)| = \frac{K_A(z) - 1}{K_A(z) + 1} < 1. \tag{22}$$

We also denote by $K_f(z)$ the smallest number for which the inequality

$$\|Df(z)\|^2 \leq K_f(z) J(z, f) \tag{23}$$

is valid. Here, $Df(z) \in \mathbb{R}^2$ is the Jacobian matrix (or the derivative) of f at z and $J(z, f) = \det(Df(z))$ is the Jacobian determinant of f.

Below, let $k \in [0, 1]$ and $K \in [1, \infty]$ be constants satisfying

$$\sup_{z \in \Omega}(|\mu(z)| + |\nu(z)|) \leq k \quad \text{and} \quad K := \frac{1+k}{1-k}. \tag{24}$$

Then (16) yields

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq k \left|\frac{\partial f}{\partial z}\right|.$$

The above ellipticity bounds have then the relation

$$K_f(z) \leq K_A(z) \leq K \quad \text{for a.e. } z \in \Omega. \tag{25}$$

A mapping $f \in W^{1,2}_{loc}(\Omega)$ satisfying (23) with $K_f(z) \leq K < \infty$ is called a *K-quasiregular mappings.* If f is a homeomorphism, we call it K-*quasiconformal.* By Stoilow's factorization Theorem B.9 any K-quasiregular mapping is a composition of holomorphic function and a K-quasiconformal mapping.

Note the following:

1. In this correspondence, ν is real valued if and only if the matrix A is symmetric.

2. A has determinant 1 if and only if $\nu = 0$ (this corresponds to the $\mathbb{C}$-linear Beltrami equation).

3. A is isotropic, that is, $A = \sigma(z)\mathbf{I}$ with $\sigma(z) \in \mathbb{R}_+$, if and only if $\mu(z) = 0$. For such A, the Beltrami equation (16) then takes the form

$$\frac{\partial f}{\partial \bar{z}} - \frac{1-\sigma}{1+\sigma}\,\overline{\frac{\partial f}{\partial z}} = 0. \tag{26}$$

2.2. **Existence and uniqueness for non-linear Beltrami equations.** Solutions to the Beltrami equation conformal near infinity are particularly useful in solving the equation.

When μ and ν as above have compact support and we have a $W^{1,2}_{loc}(\mathbb{C})$ solution to the Beltrami equation $f_{\bar{z}} = \mu f_z + \nu f_{\bar{z}}$ in $\mathbb{C}$, where $f_{\bar{z}} = \partial_{\bar{z}} f$ and $f_z = \partial_z f$, normalized by the condition

$$f(z) = z + \mathcal{O}(1/z)$$

near ∞, we call f a *principal solution.* Indeed, with the Cauchy and Beurling transform (see the appendix) we have the identities

$$\frac{\partial f}{\partial z} = 1 + \mathcal{S}\,\frac{\partial f}{\partial \bar{z}}, \tag{27}$$

and

$$f(z) = z + \mathcal{C}\left(\frac{\partial f}{\partial \bar{z}}\right)(z), \qquad z \in \mathbb{C}. \tag{28}$$

Principal solutions are necessarily homeomorphisms. In fact we have the following fundamental Measurable Riemann Mapping Theorem,

Theorem 2.4. *Let $\mu(z)$ be compactly supported measurable function defined in $\mathbb{C}$ with $\|\mu\|_{L^\infty} \le k < 1$. Then there is a unique principal solution to the Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z} \quad \text{for almost every } z \in \mathbb{C},$$

and the solution $f \in W^{1,2}_{loc}(\mathbb{C})$ is a K-quasiconformal homeomorphism of $\mathbb{C}$.

The result holds also for the general Beltrami equation with coefficients μ and ν, see Theorem 2.5 below.

In constructing the exponentially growing solutions to the divergence and Beltrami equations, the most powerful approach is by non-linear Beltrami equations which we next discuss.

When one is looking for solutions to the general nonlinear elliptic systems

$$\frac{\partial f}{\partial \bar{z}} = H\big(z, f, \frac{\partial f}{\partial z}\big), \qquad z \in \mathbb{C} \tag{29}$$

there are necessarily some constraints to be placed on the function H that we now discuss. We write

$$H : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}.$$

We will not strive for full generality, but settle for the following special case. For the most general existence results, with very weak assumptions on H, see [5]. Here we assume:

(1) The homogeneity condition, that $f_{\bar{z}} = 0$ whenever $f_z = 0$, equivalently,

$$H(z, w, 0) \equiv 0, \qquad \text{for almost every } (z, w) \in \mathbb{C} \times \mathbb{C}$$

(2) The uniform ellipticity condition, that for almost every $z, w \in \mathbb{C}$ and all $\zeta, \xi \in \mathbb{C}$,

$$|H(z, w, \zeta) - H(z, w, \xi)| \le k|\zeta - \xi|, \qquad 0 \le k < 1$$

(3) The Lipschitz continuity in the function variable,

$$|H(z, w_1, \zeta) - H(z, w_2, \zeta)| \le C|\zeta|\,|w_1 - w_2|$$

for some absolute constant C independent of z and ζ.

Theorem 2.5. *Suppose $H : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ satisfies the conditions 1–3 above and is compactly supported in the z-variable. Then the uniformly elliptic nonlinear differential equation*

$$\frac{\partial f}{\partial \bar{z}} = H\Big(z, f, \frac{\partial f}{\partial z}\Big) \tag{30}$$

admits exactly one principal solution $f \in W^{1,2}_{loc}(\mathbb{C})$.

Sketch of the proof. (For complete proof, see [5, Chapter 8]) Uniqueness is easy. Suppose that both f and g are principal solutions to (30). Then

$$\frac{\partial f}{\partial \bar{z}} = H\Big(z, f, \frac{\partial f}{\partial z}\Big), \quad \frac{\partial g}{\partial \bar{z}} = H\Big(z, g, \frac{\partial f}{\partial z}\Big).$$

We set $F = f - g$ and estimate

$$\begin{aligned} |F_{\bar z}| &= |H(z,f,f_z) - H(z,g,g_z)| \\ &\leq |H(z,f,f_z) - H(z,f,g_z)| + |H(z,f,g_z) - H(z,g,g_z)| \\ &\leq k|f_z - g_z| + C\chi_R|g_z||f-g|, \end{aligned}$$

where χ_R denotes the characteristic function of the disk $\mathbb{D}(R)$ of radius R and center zero. Put briefly, F satisfies the differential inequality

$$|F_{\bar z}| \leq k|F_z| + C\chi_R|g_z||F|$$

By assumption, the principal solutions $f, g \in W^{1,2}_{loc}(\mathbb{C})$ with

$$\lim_{z\to\infty} f(z) - g(z) = 0$$

Once we observe that

$$\sigma = C\chi_R(z)|g_z| \in L^2(\mathbb{C})$$

and has compact support, Liouville type results such as Theorem B.8 in the Appendix shows us that $F \equiv 0$, as desired.

The proof of existence we only sketch, for details, in the more general setup of Lusin measurable H, see [5, Chapter 8].

We look for a solution f in the form

$$f(z) = z + \mathcal{C}\phi, \qquad \phi \in L^p(\mathbb{C}) \quad \text{of compact support,} \tag{31}$$

where the exponent $p > 2$. Note that

$$f_{\bar z} = \phi, \qquad f_z = 1 + \mathcal{S}\phi.$$

Thus we need to solve only the following integral equation:

$$\phi = H\big(z, z + \mathcal{C}\phi, 1 + \mathcal{S}\phi\big). \tag{32}$$

To solve this equation we first associate with every given $\phi \in L^p(\mathbb{C})$ an operator $\mathbf{R} : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ defined by

$$\mathbf{R}\Phi = H\big(z, z + \mathcal{C}\phi, 1 + \mathcal{S}\Phi\big)$$

Through the ellipticity hypothesis we observe that $\mathbf{R}$ is a contractive operator on $L^p(\mathbb{C})$. Indeed, from (30) we have the pointwise inequality

$$|\mathbf{R}\Phi_1 - \mathbf{R}\Phi_2| \leq k\,|\mathcal{S}\Phi_1 - \mathcal{S}\Phi_2|.$$

Hence

$$\|\mathbf{R}\Phi_1 - \mathbf{R}\Phi_2\|_p \leq k\,\mathbf{S}_p\|\Phi_1 - \Phi_2\|_p, \qquad k\mathbf{S}_p < 1,$$

for p sufficiently close to 2. By the Banach contraction principle, $\mathbf{R}$ has a unique fixed point $\Phi \in L^p(\mathbb{C})$. In other words, with each $\phi \in L^p(\mathbb{C})$ we can associate a unique function $\Phi \in L^p(\mathbb{C})$ such that

$$\Phi = H\big(z, z + \mathcal{C}\phi, 1 + \mathcal{S}\Phi\big) \tag{33}$$

In fact, the procedure (33), $\phi \mapsto \Phi$, gives a well-defined and nonlinear operator $\mathbf{T} : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ by simply requiring that $\mathbf{T}\phi = \Phi$. Further, solving the original integral equation (32) means precisely that we have to find a fixed point for the operator $\mathbf{T}$. This, however, is more involved than in the case of the contraction $\mathbf{R}$, and one needs to invoke the celebrated Schauder fixed-point theorem, see [5, Chapter 8] for details. $\square$

2.3. **Complex Geometric Optics Solutions.** Below in this section, where we prove Theorem 2.1, *we will assume* that A is isotropic,

$$A(z) = \sigma(z)\mathbf{I}, \quad \sigma(z),\ \sigma(z) \in \mathbb{R}_+ \text{ and } c_1 \le \sigma(z) \le c_2 \text{ with } c_1, c_2 > 0.$$

Moreover, we will denote

$$\mu = \frac{1-\sigma(z)}{1+\sigma(z)}.$$

We will use the following convenient notation

$$e_\xi(z) = e^{i(z\xi + \bar{z}\bar{\xi})}, \qquad z, \xi \in \mathbb{C}. \tag{34}$$

The main emphasize in the analysis below is on isotropic conductivities, corresponding to the Beltrami equations of type (26). However, for later purposes it is useful to consider exponentially growing solutions to divergence equations with matrix coefficients, hence we are led to general Beltrami equations.

We will extend the coefficient matrix $A(z)$ to the entire plane $\mathbb{C}$ by requiring $A(z) \equiv \mathbf{I}$ when $|z| \ge 1$. Clearly, this keeps all ellipticity bounds. Moreover, then

$$\mu(z) \equiv \nu(z) \equiv 0, \qquad |z| \ge 1$$

As a first step toward Theorem 2.1, we establish the existence of a family of special solutions to (16). These, called the complex geometric optics solutions, are specified by having the asymptotics

$$f_{\mu,\nu}(z,\xi) = e^{i\xi z} M_{\mu,\nu}(z,\xi), \tag{35}$$

where

$$M_{\mu,\nu}(z,\xi) - 1 = \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } |z| \to \infty. \tag{36}$$

Theorem 2.6. *Let μ and ν be functions supported in $\mathbb{D}$ that k in (24) satisfies $k < 1$. Then for each parameter $\xi \in \mathbb{C}$ and for each $2 \le p < 1 + 1/k$, the equation*

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\,\frac{\partial f}{\partial z} + \nu(z)\overline{\frac{\partial f}{\partial z}} \tag{37}$$

admits a unique solution $f = f_{\mu,\nu} \in W^{1,p}_{loc}(\mathbb{C})$ *that has the form (35) with (36) holding. In particular,* $f(z,0) \equiv 1$.

Proof. Any solution to (37) is quasiregular. If $\xi = 0$, (35) and (36) imply that f is bounded, hence constant by the Liouville theorem.

If $\xi \neq 0$, we seek for a solution $f = f_{\mu,\nu}(z,\xi)$ of the form

$$f(z,\xi) = e^{i\xi\psi_\xi(z)}, \qquad \psi_\xi(z) = z + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } |z| \to \infty \tag{38}$$

Substituting (38) into (37) indicates that ψ_ξ is the principal solution to the quasilinear equation

$$\frac{\partial}{\partial \bar z}\psi_\xi(z) = \mu(z)\frac{\partial}{\partial z}\psi_\xi(z) - \frac{\overline{\xi}}{\xi}\, e_{-\xi}\big(\psi_\xi(z)\big)\,\nu(z)\,\overline{\frac{\partial}{\partial z}\psi_\xi(z)}. \tag{39}$$

The function $H(z,w,\zeta) = \mu(z)\zeta - (\overline{\xi}/\xi)\,\nu(z)\,e_\xi(w)\,\overline{\zeta}$ satisfies requirements 1–3 of Theorem 2.5. We thus obtain the existence and uniqueness of the principal solution ψ_ξ in $W^{1,2}_{loc}(\mathbb{C})$. Equation (39) together with Theorem B.5 yields $\psi_\xi \in W^{1,p}_{loc}(\mathbb{C})$ for all $p < 1 + 1/k$ since $|\mu(z)| \le k$ and e_ξ is unimodular.

Finally, to see the uniqueness of the complex geometric optics solution $f_{\mu,\nu}$, let $f \in W^{1,2}_{loc}(\mathbb{C})$ be a solution to (37) satisfying

$$f = \alpha e^{i\xi z}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \quad \text{as } |z| \to \infty. \tag{40}$$

Denote then

$$\mu_1(z) = \mu(z)\frac{\overline{\partial_z f(z)}}{\partial_z f(z)}$$

where $\partial_z f(z) \neq 0$ and set $\mu_1 = 0$ elsewhere. Next, let φ be the unique principal solution to

$$\frac{\partial \varphi}{\partial \bar z} = \mu_1 \frac{\partial \varphi}{\partial \bar z}. \tag{41}$$

Then the Stoilow factorization, Theorem B.9, gives $f = h \circ \varphi$, where $h : \mathbb{C} \to \mathbb{C}$ is an entire analytic function. But (40) shows that

$$\frac{h \circ \varphi(z)}{\exp(i\xi\varphi(z))} = \frac{f(z)}{\exp(i\xi\varphi(z))}$$

has the limit α when the variable $z \to \infty$. Thus

$$h(z) \equiv \alpha e^{i\xi z}.$$

Therefore $f(z) = \alpha \exp(i\xi\varphi(z))$. In particular, if we have two solutions f_1, f_2 satisfying (35), (36), then the argument gives

$$f_\varepsilon := f_1 - (1+\varepsilon) f_2 = \varepsilon e^{i\xi\varphi(z)},$$

The principal solutions are homeomorphisms with $\phi(z) = z + O(\frac{1}{z})$ as $|z| \to \infty$, where the error term $O(\frac{1}{z})$ is uniformly bounded by Koebe distortion, Theorem B.6. Letting now $\varepsilon \to 0$ gives $f_1 = f_2$. $\square$

It is useful to note that if a function f satisfies (37), then if satisfies not the same equation but the equation where ν is replaced by $-\nu$. In terms of the real and imaginary parts of $f = u + iv$, we see that

$$\frac{\partial f}{\partial \bar z} = \mu(z)\,\frac{\partial f}{\partial z} + \nu(z)\overline{\frac{\partial f}{\partial z}} \quad \text{if and only if} \tag{42}$$
$$\nabla \cdot A(z)\,\nabla u = 0 \quad \text{and} \quad \nabla \cdot A^*(z)\,\nabla v = 0,$$

where $A^*(z) = *A(z)^{-1}* = \frac{1}{\det A}A$. In case $A(z) = \sigma(z)\mathbf{I}$ is isotropic (*i.e.*, $\mu = 0$) and bounded by positive constants from above and below, we see that

$$\frac{\partial f}{\partial \bar z} = \frac{1\,-\,\sigma}{1\,+\,\sigma}\overline{\frac{\partial f}{\partial z}} \quad \Leftrightarrow \quad \nabla \cdot \sigma\,\nabla u = 0 \quad \text{and} \quad \nabla \cdot \frac{1}{\sigma}\,\nabla v = 0.$$

From these identities we obtain the complex geometric optics solutions also for the conductivity equation (1).

Corollary 2.7. *Suppose $A(z)$ is uniformly elliptic, so that (11) holds with $K \in L^\infty(\mathbb{D})$. Assume also that $A(z) = \mathbf{I}$ for $|z| \geq 1$.*

Then the equation $\nabla \cdot A(z)\nabla u(z) = 0$ admits a unique weak solution $u = u_\xi \in W^{1,2}_{loc}(\mathbb{C})$ such that

$$u(z,\xi) = e^{i\xi z}\left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right) \quad \text{as } |z| \to \infty. \tag{43}$$

Proof. Let $f_{\mu,\nu}$ be the solution considered in Theorem 2.6. Using equivalence (42) we see that the function

$$u(z,\xi) = \mathrm{Re}\ f_{\mu,\nu} + i\,\mathrm{Im}\ f_{\mu,-\nu} \tag{44}$$

satisfies the requirements stated in the claim of the corollary.

To see the uniqueness, let $u \in W^{1,2}_{loc}$ be any function satisfying the divergence equation $\nabla \cdot A(z)\nabla u(z) = 0$ with (43). Then using Theorem 2.3 for the real and imaginary parts of u, we can write it as

$$u = \mathrm{Re}\ f_+ + i\mathrm{Im}\ f_- = \frac{1}{2}(f_+ + f_- + \overline{f_+} - \overline{f_-}),$$

where $f_\pm$ are quasiregular mappings with

$$\frac{\partial f_\pm}{\partial \bar z} = \mu(z)\,\frac{\partial f_\pm}{\partial z} \pm \nu(z)\overline{\frac{\partial f_\pm}{\partial z}}$$

and where μ, ν are given by (17). Given the asymptotics (43), it is not hard to see that both f_+ and f_- satisfy (35) with (36). Therefore $f_+ = f_{\mu,\nu}$ and $f_- = f_{\mu,-\nu}$. $\square$

The exponentially growing solutions of Corollary 2.7 can be considered σ-harmonic counterparts of the usual exponential functions $e^{i\xi z}$. They are the building blocks of the *nonlinear Fourier transform* to be discussed in more detail in Section 2.6.

2.4. **The Hilbert Transform $\mathcal{H}_\sigma$.** Assume that $u \in W^{1,2}(\mathbb{D})$ is a weak solution to $\nabla \cdot \sigma(z)\nabla u(z) = 0$. Then, by Theorem 2.3, u admits a conjugate function $v \in W^{1,2}(\mathbb{D})$ such that

$$\partial_x v = -\sigma\partial_y u, \quad \partial_y v = \sigma\partial_x u.$$

Let us now elaborate on the relationship between u and v. Since the function v is defined only up to a constant, we will normalize it by assuming

$$\int_{\partial\mathbb{D}} v\, ds = 0. \tag{45}$$

This way we obtain a unique map $\mathcal{H}_\mu : W^{1/2,2}(\partial\mathbb{D}) \to W^{1/2,2}(\partial\mathbb{D})$ by setting

$$\mathcal{H}_\mu : u\big|_{\partial\mathbb{D}} \mapsto v\big|_{\partial\mathbb{D}}. \tag{46}$$

In other words, $v = \mathcal{H}_\mu(u)$ if and only if $\int_{\partial\mathbb{D}} v\, ds = 0$, and $u + iv$ has a $W^{1,2}$-extension f to the disk $\mathbb{D}$ satisfying $f_{\overline{z}} = \mu\overline{f_z}$. We call $\mathcal{H}_\mu$ the *Hilbert transform* corresponding to (37).

Since the function $g = -if = v - iu$ satisfies $g_{\overline{z}} = -\mu\overline{g_z}$, we have

$$\mathcal{H}_\mu \circ \mathcal{H}_{-\mu} u = \mathcal{H}_{-\mu} \circ \mathcal{H}_\mu u = -u + \frac{1}{2\pi}\int_{\partial\mathbb{D}} u\, ds. \tag{47}$$

So far we have defined $\mathcal{H}_\mu(u)$ only for real-valued functions u. By setting

$$\mathcal{H}_\mu(iu) = i\mathcal{H}_{-\mu}(u),$$

we extend the definition of $\mathcal{H}_\mu(\cdot)$ to all $\mathbb{C}$-valued functions in $W^{1/2,2}(\partial\mathbb{D})$. Note, however, that $\mathcal{H}_\mu$ still remains only $\mathbb{R}$-linear.

As in the case of analytic functions, the Hilbert transform defines a projection, now on the "μ-analytic" functions. That is, we define $Q_\mu : W^{1/2,2}(\partial\mathbb{D}) \to W^{1/2,2}(\partial\mathbb{D})$ by

$$Q_\mu(g) = \frac{1}{2}\left(g - i\mathcal{H}_\mu g\right) + \frac{1}{4\pi}\int_{\partial\mathbb{D}} g\, ds. \tag{48}$$

Then Q_μ is a projector in the sense that $Q_\mu^2 = Q_\mu$. Furthermore, we have the following lemma.

Lemma 2.8. *If $g \in W^{1/2,2}(\partial\mathbb{D})$, the following conditions are equivalent:*

(a) $g = f\big|_{\partial\mathbb{D}}$, where $f \in W^{1,2}(\mathbb{D})$ satisfies $f_{\overline{z}} = \mu\overline{f_z}$

(b) $Q_\mu(g)$ is a constant

Proof. Condition (a) holds if and only if $g = u + i\mathcal{H}_\mu u + ic$ for some real-valued $u \in W^{1/2,2}(\partial\mathbb{D})$ and real constant c. If g has this representation, then $Q_\mu(g) = \frac{1}{4\pi}\int_{\partial\mathbb{D}} uds + ic$. On the other hand, if $Q_\mu(g)$ is a constant, then we put $g = u + iw$ into (48) and use (47) to show that $w = \mathcal{H}_\mu u + \text{constant}$. This shows that (a) holds. □

The Dirichlet-to-Neumann map (5) and the Hilbert transform (46) are closely related, as the next lemma shows.

Theorem 2.9. *Choose the counterclockwise orientation for $\partial\mathbb{D}$ and denote by ∂_T the tangential (distributional) derivative on $\partial\mathbb{D}$ corresponding to this orientation. We then have*

$$\partial_T\mathcal{H}_\mu(u) = \Lambda_\sigma(u). \tag{49}$$

In particular, the Dirichlet-to-Neumann map Λ_σ uniquely determines $\mathcal{H}_\mu$, $\mathcal{H}_{-\mu}$ and $\Lambda_{1/\sigma}$.

Proof. By the definition of Λ_σ we have

$$\int_{\partial\mathbb{D}} \varphi\Lambda_\sigma u\, ds = \int_{\mathbb{D}} \nabla\varphi\cdot\sigma\nabla u\, dm(x)\,, \qquad \varphi \in C^\infty(\overline{\mathbb{D}}).$$

Thus, by (7) and integration by parts, we get

$$\int_{\partial\mathbb{D}} \varphi\Lambda_\sigma u\, ds = \int_{\mathbb{D}} (\partial_x\varphi\,\partial_y v - \partial_y\varphi\,\partial_x v)\; dm(x) = -\int_{\partial\mathbb{D}} v\,\partial_T\varphi\, ds,$$

and (49) follows. Next,

$$-\mu = (1 - 1/\sigma)/(1 + 1/\sigma),$$

and so $\Lambda_{1/\sigma}(u) = \partial_T\mathcal{H}_{-\mu}(u)$. Since by (47) $\mathcal{H}_\mu$ uniquely determines $\mathcal{H}_{-\mu}$, the proof is complete. □

With these identities we can now show that, for the points z that lie outside $\mathbb{D}$, the values of the complex geometric optics solutions $f_\mu(z,\xi)$ and $f_{-\mu}(z,\xi)$ are determined by the Dirichlet-to-Neumann operator Λ_σ.

Theorem 2.10. *Let σ and $\widetilde{\sigma}$ be two conductivities satisfying the assumptions of Theorem 2.1 and assume $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$. Then if μ and $\widetilde{\mu}$ are the corresponding Beltrami coefficients, we have*

$$f_\mu(z,\xi) = f_{\widetilde{\mu}}(z,\xi) \quad \text{and} \quad f_{-\mu}(z,\xi) = f_{-\widetilde{\mu}}(z,\xi) \tag{50}$$

for all $z \in \mathbb{C}\setminus\overline{\mathbb{D}}$ and $\xi \in \mathbb{C}$.

Proof. By Theorem 2.9 the condition $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$ implies that $\mathcal{H}_\mu = \mathcal{H}_{\widetilde{\mu}}$. In the same way Λ_σ determines $\Lambda_{\sigma^{-1}}$, and so it is enough to prove the first claim of (50).

Fix the value of the parameter $\xi \in \mathbb{C}$. From (48) we see that the projections $Q_\mu = Q_{\widetilde{\mu}}$, and thus by Lemma 2.8

$$Q_\mu(\widetilde{f}) = Q_{\widetilde{\mu}}(\widetilde{f}) \quad \text{is constant.}$$

Here we used the notation $\widetilde{f} = f_{\widetilde{\mu}}|_{\partial\mathbb{D}}$. Using Lemma 2.8 again, we see that there exists a function $G \in W^{1,2}(\mathbb{D})$ such that $G_{\overline{z}} = \mu\overline{G_z}$ in $\mathbb{D}$ and

$$G\big|_{\partial\mathbb{D}} = \widetilde{f}.$$

We then define $G(z) = f_{\widetilde{\mu}}(z,\xi)$ for z outside $\mathbb{D}$. Now $G \in W^{1,2}_{loc}(\mathbb{C})$, and it satisfies $G_{\overline{z}} = \mu\overline{G_z}$ in the whole plane. Thus it is quasiregular, and so $G \in W^{1,p}_{loc}(\mathbb{C})$ for all $2 \le p < 2 + 1/k$, $k = \|\mu\|_\infty$. But now G is a solution to (35) and (36). By the uniqueness part of Theorem 2.6, we obtain $G(z) \equiv f_\mu(z,\xi)$. $\square$

Similarly, the Dirichlet-to-Neumann operator determines the complex geometric optics solutions to the conductivity equation at every point z outside the disk $\mathbb{D}$.

Corollary 2.11. *Let σ and $\widetilde{\sigma}$ be two conductivities satisfying the assumptions of Theorem 2.1 and assume $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$.*
Then

$$u_\sigma(z,\xi) = u_{\widetilde{\sigma}}(z,\xi) \quad \text{for all} \quad z \in \mathbb{C}\setminus\overline{\mathbb{D}} \quad \text{and} \quad \xi \in \mathbb{C}.$$

Proof. The claim follows immediately from the previous theorem and the representation $u_\sigma(z,\xi) = \mathrm{Re}\, f_\mu(z,\xi) + i\mathrm{Im}\, f_{-\mu}(z,\xi)$. $\square$

2.5. **Dependence on Parameters.** Our strategy will be to extend the identities $f_\mu(z,\xi) = f_{\widetilde{\mu}}(z,\xi)$ and $u_\sigma(z,\xi) = u_{\widetilde{\sigma}}(z,\xi)$ from outside the disk to points z inside $\mathbb{D}$. Once we do that, Theorem 2.1 follows via the equation $f_{\overline{z}} = \mu\overline{f_z}$.

For this purpose we need to understand the ξ-dependence in $f_\mu(z,\xi)$ and the quantities controlling it. In particular, we will derive equations relating the solutions and their derivatives with respect to the ξ-variable. For this purpose we prove the following theorem.

Theorem 2.12. *The complex geometric optics solutions $u_\sigma(z,\xi)$ and $f_\mu(z,\xi)$ are (Hölder)-continuous in z and C^∞-smooth in the parameter ξ.*

The continuity in the z-variable is clear since f_μ is a quasiregular function of z. However, for analyzing the ξ-dependence we need to realize the solutions in a different manner, by identities involving linear operators that depend smoothly on the variable ξ.

Let

$$f_\mu(z,\xi) = e^{i\xi z} M_\mu(z,\xi), \quad f_{-\mu}(z,\xi) = e^{i\xi z} M_{-\mu}(z,\xi)$$

be the solutions of Theorem 2.6 corresponding to conductivities σ and σ^{-1}, respectively. We can write (8), (35) and (36) in the form

$$\frac{\partial}{\partial \bar z} M_\mu = \mu(z)\, \overline{\frac{\partial}{\partial z}(e_\xi M_\mu)}, \qquad M_\mu - 1 \in W^{1,p}(\mathbb{C}) \tag{51}$$

when $2 < p < 1+1/k$. By taking the Cauchy transform and introducing a $\mathbb{R}$-linear operator L_μ,

$$L_\mu g = \mathcal{C}\left(\mu \frac{\partial}{\partial \bar z}\left(e_{-\xi}\, \overline{g}\right)\right), \tag{52}$$

we see that (51) is equivalent to

$$(\mathbf{I} - L_\mu) M_\mu = 1. \tag{53}$$

Theorem 2.13. *Assume that $\xi \in \mathbb{C}$ and $\mu \in L^\infty(\mathbb{C})$ is compactly supported with $\|\mu\|_\infty \le k < 1$. Then for $2 < p < 1 + 1/k$ the operator*

$$\mathbf{I} - L_\mu : W^{1,p}(\mathbb{C}) \oplus \mathbb{C} \to W^{1,p}(\mathbb{C}) \oplus \mathbb{C}$$

is bounded and invertible.

Here we denote by $W^{1,p}(\mathbb{C}) \oplus \mathbb{C}$ the Banach space consisting of functions of the form $f = \text{constant} + f_0$, where $f_0 \in W^{1,p}(\mathbb{C})$.

Proof. We write $L_\mu(g)$ as

$$L_\mu(g) = \mathcal{C}\left(\mu\, e_{-\xi}\, \overline{g_z} - i\, \overline{\xi}\, \mu\, e_{-\xi}\, \overline{g}\right). \tag{54}$$

Then Theorem B.2 shows that

$$L_\mu : W^{1,p}(\mathbb{C}) \oplus \mathbb{C} \to W^{1,p}(\mathbb{C}) \tag{55}$$

is bounded. Thus we need only establish invertibility.

To this end let us assume $h \in W^{1,p}(\mathbb{C})$. Consider the equation

$$(\mathbf{I} - L_\mu)(g + C_0) = h + C_1, \tag{56}$$

where $g \in W^{1,p}(\mathbb{C})$ and C_0, C_1 are constants. Then

$$C_0 - C_1 = g - h - L_\mu(g + C_0),$$

which by (55) gives $C_0 = C_1$. By differentiating and rearranging we see that (56) is equivalent to $g_{\overline{z}} - \mu(e_{-\xi}\, \overline{g})_{\overline{z}} = h_{\overline{z}} + \mu(\overline{C}_0 e_{-\xi})_{\overline{z}}$, or in other words, to

$$g_{\overline{z}} - (\mathbf{I} - \mu e_{-\xi}\overline{\mathcal{S}})^{-1}\big(\mu (e_{-\xi})_{\overline{z}} \overline{g}\big) = (\mathbf{I} - \mu e_{-\xi}\overline{\mathcal{S}})^{-1}\big(h_{\overline{z}} + \mu(\overline{C}_0 e_{-\xi})_{\overline{z}}\big). \tag{57}$$

We analyze this by using the real linear operator R defined by

$$R(g) = \mathcal{C}\left(\mathbf{I} - \nu\overline{\mathcal{S}}\right)^{-1}(\alpha\overline{g}),$$

where $\nu(z) = \mu e_{-\xi}$ satisfies $|\nu(z)| \le k\chi_{\mathbb{D}}(z)$ and α is defined by $\alpha = \mu(e_{-\xi})_{\overline{z}} = -i\overline{\xi}\,\mu\, e_{-\xi}$. According to Theorem B.4, $\mathbf{I} - \nu\overline{\mathcal{S}}$ is invertible in $L^p(\mathbb{C})$ when $1+k < p < 1+1/k$, while the boundedness of the Cauchy transform requires $p > 2$. Therefore R is a well-defined and bounded operator on $L^p(\mathbb{C})$ for $2 < p < 1+1/k$.

Moreover, the right hand side of (57) belongs to $L^p(\mathbb{C})$ for each $h \in W^{1,p}(\mathbb{C})$. Hence this equation admits a unique solution $g \in W^{1,p}(\mathbb{C})$ if and only if the operator $\mathbf{I} - R$ is invertible in $L^p(\mathbb{C}),\ 2 < p < 1+1/k$.

To get this we will use Fredholm theory. First, Theorem B.3 shows that R is a compact operator on $L^p(\mathbb{C})$ when $2 < p < 1+1/k$. Therefore it suffices to show that $\mathbf{I} - R$ is injective. Suppose now that $g \in L^p(\mathbb{C})$ satisfies

$$g = Rg = \mathcal{C}\left(\mathbf{I} - \nu\overline{\mathcal{S}}\right)^{-1}(\alpha\overline{g}).$$

Then $g \in W^{1,p}(\mathbb{C})$ by Theorem B.2 and $g_{\overline{z}} = \left(\mathbf{I} - \nu\overline{\mathcal{S}}\right)^{-1}(\alpha\overline{g})$. Equivalently,

$$g_{\overline{z}} - \nu\overline{g_z} = \alpha\overline{g} \tag{58}$$

Thus the assumptions of Theorem B.8 are fulfilled, and we must have $g \equiv 0$. Therefore $\mathbf{I} - R$ is indeed injective on $L^p(\mathbb{C})$. By the Fredholm alternative, it therefore is invertible in $L^p(\mathbb{C})$. Therefore the operator $\mathbf{I} - L_\mu$ is invertible in $W^{1,p}(\mathbb{C}),\ 2 < p < 1+1/k$. □

A glance at (52) shows that $\xi \to L_\mu$ is an infinitely differentiable family of operators. Therefore, with Theorem 2.13, we see that $M_\mu = (\mathbf{I} - L_\mu)^{-1}1$ is C^∞-smooth in the parameter ξ. Thus we have obtained Theorem 2.12.

2.6. **Nonlinear Fourier Transform.** The idea of studying the $\overline{\xi}$-dependence of operators associated with complex geometric optics solutions was used by Beals and Coifman [13] in connection with the inverse scattering approach to KdV-equations. Here we will apply this method to the solutions u_σ to the conductivity equation (1) and show that they satisfy a simple $\overline{\partial}$-equation with respect to the parameter ξ.

We start with the representation $u_\sigma(z,\xi) = \mathrm{Re}\ f_\mu(z,\xi) + i\,\mathrm{Im}\ f_{-\mu}(z,\xi)$, where $f_{\pm\mu}$ are the solutions to the corresponding Beltrami equations; in particular, they are analytic outside the unit disk. Hence with the asymptotics (36) they admit the following power series development,

$$f_{\pm\mu}(z,\xi) = e^{i\xi z}\Big(1 + \sum_{n=1}^{\infty} b_n^{\pm}(\xi) z^{-n}\Big), \qquad |z| > 1, \tag{59}$$

where $b_n^+(\xi)$ and $b_n^-(\xi)$ are the coefficients of the series, depending on the parameter ξ. For the solutions to the conductivity equation, this gives

$$u_\sigma(z,\xi) = e^{i\xi z} + \frac{a(\xi)}{z}\, e^{i\xi z} + \frac{b(\xi)}{\bar z}\, e^{-i\bar\xi \bar z} + e^{i\xi z}\, \mathcal{O}(\frac{1}{|z|^2})$$

as $z \to \infty$, where

$$a(\xi) = \frac{b_1^+(\xi) + b_1^-(\xi)}{2}, \qquad b(\xi) = \frac{\overline{b_1^+(\xi)} - \overline{b_1^-(\xi)}}{2\,\bar z}. \tag{60}$$

Fixing the z-variable, we take the $\partial_{\bar\xi}$-derivative of $u_\sigma(z,\xi)$ and get

$$\partial_{\bar\xi} u_\sigma(z,\xi) = -i\tau_\sigma(\xi)\, e^{-i\bar\xi\bar z}\, \big(1 + \mathcal{O}(\frac{1}{|z|})\big), \tag{61}$$

with the coefficient

$$\tau_\sigma(\xi) := \overline{b(\xi)}. \tag{62}$$

However, the derivative $\partial_{\bar\xi} u_\sigma(z,\xi)$ is another solution to the conductivity equation! From the uniqueness of the complex geometric optics solutions under the given exponential asymptotics, Corollary 2.7, we therefore have the simple but important relation

$$\partial_{\bar\xi} u_\sigma(z,\xi) \;=\; -i\,\tau_\sigma(\xi)\,\overline{u_\sigma(z,\xi)} \quad \text{for all } \ \xi, z \in \mathbb{C}. \tag{63}$$

The remarkable feature of this relation is that the coefficient $\tau_\sigma(\xi)$ does not depend on the space variable z. Later, this phenomenon will become of crucial importance in solving the inverse problem.

In analogy with the one-dimensional scattering theory of integrable systems and associated inverse problems (see [13, 67, 68]), we call τ_σ the *nonlinear Fourier transform* of σ.

To understand the basic properties of the nonlinear Fourier transform, we need to return to the Beltrami equation. We will first show that the Dirichlet-to-Neumann data determines τ_σ. This is straightforward. Then the later sections are devoted to showing that the nonlinear Fourier transform τ_σ determines the coefficient σ almost everywhere. There does not seem to be any direct method for this, rather we will have to show that from τ_σ we can determine the exponentially growing solutions $f_{\pm\mu}$ defined in the entire plane. From this information the coefficient μ, and hence σ, can be found.

The non-linear Fourier transform τ_σ has many properties which are valid for the linear Fourier transform. We have the usual transformation rules under scaling and and translation,

$$\sigma_1(z) = \sigma(Rz) \quad \Rightarrow \quad \tau_{\sigma_1}(\xi) = \frac{1}{R}\tau_\sigma(\xi/R),$$

$$\sigma_2(z) = \sigma(z+p) \quad \Rightarrow \quad \tau_{\sigma_2}(\xi) = e^{i(p\xi+\bar{p}\bar{\xi})}\tau_\sigma(\xi).$$

but not much is known concerning questions such as the possibility of a Plancherel formula. However, some simple mapping properties of it can be proven. We will show that for σ as above, $\tau_\sigma \in L^\infty$. For this we need the following result, which is useful also elsewhere.

Here let $f_\mu(z,\xi) = e^{i\xi z}M_\mu(z,\xi)$ and $f_{-\mu}(z,\xi) = e^{i\xi z}M_{-\mu}(z,\xi)$ be the solutions of Theorem 2.6 corresponding to conductivities σ and σ^{-1}, respectively, which are holomorphic outside $\mathbb{D}$.

Theorem 2.14. *For every $\xi, z \in \mathbb{C}$ we have $M_{\pm\mu}(z,\xi) \neq 0$. Moreover,*

$$Re\left(\frac{M_\mu(z,\xi)}{M_{-\mu}(z,\xi)}\right) > 0\ . \tag{64}$$

Proof. First, note that (8) implies, for $M_{\pm\mu}$,

$$\frac{\partial}{\partial \bar{z}}\, M_{\pm\mu} \mp \mu e_{-\xi}\, \overline{\frac{\partial}{\partial z}\, M_{\pm\mu}} = \mp i\bar{\xi}\mu e_{-\xi}\overline{M_{\pm\mu}}. \tag{65}$$

Thus we may apply Theorem B.8 to get

$$M_{\pm\mu}(z) = \exp(\eta_\pm(z)) \neq 0, \tag{66}$$

and consequently $M_\mu/M_{-\mu}$ is well defined. Second, if (64) is not true, the continuity of $M_{\pm\mu}$ and the fact $\lim_{z\to\infty} M_{\pm\mu}(z,\xi) = 1$ imply the existence of $z_0 \in \mathbb{C}$ such that

$$M_\mu(z_0,\xi) = itM_{-\mu}(z_0,\xi)$$

for some $t \in \mathbb{R}\setminus\{0\}$ and $\xi \in \mathbb{C}$. But then, $g = M_\mu - itM_{-\mu}$ satisfies

$$\begin{aligned} \frac{\partial}{\partial \bar{z}}\, g &= \mu(z)\overline{\frac{\partial}{\partial z}\,(e_\xi g)}, \\ g(z) &= 1 - it + \mathcal{O}\left(\frac{1}{z}\right), \quad \text{as } z \to \infty. \end{aligned}$$

According to Theorem B.8, this implies

$$g(z) = (1-it)\exp(\eta(z)) \neq 0,$$

contradicting the assumption $g(z_0) = 0$. □

The boundedness of the nonlinear Fourier transform is now a simple corollary of Schwarz's lemma.

Theorem 2.15. *The functions* $f_{\pm\mu}(z,\xi) = e^{i\xi z} M_{\pm\mu}(z,\xi)$ *satisfy, for* $|z| > 1$ *and for all* $\xi \in \mathbb{C}$,

$$|m(z)| \leq \frac{1}{|z|}, \ \textit{where } m(z) = \frac{M_\mu(z,\xi) - M_{-\mu}(z,\xi)}{M_\mu(z,\xi) + M_{-\mu}(z,\xi)}. \tag{67}$$

Moreover, for the nonlinear Fourier transform τ_σ*, we have*

$$|\tau_\sigma(\xi)| \leq 1 \quad \text{for all } \xi \in \mathbb{C}. \tag{68}$$

Proof. Fix the parameter $\xi \in \mathbb{C}$. Then by Theorem 2.14, $|m(z)| < 1$ for all $z \in \mathbb{C}$. Moreover, m is holomorphic for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $m(\infty) = 0$, and thus by Schwarz's lemma we have $|m(z)| \leq 1/|z|$ for all $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$.

On the other hand, from the development (59),

$$M_\mu(z,\xi) = 1 + \sum_{n=1}^{\infty} b_n(\xi) z^{-n} \quad \text{for } |z| > 1,$$

and similarly for $M_{-\mu}(z,\xi)$. We see that

$$\tau_\sigma(\xi) = \frac{1}{2}\left(\overline{b_1^+(\xi)} - \overline{b_1^-(\xi)}\right) = \lim_{z\to\infty} \overline{z\, m(z)}\,.$$

Therefore the second claim also follows. □

With these results the Calderón problem reduces to the question whether we can invert the nonlinear Fourier transform.

Theorem 2.16. *The operator* Λ_σ *uniquely determines the nonlinear Fourier transform* τ_σ.

Proof. The claim follows immediately from Theorem 2.10, from the development (59) and from the definition (62) of τ_σ. □

From the relation $-\mu = (1 - 1/\sigma)/(1 + 1/\sigma)$ we see the symmetry

$$\tau_\sigma(\xi) = -\tau_{1/\sigma}(\xi).$$

It follows that the functions

$$u_1 = \operatorname{Re} f_\mu + i \operatorname{Im} f_{-\mu} = u_\sigma \quad \text{and} \quad u_2 = i \operatorname{Re} f_{-\mu} - \operatorname{Im} f_\mu = i u_{1/\sigma} \tag{69}$$

form a "primary pair" of complex geometric optics solutions:

Corollary 2.17. *The functions* $u_1 = u_\sigma$ *and* $u_2 = iu_{1/\sigma}$ *are complex-valued* $W^{1,2}_{loc}(\mathbb{C})$*-solutions to the conductivity equations*

$$\nabla \cdot \sigma \nabla u_1 = 0 \quad \textit{and} \quad \nabla \cdot \frac{1}{\sigma} \nabla u_2 = 0, \tag{70}$$

respectively. In the ξ*-variable they are solutions to the same* $\partial_{\overline{\xi}}$*-equation,*

$$\frac{\partial}{\partial \overline{\xi}}\, u_j(z,\xi) = -i\, \tau_\sigma(\xi)\, \overline{u_j}(z,\xi), \qquad j = 1,2, \tag{71}$$

and their asymptotics, as $|z| \to \infty$, are

$$u_\sigma(z,\xi) = e^{i\xi z}\left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right), \qquad u_{1/\sigma}(z,\xi) = e^{i\xi z}\left(i + \mathcal{O}\left(\frac{1}{|z|}\right)\right).$$

2.7. **Subexponential Growth.** A basic difficulty in the solution to Calderón's problem is to find methods to control the asymptotic behavior in the parameter ξ for complex geometric optics solutions. If we knew that the assumptions of the Liouville type Theorem B.8 were valid in (71), then the equation, hence the Dirichlet-to-Neumann map, would uniquely determine $u_\sigma(z,\xi)$ with $u_{1/\sigma}(z,\xi)$, and the inverse problem could easily be solved. However, we only know from Theorem 2.15 that $\tau_\sigma(\xi)$ is bounded in ξ. It takes considerably more effort to prove the counterpart of the Riemann-Lebesgue lemma, that is,

$$\tau_\sigma(\xi) \to 0, \qquad \text{as } \xi \to \infty.$$

Indeed, this will be one of the consequences of the results in the present section.

It is clear that some control of the parameter ξ is needed for $u_\sigma(z,\xi)$. Within the category of conductivity equations with L^∞-coefficients σ, the complex analytic and quasiconformal methods provide by most powerful tools. Therefore we return to the Beltrami equation. The purpose of this section is to study the ξ-behavior in the functions $f_\mu(z,\xi) = e^{i\xi z}M_\mu(z,\xi)$ and to show that for a fixed z, $M_\mu(z,\xi)$ grows at most subexponentially in ξ as $\xi \to \infty$. Subsequently, the result will be applied to $u_j(z,\xi)$.

For some later purposes we will also need to generalize the situation a bit by considering complex Beltrami coefficients μ_λ of the form $\mu_\lambda = \lambda\mu$, where the constant $\lambda \in \partial\mathbb{D}$ and μ is as before. Exactly as in Theorem 2.6, we can show the existence and uniqueness of $f_{\lambda\mu} \in W^{1,p}_{loc}(\mathbb{C})$ satisfying

$$\frac{\partial}{\partial \bar z} f_{\lambda\mu} = \lambda\mu \overline{\frac{\partial}{\partial z} f_{\lambda\mu}} \quad \text{in } \mathbb{C}, \tag{72}$$

$$f_{\lambda\mu}(z,\xi) = e^{i\xi z}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \quad \text{as } |z| \to \infty. \tag{73}$$

In fact, we have that the function $f_{\lambda\mu}$ admits a representation of the form

$$f_{\lambda\mu}(z,\xi) = e^{i\,\xi\,\varphi_\lambda(z,\xi)}, \tag{74}$$

where for each fixed $\xi \in \mathbb{C}\setminus\{0\}$ and $\lambda \in \partial\mathbb{D}$, $\varphi_\lambda(z,\xi) = z + \mathcal{O}\left(\frac{1}{z}\right)$ for $z \to \infty$. The principal solution $\varphi = \varphi_\lambda(z,\xi)$ satisfies the nonlinear equation

$$\frac{\partial}{\partial \bar z}\, \varphi(z) = \kappa_{\lambda,\xi}\, e_{-\xi}\big(\varphi(z)\big)\, \mu(z)\, \overline{\frac{\partial}{\partial z}\, \varphi(z)} \tag{75}$$

where $\kappa = \kappa_{\lambda,\xi} = -\lambda\, \overline{\xi}^{\,2} |\xi|^{-2}$ is constant in z with $|\kappa_{\lambda,\xi}| = 1$.

The main goal of this section is to show the following theorem.

Theorem 2.18. *If $\varphi = \varphi_\lambda$ and $f_{\lambda\mu}$ are as in (72)–(75), then*

$$\lim_{\xi\to\infty} \varphi_\lambda(z,\xi) = z$$

uniformly in $z \in \mathbb{C}$ and $\lambda \in \partial\mathbb{D}$.

From the theorem we have the immediate consequence,

Corollary 2.19. *If $\sigma, \sigma^{-1} \in L^\infty(\mathbb{C})$ with $\sigma(z) = 1$ outside a compact set, then* $\lim_{\xi\to\infty} \tau_\sigma(\xi) = 0$.

Proof of Corollary 2.19. Let $\lambda = 1$. The principal solutions in (74) have the development

$$\varphi(z,\xi) = z + \sum_{n=1}^{\infty} \frac{c_n(\xi)}{z^n}, \qquad |z| > 1.$$

By Cauchy integral formula and Theorem 2.18 we have $\lim_{\xi\to\infty} c_n(\xi) = 0$ for all $n \in \mathbb{N}$. Comparing now with (59)-(62) proves the claim. $\square$

It remains to prove Theorem 2.18, which will the rest of this section. We shall split the proof up into several lemmas.

Lemma 2.20. *Suppose $\varepsilon > 0$ is given. Suppose also that for $\mu_\lambda(z) = \lambda\mu(z)$, we have*

$$f_n = \mu_\lambda S_n \mu_\lambda S_{n-1} \mu_\lambda \cdots \mu_\lambda S_1 \mu_\lambda, \tag{76}$$

where $S_j : L^2(\mathbb{C}) \to L^2(\mathbb{C})$ are Fourier multiplier operators, each with a unimodular symbol. Then there is a number $R_n = R_n(k,\varepsilon)$ depending only on $k = \|\mu\|_\infty$, n and ε such that

$$|\widehat{f_n}(\eta)| < \varepsilon \quad \text{for } |\eta| > R_n. \tag{77}$$

Proof. It is enough to prove the claim for $\lambda = 1$. By assumption,

$$\widehat{S_j g}(\eta) = m_j(\eta)\widehat{g}(\eta),$$

where $|m_j(\eta)| = 1$ for $\eta \in \mathbb{C}$. We have by (76),

$$\|f_n\|_{L^2} \le \|\mu\|_{L^\infty}^n \|\mu\|_{L^2} \le \sqrt{\pi} k^{n+1} \tag{78}$$

since $\mathrm{supp}(\mu) \subset \mathbb{D}$. Choose ρ_n so that

$$\int_{|\eta|>\rho_n} |\widehat{\mu}(\eta)|^2 \, dm(\eta) < \varepsilon^2. \tag{79}$$

After this, choose $\rho_{n-1}, \rho_{n-2}, \ldots, \rho_1$ inductively so that for $l = n - 1, \ldots, 1$,

$$\pi \int_{|\eta|>\rho_l} |\widehat{\mu}(\eta)|^2 \, dm(\eta) \leq \varepsilon^2 \left(\prod_{j=l+1}^{n} \pi\rho_j \right)^{-2}. \tag{80}$$

Finally, choose ρ_0 so that

$$|\widehat{\mu}(\eta)| < \varepsilon\pi^{-n} \left(\prod_{j=1}^{n} \rho_j \right)^{-1} \qquad \text{when} \quad |\eta| > \rho_0. \tag{81}$$

All these choices are possible since $\mu \in L^1(\mathbb{C}) \cap L^2(\mathbb{C})$.

Now, we set $R_n = \sum_{j=0}^{n} \rho_j$ and claim that (77) holds for this choice of R_n. Hence assume that $|\eta| > \sum_{j=0}^{n} \rho_j$. We have

$$|\widehat{f}_n(\eta)| \leq \left(\int_{|\eta-\zeta|\leq\rho_n} + \int_{|\eta-\zeta|\geq\rho_n} \right) |\widehat{\mu}(\eta - \zeta)| \, |\widehat{f}_{n-1}(\zeta)| \, dm(\zeta). \tag{82}$$

But if $|\eta - \zeta| \leq \rho_n$, then $|\zeta| > \sum_{j=0}^{n-1} \rho_j$. Thus, if we denote

$$\Delta_n = \sup \left\{ |\widehat{f}_n(\eta)| \ : \ |\eta| > \sum_{j=0}^{n} \rho_j \right\},$$

it follows from (82) and (78) that

$$\begin{aligned} \Delta_n &\leq \Delta_{n-1}(\pi\rho_n^2)^{1/2} \|\mu\|_{L^2} + \left(\int_{|\zeta|\geq\rho_n} |\widehat{\mu}(\zeta)|^2 \, dm(\zeta) \right)^{1/2} \|\widehat{f}_{n-1}\|_{L^2} \\ &\leq \pi\rho_n k \, \Delta_{n-1} + k^n \left(\pi \int_{|\zeta|\geq\rho_n} |\widehat{\mu}(\zeta)|^2 \, dm(\zeta) \right)^{1/2} \end{aligned}$$

for $n \geq 2$. Moreover, the same argument shows that

$$\Delta_1 \leq \pi\rho_1 \, k \, \sup\{|\widehat{\mu}(\eta)| \ : \ |\eta| > \rho_0\} + k \left(\pi \int_{|\zeta|>\rho_1} |\widehat{\mu}(\zeta)|^2 \, dm(\zeta) \right)^{1/2}.$$

In conclusion, after iteration we will have

$$\begin{aligned}\Delta_n \quad &\leq \quad (k\pi)^n \left(\prod_{j=1}^{n} \rho_j \right) \sup\{|\widehat{\mu}(\eta)| \; : \; |\eta| > \rho_0\} \\ &\qquad + k^n \sum_{l=1}^{n} \left(\prod_{j=l+1}^{n} \pi \rho_j \right) \left(\pi \int_{|\zeta|>\rho_l} |\widehat{\mu}(\zeta)|^2 \, dm(\zeta) \right)^{1/2}.\end{aligned}$$

With the choices (79)–(81), this leads to

$$\Delta_n \leq (n+1) k^n \varepsilon \leq \frac{\varepsilon}{1-k},$$

which proves the claim. □

Our next goal is to use Lemma 2.20 to prove the asymptotic result required in Theorem 2.18 for the solution of a closely related linear equation.

Theorem 2.21. *Suppose $\psi \in W^{1,2}_{loc}(\mathbb{C})$ satisfies*

$$\frac{\partial \psi}{\partial \bar{z}} \; = \; \kappa \, \mu(z) \, e_{-\xi}(z) \, \frac{\partial \psi}{\partial z} \quad \text{in } \mathbb{C}, \tag{83}$$

$$\psi(z) \; = \; z + \mathcal{O}\left(\frac{1}{z} \right) \quad \text{as } z \to \infty, \tag{84}$$

where κ is a constant with $|\kappa| = 1$.

Then $\psi(z, \xi) \to z$, uniformly in $z \in \mathbb{C}$ and $\kappa \in \partial \mathbb{D}$, as $\xi \to \infty$.

To prove Theorem 2.21 we need some preparation. First, since the L^p-norm of the Beurling transform, denoted as $\mathbf{S}_p$, tends to 1 when $p \to 2$, we can choose a $\delta_k > 0$ so that $k \mathbf{S}_p < 1$ whenever $2 - \delta_k \leq p \leq 2 + \delta_k$. With this notation we then have the following lemma.

Lemma 2.22. *Let $\psi = \psi(\cdot, \xi)$ be the solution of* (83) *and let $\varepsilon > 0$. Then $\psi_{\bar{z}}$ can be decomposed as $\psi_{\bar{z}} = g + h$, where*

(1) $\|h(\cdot, \xi)\|_{L^p} < \varepsilon$ *for* $2 - \delta_k \leq p \leq 2 + \delta_k$ *uniformly in* ξ,
(2) $\|g(\cdot, \xi)\|_{L^p} \leq C_0 = C_0(k)$ *uniformly in* ξ,
(3) $\widehat{g}(\eta, \xi) \to 0$ *as* $\xi \to \infty$.

In statement 3 convergence is uniform on compact subsets of the η-plane and also uniform in $\kappa \in \partial \mathbb{D}$. Here, the Fourier transform is taken with respect to the first variable only.

Proof. We may solve (83) using a Neumann series, which will converge in L^p,

$$\frac{\partial \psi}{\partial \bar{z}} = \sum_{n=0}^{\infty} \left(\kappa \, \mu \, e_{-\xi} \, \mathcal{S} \right)^n \left(\kappa \, \mu \, e_{-\xi} \right).$$

Let

$$h = \sum_{n=n_0}^{\infty} (\kappa\, \mu e_{-\xi}\, \mathcal{S})^n\, (\kappa\, \mu\, e_{-\xi})\,.$$

Then

$$\|h\|_{L^p} \le \pi^{1/p} \frac{k^{n_0+1}\mathbf{S}_p^{n_0}}{1 - k\mathbf{S}_p}$$

and we obtain the first statement by choosing n_0 large enough.

The remaining part clearly satisfies the second statement with a constant C_0 that is independent of ξ and λ. To prove statement 3 we first note that

$$\mathcal{S}(e_{-\xi}\, \phi) = e_{-\xi}\, S_\xi\, \phi,$$

where $\widehat{(S_\xi \phi)}(\eta) = m(\eta - \xi)\, \widehat{\phi}(\eta)$ and $m(\eta) = \eta/\overline{\eta}$. Consequently,

$$(\mu e_{-\xi}\, \mathcal{S})^n \mu\, e_{-\xi} = e_{-(n+1)\xi}\, \mu S_{n\xi}\, \mu S_{(n-1)\xi} \cdots \mu S_\xi\, \mu,$$

and so

$$g = \sum_{j=1}^{n_0} \kappa^j\, e_{-j\xi}\, \mu S_{(j-1)\xi}\, \mu \cdots \mu S_\xi \mu.$$

Therefore,

$$g = \sum_{j=1}^{n_0} e_{-j\xi} G_j,$$

where by Lemma 2.20, $|\widehat{G}_j(\eta)| < \widetilde{\varepsilon}$ whenever $|\eta| > R = \max_{j \le n_0} R_j$. As $\widehat{(e_{j\xi} G_j)}(\eta) = \widehat{G}_j(\eta + j\xi)$, for any fixed compact set K_0, we can take ξ so large that $j\xi + K_0 \subset \mathbb{C} \setminus \mathbb{D}(0, R)$ for each $1 \le j \le n_0$. Then

$$\sup_{\eta \in K_0} |\widehat{g}(\eta, \xi)| \le n_0 \widetilde{\varepsilon}.$$

This proves the lemma. □

Proof of Theorem 2.21. We show first that when $\xi \to \infty$, $\psi_{\bar{z}} \to 0$ weakly in $L^p(\mathbb{C})$, $2 - \delta_k \le p \le 2 + \delta_k$. For this suppose that $f_0 \in L^q(\mathbb{C})$, $q = p/(p-1)$, is fixed and choose $\varepsilon > 0$. Then there exists $f \in C_0^\infty(\mathbb{C})$ such that $\|f_0 - f\|_{L^q(\mathbb{C})} < \varepsilon$, and so by Lemma 2.22,

$$|\langle f_0, \psi_{\bar{z}} \rangle| \le \varepsilon C_1 + \Big| \int_{\mathbb{C}} \widehat{f}(\eta) \widehat{g}(\eta, \xi)\, dm(\eta) \Big|.$$

First choose R so large that

$$\int_{\mathbb{C} \setminus \mathbb{D}(0,R)} |\widehat{f}(\eta)|^2\, dm(\eta) \le \varepsilon^2$$

and then $|\xi|$ so large that $|\widehat{g}(\eta,\xi)| \le \varepsilon/(\sqrt{\pi}R)$ for all $\eta \in \mathbb{D}(R)$. Now,

$$\begin{aligned} \left|\int_{\mathbb{C}} \widehat{f}(\eta)\widehat{g}(\eta,\xi)\,d\eta\right| &\le \int_{\mathbb{D}(R)} \widehat{f}(\eta)\widehat{g}(\eta,\xi)\,d\eta + \int_{\mathbb{C}\setminus\mathbb{D}(R)} \widehat{f}(\eta)\widehat{g}(\eta,\xi)\,d\eta \\ &\le \varepsilon(\|f\|_{L^2(\mathbb{C})} + \|g\|_{L^2(\mathbb{C})}) \le C_2(f)\varepsilon. \end{aligned} \tag{85}$$

The bound is the same for all κ, hence

$$\lim_{|\xi|\to\infty} \sup_{\kappa\in\partial\mathbb{D}} |\langle f_0, \psi_{\bar z}\rangle| = 0. \tag{86}$$

To prove the uniform convergence of ψ itself, we write

$$\psi(z,\xi) = z - \frac{1}{\pi}\int_{\mathbb{D}} \frac{1}{\zeta - z}\frac{\partial}{\partial\overline{\zeta}}\,\psi(\zeta,\xi)\,dm(\zeta). \tag{87}$$

Here note that $\operatorname{supp}(\psi_{\bar z}) \subset \mathbb{D}$ and $\chi_{\mathbb{D}}(\zeta)/(\zeta - z) \in L^q(\mathbb{C})$ for all $q < 2$. Thus by the weak convergence we have for each fixed $z \in \mathbb{C}$

$$\lim_{\xi\to\infty} \psi(z,\xi) = z, \quad \text{uniformly in } \kappa \in \partial\mathbb{D}. \tag{88}$$

On the other hand, as

$$\sup_{\xi}\left\|\frac{\partial\psi}{\partial\bar z}\right\|_{L^p(\mathbb{C})} \le C_0 = C_0(p, \|\mu\|_\infty) < \infty$$

for all z sufficiently large, $|\psi(z,\xi) - z| < \varepsilon$, uniformly in $\xi \in \mathbb{C}$ and $\kappa \in \partial\mathbb{D}$. Moreover, (87) shows also that the family $\{\psi(\cdot,\xi) : \xi \in \mathbb{C},\ \kappa \in \partial\mathbb{D}\}$ is equicontinuous. Combining all these observations shows that the convergence in (88) is uniform in $z \in \mathbb{C}$ and $\kappa \in \partial\mathbb{D}$. □

Finally, we proceed to the nonlinear case: Assume that φ_λ satisfies (72) and (74). Since φ is a (quasiconformal) homeomorphism, we may consider its inverse $\psi_\lambda : \mathbb{C} \to \mathbb{C}$,

$$\psi_\lambda \circ \varphi_\lambda(z) = z, \tag{89}$$

which also is quasiconformal. By differentiating (89) with respect to z and $\bar z$ we find that ψ satisfies

$$\frac{\partial}{\partial\bar z}\psi_\lambda(z,\xi) = -\frac{\overline{\xi}}{\xi}\lambda\left(\mu(\psi_\lambda(z,\xi))\right)e_{-\xi}(z)\,\frac{\partial}{\partial z}\,\psi_\lambda(z,\xi) \quad \text{and} \tag{90}$$

$$\psi_\lambda(z,\xi) = z + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \to \infty. \tag{91}$$

Proof of Theorem 2.18. It is enough to show that

$$\lim_{\xi\to\infty} \psi_\lambda(z,\xi) = z. \tag{92}$$

uniformly in z and λ. For this we introduce the notation
(93)
$$\Sigma_k = \{g \in W^{1,2}_{loc}(\mathbb{C}) : g_{\bar z} = \nu\, g_z,\ |\nu| \le k\chi_{\mathbb{D}},\ g = z + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \to \infty\}.$$

Note that all mappings $g \in \Sigma_k$ are principal solutions of Beltrami equations and hence homeomorphisms $g : \mathbb{C} \to \mathbb{C}$.

The support of the coefficient $\mu \circ \psi_\lambda$ in (90) need no longer be contained in $\mathbb{D}$. However, by Koebe distortion theorem, see e.g. [5, p. 44], $\varphi_\lambda(\mathbb{D}) \subset \mathbb{D}$ and thus $\operatorname{supp}(\mu \circ \psi_\lambda) \subset \mathbb{D}$. Accordingly, $\psi_\lambda \in \Sigma_k$.

Since normalized quasiconformal mappings form a normal family, we see that the family Σ_k is compact in the topology of uniform convergence. Given sequences $\xi_n \to \infty$ and $\lambda_n \in \partial\mathbb{D}$, we may pass to a subsequence and assume that $\kappa_{\lambda_n,\xi_n} = -\lambda_n\, \overline{\xi}_n^{\,2} |\xi_n|^{-2} \to \kappa \in \partial\mathbb{D}$ as $n \to \infty$ and that the corresponding mappings satisfy $\lim_{n\to\infty} \psi_{\lambda_n}(\cdot, \xi_n) = \psi_\infty$ uniformly, where the limit satisfies $\psi_\infty \in \Sigma_k$. To prove Theorem 2.18 it is enough to show that for any such sequence $\psi_\infty(z) \equiv z$.

Let ψ_∞ be an arbitrary above obtained limit function. We consider the $W^{1,2}_{loc}$-solution $\Phi(z) = \Phi_\lambda(z, \xi)$ of
$$\begin{aligned} \frac{\partial \Phi}{\partial \bar z} &= \kappa\,(\mu \circ \psi_\infty)\, e_{-\xi}\, \frac{\partial \Phi}{\partial \bar z}, \\ \Phi(z) &= z + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \to \infty. \end{aligned}$$

Observe that this equation is a linear Beltrami equation which by Theorem 2.5 has a unique solution $\Phi \in \Sigma_k$ for each $\xi \in \mathbb{C}$ and $|\lambda| = 1$. According to Theorem 2.21,

$$\Phi_\lambda(z, \xi) \to z \quad \text{as } \xi \to \infty. \tag{94}$$

Further, when $2 < p < 1 + 1/k$, by Lemma B.7,
$$\begin{aligned} &|\psi_{\lambda_n}(z, \xi_n) - \Phi_\lambda(z, \xi_n)| \\ &\quad = \frac{1}{\pi} \left| \int_{\mathbb{D}} \frac{1}{\zeta - z} \frac{\partial}{\partial \bar z} \big(\psi_{\lambda_n}(\zeta, \xi_n) - \Phi_\lambda(\zeta, \xi_n)\big)\, dm(\zeta) \right| \\ &\quad \le C_1 \left\| \frac{\partial}{\partial \bar z} \big(\psi_{\lambda_n}(\zeta, \xi_n) - \Phi_\lambda(\zeta, \xi_n)\big) \right\|_{L^p(\mathbb{D}(2))} \\ &\quad \le C_2 \big|\kappa_{\lambda_n,\xi_n} - \kappa\big| \\ &\qquad + C_2 \left(\int_{\mathbb{D}(2)} \big|\mu\left(\psi_{\lambda_n}(\zeta, \xi_n)\right) - \mu\left(\psi_\infty(\zeta)\right)\big|^{\frac{p(1+\varepsilon)}{\varepsilon}}\, dm(\zeta) \right)^{\frac{\varepsilon}{p(1+\varepsilon)}}. \end{aligned} \tag{95}$$

Finally, we apply the higher-integrability results for quasiconformal mappings, such as Theorem B.5: For all $2 < p < 1 + 1/k$ and $g = \psi^{-1}$,

$\psi \in \Sigma_k$, we have the estimate for the Jacobian $J(z,g)$,

$$\int_{\mathbb{D}} J(z,g)^{p/2}\,dm(z) \le \int_{\mathbb{D}} \left|\frac{\partial g}{\partial z}\right|^p dm(z) \le C(k) < \infty, \tag{96}$$

where $C(k)$ depends only on k. We use this estimate in the cases $\psi(z)$ is equal to $\psi_{\lambda_n}(z,\xi_n)$ or ψ_∞. Then, we see for any $\gamma \in C_0^\infty(\mathbb{D})$ that

$$\begin{aligned}\int_{\mathbb{D}(2)} |\mu(\psi(y)) - \gamma(\psi(y))|^{\frac{p(1+\varepsilon)}{\varepsilon}}\,dy \;&=\; \int_{\mathbb{D}} |\mu(z)-\gamma(z)|^{\frac{p(1+\varepsilon)}{\varepsilon}} J(z,g)\,dm(z)\\ \le \left(\int_{\mathbb{D}} |\mu(z)-\gamma(z)|^{\frac{p^2(1+\varepsilon)}{\varepsilon(p-2)}}\,dm(z)\right)^{(p-2)/p} &\left(\int_{\mathbb{D}} J(z,g)^{p/2}\,dm(z)\right)^{2/p}.\end{aligned}$$

Since μ can be approximated in the mean by $\gamma \in C_0^\infty(\mathbb{D})$, the last term can be made arbitrarily small. By uniform convergence $\psi_{\lambda_n}(z,\xi_n)) \to \psi_\infty(z)$ we see that $\gamma(\psi_{\lambda_n}(z,\xi_n)) \to \gamma(\psi_\infty(z))$ uniformly in z as $n \to \infty$. Also, $\kappa_{\lambda_n,\xi_n} \to \kappa$. Using these we see that right hand side of (95) converges to zero. In view of (94) and (95), we have established that

$$\lim_{n\to\infty} \psi_{\lambda_n}(z,\xi_n) = z$$

and thus $\psi_\infty(z) \equiv z$. The theorem is proved. □

2.8. **Completion of the proof of Theorem 2.1.** The Jacobian $J(z,f)$ of a quasiregular map can vanish only on a set of Lebesque measure zero. Since $J(z,f) = |f_z|^2 - |f_{\bar z}|^2 \le |f_z|^2$, this implies that once we know the values $f_\mu(z,\xi)$ for every $z \in \mathbb{C}$, then we can recover the values $\mu(z)$ and hence $\sigma(z)$ almost everywhere, from f_μ by the formulas

$$\frac{\partial f_\mu}{\partial \bar z} = \mu(z)\,\overline{\frac{\partial f_\mu}{\partial z}} \quad \text{and} \quad \sigma = \frac{1-\mu}{1+\mu}. \tag{97}$$

On the other hand, considering the functions

$$u_1 := u_\sigma = \mathrm{Re}\ f_\mu + i\,\mathrm{Im}\ f_{-\mu} \quad \text{and} \quad u_2 := iu_{1/\sigma} = i\,\mathrm{Re}\ f_{-\mu} - \mathrm{Im}\ f_\mu$$

that were described in Corollary 2.17, it is clear that the pair $\{u_1(z,\xi), u_2(z,\xi)\}$ determines the pair $\{f_\mu(z,\xi), f_{-\mu}(z,\xi)\}$, and vice versa. Therefore to prove Theorem 2.1 it will suffice to establish the following result.

Theorem 2.23. *Assume that $\Lambda_\sigma = \Lambda_{\widetilde\sigma}$ for two scalar conductivities σ and $\widetilde\sigma$ for which $\sigma, \widetilde\sigma, 1/\sigma, 1/\widetilde\sigma \in L^\infty(\mathbb{D})$. Then for all $z, \xi \in \mathbb{C}$,*

$$u_\sigma(z,\xi) = u_{\widetilde\sigma}(z,\xi) \quad \textit{and} \quad u_{1/\sigma}(z,\xi) = u_{1/\widetilde\sigma}(z,\xi).$$

For the proof of the theorem, our first task it to determine the asymptotic behavior of $u_\sigma(z,\xi)$. We state this as a separate result.

Lemma 2.24. *We have* $u_\sigma(z,\xi) \neq 0$ *for every* $(z,\xi) \in \mathbb{C}\times\mathbb{C}$. *Furthermore, for each fixed* $\xi \neq 0$, *we have with respect to* z

$$u_\sigma(z,\xi) = \exp(i\xi z + v(z)),$$

where $v = v_\xi \in L^\infty(\mathbb{C})$. *On the other hand, for each fixed* z *we have with respect to* ξ

$$u_\sigma(z,\xi) = \exp(i\xi z + \xi\varepsilon(\xi)), \tag{98}$$

where $\varepsilon(\xi) \to 0$ *as* $\xi \to \infty$.

Proof. For the first claim we write

$$\begin{aligned} u_\sigma &= \frac{1}{2}\left(f_\mu + f_{-\mu} + \overline{f_\mu} - \overline{f_{-\mu}}\right) \\ &= f_\mu \left(1 + \frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}}\right)^{-1} \left(1 + \frac{\overline{f_\mu} - \overline{f_{-\mu}}}{f_\mu + f_{-\mu}}\right). \end{aligned}$$

Each factor in the product is continuous and nonvanishing in z by Theorem 2.14. Taking the logarithm and using $f_{\pm\mu}(z,\xi) = e^{i\xi z}(1 + \mathcal{O}_\xi(1/z))$ we obtain

$$u_\sigma(z,\xi) = \exp\left(i\xi z + \mathcal{O}_\xi\left(\frac{1}{z}\right)\right).$$

Here, $\mathcal{O}_\xi(1/z))$ denotes a function $g(z,\xi)$ satisfying for each ξ an estimate $|g(z,\xi)| \le C_\xi 1/|z|$ with some $C_\xi > 0$. For the ξ-asymptotics we apply Theorem 2.18, which governs the growth of the functions f_μ for $\xi \to \infty$. We see that for (98) it is enough to show that

$$\inf_t \left| \frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} + e^{it} \right| \ge e^{-|\xi|\varepsilon(\xi)}. \tag{99}$$

For this, define

$$\Phi_t = e^{-it/2}(f_\mu \cos t/2 + i f_{-\mu} \sin t/2).$$

Then for each fixed ξ,

$$\Phi_t(z,\xi) = e^{i\xi z}\left(1 + \mathcal{O}_\xi\left(\frac{1}{z}\right)\right) \quad \text{as } z \to \infty,$$

and

$$\frac{\partial}{\partial \bar z}\,\Phi_t = \mu e^{-it}\overline{\frac{\partial}{\partial z}\,\Phi_t}.$$

Thus for $\lambda = e^{-it}$, the mapping $\Phi_t = f_{\lambda\mu}$ is precisely the exponentially growing solution satisfying the equations (72) and (73). A simple computation shows that

$$\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} + e^{it} = \frac{2e^{it}\,\Phi_t}{f_\mu + f_{-\mu}} = \frac{f_{\lambda\mu}}{f_\mu}\,\frac{2e^{it}}{1 + M_{-\mu}/M_\mu}. \tag{100}$$

By Theorem 2.18,

$$e^{-|\xi|\varepsilon_1(\xi)} \le |M_{\pm\mu}(z,\xi)| \le e^{|\xi|\varepsilon_1(\xi)} \tag{101}$$

and

$$e^{-|\xi|\varepsilon_2(\xi)} \le \inf_{\lambda\in\partial\mathbb{D}} \left|\frac{f_{\lambda\mu}(z,\xi)}{f_\mu(z,\xi)}\right| \le \sup_{\lambda\in\partial\mathbb{D}} \left|\frac{f_{\lambda\mu}(z,\xi)}{f_\mu(z,\xi)}\right| \le e^{|\xi|\varepsilon_2(\xi)}, \tag{102}$$

where $\varepsilon_j(\xi) \to 0$ as $\xi \to \infty$. Since $\mathrm{Re}\,(M_{-\mu}/M_\mu) > 0$, the inequality (99) follows. Thus the lemma is proven. □

As discussed earlier, the functions $u_1 = u_\sigma$ and $u_2 = iu_{1/\sigma}$ satisfy a $\partial_{\bar\xi}$-equation as a function of the parameter ξ, but unfortunately, for a fixed z the asymptotics in (98) are not strong enough to determine the individual solution $u_j(z,\xi)$. However, if we consider the entire family $\{u_j(z,\xi) : z \in \mathbb{C}\}$, then, somewhat surprisingly, uniqueness properties do arise.

To consider the uniqueness properties, assume that the Dirichlet-to-Neumann operators are equal for the conductivities σ and $\widetilde{\sigma}$. By Lemma 2.24, we have that $u_\sigma(z,\xi) \neq 0$ and $u_{\widetilde\sigma}(z,\xi) \neq 0$ at every point (z,ξ). Therefore their logarithms, denoted by δ_σ and $\delta_{\widetilde\sigma}$, respectively, are well defined. Moreover, for each fixed $z \in \mathbb{C}$,

$$\delta_\sigma(z,\xi) = \log u_\sigma(z,\xi) = i\xi z + \xi\varepsilon_1(\xi), \tag{103}$$

$$\delta_{\widetilde\sigma}(z,\xi) = \log u_{\widetilde\sigma}(z,\xi) = i\xi z + \xi\varepsilon_2(\xi), \tag{104}$$

where $\varepsilon_j(\xi) \to 0$ as for $|\xi| \to \infty$. Moreover, by Theorem 2.6,

$$\delta_\sigma(z,0) \equiv \delta_{\widetilde\sigma}(z,0) \equiv 0$$

for all $z \in \mathbb{C}$.

In addition, for each fixed $\xi \neq 0$ the function $z \mapsto \delta_\sigma(z,\xi)$ is continuous. By Lemma 2.24, we can write

$$\delta_\sigma(z,\xi) = i\xi z\left(1 + \frac{v_\xi(z)}{i\xi z}\right), \tag{105}$$

where $v_\xi \in L^\infty(\mathbb{C})$ for each fixed $\xi \in \mathbb{C}$. This means that that $\delta_\sigma(z,\xi)$ is close to a multiple of the identity for large $|z|$. Using an elementary homotopy argument, (105) yields that for any fixed $\xi \neq 0$ the map $z \mapsto \delta_\sigma(z,\xi)$ is surjective $\mathbb{C} \to \mathbb{C}$.

To prove the theorem it suffices to show that, if $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$, then

$$\delta_{\widetilde{\sigma}}(z,\xi) \neq \delta_\sigma(w,\xi) \quad \text{for } z \neq w \text{ and } \xi \neq 0. \tag{106}$$

Indeed, if the claim (106) is established, then (106) and the surjectivity of $z \mapsto \delta_\sigma(z,\xi)$ show that we necessarily have $\delta_\sigma(z,\xi) = \delta_{\widetilde{\sigma}}(z,\xi)$ for all $\xi, z \in \mathbb{C}$. Hence $u_{\widetilde{\sigma}}(z,\xi) = u_\sigma(z,\xi)$.

We are now at a point where the $\partial_{\overline{\xi}}$-method and (71) can be applied. Substituting $u_\sigma = \exp(\delta_\sigma)$ in this identity shows that $\xi \mapsto \delta_\sigma(z,\xi)$ and $\xi \mapsto \delta_{\widetilde{\sigma}}(w,\xi)$ both satisfy the $\partial_{\overline{\xi}}$-equation

$$\frac{\partial \delta}{\partial \overline{\xi}} = -i\tau(\xi)\, e^{(\overline{\delta}-\delta)}, \qquad \xi \in \mathbb{C}, \tag{107}$$

where by Theorem 2.10 and the assumption $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$, the coefficient $\tau(\xi)$ is the same for both functions δ_σ and $\delta_{\widetilde{\sigma}}$. A simple computations shows then that the difference

$$g(\xi) := \delta_{\widetilde{\sigma}}(w,\xi) - \delta_\sigma(z,\xi)$$

thus satisfies the identity

$$\frac{\partial g}{\partial \overline{\xi}} = -i\tau(\xi)\, e^{(\overline{\delta}-\delta)} \left[e^{(\overline{g}-g)} - 1\right].$$

In particular,

$$\left|\frac{\partial g}{\partial \overline{\xi}}\right| \leq |\overline{g} - g| \leq 2|g|. \tag{108}$$

Using (103) we we see that $g(\xi) = i(w-z)\xi + \xi\varepsilon(\xi)$ wheree $\varepsilon(\xi) \to 0$ as $\xi \to \infty$. Appling Theorem A.1 (with respect to ξ) we see that for $w \neq z$ the function g vanishes only at $\xi = 0$. This proves (106).

According to Theorem 2.9 (or by the identity $\tau_\sigma = -\tau_{1/\sigma}$), if $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$, the same argument works to show that $u_{1/\widetilde{\sigma}}(z,\xi) = u_{1/\sigma}(z,\xi)$ as well. Thus Theorem 2.23 is proved. As the pair $\{u_1(z,\xi), u_2(z,\xi)\}$ pointwise determines the pair $\{f_\mu(z,\xi), f_{-\mu}(z,\xi)\}$, we find via (97) that $\sigma \equiv \widetilde{\sigma}$. Therefore the proof of Theorem 2.1 is complete. □

3. Invisibility cloaking and the borderlines of visibility and invisibility

Next we consider the anisotropic conductivity equation in $\Omega \subset \mathbb{R}^2$,

$$\nabla \cdot \sigma \nabla u = \sum_{j,k=1}^{2} \frac{\partial}{\partial x^j}\left(\sigma^{jk}(x) \frac{\partial}{\partial x^k} u(x)\right) \;=\; 0 \text{ in } \Omega, \tag{109}$$

where the conductivity $\sigma = [\sigma^{jk}(x)]_{j,k=1}^2$ is a measurable function whose values are symmetric, positive definite matrixes. We say that a conductivity σ is *regular* if there are $c_1, c_2 > 0$ such that

$$c_1\mathbf{I} \leq \sigma(x) \leq c_2\mathbf{I}, \quad \text{for a.e. } x \in \Omega.$$

If conductivity is not regular, it is said to be *degenerate*. We will consider uniqueness results for the inverse problem in classes of degenerate conductivities both in the isotropic and the anisotropic case. We will also construct counterexamples for the uniqueness of the inverse problem having a close connection to the invisibility cloaking, a very topical subject in recent studies in mathematics, physics, and material science [2, 28, 36, 60, 55, 61, 71, 79]. By invisibility cloaking we mean the possibility, both theoretical and practical, of shielding a region or object from detection via electromagnetic fields.

The counterexamples for inverse problems and the proposals for invisibility cloaking are closely related. In 2003, before the appearance of practical possibilities for cloaking, it was shown in [35, 36] that passive objects can be coated with a layer of material with a degenerate conductivity which makes the object undetectable by the electrostatic boundary measurements. These constructions were based on the blow up maps and gave counterexamples for the uniqueness of inverse conductivity problem in the three and higher dimensional cases. In two dimensional case, the mathematical theory of the cloaking examples for conductivity equation have been studied in [45, 46, 52, 64].

The interest in cloaking was raised in particular in 2006 when it was realized that practical cloaking constructions are possible using so-called metamaterials which allow fairly arbitrary specification of electromagnetic material parameters. The construction of Leonhardt [55] was based on conformal mapping on a non-trivial Riemannian surface. At the same time, Pendry et al [71] proposed a cloaking construction for Maxwell's equations using a blow up map and the idea was demonstrated in laboratory experiments [72]. There are also other suggestions for cloaking based on active sources [61] or negative material parameters [2, 60].

Let $\Sigma = \Sigma(\Omega)$ be the class of measurable matrix valued functions $\sigma : \Omega \to M$, where M is the set of symmetric non-negative definite matrices. Instead of defining the Dirichlet-to-Neumann operator which may not be well defined for these conductivities, we consider the corresponding quadratic forms.

Definition 3.1. Let $h \in H^{1/2}(\partial\Omega)$. The Dirichlet-to-Neumann quadratic form corresponding to the conductivity $\sigma \in \Sigma(\Omega)$ is given by

(110)
$$Q_\sigma[h] = \inf A_\sigma[u], \quad \text{where } A_\sigma[u] = \int_\Omega \sigma(z)\nabla u(z) \cdot \nabla u(z)\, dm(z),$$

and the infimum is taken over real valued $u \in L^1(\Omega)$ such that $\nabla u \in L^1(\Omega)^3$ and $u|_{\partial\Omega} = h$. In the case where $Q_\sigma[h] < \infty$ and $A_\sigma[u]$ reaches its minimum at some u, we say that u is a $W^{1,1}(\Omega)$ solution of the conductivity problem.

In the case when σ is smooth, bounded from below and above by positive constants, $Q_\sigma[h]$ is the quadratic form corresponding the Dirichlet-to-Neumann map (5),

$$Q_\sigma[h] = \int_{\partial\Omega} h\, \Lambda_\sigma h\, ds, \tag{111}$$

where ds is the length measure on $\partial\Omega$. Physically, $Q_\sigma[h]$ corresponds to the power needed to keep voltage h at the boundary. As discussed above, for smooth conductivities bounded from below, for every $h \in H^{1/2}(\partial\Omega)$ the integral $A_\sigma[u]$ always has a unique minimizer $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = h$. It is also a distributional solution to (4). Conversely, for functions $u \in H^1(\Omega)$ the traces lie in $H^{1/2}(\partial\Omega)$. As we mostly consider conductivities which are bounded from below and above near the boundary we chose to consider the $H^{1/2}$-boundary values also in the general case. We interpret that the Dirichlet-to-Neumann form corresponds to the idealization of the boundary measurements for $\sigma \in \Sigma(\Omega)$.

Next we present few examples where the solutions u turn out to be non-smooth or do not exist.

Example 1. Let us consider 1-dimensional conductivity equation on interval $I = [0,1]$. Let $(q_j)_{j=1}^\infty$ be a sequence containing all rational numbers $\mathbb{Q} \cap (0,1)$ so that each number appears only once in the sequence. Let $a_j = 2^{-1}j^{-4}$, $K_j = (q_j - 2^{-j-2}, q_j + 2^{-j-2}) \cap I$, and define conductivity

$$\sigma(x) = 1 + \sum_{j=1}^\infty \sigma_j(x), \quad \sigma_j(x) = \frac{a_j}{|x-q_j|}\chi_{K_j}(x). \tag{112}$$

Note that the set K_j has the measure $|K_j| \le 2^{-j-1}$. As $|\bigcup_{j\ge l} K_j| \le 2^{-l}$, we see that the series (112) has only finitely many nonzero terms for

$x \in \bigcap_{l\geq 1} \bigcup_{j\geq l} K_j$ and in particular, the sum $\sigma(x)$ is finite and positive function a.e. Now, assume that $u \in C^1(I)$ is a function for which

$$\int_I \sigma(x)|u'(x)|^2 dm(x) < \infty.$$

If $u'(q_j) \neq 0$, we see that there is an open non-empty interval $I_j \subset I$ containing q_j such that $|u'(x)| \geq t > 0$ for all $x \in I_j$, and

$$\int_I \sigma(x)|u'(x)|^2 dm(x) \geq \int_{I_j} \sigma_j(x)|u'(x)|^2 dm(x) = \infty.$$

This implies that $u'(q_j) = 0$ for all q_j, and as $\{q_j\}$ is dense in I, we see that u vanishes identically. Thus if the minimization (110) is taken only over $u \in C^1(I)$ with $u(0) = f_0$ and $u(1) = f_1$, the Dirichlet-to-Neumann form is infinite for all non-constant boundary values $f_0 \neq f_1$. However, if the infimum is taken over all $u \in W^{1,1}(I)$, we see that the function

$$u_0(x) = \frac{|[0,x] \setminus K|}{|I \setminus K|}, \quad K = \bigcup_{j=1}^{\infty} K_j, \quad 0 < |K| < \frac{1}{2}$$

satisfies $u'(x) = 0$ for $x \in K$ and

$$\int_I \sigma(x)|u_0'(x)|^2 dm(x) = 1, \quad u_0(0) = 0,\ u_0(1) = 1.$$

Using functions $f_0 + (f_1 - f_0)u_0(x)$ we see that the Dirichlet-to-Neumann form for σ defined as a minimization over all $W^{1,1}$-functions is finite for all boundary values. Later we will show also examples of conductivities encountered in cloaking where the solution of the conductivity problem will be in $W^{1,p}$ for all $p < 2$ but not in $W^{1,2}$. This is another reason why $W^{1,1}$ is a convenient class to consider the minimisation.

Example 2. Consider in the disc $\mathbb{D}(2)$ a strongly twisting map,

$$G(re^{i\theta}) = re^{i(\theta + t(r))}, \quad 0 < r \leq 2,$$

where $t(r) = \exp(r^{-1} - 2^{-1})$. When $\gamma = 1$ is the homogeneous conductivity, let σ be the conductivity in $\mathbb{D}(2)$ such that $\sigma = G_*\gamma$ in the set $\mathbb{D}(2) \setminus \{0\}$. We see that if the problem (110) has a minimizer $u \in W^{1,1}(D)$ with the boundary value f for which $A_\sigma(u) < \infty$, then it has to satisfy $\nabla \cdot \sigma \nabla u = 0$ in the set $\mathbb{D}(2) \setminus \{0\}$. Then $v = u \circ G$ is harmonic function in $\mathbb{D}(2) \setminus \{0\}$ having boundary value $f \in H^{1/2}(\partial \mathbb{D}(2))$ and finite norm in $H^1(\mathbb{D}(2) \setminus \{0\})$. This implies that v can be extended to a harmonic function in the whole disc $\mathbb{D}(2)$, see e.g. [44]. Thus, if the problem (110) has a minimizer $u \in W^{1,1}(\mathbb{D}(2))$ for $f(x_1, x_2) = x_1$ we see that $v(x_1, x_2) = x_1$ and $u = v \circ F$, where $F := G^{-1}$. Then, by the chain rule we have $\nabla u(x) = DF(x)^t\,(Dv)(F(x)) \notin L^1(\mathbb{D}(2)\setminus\{0\})$. This shows that the minimizer u does not exists in the space $W^{1,1}(\mathbb{D}(2))$. Thus for a general degenerate conductivity it is reasonable to define the boundary measurements using infimum of a quadratic form instead of a distributional solution of the differential equation $\nabla\cdot\sigma\nabla u = 0$.

3.0.1. *Existence results for solutions with degenerate conductivities.* As seen in the above examples, if σ is unbounded it is possible that $Q_\sigma[h] = \infty$. Moreover, even if $Q_\sigma[h]$ is finite, the minimization problem in (110) may generally have no minimizer and even if they exist the minimizers need not be distributional solutions to (4). However, if the singularities of σ are not too strong, minimizers satisfying (4) do always exist. Below we will consider singular conductivity of exponentially integrable ellipticity function $K_\sigma(z)$ and show that for such conductivities solutions exists. To study of these solutions, we consider the *regularity gauge*

$$Q(t) = \frac{t^2}{\log(e+t)}, \quad t \geq 0. \tag{113}$$

We say accordingly that f belongs to the Orlicz space $W^{1,Q}(\Omega)$ if f and its first distributional derivatives are in $L^1(\Omega)$ and

$$\int_\Omega \frac{|\nabla f(z)|^2}{\log(e+|\nabla f(z)|)}\; dm(z) < \infty.$$

In [8] the following existence result for solutions corresponding to singular conductivity of exponentially integrable ellipticity is proven:

Theorem 3.2. *Let $\sigma(z)$ be a measurable symmetric matrix valued function. Suppose further that for some $p > 0$,*

$$\int_\Omega \exp(p\,[\mathrm{trace}(\sigma(z)) + \mathrm{trace}(\sigma(z)^{-1})])\; dm(z) = C_1 < \infty. \tag{114}$$

Then, if $h \in H^{1/2}(\partial\Omega)$ is such that $Q_\sigma[h] < \infty$ and $X = \{v \in W^{1,1}(\Omega);\ v|_{\partial\Omega} = h\}$, there is a unique $w \in X$ such that

$$A_\sigma[w] = \inf\{A_\sigma[v]\ ;\ v \in X\}. \tag{115}$$

Moreover, w satisfies the conductivity equation

$$\nabla\cdot\sigma\nabla w = 0 \quad \text{in } \Omega \tag{116}$$

in sense of distributions, and it has the regularity $w \in W^{1,Q}(\Omega) \cap C(\Omega)$.

Let $F : \Omega_1 \to \Omega_2$, $y = F(x)$ be an orientation preserving homeomorphism between domains $\Omega_1, \Omega_2 \subset \mathbb{C}$ for which F and its inverse F^{-1}

are at least $W^{1,1}$-smooth and let $\sigma(x) = [\sigma^{jk}(x)]_{j,k=1}^2 \in \Sigma(\Omega_1)$ be a conductivity on Ω_1. Then the map F pushes σ forward to a conductivity $(F_*\sigma)(y)$, defined on Ω_2 and given by

$$(F_*\sigma)(y) = \frac{1}{\det DF(x)} DF(x)\,\sigma(x)\,DF(x)^t, \quad x = F^{-1}(y). \tag{117}$$

The main methods for constructing counterexamples to Calderón's problem are based on the following principle.

Proposition 3.3. *Assume that $\sigma, \widetilde{\sigma} \in \Sigma(\Omega)$ satisfy (114), and let $F : \Omega \to \Omega$ be a homeomorphism so that F and F^{-1} are $W^{1,Q}$-smooth and C^1-smooth near the boundary, and $F|_{\partial\Omega} = id$. Suppose that $\widetilde{\sigma} = F_*\sigma$. Then $Q_\sigma = Q_{\widetilde{\sigma}}$.*

This proposition generalizes the previously known results [47] to less smooth diffeomorphisms and conductivities.

Sketch of the proof. Two implications of the assumptions for F are essential in the proof. First one is that as F is a homeomorphism satisfying $F \in W^{1,Q}(\Omega)$, it satisfies the condition $\mathcal{N}$, that is, for any measurable set $E \subset \Omega$ we have $|E| = 0 \Rightarrow |F(E)| = 0$, see e.g. [5, Thm. 19.3.2]. Also F^{-1} satisfies this condition. These imply that we have the area formula

$$\int_\Omega H(y)\,dm(y) = \int_\Omega H(F(x))\det(DF(x))\,dm(x) \tag{118}$$

for $H \in L^1(\Omega)$.

The second implication is that by Gehring-Lehto theorem, see [5, Cor. 3.3.3], a homeomorphism $F \in W^{1,1}_{loc}(\Omega)$ is differentiable almost everywhere in Ω, say in the set $\Omega \setminus A$, where A has Lebesgue measure zero. This pointwise differentiability at almost every point is essential in using the chain rule.

Let $h \in H^{1/2}(\partial\Omega)$ and assume that $Q_{\widetilde{\sigma}}[h] < \infty$. By Theorem 3.2 there is $\widetilde{u} \in W^{1,1}(\Omega)$ solving

$$\nabla \cdot \widetilde{\sigma}\nabla\widetilde{u} = 0, \quad \widetilde{u}|_{\partial\Omega} = h. \tag{119}$$

We define $u = \widetilde{u} \circ F : \Omega \to \mathbb{C}$. As F is C^1-smooth near the boundary we see that $u|_{\partial\Omega} = h$.

By Stoilow factorization theorem, see Theorem B.9, $\widetilde{u}$ can be written in the form $\widetilde{u} = \widetilde{w} \circ \widetilde{G}$ where $\widetilde{w}$ is harmonic and $\widetilde{G} \in W^{1,1}_{loc}(\mathbb{C})$ is a homeomorphism $\widetilde{G} : \mathbb{C} \to \mathbb{C}$. By Gehring-Lehto theorem $\widetilde{G}$ and the solution $\widetilde{u}$ are differentiable almost everywhere, say in the set $\Omega \setminus A'$, where A' has Lebesgue measure zero.

Since F^{-1} has the property $\mathcal{N}$, we see that $A'' = A' \cup F^{-1}(A') \subset \Omega$ has measure zero, and for $x \in \Omega \setminus A''$ the chain rule gives

$$\nabla u(x) = DF(x)^t\,(\nabla \widetilde{u})(F(x)). \tag{120}$$

Using this, the area formula and the definition (117) of $F_*\sigma$ one can show that

$$Q_\sigma[h] = \int_\Omega \nabla u(x) \cdot \sigma(x) \nabla u(x)\, dm(x)$$
$$= \int_\Omega DF(x)^t \nabla \widetilde{u}(F(x)) \cdot \frac{\sigma(x)}{\det\,(DF(x))} DF(x)^t \nabla \widetilde{u}(F(x)) \det\,(DF(x)) dm(x)$$
$$= \int_\Omega \nabla \widetilde{u}(y) \cdot \widetilde{\sigma}(y) \nabla \widetilde{u}(y)\, dy = Q_{\widetilde{\sigma}}[h].$$

□

Let us next consider various counterexamples for the solvability of inverse conductivity problem with degenerate conductivities.

3.1. **Counterexample 1: Invisibility cloaking.** We consider here invisibility cloaking in general background σ, that is, we aim to coat an arbitrary body with a layer of exotic material so that the coated body appears in measurements the same as the background conductivity σ. Usually one is interested in the case when the background conductivity σ is equal to the constant $\gamma = 1$. However, we consider here a more general case and assume that σ is a L^∞-smooth conductivity in $\overline{\mathbb{D}}(2)$, $\sigma(z) \geq c_0 I$, $c_0 > 0$. Here, $\mathbb{D}(\rho)$ is an open 2-dimensional disc of radius ρ and center zero and $\overline{\mathbb{D}}(\rho)$ is its closure. Consider a homeomorphism

$$F : \overline{\mathbb{D}}(2) \setminus \{0\} \to \overline{\mathbb{D}}(2) \setminus \mathcal{K} \tag{121}$$

where $\mathcal{K} \subset \mathbb{D}(2)$ is a compact set which is the closure of a smooth open set and suppose $F : \overline{\mathbb{D}}(2) \setminus \{0\} \to \overline{\mathbb{D}}(2) \setminus \mathcal{K}$ and its inverse F^{-1} are C^1-smooth in $\overline{\mathbb{D}}(2) \setminus \{0\}$ and $\overline{\mathbb{D}}(2) \setminus \mathcal{K}$, correspondingly. We also require that $F(z) = z$ for $z \in \partial\mathbb{D}(2)$. The standard example of invisibility cloaking is the case when $\mathcal{K} = \overline{\mathbb{D}}(1)$ and the map

$$F_0(z) = (\frac{|z|}{2} + 1)\frac{z}{|z|}. \tag{122}$$

Using the map (121), we define a singular conductivity

$$\widetilde{\sigma}(z) = \begin{cases} (F_*\sigma)(z) & \text{for } z \in \mathbb{D}(2) \setminus \mathcal{K}, \\ \eta(z) & \text{for } z \in \mathcal{K}, \end{cases} \tag{123}$$

where $\eta(z) = [\eta^{jk}(x)]$ is any symmetric measurable matrix satisfying $c_1 I \leq \eta(z) \leq c_2 I$ with $c_1, c_2 > 0$. The conductivity $\widetilde{\sigma}$ is called the cloaking conductivity obtained from the transformation map F and background conductivity σ and $\eta(z)$ is the conductivity of the cloaked (i.e. hidden) object.

In particular, choosing σ to be the constant conductivity $\sigma = 1$, $\mathcal{K} = \overline{\mathbb{D}}(1)$, and F to be the map F_0 given in (122), we obtain the standard example of the invisibility cloaking. In dimensions $n \geq 3$ it shown in 2003 in [35, 36] that the Dirichlet-to-Neumann map corresponding to $H^1(\Omega)$ solutions for the conductivity (123) coincide with the Dirichlet-to-Neumann map for $\sigma = 1$. In 2008, the analogous result was proven in the two-dimensional case in [45]. For cloaking results for the Helmholtz equation with frequency $k \neq 0$ and for Maxwell's system in dimensions $n \geq 3$, see results in [28]. We note also that John Ball [12] has used the push forward by the analogous radial blow-up maps to study the discontinuity of the solutions of partial differential equations, in particular the appearance of cavitation in the non-linear elasticity.

In [8] the following generalization of [35, 36, 45] is proven for cloaking in the context where measurements given in Definition 3.1.

Theorem 3.4. *(i) Let $\sigma \in L^\infty(\mathbb{D}(2))$ be a scalar conductivity, $\sigma(x) \geq c_0 > 0$, $\mathcal{K} \subset \mathbb{D}(2)$ be a relatively compact open set with smooth boundary and $F : \overline{\mathbb{D}}(2) \setminus \{0\} \to \overline{\mathbb{D}}(2) \setminus \mathcal{K}$ be a homeomorphism. Assume that F and F^{-1} are C^1-smooth in $\overline{\mathbb{D}}(2) \setminus \{0\}$ and $\overline{\mathbb{D}}(2) \setminus \mathcal{K}$, correspondingly and $F|_{\partial\mathbb{D}(2)} = id$. Moreover, assume there is $C_0 > 0$ such that $\|DF^{-1}(x)\| \leq C_0$ for all $x \in \overline{\mathbb{D}}(2) \setminus \mathcal{K}$. Let $\widetilde{\sigma}$ be the conductivity defined in (123). Then the boundary measurements for $\widetilde{\sigma}$ and σ coincide in the sense that $Q_{\widetilde{\sigma}} = Q_\sigma$.*

(ii) Let $\widetilde{\sigma}$ be a cloaking conductivity of the form (123) obtained from the transformation map F and the background conductivity σ where F and σ satisfy the conditions in (i). Then

$$\operatorname{trace}(\widetilde{\sigma}) \notin L^1(\mathbb{D}(2) \setminus \mathcal{K}). \tag{124}$$

Sketch of the proof. We consider the case when $F = F_0$ is given by (122) and $\sigma = 1$ is constant function.

(i) For $0 \leq r \leq 2$ and a conductivity η we define the quadratic form $A_\eta^r : W^{1,1}(\mathbb{D}(2)) \to \mathbb{R}_+ \cup \{0, \infty\}$,

$$A_\eta^r[u] = \int_{\mathbb{D}(2)\setminus\mathbb{D}(r)} \eta(x)\nabla u \cdot \nabla u \, dm(x).$$

Considering F_0 as a change of variables similarly to Proposition 3.3, we see that

$$A_{\widetilde{\sigma}}^r[u] = A_\gamma^\rho[v], \quad u = v \circ F_0,\ \rho = 2(r-1),\ r > 1.$$

Now for the conductivity $\gamma = 1$ the minimization problem (110) is solved by the unique minimizer u satisfying

$$\Delta u = 0 \quad \text{in } \mathbb{D}(2), \quad u|_{\partial \mathbb{D}(2)} = f.$$

The solution u is C^∞-smooth in $\mathbb{D}(2)$ and we see that $v = u \circ F_0$ is a $W^{1,1}$-function on $\mathbb{D}(2) \setminus \overline{\mathbb{D}}(1)$ which trace on $\partial\mathbb{D}(1)$ is equal to the constant function $h(x) = u(0)$ on $\partial\mathbb{D}(1)$. Let $\widetilde{v}$ be a function that is equal to v in $\mathbb{D}(2) \setminus \overline{\mathbb{D}}(1)$ and has the constant value $u(0)$ in $\overline{\mathbb{D}}(1)$. Then $\widetilde{v} \in W^{1,1}(\mathbb{D}(2))$ and

$$Q_{\widetilde{\sigma}}[f] \le A^1_{\widetilde{\sigma}}[v] = \lim_{r \to 1} A^r_{\widetilde{\sigma}}[v] = \lim_{\rho \to 0} A^\rho_\gamma[u] = Q_\gamma[f]. \tag{125}$$

To construct an inequality opposite to (125), let η_ρ be a conductivity which coincides with $\widetilde{\sigma}$ in $\mathbb{D}(2) \setminus \mathbb{D}(\rho)$ and is 0 in $\mathbb{D}(\rho)$. For this conductivity the minimization problem (110) has a minimizer that in $\mathbb{D}(2) \setminus \overline{\mathbb{D}}(\rho)$ coincides with the solution of the boundary value problem

$$\Delta u = 0 \quad \text{in } \mathbb{D}(2) \setminus \overline{\mathbb{D}}(\rho), \quad u|_{\partial\mathbb{D}(2)} = f, \quad \partial_\nu u|_{\partial\mathbb{D}(\rho)} = 0$$

and is arbitrary $W^{1,1}$-smooth extension of u to $\mathbb{D}(\rho)$. Then $\eta_\rho(x) \le \widetilde{\sigma}(x)$ for all $x \in \mathbb{D}(2)$ and thus $Q_{\eta_\rho}[f] \le Q_{\widetilde{\sigma}}[f]$. It is not difficult to see that

$$\lim_{\rho \to 0} Q_{\eta_\rho}[f] = Q_\gamma[f],$$

that is, the effect of an insulating disc of radius ρ in the boundary measurements vanishes as $\rho \to 0$. These and (125) yield $Q_{\widetilde{\sigma}}[f] = Q_\gamma[f]$. This proves (i).

(ii) Assume that (124) is not valid, i.e., $\mathrm{trace}(\widetilde{\sigma}) \in L^1(\mathbb{D}(2) \setminus \overline{\mathbb{D}}(1))$. As $\sigma = 1$ and $\det(\widetilde{\sigma}) = 1$, simple linear algebra yields that $K_{\widetilde{\sigma}} \in L^1(\mathbb{D}(2) \setminus \overline{\mathbb{D}}(1))$ and

$$\|\widetilde{\sigma}(y)\| = \frac{\|DF(x) \cdot \sigma(x) \cdot DF(x)^t\|}{J(x,F)} \ge \frac{\|DF(x)\|^2}{J(x,F)} = K_F(x), \quad x = F^{-1}(y).$$

Then $G = F^{-1}$ satisfies $K_G = K_F \circ F^{-1} \in L^1(\mathbb{D}(2) \setminus \overline{\mathbb{D}}(1))$ which yields that $F \in W^{1,2}(\mathbb{D}(2) \setminus \{0\})$ and $\|DF\|_{L^2(\mathbb{D}(2)\setminus\{0\})} \le 2\|K_G\|_{L^1(\mathbb{D}(2)\setminus\overline{\mathbb{D}}(1))}$, see e.g. [5, Thm. 21.1.4]. By the removability of singularities in Sobolev spaces, see [44], this implies that $F : \mathbb{D}(2) \setminus \{0\} \to \mathbb{D}(2) \setminus \overline{\mathbb{D}}(1)$ can be extended to a function $F^{ext} : \mathbb{D}(2) \to \mathbb{C}$, $F^{ext} \in W^{1,2}(\mathbb{D}(2))$. It follows from this and the continuity theorem of finite distortion maps [5, Thm. 20.1.1] that $F^{ext} : \mathbb{D}(2) \to \mathbb{C}$ is continuous, which is not possible. Thus (124) has to be valid. □

The result (124) is optimal in the following sense. When F is the map F_0 in (122) and $\sigma = 1$, the eigenvalues of the cloaking conductivity $\widetilde{\sigma}$ in $\mathbb{D}(2) \setminus \overline{\mathbb{D}}(1)$ behaves asymptotically as $(|z| - 1)$ and $(|z| - 1)^{-1}$ as

$|z| \to 1$. This cloaking conductivity has so strong degeneracy that (124) holds. On the other hand,

$$\mathrm{trace}(\widetilde{\sigma}) \in L^1_{weak}(\mathbb{D}(2)). \tag{126}$$

where L^1_{weak} is the weak-L^1 space. We note that in the case when $\sigma = 1$, $\det(\widetilde{\sigma})$ is identically 1 in $\mathbb{D}(2) \setminus \overline{\mathbb{D}}(1)$.

The formula (126) for the blow up map F_0 in (122) and Theorem 3.4 identify the *borderline of the invisibility* for the trace of the conductivity: Any cloaking conductivity $\widetilde{\sigma}$ satisfies $\mathrm{trace}(\widetilde{\sigma}) \notin L^1(\mathbb{D}(2))$ and there is an example of a cloaking conductivity for which $\mathrm{trace}(\widetilde{\sigma}) \in L^1_{weak}(\mathbb{D}(2))$. Thus the borderline of invisibility is the same as the border between the space L^1 and the weak-L^1 space.

3.2. **Counterexample 2: Illusion of a non-existent obstacle.** Next we consider new counterexamples for the inverse problem which could be considered as creating an illusion of a non-existing obstacle. The example is based on a radial shrinking map, that is, a mapping $\mathbb{D}(2) \setminus \overline{\mathbb{D}}(1) \to \mathbb{D}(2) \setminus \{0\}$. The suitable maps are the inverse maps of the blow-up maps $F_1 : \mathbb{D}(2) \setminus \{0\} \to \mathbb{D}(2) \setminus \overline{\mathbb{D}}(1)$ which are constructed by Iwaniec and Martin [39] and have the optimal smoothness. Using the properties of these maps and defining a conductivity $\sigma_1 = (F_1^{-1})_* 1$ on $\mathbb{D}(2) \setminus \{0\}$ we will later prove the following result.

Theorem 3.5. *Let γ_1 be a conductivity in $\mathbb{D}(2)$ which is identically 1 in $\mathbb{D}(2) \setminus \overline{\mathbb{D}}(1)$ and zero in $\mathbb{D}(1)$ and $\mathcal{A} : [1,\infty] \to [0,\infty]$ be any strictly increasing positive smooth function with $\mathcal{A}(1) = 0$ which is sub-linear in the sense that*

$$\int_1^\infty \frac{\mathcal{A}(t)}{t^2} dt < \infty. \tag{127}$$

Then there is a conductivity $\sigma_1 \in \Sigma(B_2)$ satisfying $\det(\sigma_1) = 1$ and

$$\int_{\mathbb{D}(2)} \exp(\mathcal{A}(\mathrm{trace}(\sigma_1) + \mathrm{trace}(\sigma_1^{-1})))\, dm(z) < \infty, \tag{128}$$

such that $Q_{\sigma_1} = Q_{\gamma_1}$, i.e., the boundary measurements corresponding to σ_1 and γ_1 coincide.

Sketch of the proof. Following [39, Sect. 11.2.1], there is $k(s)$ satisfies the relation

$$k(s) e^{\mathcal{A}(k(s))} = \frac{e}{s^2}, \quad 0 < s < 1$$

that is strictly decreasing function and satisfies $k(s) \le s^{-1}$ and $k(1) = 1$. Then

$$\rho(t) = \exp\left(\int_0^t \frac{ds}{sk(s)}\right)$$

is a function for which $\rho(0) = 1$. Then by defining fucntion the maps $h(t) = 2\rho(\frac{|x|}{2})/\rho(1)$ and

$$F_h : \mathbb{D}(2) \setminus \{0\} \to \mathbb{D}(2) \setminus \overline{\mathbb{D}}(1), \quad F_h(x) = h(t)\frac{x}{|x|} \tag{129}$$

and $\sigma_1 = (F_h)_*\gamma_1$, we obtain a conductivity that satisfies conditions of the statement.

Finally, the identity $Q_{\sigma_1} = Q_{\gamma_1}$ follows considering F_h as a change of variables similarly to the proof of Proposition 3.3. $\square$

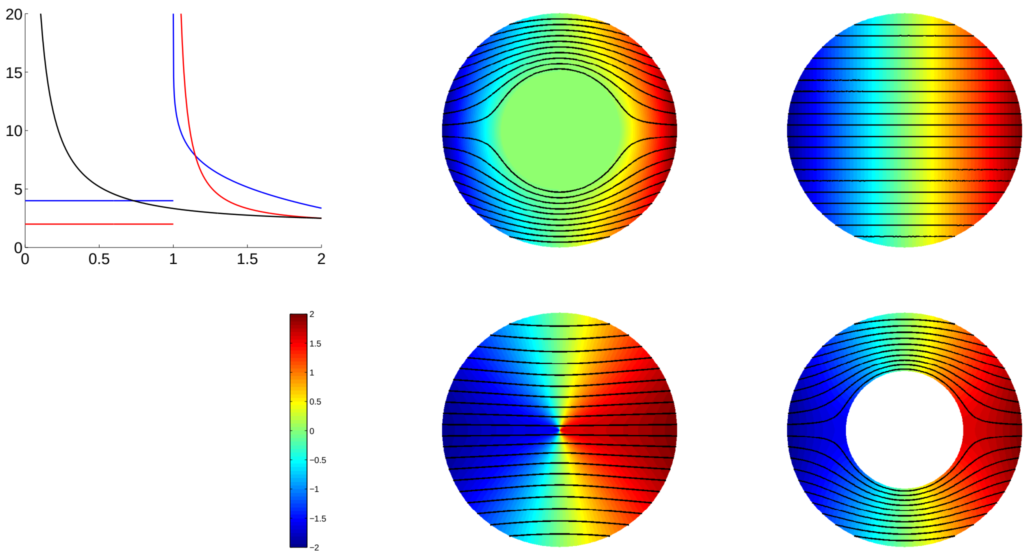

Figure 1. **Left**. trace(σ) *of three radial and singular conductivities on the positive x axis. The curves correspond to the invisibility cloaking conductivity (red), with the singularity $\sigma^{22}(x, 0) \sim (|x| - 1)^{-1}$ for $|x| > 1$, a visible conductivity (blue) with a* $\log\log$ *type singularity at $|x| = 1$, and an electric hologram (black) with the conductivity having the singularity $\sigma^{11}(x, 0) \sim |x|^{-1}$.* **Right, Top.** *All measurements on the boundary of the invisibility cloak (left) coincide with the measurements for the homogeneous disc (right). The color shows the value of the solution u with the boundary value $u(x, y)|_{\partial\mathbb{D}(2)} = x$ and the black curves are the integral curves of the current $-\sigma\nabla u$.* **Right, Bottom.**

All measurements on the boundary of the electric hologram (left) coincide with the measurements for an isolating disc covered with the homogeneous medium (right). The solutions and the current lines corresponding to the boundary value $u|_{\partial\mathbb{D}(2)} = x$ *are shown.*

We observe that for instance the function $\mathcal{A}_0(t) = t/(1+\log t)^{1+\varepsilon}$ satisfies (127) and for such weight function $\sigma_1 \in L^1(B_2)$.

Note that γ_1 corresponds to the case when $\mathbb{D}(2)$ is a perfect insulator which is surrounded with constant conductivity 1. Thus Theorem 3.5 can be interpreted by saying that there is a relatively weakly degenerated conductivity satisfying integrability condition (128) that creates in the boundary observations an illusion of an obstacle that does not exists. Thus the conductivity can be considered as "electric hologram". As the obstacle can be considered as a "hole" in the domain, we can say also that even the topology of the domain can not be detected. In other words, Calderón's program to image the conductivity inside a domain using the boundary measurements can not work within the class of degenerate conductivities satisfying (127) and (128).

3.3. **Positive results for Calderón's inverse problem.** In this section we formulate positive results for uniqueness of the inverse problems. Proof of the results can be found in [8].

For inverse problems for anisotropic conductivities where both the trace and the determinant of the conductivity are degenerate the following result holds.

Theorem 3.6. *Let* $\Omega \subset \mathbb{C}$ *be a bounded simply connected domain with smooth boundary. Let* $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ *be matrix valued conductivities in* Ω *which satisfy the integrability condition*

$$\int_\Omega \exp(p(\mathrm{trace}(\sigma(z)) + \mathrm{trace}(\sigma(z)^{-1})))\, dm(z) < \infty$$

for some $p > 1$. *Moreover, assume that*

$$\int_\Omega \mathcal{E}(q \det \sigma_j(z))\, dm(z) < \infty, \quad \textit{for some } q > 0, \tag{130}$$

where $\mathcal{E}(t) = \exp(\exp(\exp(t^{1/2} + t^{-1/2})))$ *and* $Q_{\sigma_1} = Q_{\sigma_2}$. *Then there is a* $W^{1,1}_{loc}$*-homeomorphism* $F : \Omega \to \Omega$ *satisfying* $F|_{\partial\Omega} = \mathrm{id}$ *such that*

$$\sigma_1 = F_* \sigma_2. \tag{131}$$

Equation (131) can be stated as saying that σ_1 and σ_2 are the same up to a change of coordinates, that is, the invariant manifold structures corresponding to these conductivities are the same, cf. [53, 51].

In the case when the conductivities are isotropic one can improve the result of Theorem 3.6 as follows.

Theorem 3.7. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with smooth boundary. If $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ are isotropic conductivities, i.e., $\sigma_j(z) = \gamma_j(z)I$, $\gamma_j(z) \in [0,\infty]$ satisfying for some $q > 0$*

$$\int_\Omega \exp\big(\exp\big[q(\gamma_j(z) + \frac{1}{\gamma_j(z)})\big]\big)\, dm(z) < \infty \tag{132}$$

and $Q_{\sigma_1} = Q_{\sigma_2}$, then $\sigma_1 = \sigma_2$.

Let us next consider anisotropic conductivities with bounded determinant but more degenerate ellipticity function $K_\sigma(z)$ and ask how far can we then generalize Theorem 3.6. Motivated by the counterexample given in Theorem 3.5 we consider the following class: We say that $\sigma \in \Sigma(\Omega)$ has an exponentially degenerated anisotropy with a weight $\mathcal{A}$ and denote $\sigma \in \Sigma_\mathcal{A} = \Sigma_\mathcal{A}(\Omega)$ if $\sigma(z) \in \mathbb{R}^{2\times 2}$ for a.e. $z \in \Omega$ and

$$\int_\Omega \exp(\mathcal{A}(\mathrm{trace}(\sigma) + \mathrm{trace}(\sigma^{-1})))\, dm(z) < \infty. \tag{133}$$

In view of Theorem 3.5, for obtaining uniqueness for the inverse problem we need to consider weights that are strictly increasing positive smooth functions $\mathcal{A} : [1,\infty] \to [0,\infty]$, $\mathcal{A}(1) = 0$, with

$$\int_1^\infty \frac{\mathcal{A}(t)}{t^2}\, dt = \infty \qquad \text{and} \qquad t\mathcal{A}'(t) \to \infty, \ \text{ as } t \to \infty. \tag{134}$$

We say that $\mathcal{A}$ has almost linear growth if (134) holds.

Note in particular that affine weights $\mathcal{A}(t) = pt - p$, $p > 0$ satisfy the condition (134). To develop uniqueness results for inverse problems within the class $\Sigma_\mathcal{A}$, one needs to find the right Sobolev-Orlicz regularity for the solutions u of finite energy, i.e., for solutions satisfying $A_\sigma[u] < \infty$. For this, we use the counterpart of the gauge $Q(t)$ defined at (113). In the case of a general weight $\mathcal{A}$ we define

$$P(t) = \begin{cases} t^2, & \text{for } \ 0 \le t < 1, \\ \frac{t^2}{\mathcal{A}^{-1}(\log(t^2))}, & \text{for } \ t \ge 1 \end{cases} \tag{135}$$

where $\mathcal{A}^{-1}$ is the inverse function of $\mathcal{A}$. We note that the condition $\int_1^\infty \frac{\mathcal{A}(t)}{t^2}\, dt = \infty$ is equivalent to

$$\int_1^\infty \frac{P(t)}{t^3}\, dt = \frac{1}{2}\int_1^\infty \frac{\mathcal{A}'(t)}{t}\, dt = \frac{1}{2}\int_1^\infty \frac{\mathcal{A}(t)}{t^2}\, dt = \infty \tag{136}$$

where we have used the substitution $\mathcal{A}(s) = \log(t^2)$. A function $u \in W^{1,1}_{loc}(\Omega)$ is in the Orlicz space $W^{1,P}(\Omega)$ if

$$\int_\Omega P\big(|\nabla u(z)|\big)\, dm(z) < \infty.$$

When $\mathcal{A}$ satisfies the almost linear growth condition (134) and P is as above one can show for $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$ and $u \in W^{1,1}_{loc}(\Omega)$ an inequality

(137)

$$\int_\Omega P\big(|\nabla u|\big)\, dm(z) \le 2\int_\Omega e^{\mathcal{A}\left(\operatorname{tr}\sigma + \operatorname{tr}(\sigma^{-1})\right)}\, dm(z) + 2\int_\Omega \nabla u\cdot\sigma\nabla u\, dm(z).$$

This implies that any solution u of the conductivity equation (4) with $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$ satisfies $u \in W^{1,P}(\Omega)$.

The Sobolev-Orlicz gauge $P(t)$ is essential also in the study of the counterexamples for solvability of the inverse problem and the optimal smoothness of conductivities corresponding to electric holograms: Assume that $G : \mathbb{D}(2)\setminus\overline{\mathbb{D}}(1) \to \mathbb{D}(2)\setminus\{0\}$ is a homeomorphic map which produces a hologram conductivity $\widetilde{\sigma} = G_*1$ in $\mathbb{D}(2)\setminus\{0\}$. Assume also that G and its inverse map, denoted $F = G^{-1}$, are $W^{1,1}_{loc}$-smooth. By the definition of the push forward of a conductivity (117), we see that

$$K_{\widetilde{\sigma}}(z) = K_F(z), \quad z \in \mathbb{D}(2)\setminus\{0\}.$$

This implies that F satisfies a Beltrami equation

$$\partial_{\bar z}F(z) = \widetilde{\mu}(z)\partial_z F(z), \quad z \in \mathbb{D}(2)\setminus\{0\}$$

where $K_{\widetilde{\mu}}(z) = K_{\widetilde{\sigma}}(z)$. By Theorem 2.3, the functions $w_1 = \operatorname{Re} F$ and $w_2 = \operatorname{Im} F$ satisfy a conductivity equation with a conductivity $A(z)$ with $K_A(z) = K_{\widetilde{\mu}}(z)$. Thus, if it happens that $\widetilde{\sigma} \in \Sigma_{\mathcal{A}}(\mathbb{D}(2))$ where $\mathcal{A}$ satisfies the almost linear growth condition (134), so that P satisfies condition (136), we see using (137) that $w_1, w_2 \in W^{1,P}(\mathbb{D}(2)\setminus\{0\})$. By using Stoilow factorization, Theorem B.9, we see that F can be written in the form $F(z) = \phi(f(z))$ where $f : \mathbb{C} \to \mathbb{C}$ is a homeomorphism and $\phi : f(\mathbb{D}(2)\setminus\{0\}) \to \mathbb{C}$ is analytic. As F and thus ϕ are bounded, we see that ϕ can be extended to an analytic function $\widetilde{\phi} : f(\mathbb{D}(2)) \to \mathbb{C}$ and thus also F can then be extended to a continuous function to $\widetilde{F} : \mathbb{D}(2) \to \mathbb{C}$. However, this is not possible as $F : \mathbb{D}(2)\setminus\{0\} \to \mathbb{D}(2)\setminus\overline{\mathbb{D}}(1)$ is a homeomorphism. This proves that no electric hologram conductivity $\widetilde{\sigma}$ can be in $\Sigma_{\mathcal{A}}(\mathbb{D}(2))$ where $\mathcal{A}$ satisfies the almost linear growth condition (134).

The above non-existence of electric hologram conductivities in $\Sigma_{\mathcal{A}}(\mathbb{D}(2))$ motivates the following sharp result for the uniqueness of the inverse problem for singular anisotropic conductivities with a determinant bounded from above and below by positive constants.

Theorem 3.8. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with smooth boundary and $\mathcal{A} : [1,\infty) \to [0,\infty)$ be a strictly increasing smooth function satisfying the almost linear growth condition (134). Let $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ be matrix valued conductivities in Ω which satisfy the integrability condition*

$$\int_\Omega \exp(\mathcal{A}(\mathrm{trace}(\sigma(z)) + \mathrm{trace}(\sigma(z)^{-1}))) \, dm(z) < \infty. \tag{138}$$

Moreover, suppose that $c_1 \le \det(\sigma_j(z)) \le c_2$, $z \in \Omega$, $j = 1,2$ for some $c_1, c_2 > 0$ and $Q_{\sigma_1} = Q_{\sigma_2}$. Then there is a $W^{1,1}_{loc}$-homeomorphism $F : \Omega \to \Omega$ satisfying $F|_{\partial\Omega} = \mathrm{id}$ such that

$$\sigma_1 = F_* \sigma_2.$$

We note that the determination of σ from Q_σ in Theorems 3.6, 3.7, and 3.8 is constructive in the sense that one can write an algorithm which constructs σ from Λ_σ. For example, for the non-degenerate scalar conductivities such a construction has been numerically implemented in [9].

Let us next discuss the borderline of the visibility somewhat formally. Below we say that a conductivity is visible if there is an algorithm which reconstructs the conductivity σ from the boundary measurements Q_σ, possibly up to a change of coordinates. In other words, for visible conductivities one can use the boundary measurements to produce an image of the conductivity in the interior of Ω in some deformed coordinates. For simplicity, let us consider conductivities with $\det \sigma$ bounded from above and below. Then, Theorems 3.5 and 3.8 can be interpreted by saying that the almost linear growth condition (134) for the weight function $\mathcal{A}$ gives the *borderline of visibility* for the trace of the conductivity matrix: If $\mathcal{A}$ satisfies (134), the conductivities satisfying the integrability condition (138) are visible. However, if $\mathcal{A}$ does not satisfy (134) we can construct a conductivity in Ω satisfying the integrability condition (138) which appears as if an obstacle (which does not exist in reality) would have included in the domain.

Thus the borderline of the visibility is between any spaces $\Sigma_{\mathcal{A}_1}$ and $\Sigma_{\mathcal{A}_2}$ where $\mathcal{A}_1$ satisfies condition (134) and $\mathcal{A}_2$ does not satisfy it. Example of such gauge functions are $\mathcal{A}_1(t) = t(1+\log t)^{-1}$ and $\mathcal{A}_2(t) = t(1+\log t)^{-1-\varepsilon}$ with $\varepsilon > 0$.

Summarizing, in terms of the trace of the conductivity, the above results identify the borderline of visible conductivities and the borderline of invisibility cloaking conductivities. Moreover, these borderlines are not the same and between the visible and the invisibility cloaking conductivities there are conductivities creating electric holograms.

Finally, let us comment the techniques needed to prove the above uniqueness results. The degeneracy of the conductivity causes that the exponentially growing solutions, the standard tools used to study Calderón's inverse problem, can not be constructed using purely microlocal or functional analytic methods. Instead, one needs to use the topological properties of the solutions: By Stoilow's theorem the solutions Beltrami equations are compositions of analytic functions and homeomorphisms. Using this, the continuity properties of the weakly monotone maps, and the Orlicz-estimates holding for homeomorphisms one can prove the existence of the exponentially growing solutions for Beltrami equations. Combining solutions of the appropriate Beltrami equations, see (44), one obtains exponentially growing solutions for conductivity equation in the Sobolev-Orlicz space $W^{1,Q}$ for isotropic conductivity and in $W^{1,P}$ for anisotropic conductivity.

Using these results one can obtain subexponential asymptotics for the families of exponentially growing solutions needed to apply similar $\overline{\partial}$ technique that were used to solve the inverse problem for the non-degenerate conductivity.

Appendix A. Argument Principle

The solution to the Calderón problem combines analysis with topological arguments that are specific to two dimensions. For instance, we need a version of the argument principle, which we here consider.

Theorem A.1. *Let $F \in W^{1,p}_{\text{loc}}(\mathbb{C})$ and $\gamma \in L^p_{\text{loc}}(\mathbb{C})$ for some $p > 2$. Suppose that, for some constant $0 \le k < 1$, the differential inequality*

$$\left|\frac{\partial F}{\partial \bar z}\right| \le k \left|\frac{\partial F}{\partial z}\right| + \gamma(z)\,\big|F(z)\big| \tag{139}$$

holds for almost every $z \in \mathbb{C}$ and assume that, for large z, $F(z) = \lambda z + \varepsilon(z)z$, where the constant $\lambda \neq 0$ and $\varepsilon(z) \to 0$ as $|z| \to \infty$.

Then $F(z) = 0$ at exactly one point, $z = z_0 \in \mathbb{C}$.

Proof. The continuity of $F(z) = \lambda z + \varepsilon(z)z$ and an elementary topological argument show that F is surjective, and consequently there exists at least one point $z_0 \in \mathbb{C}$ such that $F(z_0) = 0$.

To show that F can not have more zeros, let $z_1 \in \mathbb{C}$ and choose a large disk $B = \mathbb{D}(R)$ containing both z_1 and z_0. If R is so large that $\varepsilon(z) < \lambda/2$ for $|z| = R$, then $F\big|_{\{|z|=R\}}$ is homotopic to the identity relative to $\mathbb{C}\setminus\{0\}$. Next, we express (139) in the form

$$\frac{\partial F}{\partial \bar z} = \nu(z)\frac{\partial F}{\partial z} + A(z)\,F, \tag{140}$$

where $|\nu(z)| \le k < 1$ and $|A(z)| \le \gamma(z)$ for almost every $z \in \mathbb{C}$. Now $A\chi_B \in L^r(\mathbb{C})$ for all $2 \le r \le p' = \min\{p, 1+1/k\}$, and we obtain from Theorem B.4 that $(\mathbf{I}-\nu\mathcal{S})^{-1}(A\chi_B) \in L^r(\mathbb{C})$ for all $p'/(p'-1) < r < p'$.

Next, we define $\eta = \mathcal{C}\big((\mathbf{I}-\nu\mathcal{S})^{-1}(A\chi_B)\big)$. Then by Theorem B.3 we have $\eta \in C_0(\mathbb{C})$, and we also have

$$\frac{\partial \eta}{\partial \bar z} - \nu\frac{\partial \eta}{\partial z} = A(z), \qquad z \in B. \tag{141}$$

Therefore simply by differentation we see that the function

$$g = e^{-\eta}F \tag{142}$$

satisfies

$$\frac{\partial g}{\partial \bar z} - \nu\frac{\partial g}{\partial z} = 0, \qquad z \in B. \tag{143}$$

Since η has derivatives in $L^r(\mathbb{C})$, we have $g \in W^{1,r}_{loc}(\mathbb{C})$. As $r \ge 2$, the mapping g is quasiregular in B. The Stoilow factorization theorem gives $g = h\circ\psi$, where $\psi : B \to B$ is a quasiconformal homeomorphism and h is holomorphic, both continuous up to the boundary.

Since η is continuous, (142) shows that $g\big|_{|z|=R}$ is homotopic to the identity relative to $\mathbb{C}\setminus\{0\}$, as is the holomorphic function h. Therefore the argument principle shows that h has precisely one zero in $B = \mathbb{D}(R)$. Already, $h(\psi(z_0)) = e^{-\eta(z_0)}F(z_0) = 0$, and there can be no further zeros for F either. This finishes the proof. $\square$

Appendix B. Some Background in complex analysis and quasiconformal mappings.

Here we collect, without proof, some basic facts related to quasiconformal mappings. The proofs can be found e.g. in [5].

We start with harmonic analysis, where we often need refine estimates of the Cauchy transform.

Definition B.1. *The Cauchy transform is defined by the rule*

$$(\mathcal{C}\phi)(z) := \frac{1}{\pi}\int_{\mathbb{C}} \frac{\phi(\tau)}{z-\tau}\,d\tau. \tag{144}$$

Theorem B.2. *Let $1 < p < \infty$. If $\phi \in L^p(\mathbb{C})$ and $\phi(\tau) = 0$ for $|\tau| \ge R$, then*

(1) $\|\mathcal{C}\phi\|_{L^p(D_{2R})} \le 6R\,\|\phi\|_p$,

(2) $\|\mathcal{C}\phi(z) - \frac{1}{\pi z}\int \phi\|_{L^p(\mathbb{C}\setminus D_{2R})} \leq \frac{2\,R}{(p-1)^{1/p}}\|\phi\|_p$.

Thus, in particular, for $1 < p \leq 2$,

$$\|\mathcal{C}\phi\|_{L^p(\mathbb{C})} \leq \frac{8\,R}{(p-1)^{1/p}}\|\phi\|_p \quad \textit{provided} \int_{\mathbb{C}} \phi(z)dm(z) = 0.$$

For $p > 2$ *the vanishing condition for the integral over* $\mathbb{C}$ *is not needed, and we have*

$$\|\mathcal{C}\phi\|_{L^p(\mathbb{C})} \leq (6 + 3(p-2)^{-1/p})\; R\,\|\phi\|_p, \qquad p > 2.$$

Concerning compactness, we have

Theorem B.3. *Let* Ω *be a bounded measurable subset of* $\mathbb{C}$. *Then the following operators are compact*

(1) $\chi_\Omega \circ \mathcal{C} : L^p(\mathbb{C}) \to C^\alpha(\Omega)$, *or* $2 < p \leq \infty$ *and* $0 \leq \alpha < 1 - \frac{2}{p}$
(2) $\chi_\Omega \circ \mathcal{C} : L^p(\mathbb{C}) \to L^s(\mathbb{C})$, *for* $1 \leq p \leq 2$, *and* $1 \leq s < \frac{2p}{2-p}$.

The fundamental operator in the theory of planar quasiconformal mappings is the Beurling transform,

$$(\mathcal{S}\phi)(z) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\phi(\tau)}{(z-\tau)^2}\, d\tau. \tag{145}$$

The importance of the Beurling transform in complex analysis is furnished by the identity

$$\mathcal{S} \circ \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z}, \tag{146}$$

initially valid for functions contained in the space $C_0^\infty(\mathbb{C})$. Moreover, $\mathcal{S}$ extends to abounded operator on $L^p(\mathbb{C})$, $1 < p < \infty$; on $L^2(\mathbb{C})$ it is an isometry. We denote by

$$\mathbf{S}_p := \|\mathcal{S}\|_{L^p(\mathbb{C})\to L^p(\mathbb{C})}$$

the norm of this operator. By Riesz-Thorin interpolation, $\mathbf{S}_p \to 1$ as $p \to 2$.

In other words, $\mathcal{S}$ intertwines the Cauchy-Riemann operators $\frac{\partial}{\partial \bar{z}}$ and $\frac{\partial}{\partial z}$, a fact that explains the importance of the operator in complex analysis. For instance we have [5, p.363] the following result.

Theorem B.4. *Let* μ *be measurable with* $\|\mu\|_\infty \leq k < 1$. *Then the operator* $\mathbf{I} - \mu\mathcal{S}$ *is invertible on* $L^p(\mathbb{C})$ *whenever* $\|\mu\|_\infty \leq k < 1$ *and* $1 + k < p < 1 + 1/k$.

The result has important consequences on the regularity of elliptic systems. In fact, it is equivalent to the improved Sobolev regularity of quasiregular mappings.

Theorem B.5. *Let $\mu, \nu \in L^\infty(\mathbb{C})$ with $|\mu| + |\nu| \le k < 1$ almost everywhere. Then the equation*

$$\frac{\partial f}{\partial \bar z} - \mu(z)\frac{\partial f}{\partial z} - \nu(z)\overline{\frac{\partial f}{\partial z}} = h(z)$$

has a solution f, locally integrable with gradient in $L^p(\mathbb{C})$, whenever $1 + k < p < 1 + 1/k$ and $h \in L^p(\mathbb{C})$. Further, f is unique up to an additive constant.

We will also need a simple version of the Koebe distortion theorem.

Lemma B.6. [5, p. 42] *If $f \in W^{1,1}_{loc}(\mathbb{C})$ is a homeomorphism analytic outside the disk $\mathbb{D}(r)$ with $|f(z) - z| = o(1)$ at ∞, then*

$$|f(z)| < |z| + 3r, \quad \textit{for all } z \in \mathbb{C}. \tag{147}$$

Next, we have the continuous dependence of the quasiconformal mappings on the complex dilatation.

Lemma B.7. *Suppose $|\mu|, |\nu| \le k\chi_{\mathbb{D}(r)}$, where $0 \le k < 1$. Let $f, g \in W^{1,2}_{loc}(\mathbb{C})$ be the principal solutions to the equations*

$$\frac{\partial f}{\partial \bar z} = \mu(z)\frac{\partial f}{\partial z}, \qquad \frac{\partial g}{\partial \bar z} = \nu(z)\frac{\partial g}{\partial z}.$$

If for a number s we have $2 \le p < ps < P(k)$, then

$$\|f_{\bar z} - g_{\bar z}\|_{L^p(\mathbb{C})} \le C(p, s, k)\, r^{2/ps}\, \|\mu - \nu\|_{L^{ps/(s-1)}(\mathbb{C})}.$$

To prove uniqueness, Liouville type result are often valuable. Here we have collected a number of such results.

Theorem B.8. *Suppose that $F \in W^{1,q}_{loc}(\mathbb{C})$ satisfies the distortion inequality*

$$|F_{\bar z}| \le k|F_z| + \sigma(z)|F|, \qquad 0 \le k < 1, \tag{148}$$

where $\sigma \in L^2(\mathbb{C})$ and the Sobolev regularity exponent q lies in the critical interval $1 + k < q < 1 + 1/k$. Then $F = e^\theta g$, where g is quasiregular and $\theta \in VMO$. If $\sigma \in L^{2\pm}(\mathbb{C})$, then θ is continuous, and if furthermore F is bounded, then $F = C_1 e^\theta$.

In addition, if one of the following additional hypotheses holds,

1. *σ has compact support and $\lim_{z\to\infty} F(z) = 0$, or*
2. *$F \in L^p(\mathbb{C})$ for some $p > 0$ and $\limsup_{z\to\infty} |F(z)| < \infty$,*

then $F \equiv 0$.

Here we used the notation

$$L^{2\pm}(\mathbb{C}) = \{f : f \in L^s(\mathbb{C}) \cap L^t(\mathbb{C}) \text{ for some } s < 2 < t\}.$$

Finally, we formulate a generalization of the Stoilow factorization theorem for the solutions of Beltrami equation in the space $W^{1,P}_{loc}(\Omega)$.

Theorem B.9. *Let $\mathcal{A}$ satisfy the almost linear growth condition (134). Suppose the Beltrami coefficient, with $|\mu(z)| < 1$ almost everywhere, is compactly supported and the associated distortion function $K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$ satisfies*

$$e^{\mathcal{A}(K_\mu(z))} \in L^1_{loc}(\mathbb{C}) \tag{149}$$

Then the Beltrami equation $f_{\bar z}(z) = \mu(z)\, f_z(z)$ admits a unique principal solution $f \in W^{1,P}_{loc}(\mathbb{C})$ with $P(t)$ as in (135). Moreover, any solution $h \in W^{1,P}_{loc}(\Omega)$ to this Beltrami equation in a domain $\Omega \subset \mathbb{C}$ admits a factorization

$$h = \phi \circ f,$$

where ϕ is holomorphic in $f(\Omega)$.

References

[1] L. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand, Princeton 1966; Reprinted by American Mathematical Society, 2006.

[2] A. Alu and N. Engheta, Achieving transparency with plasmonic and metamaterial coatings, *Phys. Rev. E*, **72** (2005), 016623

[3] A. Alu and N. Engheta, Cloaking a sensor. *Phys. Rev. Lett.* **102**, 233901 (2009).

[4] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math., **173**, (1994), 37–60.

[5] K. Astala, T. Iwaniec and G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton Mathematical Series 48, Princeton University Press, 2009.

[6] K. Astala, T. Iwaniec and E. Saksman, *Beltrami operators in the plane*, Duke Math. J. 107 (2001), 27–56.

[7] K. Astala, M. Lassas and L. Päivärinta, *Calderón's inverse conductivity problem in the plane*, Comm. Partial Differential Equations, **30** (2005), 207–224.

[8] K. Astala, M. Lassas and L. Päivärinta, *The borderlines of the invisibility and visibility for Calderón's inverse problem.* arXiv:1109.2749v1

[9] K. Astala, J. Mueller, L. Päivärinta, A. Peramaki, and S. Siltanen, Direct electrical impedance tomography for nonsmooth conductivities. *Inv. Prob. Imag.* **5** (2011), 531–549.

[10] K. Astala and V. Nesi, *Composites and quasiconformal mappings: new optimal bounds in two dimensions,* Calculus of Variations and Partial Differential Equations, **18**, (2003), 335–355.

[11] K. Astala and L. Päivärinta, *Calderón's inverse problem for anisotropic conductivity in the plane*, Ann. of Math., **163** (2006), 265–299.

[12] J. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. *Phil. Trans. Royal Soc. London A* **306** (1982), 557–611.

[13] R. Beals and R. Coifman, *The spectral problem for the Davey-Stewartson and Ishimori hierarchies*, Nonlinear Evolution Equations: Integrability and Spectral Methods, Manchester University Press, 1988, pp. 15–23.

[14] M. Boiti, J. Leon, M. Manna and F. Pempinelli, On a spectral transform of a KdV-like equation related to the Schrödinger operator in the plane. *Inverse Problems* **3**, (1987), 25–36

[15] B. Bojarski, *Homeomorphic solutions of Beltrami systems*, Dokl. Akad. Nauk. SSSR, **102**, (1955), 661–664.

[16] L. Borcea, *Electrical impedance tomography*, Inverse problems 18 (2002) R99-R136.

[17] R. Brown and R. Torres, Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in $L^p, p > 2n$, *J. Fourier Analysis Appl.*, **9**(2003), 1049–1056.

[18] R. Brown and G. Uhlmann, Uniqueness in the inverse conductivity problem with less regular conductivities in two dimensions, *Comm. PDE*, **22**(1997), 1009–10027.

[19] A.P. Calderón, On an inverse boundary value problem, *Seminar on Numerical Analysis and its Applications to Continuum Physics*, 65–73, Soc. Brasil. Mat., Rio de Janeiro, 1980.

[20] H. Chen and C. Chan, Acoustic cloaking in three dimensions using acoustic metamaterials. *Appl. Phys. Lett.* **91** (2007), 183518

[21] H. Chen and C. Chan, Transformation media that rotate electromagnetic fields. *Appl. Phys. Lett.* **90** (2007), 241105.

[22] M. Cheney, D. Isaacson and J. Newell, *Electrical impedance tomography.* SIAM Review, **41**, (1999), 85-101.

[23] S. Cummer and D. Schurig. One path to acoustic cloaking. *New J. Phys.* **9**, 45 (2007).

[24] G. David, *Solutions de l'equation de Beltrami avec* $\|\mu\|_\infty = 1$, Ann. Acad. Sci. Fenn. Ser. AI Math., **13**, (1988), 25–70.

[25] A. Dijkstra, B. Brown, N. Harris, D. Barber and D. Endbrooke, *Review: Clinical applications of electrical impedance tomography*, J. Med. Eng. Technol., **17** (1993), 89–98.

[26] D. Faraco, P. Koskela and X.Zhong, *Mappings of finite distortion: the degree of regularity,* Adv. Math. 190 (2005) 300–318.

[27] T. Ergin et al, Three-Dimensional Invisibility Cloak at Optical Wavelengths, *Science* **328** (2010), 337–339.

[28] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann: Full-wave invisibility of active devices at all frequencies. *Comm. Math. Phys.* **275** (2007), 749–789.

[29] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Electromagnetic wormholes via handle body constructions, *Comm. Math. Phys.* **281** (2008), 369–385.

[30] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Cloaking a sensor via transformation optics. *Physical Review E* **83**, 016603 (2011).

[31] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Electromagnetic wormholes and virtual magnetic monopoles from metamaterials. *Phys. Rev. Lett.* **99**, 183901 (2007).

[32] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Approximate quantum cloaking and almost-trapped states. *Phys. Rev. Lett.* **101**, 220404 (2008).

[33] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Schrödinger's Hat: Electromagnetic, acoustic and quantum amplifiers via transformation optics, arXiv:1107.4685 (2011).

[34] A. Greenleaf, M. Lassas, and G. Uhlmann, The Calderón problem for conormal potentials, I: Global uniqueness and reconstruction, *Comm. Pure Appl. Math* **56** (2003), 328–352.

[35] A. Greenleaf, M. Lassas, and G. Uhlmann, Anisotropic conductivities that cannot detected in EIT, *Physiolog. Meas.* **24** (2003), 413–420.

[36] A. Greenleaf, M. Lassas, G. Uhlmann, On nonuniqueness for Calderón's inverse problem, *Math. Res. Lett.* **10** (2003), 685–693.

[37] B. Haberman, D. Tataru, Uniqueness in Calderón's problem with Lipschitz conductivities, preprint 2011, arXiv:1108.6068.

[38] O. Imanuvilov, G. Uhlmann, M. Yamamoto, The Calderón Problem with Partial Data in Two Dimensions. *J. American Math. Soc.* **23** (2010), 655–691.

[39] T. Iwaniec and G.J. Martin *Geometric Function Theory and Nonlinear Analysis*, Oxford University Press, 2001.

[40] T. Iwaniec and G.J. Martin *The Beltrami equation,* Mem. Amer. Math. Soc. 191, 2008.

[41] T. Iwaniec, T, P. Koskela and G.Martin *Mappings of BMO-distortion and Beltrami-type operators.* J. Anal. Math. 88, (2002), 337–381.

[42] T. Iwaniec, T, P. Koskela, G. Martin and C. Sbordone, *Mappings of finite distortion: $L^n \log^\chi L$-integrability*, J. London Math. Soc. (2) 67, (2003), 123–136.

[43] C. Kenig, J. Sjöstrand, G. Uhlmann, The Calderón Problem with Partial Data. *Ann. of Math.* **165** (2007), 567–591.

[44] T. Kilpeläinen, J. Kinnunen, M. Martio, Sobolev spaces with zero boundary values on metric spaces. *Potential Anal.* **12** (2000), 233–247.

[45] R. Kohn, H. Shen, M. Vogelius, and M. Weinstein, Cloaking via change of variables in Electrical Impedance Tomography, *Inverse Problems* **24** (2008), 015016 (21pp)

[46] R. Kohn, D. Onofrei, M. Vogelius and M. Weinstein, Cloaking via change of variables for the Helmholtz equation, *Comm. Pure Appl. Math.* **63** (2010), 973–1016.

[47] R. Kohn R. and Vogelius, M., Identification of an unknown conductivity by means of measurements at the boundary, in *Inverse Problems, SIAM-AMS Proc.*, **14** (1984).

[48] N. Landmy, et al, Perfect Metamaterial Absorber. *Phys. Rev. Lett* **100** (2008), 207402.

[49] M. Lassas, J. Mueller, S. Siltanen: Mapping properties of the nonlinear Fourier transform in dimension two. *Communications in Partial Differential Equations* **32** (2007), 591-610.

[50] M. Lassas, J. Mueller, S. Siltanen: Mapping properties of the nonlinear Fourier transform in dimension two. *Communications in Partial Differential Equations* **32** (2007), 591-610.

[51] M. Lassas and G. Uhlmann, Determining Riemannian manifold from boundary measurements, *Ann. Sci. École Norm. Sup.*, **34** (2001), 771–787.

[52] M. Lassas, T. Zhou: Two dimensional invisibility cloaking for Helmholtz equation and non-local boundary conditions. *Mathematical Research Letters* **18** (2011), 473–488.

[53] J. Lee and G. Uhlmann, Determining anisotropic real-analytic conductivities by boundary measurements, *Comm. Pure Appl. Math.*, **42** (1989), 1097–1112.

[54] O. Lehto, K. Virtanen, *Quasiconformal mappings in the plane*, Second edition, Springer-Verlag, 1973.

[55] U. Leonhardt, Optical Conformal Mapping, *Science* **312** (23 June, 2006), 1777–1780.

[56] U. Leonhardt and T. Philbin, General relativity in electrical engineering, *New J. Phys.*, **8** (2006), 247.

[57] U. Leonhardt and T. Tyc, Superantenna made of transformation media, *New J. Phys.* **10** (2008) 115026.

[58] H. Liu and T. Zhou, On Approximate Electromagnetic Cloaking by Transformation Media, *SIAM J. Appl. Math* **71** (2011), 218–241.

[59] O. Martio and J. Maly, *Lusin's condition (N) and mappings of the class* $W^{1,n}$, J. Reine Angew. Math. 458, (1995), 19–36.

[60] G. Milton and N.-A. Nicorovici, On the cloaking effects associated with anomalous localized resonance, *Proc. Royal Soc. A* **462** (2006), 3027–3059.

[61] G. Milton, D. Onofrei, F. Vasquez, Active exterior cloaking for the 2D Laplace and Helmholtz Equations. *Phys. Rev. Lett.* **103** (2009), 073901.

[62] A. Nachman, Reconstructions from boundary measurements, *Ann. of Math.* **128** (1988), 531–576.

[63] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, *Ann. of Math.* **143** (1996), 71–96.

[64] H.-M. Nguyen, Approximate cloaking for the Helmholtz equation via transformation optics and consequences for perfect cloaking, preprint.

[65] L. Päivärinta, A. Panchenko and G. Uhlmann, Complex geometrical optics for Lipschitz conductivities, *Revista Matematica Iberoamericana*, **19**(2003), 57–72.

[66] R. Brown, *Estimates for the scattering map associated with a two-dimensional first-order system*, J. Nonlinear Sci., **11** (2001), 459–471.

[67] R. Brown and G. Uhlmann, *Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions*, Comm. Partial Differential Equations, **22** (1997), 1009–1027.

[68] A. Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem.* Ann. of Math., **143** (1996), 71–96.

[69] L. Päivärinta, A. Panchenko and G. Uhlmann, *Complex geometrical optics solutions for Lipschitz conductivities*, Rev. Math. Iberoamericana, **19** (2003), 57–72.

[70] D. Dos Santos Ferreira, C. Kenig, S. Salo, and G. Uhlmann, Limiting Carleman weights and anisotropic inverse problems. *Invent. Math.* **178** (2009), 119–171.

[71] J.B. Pendry, D. Schurig, and D.R. Smith, Controlling electromagnetic fields, *Science* **312** (23 June, 2006), 1780–1782.

[72] D. Schurig, J. Mock, B. Justice, S. Cummer, J. Pendry, A. Starr, and D. Smith, Metamaterial electromagnetic cloak at microwave frequencies, *Science* **314** (2006), 5801, 977–980.

[73] J. Sylvester, An anisotropic inverse boundary value problem, *Comm. Pure Appl. Math.* **43** (1990), 201–232.

[74] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math., **125** (1987), 153–169.

[75] T. Tao and Ch. Thiele, *Nonlinear Fourier analysis*, Lecture Notes; LAS Park City Summer School, 2003.

[76] T. Tsai, The Schrödinger operator in the plane. *Inverse Problems* **9** (1993), 763-787.
[77] J. Valentine, J. Li, T.Zentgraf, G. Bartal and X. Zhang, An optical cloak made of dielectrics, *Nature Materials*, **8** (2009), 568.
[78] A. Veselov, S. Novikov Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formulas and evolution equations *Sov. Math. Dokl.* **30** (1984), 558-591.
[79] R. Weder, A Rigorous Time-Domain Analysis of Full–Wave Electromagnetic Cloaking (Invisibility), *Journal of Physics A: Mathematical and Theoretical*, **41** (2008), 065207 (21pp).
[80] B. Wood and J. Pendry, Metamaterials at zero frequency *J. Phys.: Condens. Matter* **19** (2007), 076208 (9pp).
[81] E. Zeidler, *Nonlinear functional analysis and its applications.* I. Fixed-point theorems. Springer-Verlag, 1986.
[82] S. Zhang, D. Genov, C. Sun, X. Zhang. Cloaking of matter waves. *Phys. Rev. Lett.* **100** (2008), 123002.

Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68 (Gustaf Hällstrmin katu 2b), FI-00014 University of Helsinki, Finland.