

Gauge-string dualities and QCD

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Plan

1. Example: free energy of hot quark-gluon plasma
2. QCD and $\mathcal{N} = 4$ Supersymmetric Yang Mills
3. AdS/CFT
4. Classical gravity
5. Hot quark-gluon plasma: pressure, viscosity
6. $V_{Q\bar{Q}}(L)$, Wilson loops, quark energy loss and wake

1. Gauge-string duality

Could one "solve" QCD by rewriting it in a different and solvable form?

AdS/CFT duality

The rewriting succeeds for a **Conformal** Field Theory, $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills in 4d.

Get classical gravity in a 10d space with **Anti de Sitter** $_5 \times S_5$ metric, if

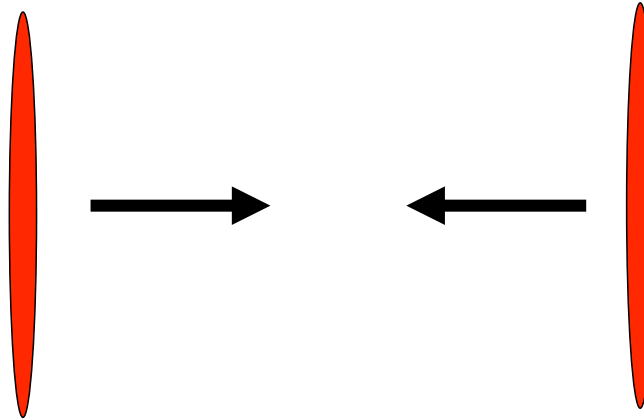
$$N_c \gg 1, \quad g^2 N_c \gg 1$$

When is QCD

- **conformally invariant** (no scales, Λ_{QCD} , hadrons) and
- **strongly coupled** ($3 \gg 1, \quad 4 \cdot 3 \gg 1$) ?

2. An example: $p(T)/p_{\text{ideal}}$.

Data: Collide two Gold nuclei at $A \times 100 + 100$ GeV:



prove that the

$$\frac{dN}{dy} \approx 1000$$

produced hadrons

- came from hot ($T \lesssim 1 \text{ GeV} \approx 5 T_c$) QCD matter which
- expanded nearly ideally $\epsilon(\tau) \sim 1/\tau^{4/3}$ for $\tau \gtrsim 0.2 \text{ fm}$

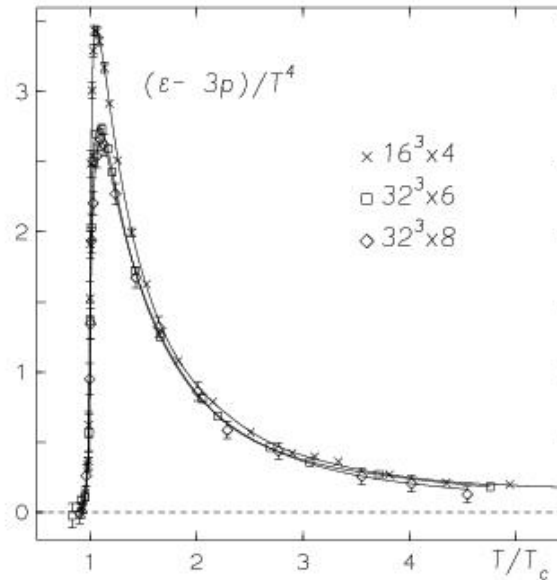
This matter for $T \gtrsim 1.5 T_c$ is a candidate for AdS/CFT application!

Hadrons, existence of phase transition with $T_c \approx \Lambda_{\text{QCD}} \approx 200 \text{ MeV}$ break conformal invariance.

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Deviations from conf inv are measured by $\epsilon - 3p$ ($=0$ for massless photon gas).

QCD matter is conformally invariant soon above T_c :

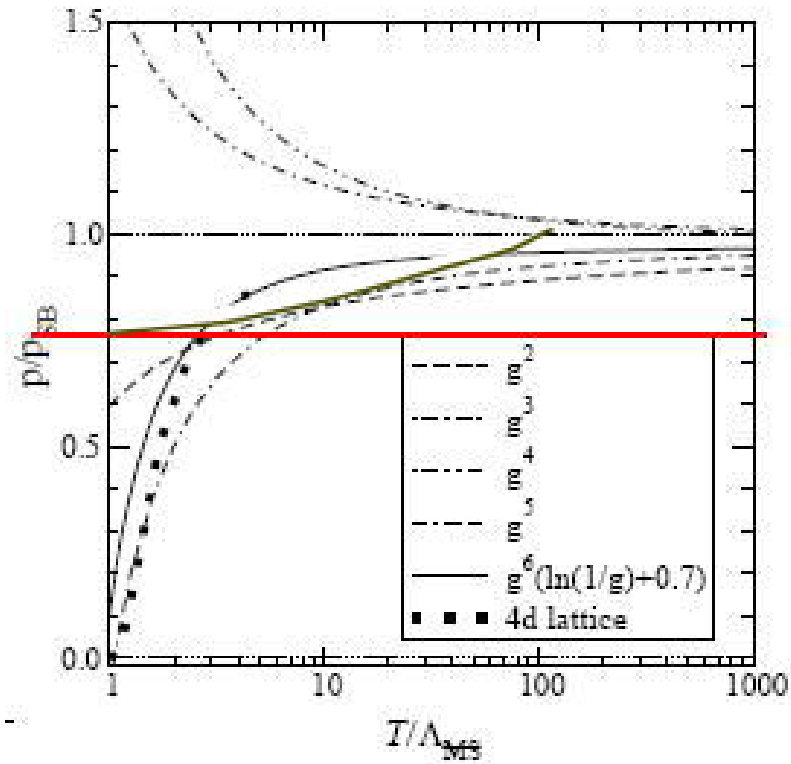


Prediction from $\text{AdS}_5 \times \text{S}_5$ is

$$p(T) = \frac{\pi^2 N_c^2}{6} T^4 \left[\frac{3}{4} + \frac{45\zeta(3)}{32} \frac{1}{(g^2 N_c)^{3/2}} + \dots \right] \equiv p_{\text{SB}} \left[\frac{3}{4} + \left(\frac{1.4}{g^2 N_c} \right)^{3/2} + \dots \right]$$

Comparison between theory and "experiment"

Is $p(T) = p_{\text{SB}}[0.75 + (0.15 \text{Log} \frac{T}{T_c})^{1.5} + ..]$ also for conf inv strongly coupled QCD?



Points: Lattice Monte Carlo, Curves: Perturbation theory up to $g^6 \log g$

Red: The famous 3/4 (curve: with correction term)

Wrong for $T \lesssim 3T_c$ (not conformal) and $T \gtrsim 100T_c$ (not strongly coupled)

Anyway this (and related result on shear viscosity η - coming later) has caused lots of excitement:

[Polchinski, cosmicvariance.com/2006/12/07/:](http://cosmicvariance.com/2006/12/07/)

Physicists have found that some of the properties of this plasma are better modeled (via duality) as a tiny black hole in a space with extra dimensions than as the expected clump of elementary particles in the usual four dimensions of spacetime.

Back to theory and explaining some catch words:

3. Quantum Chromodynamics

$$S_{\text{QCDm}=0}[A_\mu^a, \psi_f, \bar{\psi}_f] = \int d^d x \left[-\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \sum_f \bar{\psi}_f (i\gamma^\mu (\partial_\mu + igA_\mu)) \psi_f \right]$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc} A_\mu^b A_\nu^c$$

Adjoint gluons A_μ^a , $a = 1, \dots, N_c^2 - 1$, fundamental quarks ψ^i , $i = 1, \dots, N_c$

Symmetries:

- local $SU(N_c)$,
- conformal symmetry $O(2,4)$ (classically),
- chiral symmetry (if $m_q = 0$).

$$\mu \frac{\partial g(\mu)}{\partial \mu} = \beta(g) = -\beta_0 g^3 - \dots \Rightarrow \text{coupling runs}$$

What is the invariance group of Maxwell's equations?

1. Lorentz 1892: Lorentz transformations + Translations (6+4 parameters, Poincaré group)
2. Cunningham & Bateman 1909: There is more: dilatations (1 parameter) and "special conformal transformations" (4 parameters)

Conformal group $O(2,4)$ (15 parameters)

Running of $g(\mu)$ spoils classical conformal invariance of QCD: a scale Λ_{QCD} is introduced

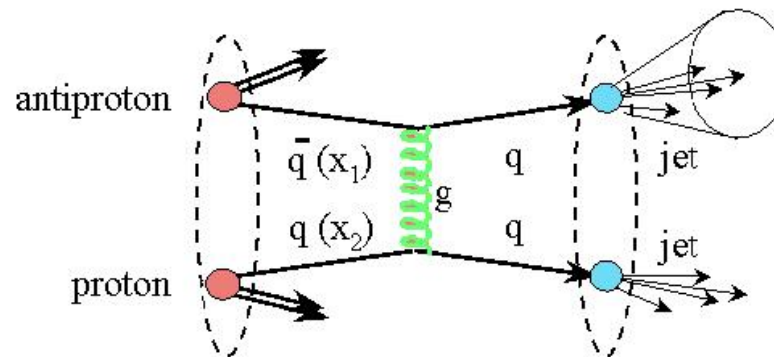
Solving, testing QCD

1. Non-perturbative: Spectrum, hadron masses,

Analytically: Million \$ problem, *prove* QCD ($N_f = 0$) has a "gap", put in massless stuff (gluons), get out massive stuff (glueballs)

Numerically: Lattice Monte Carlo. Solve QCD rigorously from 1st principles.

2. Perturbative: Collision physics



Beautiful physical theory – but complicated.

Modify it somewhat.

4. $\mathcal{N} = 4$ supersymmetric Yang-Mills

Add to A_μ^a (=1 massless adjoint vector) 4 adjoint fermions and 6 adjoint scalars with carefully tuned couplings: ¹

$$S[A_\mu^a, \phi_i^a, \psi^a, \bar{\psi}^a] = \frac{1}{2g^2} \int d^4x \left\{ \frac{1}{2} F_{\mu\nu}^a{}^2 + (\partial_\mu \phi_i^a + f_{abc} A_\mu^b \phi_i^c)^2 + \bar{\psi}^a i \gamma^\mu (\partial_\mu \psi^a + f_{abc} A_\mu^b \psi^c) \right. \\ \left. + i f_{abc} \bar{\psi}^a \Gamma^i \phi_i^b \psi^c - \sum_{i < j} f_{abc} f_{ade} \phi_i^b \phi_j^c \phi_i^d \phi_j^e + \partial_\mu \bar{c}^a (\partial_\mu c^a + f_{abc} A_\mu^b c^c) + \xi (\partial_\mu A_\mu^a)^2 \right\}$$

Compute the beta function: ²

$$\mu \frac{\partial g(\mu)}{\partial \mu} = \beta(g) = 0 !!$$

This theory is

1. Supersymmetric (2+6 bosonic = 2·4 fermionic dofs $\times N_c^2 - 1$)
2. Conformally invariant on quantum level! Coupling does not run! Coupling is just a number!

¹Brink-Schwarz-Scherk, NPB121(1977)77

²D.R.T.Jones, Phys.Lett.B100(1977)199(2loop), Brink et al, 1983(anyloop)

What is this \mathcal{N} in $\mathcal{N} = 4$?

\mathcal{N} is the number of supersymmetry generators Q_a^i , $a = 1, 2$ (Weyl spinor index, representation of Lorentz group),

$$Q_a^i, \quad i = 1, \dots, \mathcal{N}.$$

SuSy algebra:

$$[P_\mu, P_\nu] = 0, \quad [P_\lambda, L_{\mu\nu}] = \dots, \quad \{Q_a^i, \bar{Q}_{\dot{a}}^j\} = \delta_{ij} 2\sigma_{a\dot{a}}^\mu P_\mu, \quad \{Q_a^i, Q_{\dot{a}}^j\} = 0,$$

Bigger \mathcal{N} , bigger SuSy multiplets: $(1, \frac{1}{2})$, $(1, \frac{1}{2}, 0)$, $(1, \frac{1}{2}, 0, -\frac{1}{2}, -1)$, ...

$\mathcal{N} = 4$ SYM thus misses some essential properties of QCD:

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- scale invariance \Rightarrow no confinement,

$$V(R) \sim -\frac{\sqrt{g^2 N_c}}{R}, \quad \lambda \equiv g^2 N_c \gg 1$$

- no "hadrons"
- no finite T phase transition

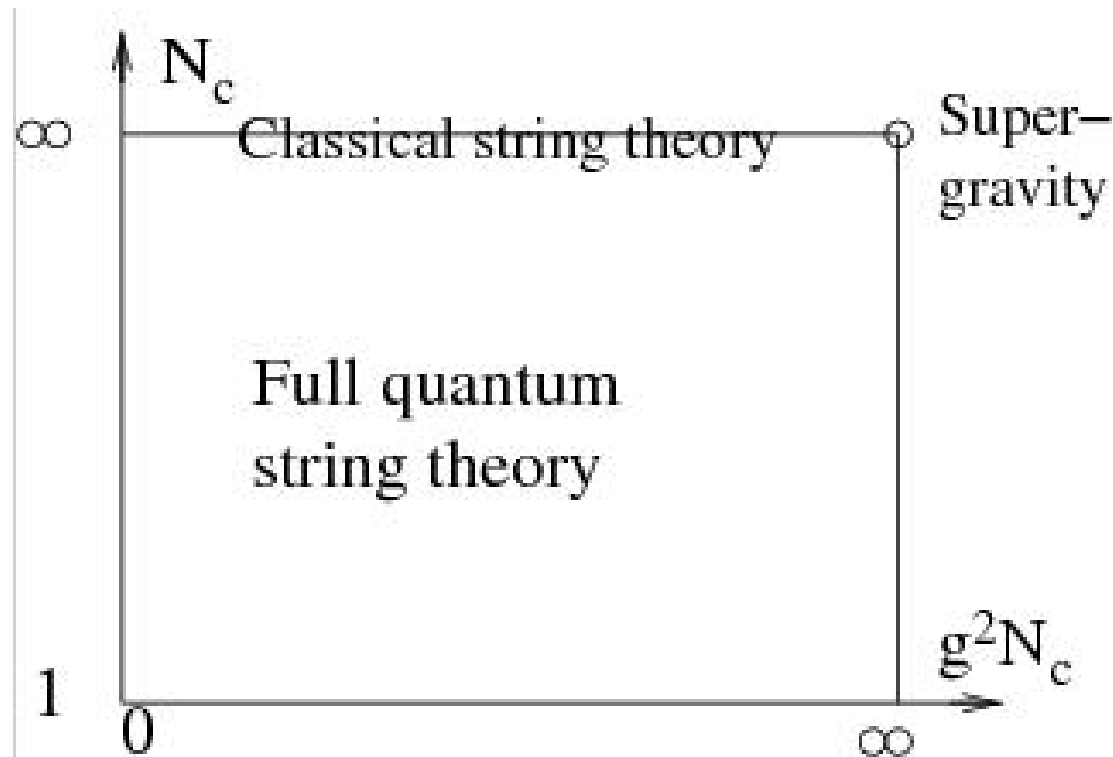
but can be related to string theory:

$$\text{Tension} = \frac{\text{Energy}}{\text{Length}} = \frac{1}{2\pi\alpha'} \equiv \frac{1}{\pi l_s^2}$$

5. AdS/CFT duality

IIB string theory living in a 10 dimensional space with the metric $G_{MN} = \text{AdS}_5 \times S^5$ is the same as $\mathcal{N} = 4$ SYM living on the 4d boundary of AdS_5 if

$$\mathcal{L}^2 = \sqrt{g^2 N_c \alpha'}, \quad \mathcal{L} = \text{"radius" of AdS}$$



Note two levels of simplifying string theory:

1. $N_c \gg 1$

Color algebra: planar diagrams are leading for $N_c \gg 1$. Diagrams with holes \Rightarrow string loops. For $N_c \gg 1$ no string loops \Rightarrow classical string theory.

Nobody knows how to compute NLO corrections in $1/N_c!$

2. $g^2 N_c \gg 1$

String theory has one parameter $\alpha' = 1/\text{Tension}$. When $\alpha' \rightarrow 0$ Tension $\rightarrow \infty$, strings shrink to points, get supergravity (since there are massless spin 2 excitations). For $g^2 N_c = \mathcal{L}^4/\alpha'^2 \gg 1$ we get classical gravity!

NLO corrections in $1/g^2 N_c$ can be computed!

What is AdS?

6. Some Classical Gravity

Einstein-Hilbert (some coordinates $x^\mu = x^0, x^1, \dots, x^{d-1}$):

$$S[g_{\mu\nu}] = \frac{1}{16\pi G} \int d^d x \sqrt{g} (R + 2\Lambda), \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad g = |\det g_{\mu\nu}| \quad g^{\mu\alpha} g_{\alpha\nu} = \delta_\nu^\mu,$$

$$g_{\mu\nu} \Rightarrow R^\alpha_{\mu\beta\nu}, \quad R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}, \quad \dim R = 1/\text{length}^2 = \text{GeV}^2, \quad \dim G = \text{GeV}^{d-2}.$$

EOM from $\delta S / \delta g_{\mu\nu} = 0$:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 0 \left(= 8\pi G T_{\mu\nu}, \quad T_{\mu\nu} = \frac{-2 \delta S_{\text{matter}}}{\sqrt{g} \delta g^{\mu\nu}} \right).$$

For d -dimensional anti-de Sitter (AdS) space of radius \mathcal{L} :

$$\Lambda = \frac{(d-1)(d-2)}{\mathcal{L}^2} \Rightarrow R_{\mu\nu} = -\frac{(d-1)}{\mathcal{L}^2} g_{\mu\nu}$$

There is a length scale \mathcal{L} associated with AdS!

Special solutions

1. Black hole in our world, $d = 4$, solution of

$$R_{\mu\nu} = 0 \quad \text{or of} \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

which is asymptotically flat ($\eta_{\mu\nu}$) and regular on and outside an event horizon (coordinates t, r, θ, ϕ):

$$ds^2 = - \left(1 - \frac{r_s}{r}\right) dt^2 + \frac{1}{1 - r_s/r} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

$$T_{\text{Hawk}} = \frac{1}{4\pi r_s} = \frac{M_{\text{Pl}}^2}{8\pi M}, \quad S_{\text{BH}} = \frac{A}{4G} = 4\pi \frac{M^2}{M_{\text{Pl}}^2}.$$

2. AdS₅, a solution of

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{6}{\mathcal{L}^2} g_{\mu\nu}$$

With coordinates t, x^1, x^2, x^3, z

$$ds^2 = \frac{\mathcal{L}^2}{z^2} (-dt^2 + d\mathbf{x}^2 + dz^2) \quad z = 0 \text{ is } \mathbf{boundary}, \quad z > 0 \text{ is } \mathbf{bulk}$$

Note how a distance scale \mathcal{L} has entered!

Symmetry of AdS₅

AdS₅ can be represented as the surface

$$-t_1^2 - t_2^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = -\mathcal{L}^2$$

in a flat 6 dimensional space with metric

$$ds^2 = -dt_1^2 - dt_2^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

2d sphere S₂ is the surface

$$x_1^2 + x_2^2 + x_3^2 = R^2$$

in flat R₃ with metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2.$$

AdS₅ has the symmetry O(2,4) like $\mathcal{N} = 4$ SYM!

3. AdS₅ black hole:

$$ds^2 = \frac{\mathcal{L}^2}{\tilde{z}^2} \left[- \left(1 - \frac{\tilde{z}^4}{z_0^4} \right) dt^2 + d\mathbf{x}^2 + \frac{d\tilde{z}^2}{1 - \tilde{z}^4/z_0^4} \right]$$

with temperature

$$T_{\text{Hawk}} = \frac{1}{\pi z_0}$$

and entropy

$$S = \frac{A}{4G_5} = V_3 \cdot \frac{\pi^2 N_c^2}{2} T^3 \quad (\text{ultimately})$$

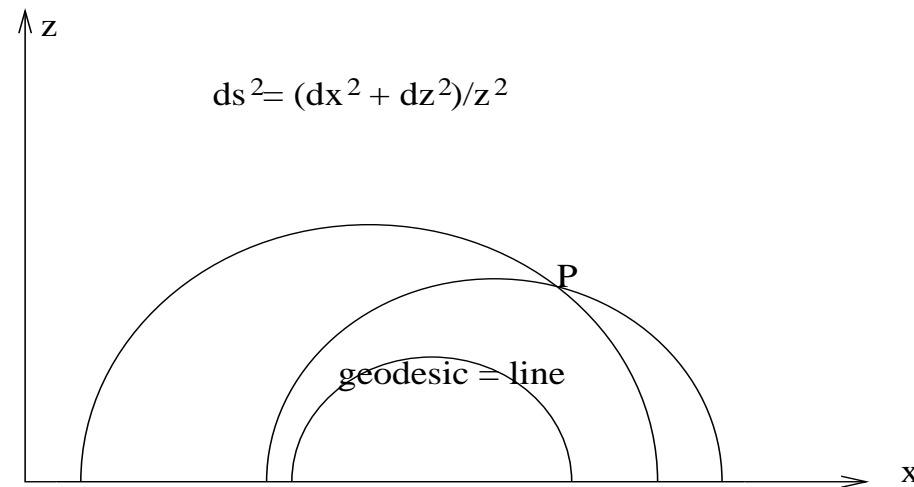
Equivalently

$$ds^2 = \frac{\mathcal{L}^2}{z^2} \left[- \frac{(1 - z^4/(4z_0^4))^2}{1 + z^4/(4z_0^4)} dt^2 + \left(1 + \frac{z^4}{4z_0^4} \right) (dx_1^2 + dx_2^2 + dx_3^2) + dz^2 \right]$$

What is this \mathcal{L}^2/z^2 in the AdS metric?

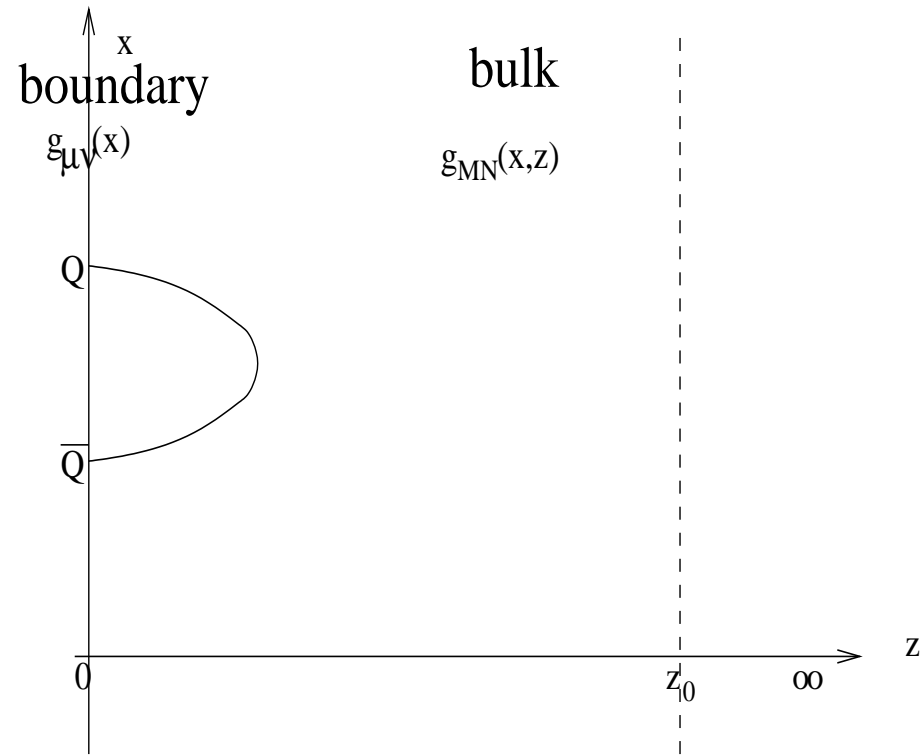
Poincare plane: model of non-Euclidian geometry ³

$$ds^2 = \frac{1}{z^2}(dx^2 + dz^2)$$



$$s = \int ds = \int dz \frac{1}{z} \sqrt{1 + x'(z)^2} \equiv \int dz L[x'(z)] \Rightarrow \frac{d}{dz} \left[\frac{x'(z)}{z\sqrt{1 + x'^2}} \right] = 0 \Rightarrow (x-a)^2 + z^2 = c^2$$

³<http://en.wikipedia.org/wiki/Poincarémetric>



Master formula:

$$\langle \exp \left[\int d^4x O(x) \phi(x, 0) \right] \rangle_{\text{FT}} = \exp \left\{ - \int d^4x \int_0^{z_0} dz \mathcal{L}_{\text{class}}[\phi(x, z)] \right\}$$

7. Hot SYM matter, pressure

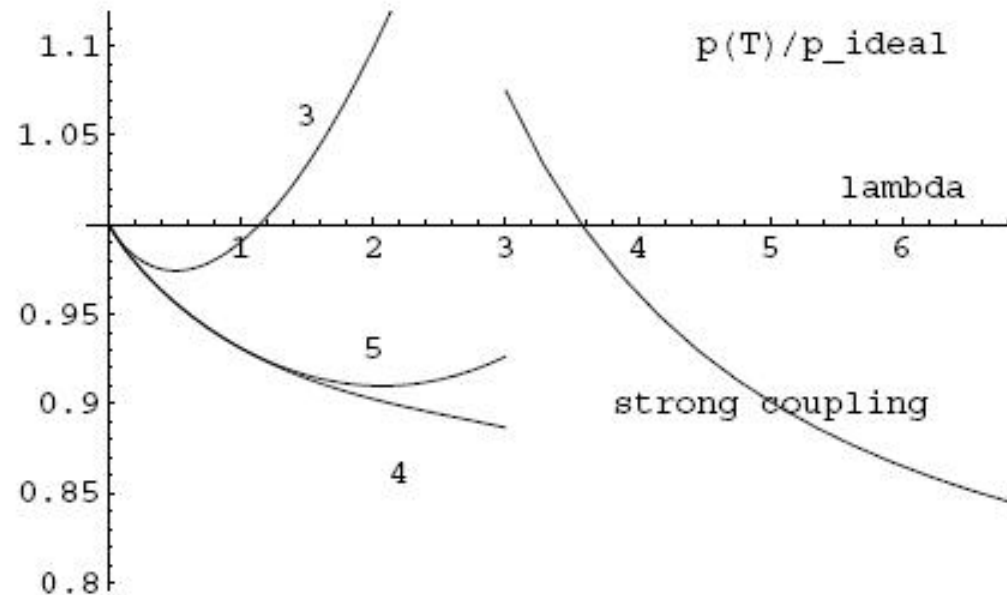
Weak coupling limit:

$$p(T) = (g_B + \frac{7}{8} g_F) \frac{\pi^2}{90} T^4 = (8 + 7) d_A \frac{\pi^2}{90} T^4 = \frac{\pi^2 (N_c^2 - 1)}{6} T^4 (1 + g^2 + g^3 + \dots)$$

(one vector = 2, six scalars = 6, four fermions = 8, all adjoint).

Strong coupling result from $\text{AdS}_5 \times S^5$:

$$p(T) = \frac{\pi^2 N_c^2}{6} T^4 \left[\frac{3}{4} + \frac{45\zeta(3)}{32} \frac{1}{(g^2 N_c)^{3/2}} + \dots \right]$$



How do you derive the result?

Expect $T_{\mu\nu}(x)$ to be related to $g_{MN}(x, z)$.

Method: write the 5d metric in the form

$$g_{MN} = \frac{\mathcal{L}^2}{z^2} \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}$$

and expand near $z = 0$:

$$g_{\mu\nu}(x, z) = g_{\mu\nu}^{(0)}(x) + g_{\mu\nu}^{(2)}(x)z^2 + g_{\mu\nu}^{(4)}(x)z^4 + \dots$$

Then

$$T_{\mu\nu} = \frac{\mathcal{L}^3}{4\pi G_5} [g_{\mu\nu}^{(4)} + \dots]$$

Bulk black hole metric was

$$ds^2 = \frac{\mathcal{L}^2}{z^2} \left[-\frac{(1 - z^4/(4z_0^4))^2}{1 + z^4/(4z_0^4)} dt^2 + \left(1 + \frac{z^4}{4z_0^4}\right) (dx_1^2 + dx_2^2 + dx_3^2) + dz^2 \right]$$

$$\equiv \frac{\mathcal{L}^2}{z^2} \left\{ \left[g_{\mu\nu}(x, 0) + \underbrace{g_{\mu\nu}^{(4)}(x)}_{\sim \tilde{T}_{\mu\nu}} z^4 + \dots \right] dx^\mu dx^\nu + dz^2 \right\}$$

$$\Rightarrow g_{\mu\nu}^{(4)} = \text{diag}(3, 1, 1, 1) \frac{1}{z_0^4}, \quad \frac{1}{z_0} = \pi T.$$

Magnitude: Relating string theory \rightarrow supergravity

$$16\pi G_{10} = 16\pi G_5 \mathcal{L}^5 \pi^3 = (2\pi)^7 \alpha'^4 g_s^2, \quad g_s = g^2/4\pi \quad \text{nontrivial!!}$$

$g_s =$ closed string coupling, one handle costs g_s^2 . $\mathcal{L}^4 = g^2 N_c \alpha'^2$

$$\Rightarrow \frac{\mathcal{L}^3}{G_5} = \frac{2N_c^2}{\pi}$$

$$\Rightarrow T_{\mu\nu} = \frac{\mathcal{L}^3}{4\pi G_5} g_{\mu\nu}^{(4)} = \frac{N_c^2}{2\pi^2} g_{\mu\nu}^{(4)} = \begin{pmatrix} 3aT^4 & 0 & 0 & 0 \\ 0 & aT^4 & 0 & 0 \\ 0 & 0 & aT^4 & 0 \\ 0 & 0 & 0 & aT^4 \end{pmatrix} \quad a = \frac{\pi^2 N_c^2}{8}$$

8. Hot SYM matter, viscosity

Reminder: Reynolds number

$$Re = \rho L V / \eta.$$

Puzzle of "small" η : Solutions of Navier-Stokes flow equations (η included) do not go to those of Euler flow ($\eta = 0$) when η is "small"; get turbulence for large Re .

Weak coupling kinetic theory:

$$\eta = p\tau_c \sim \frac{T^4}{nv\sigma} \sim \frac{T^3}{g^4}, \quad \text{parametrically large}$$

but

$$\frac{\eta}{s} \approx \frac{p\tau_c}{s} \approx \frac{T^4\tau_c}{T^3} = T\tau_c \gtrsim \hbar \quad \text{uncertainty principle}$$

Experimental fact: QCD matter observed in heavy ion collisions at RHIC/BNL has T up to $5T_c$ (strongly coupled!!) and flows nearly ideally.

Seems paradoxical: weakly coupled fluid has a "large" viscosity!

Bjorken flow: $v(t, x) = x/t$,

$$T(\tau) = \left(T_i + \frac{1}{6\pi\tau_i} \right) \left(\frac{\tau_i}{\tau} \right)^{1/3} - \frac{1}{6\pi\tau}.$$

Strong coupling result:

$$\eta = \frac{\pi}{8} N_c^2 T^3 \left[1 + \frac{75\zeta(3)}{4\lambda^{3/2}} + \dots \right] \quad 1 + \left(\frac{8.0}{\lambda} \right)^{3/2}$$

$$\frac{\eta}{s} = \frac{\hbar}{4\pi} \left[1 + \frac{135\zeta(3)}{8\lambda^{3/2}} + \dots \right] \quad 1 + \left(\frac{7.4}{\lambda} \right)^{3/2}$$

From the correlator:

$$\eta = \lim_{\omega \rightarrow 0} \frac{1}{2\omega} \int dt d^3x e^{i\omega t} \langle T_{12}(x) T_{12}(0) \rangle \quad \int d^4x T_1^2(x) g_2^1(x, z=0)$$

Lower limit for all physical systems:

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi}$$

= holds for systems having a gravity dual.

Air ($\eta \sim 10^{-5}$, $s = S/V \sim N/V \sim 1\text{kg}/m_p/m^3 \sim 10^{27}/m^3$):

$$\frac{\eta}{s} \approx \frac{10^{-5}}{10^{27}} \gg \frac{10^{-35}}{4\pi}$$

Conformal hydrodynamics

Navier-Stokes + Bjorken:

$$\tau \epsilon'(\tau) = -\frac{4}{3} \epsilon + \frac{4}{3} \frac{\eta}{\tau}$$

But there is more:⁴ 2nd order causal hydro:

$$\tau \epsilon'(\tau) = -\frac{4}{3} \epsilon + \Phi$$

$$\tau_{\text{II}} \partial_{\tau} \Phi = \frac{4}{3} \frac{\eta}{\tau} - \Phi - \frac{4}{3} \frac{\tau_{\text{II}}}{\tau} \Phi - \frac{1}{2} \frac{\lambda_1}{\eta^2} \Phi^2$$

$$\tau_{\text{II}} = \frac{2 - \log 2}{2\pi T}, \quad \lambda_1 = \frac{\eta}{2\pi T}$$

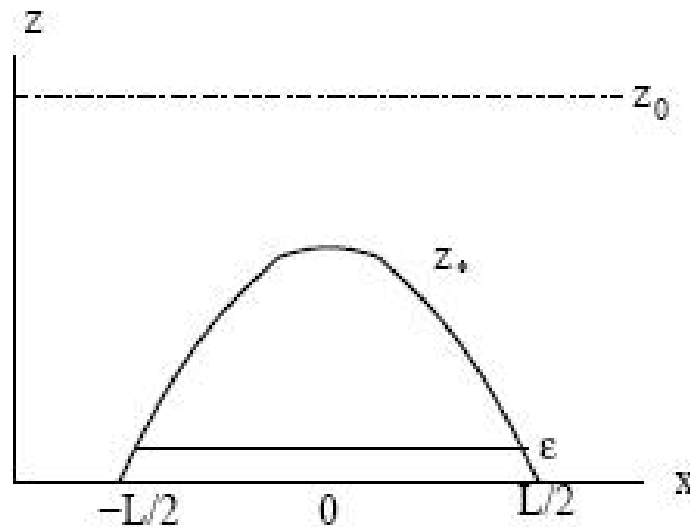
⁴Baier-Romatschke-Son-Starinets-Stephanov, 0712.245

9. Wilson loops, $Q\bar{Q}$ potential

$P \exp [ig \int_C A^\mu dx_\mu]$ $C =$ closed loop, Tr is gauge invariant

Expectation value of a Wilson loop in the boundary field theory = Extremal action of the string hanging from the loop in the 5th dimension.

Take Q at $x = -L/2$, \bar{Q} at $L/2$. How deep does the string connecting them hang in the z direction, i.e., what is $z = z(x, t) = z(x)$ for the extremal configuration?

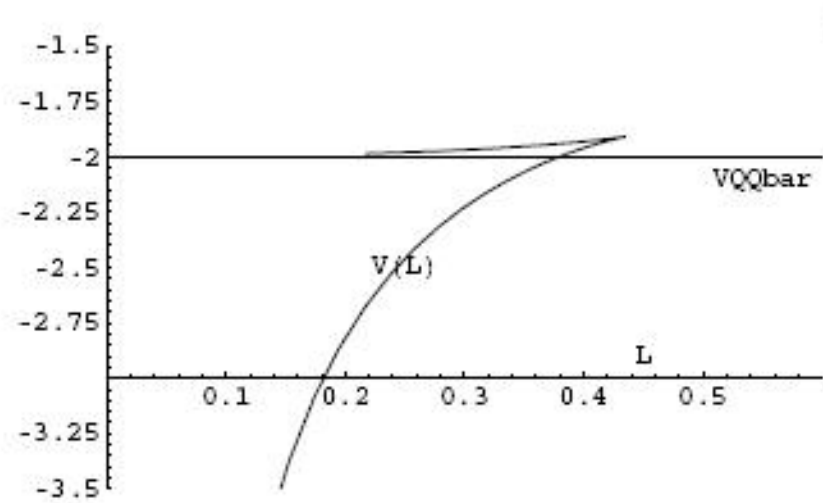


Metric:

$$ds^2 = \frac{\mathcal{L}^2}{z^2} \left[- \left(1 - \frac{z^4}{z_0^4} \right) dt^2 + d\mathbf{x}^2 + \frac{dz^2}{1 - z^4/z_0^4} \right]$$

Extremize:

$$V = \frac{\mathcal{L}^2}{\pi\alpha'} \int_{\epsilon}^{z_*} \frac{dz}{z^2} \sqrt{1 + \left(1 - \frac{z^4}{z_0^4} \right) x'(z)^2}$$



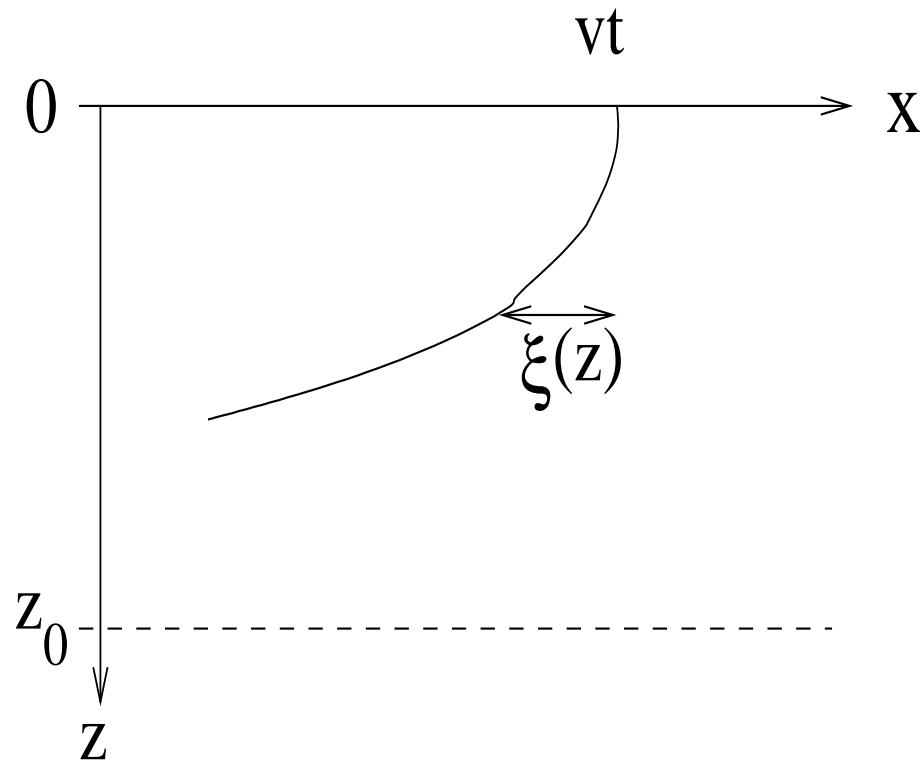
Small L : $V(L) = -\frac{0.2285\sqrt{g^2 N_c}}{L}$

$L < L_{\max}$: $V(L) = -\frac{4}{3} \frac{\alpha_s}{L} + \sigma L$

$L > L_{\max}$ dissociation into separate Q, \bar{Q} .

10. Q in motion, dangling string

Extremize for $x(t, z) = vt + \xi(z)$ $\frac{1}{2\pi\alpha'} \int dt dx \frac{\mathcal{L}^2}{z^2} \sqrt{1 - \frac{z^4}{z_0^4} + z'^2 - \dot{z}^2 / (1 - z^4/z_0^4)}$

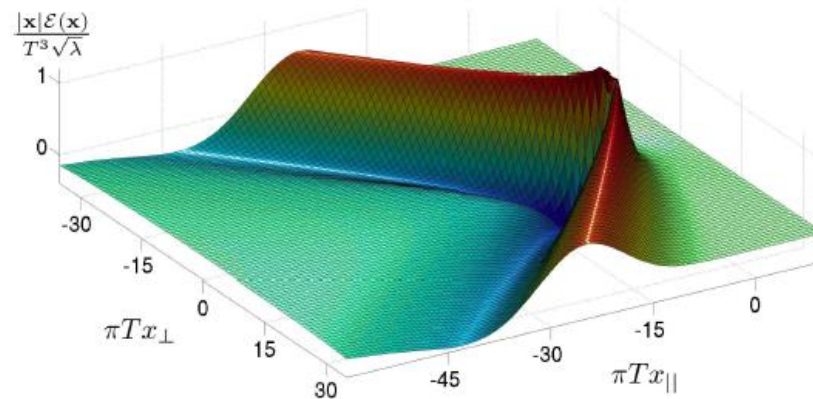
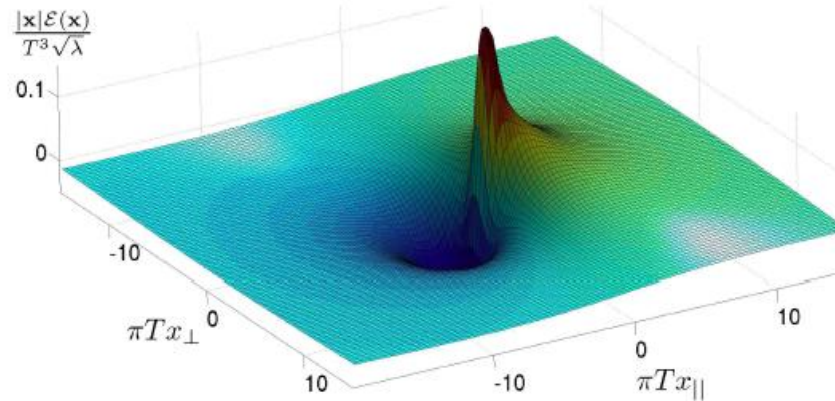


$$\frac{\xi(z)}{z_0} = v \left(\frac{1}{4} \log \frac{z_0+z}{z_0-z} - \frac{1}{2} \arctan \frac{z}{z_0} \right) = v \frac{1}{3} z^3 + \dots$$

$$\text{Drag force } \frac{dp}{dt} = -\frac{1}{2} \pi v \gamma \sqrt{g^2 N_c T^2} = -\mu p$$

Quark is forced to move: where does this energy go?

- Take the T_{MN} of the dangling string as source of gravity in 5d
- Solve linearised Einstein equations for $g_{MN} = g_{MN}^{AdSBH} + h_{MN}(x, z)$
- Compute $T_{\mu\nu}(x)$: $v = \frac{1}{4}, \frac{3}{4}$ towards top right



Conclusions

AdS/CFT is a vast field of theoretical research, based on beautifying assumptions.

Increases quality of life for theorists.

Nature is not conformal, nor susy, nor is there evidence of extra dimensions

Unclear if it ever will develop into a quantitative method of computation