ITP/Santa Barbara, April 2002

The free energy of hot QCD

K. Kajantie (Helsinki)

1. Free energy of QCD up to $T \approx 10T_c$ from 4d $(\tau, \mathbf{x})$ lattice Monte Carlo: $f(T, T/\Lambda_{\text{QCD}}; N_c, N_f)$

2. Free energy of QCD from $\approx 3T_c$ to T_{Planck} from effective theories and 3d $(\mathbf{x})$ lattice Monte Carlo

3. Effective degrees of freedom in equilibrium QCD plasma at $T \gtrsim 3T_c$ are gluons with up to 4loop interactions; the nonperturbative constant seems small for p (is large for, say, Debye mass)

Note: Same methods solve electroweak *phase transition*

Joint work with K. Kajantie, M. Laine, K. Rummukainen, Y. Schröder, hep-ph/0007109 (PRL),hep-ph/0109100 (PRD)

1. General definition

We know QCD:

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \sum_{N_f}\bar{\psi}_f\gamma_\mu(\partial_\mu + igT_aA^a_\mu)\psi_f,\\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf_{abc}A^b_\mu A^c_\nu\end{aligned}$$

Its free energy density $(= f = -p)$ is

$$e^{-f(T,g)V/T} = \int \mathcal{D}A^a_\mu(\tau,\mathbf{x})\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\int_0^{1/T}d\tau\int d^3x\,\mathcal{L}[A^a_\mu,\bar{\psi},\psi]}$$

so just compute this!

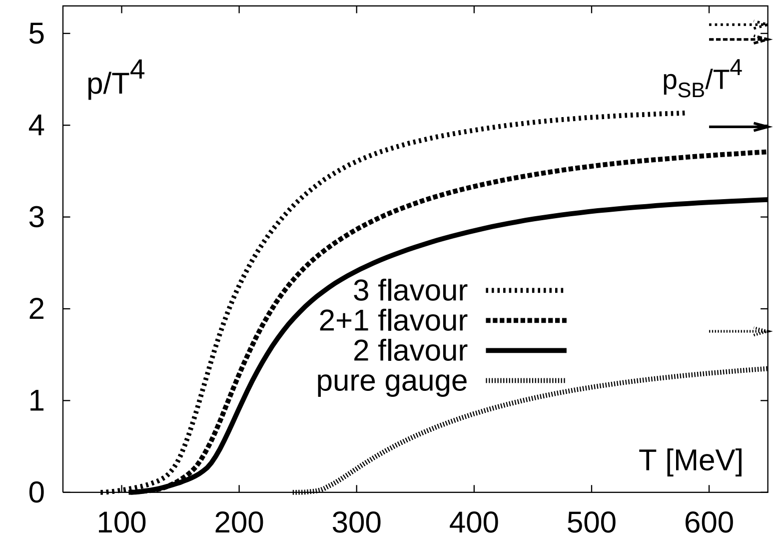
b.c. over τ are periodic (A^a_μ) or anti-periodic in the Grassmannian $(\bar{\psi},\psi)$.

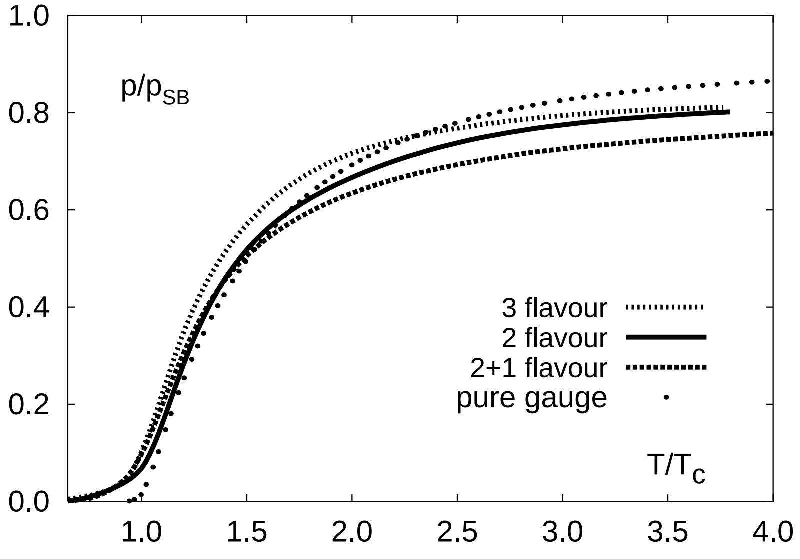
Ideal gas:

$$-f(T,g=0) \equiv p_{\rm SB}(T) = (16 + \frac{21}{2}N_f)\frac{\pi^2}{90}T^4.$$

A truly first-principle method for computing f is numerical lattice Monte Carlo:

Lattice: measure derivatives of F as expectation values, integrate back F by choosing $F/T = 0$ deep in hadronic phase (data from Bielefeld):





Note: at $T = T_c$

- $p(T_c) \approx 0$ for pure glue since glueballs very heavy
- $p(T_c) = p$ of (massless) pion gas for $N_f = 2 \ldots 3$
- $p'(T_c)$ discontinuous $\Rightarrow$ genuine phase transition

Related catch words:

Casimir effect:

Finite range in coordinates, now $0 < \tau < \hbar/T$

$$\Rightarrow \epsilon \sim T^4$$

Usually ideal gas in domain of some shape, now emphasis is on interactions

Cosmological constant:

The value of the functional integral is $\sim$ the "cosmological constant" of our theory: we can only determine how it varies, but there is an overall unknown additive scale.

Two ranges of different equilibrium physics:

- $T_c \lesssim T \lesssim \mathrm{few} \times T_c$: Relevant dofs (?): T-modified hadrons, instantons, monopoles, Polyakov line,...
- $T \gtrsim \mathrm{few} \times T_c$: Relevant dofs (?): collective modes, *interacting* gluons

Static and dynamic dofs may be different!

The 1st principle 4dim lattice computation in practice only is feasible up to some $10T_c$:

$$a \ll \frac{1}{T} \ll \frac{1}{T_c} \approx \frac{1}{\Lambda_{\text{QCD}}} \approx 1\text{fm} \ll Na$$

At $10T_c$ and if $\ll$ is a factor 1/4:

$$0.025\text{fm} < 0.1\text{fm} < 1\text{fm} < 4\text{fm}$$

Intractable multiscale problem!

So what to do at $T > 2...3T_c$?

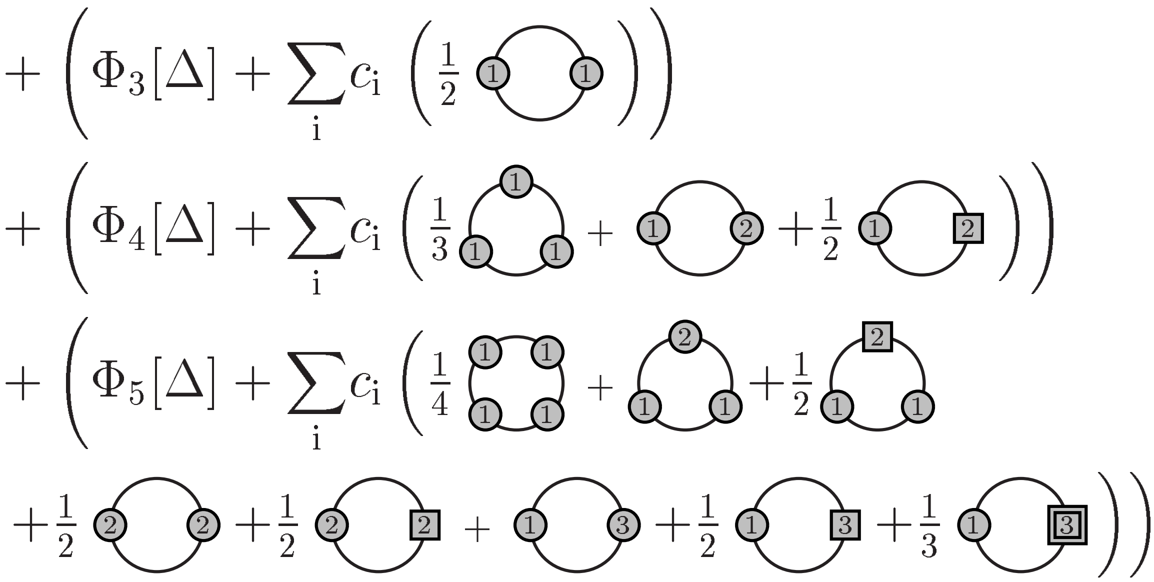
Why care about this? Just structureless ≈20% below S-B! But: this is most fundamental quantity in hot QCD, want to understand everything!

- 4d perturbation theory: sooner or later you hit the infrared wall; must use experiment (like in jet physics) or computers

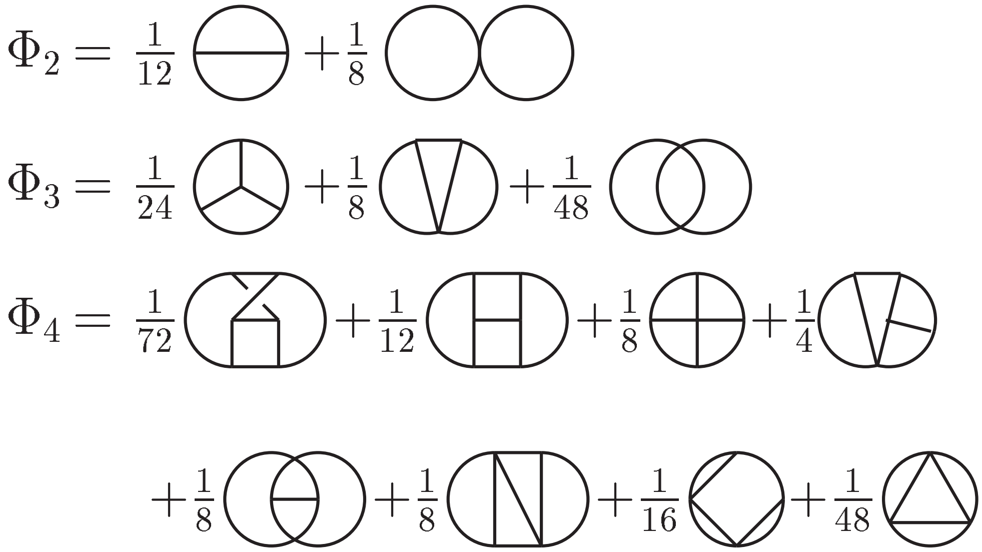
- Organise the computation in a part in which perturbation theory works (derive 3d effective theories by integrating over excitations of mass $\sim \pi T, gT$) and a part which has to be treated numerically

Topologies of diagrams one has to evaluate (hepph0109100):

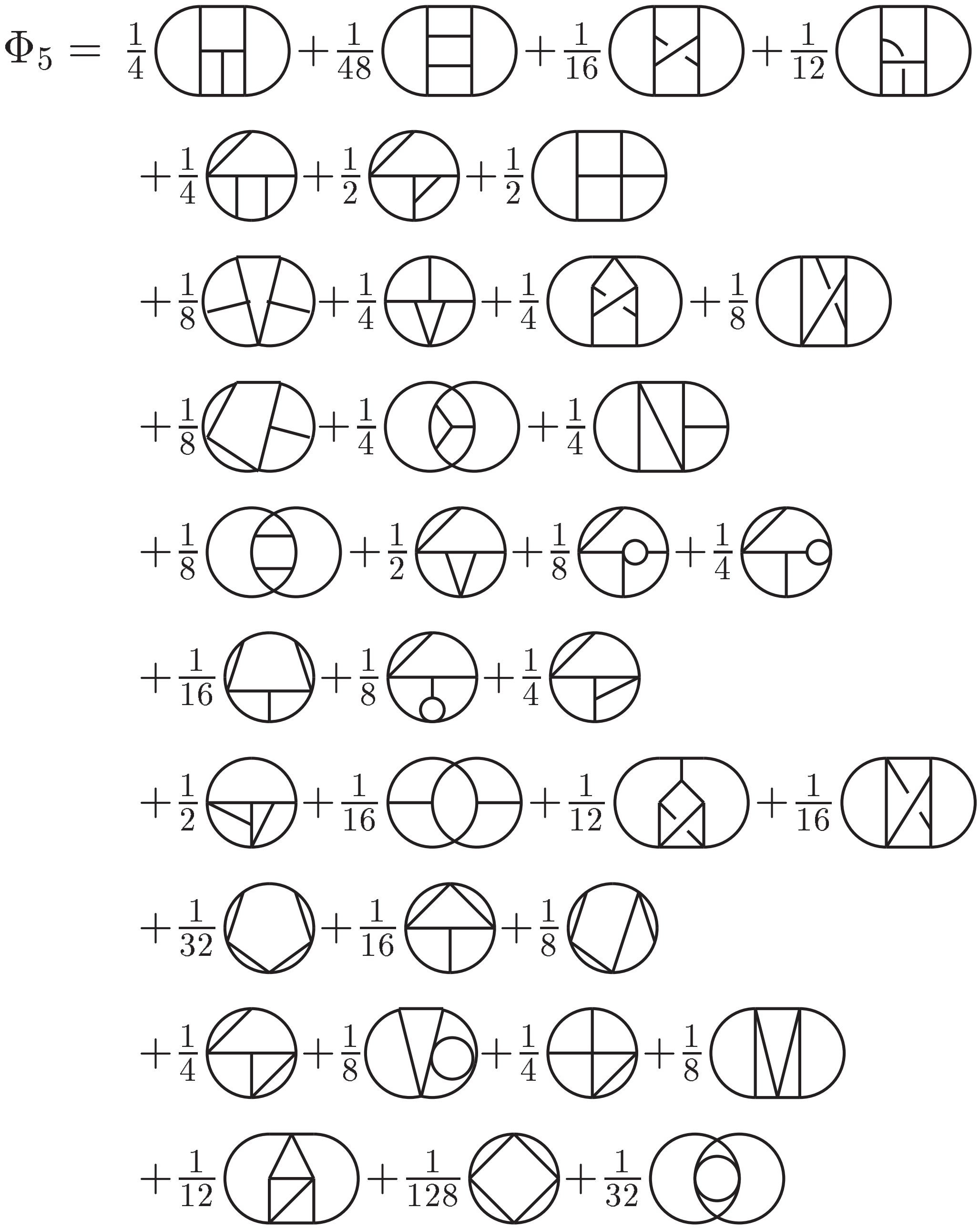
$$-F \;=\; -F_0 + \Phi_2[\Delta] \qquad c_{\text{boson}} = \tfrac{1}{2}, \quad c_{\text{fermion}} = -1$$



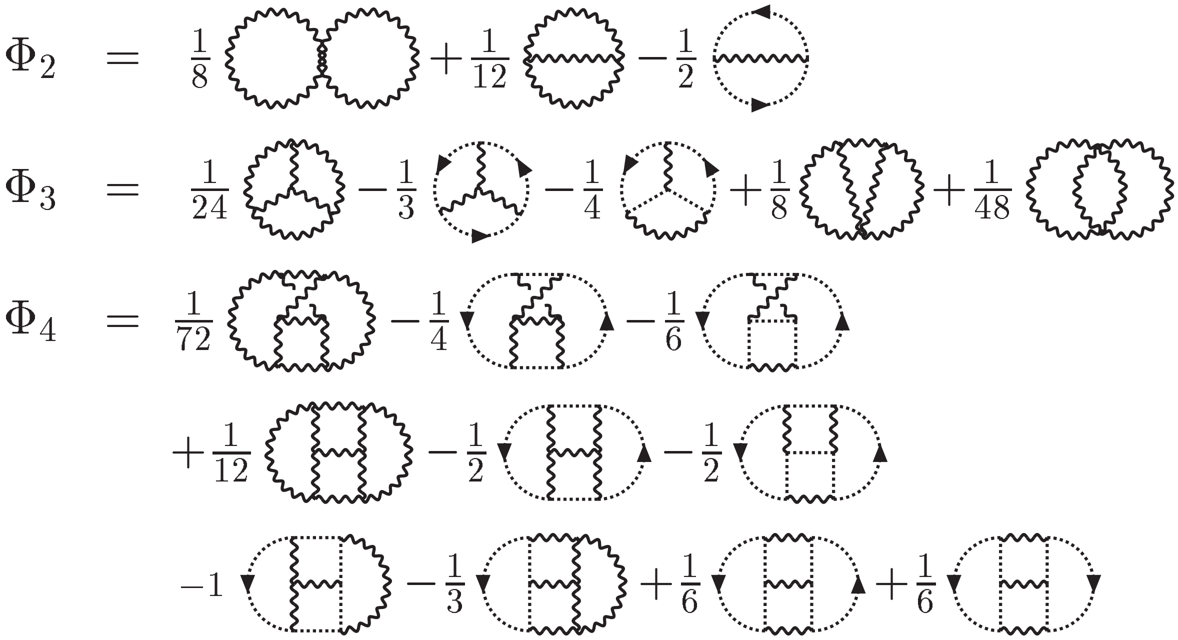
The skeleton topologies are (up to 4loop):



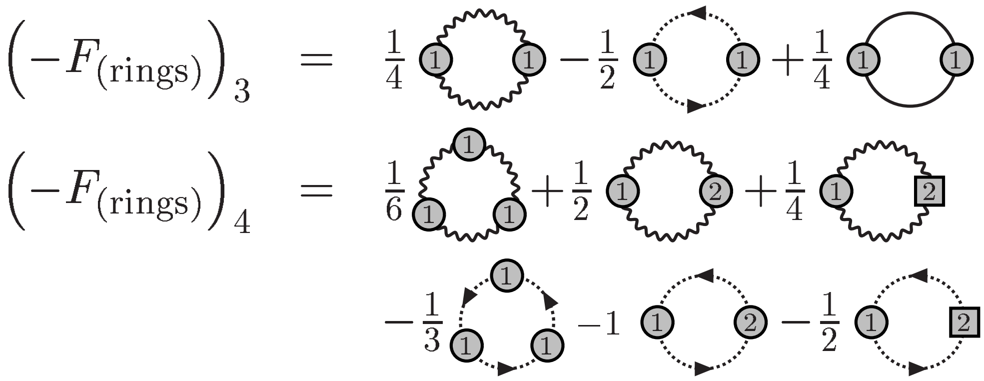
Here are the 5loop skeletons, for your enjoyment;



QCD skeletons:



and ring diags:

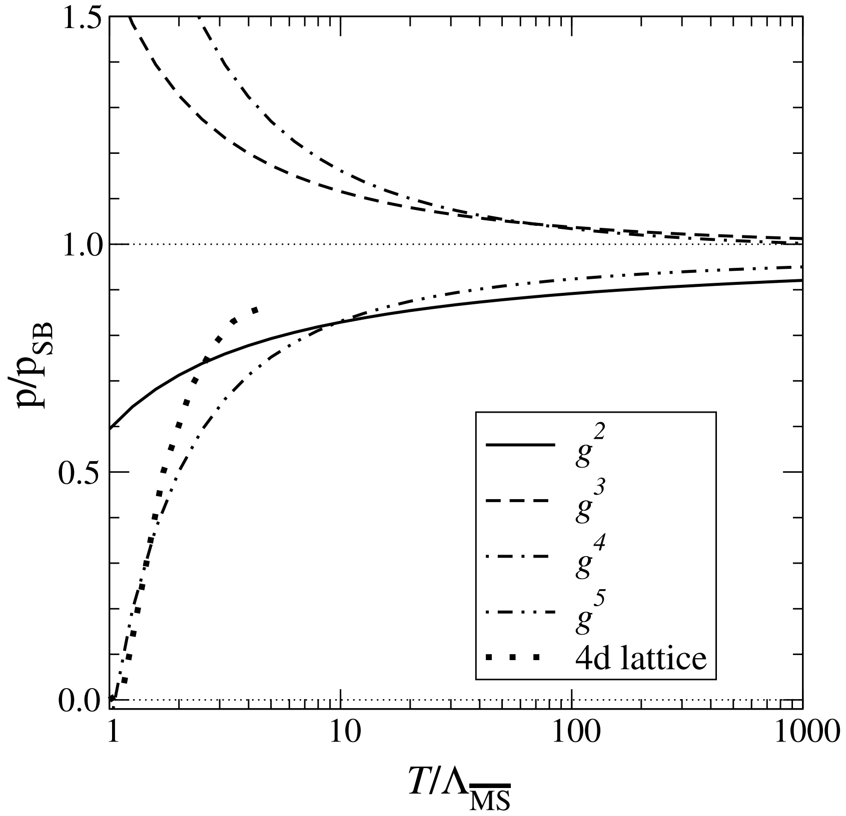


4d perturbation theory in g hits the infrared wall at order g^6:

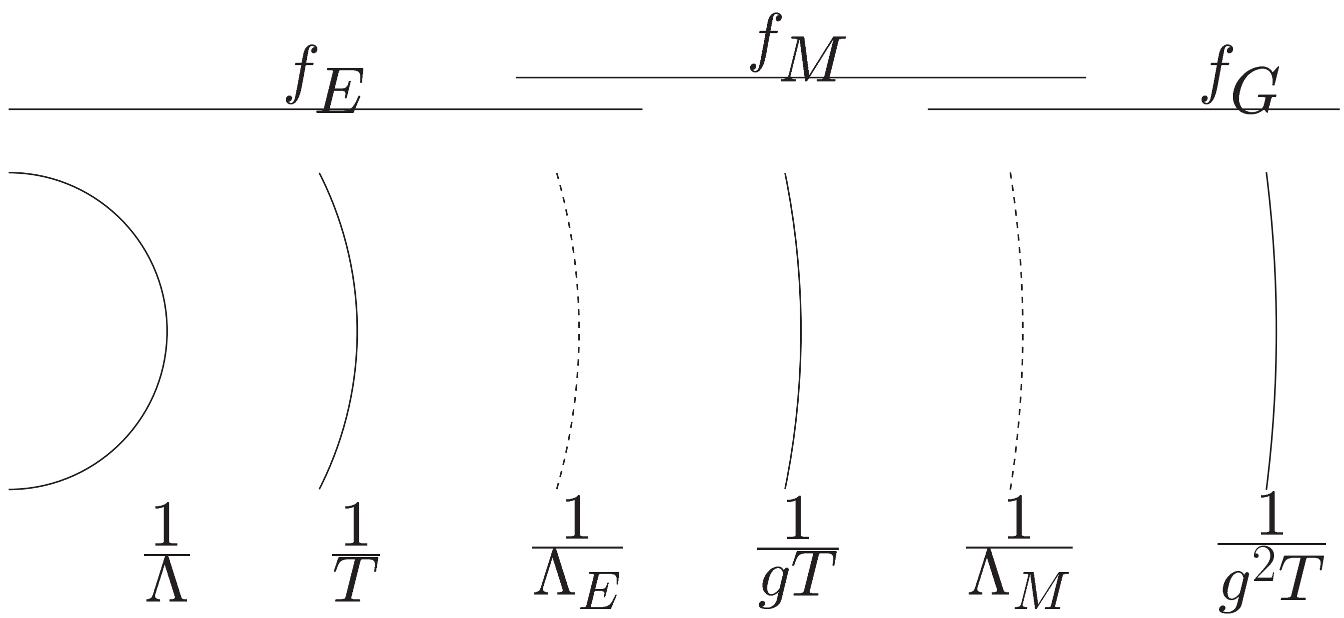
$$\frac{p}{p_{\text{SB}}} = 1 + c_1 g^2 + c_2 g^3 + c_3 g^4 \ln\frac{1}{g} + c_4 g^4 + c_5 g^5 + c_6 g^6 \ln\frac{1}{g} + c_7 g^6 + \ldots$$

c_1: Shuryak '78 (2-loop)
c_2: Kapusta '79 (resummed 2-loop)
c_3: Toimela '83 (resummed 2-loop)
c_4: Arnold, Zhai '94 (resummed 3-loop)
c_5: Kastening, Zhai; Braaten, Nieto '95 (resummed 3-loop)
c_6: perturbatively computable; not computed yet (4-loop)
c_7: numerically computable

$N_f = 0$:



Relevant distance and matching scales:



$\Lambda =$ ultraviolet scale for f_E (theory at scale T)

$\Lambda_E =$ infrared scale for f_E **AND** UV scale for f_M

$\Lambda_M =$ IR scale for f_M **AND** UV scale for f_G

Note:

- Matching scales Λ_E, Λ_M cancel
- No IR scale for f_G
- Big technical problem: dim reg in MSbar has UV scale = IR scale!

Braaten-Nieto decomposition (omit $\lambda_E \sim g^4$):PRD53(96)3421

$$
\begin{aligned}
F/T &= f_E + f_M + f_G \\
&= f_E(\Lambda; T, g(\Lambda); \Lambda_E) + \\
&\quad + f_M(\Lambda_E; m_E(\Lambda, g^2(\Lambda), \Lambda_E), g_E^2(\Lambda, g^2(\Lambda)); \Lambda_M) + \\
&\quad + f_G(\Lambda_M; g_M^2(\frac{g_E^2}{m_E})) \\
&= T^3[1 + g^2(\Lambda) + (2\beta_0 \log\frac{\Lambda}{T} + \log\frac{T}{\Lambda_E})g^4(\Lambda) + \\
&\quad + (\ldots + 2\beta_1 \log\frac{\Lambda}{T} + a\log\frac{T}{\Lambda_E})g^6(\Lambda) + \ldots] \\
&\quad + \Lambda_E^3 + \Lambda_E^2 g_E^2 + \Lambda_E m_E^2 + \Lambda_E g_E^4 + \\
&\quad + m_E^3 + \log\frac{\Lambda_E}{m_E} \cdot m_E^2 g_E^2 + m_E g_E^4 + \\
&\qquad\qquad + (a\log\frac{\Lambda_E}{m_E} + b\log\frac{m_E}{\Lambda_M})g_E^6 \\
&\quad + \Lambda_M^3 + \Lambda_M^2 g_M^2 + \Lambda_M g_M^4 + b\log\frac{\Lambda_M}{g_M^2} \cdot g_M^6
\end{aligned}
$$

We need a and b = 4loop UV divergences of f_M, f_G.

3d effective theory for f_M

is a 3d SU(3) gauge + adjoint Higgs theory:

$$\exp[-V f_M(T/\Lambda_{\overline{\rm MS}})] = \int \mathcal{D}A_i^a \mathcal{D}A_0^a \exp\{-\int d^3x \times$$
$$[\frac{1}{4}F_{ij}^a F_{ij}^a + \frac{1}{2}(D_iA_0)^a(D_iA_0)^a +$$
$$+ \underbrace{\frac{1}{16\pi^2}(22\log\frac{5.371T}{\Lambda_{\overline{\rm MS}}} + 9)}_{\equiv y/2 \sim 1/g^2} A_0^a A_0^a +$$
$$+ \underbrace{\frac{3}{44\log(5.371T/\Lambda_{\overline{\rm MS}})}}_{\equiv x/4}(A_0^a A_0^a)^2 + \frac{\cdots}{T}]\}$$

A_i, A_0,$\mathbf{x}$, dimensionless, $D_i = \partial_i + iA_i$

The coefficients are determined in next-to-leading-order optimised perturbation theory.

Challenging problem: compute them to NNLO!

Note: $f_M(x,y)$ is defined for any x, y; 4d physics or $f_M(T/\Lambda_{\overline{\rm MS}})$ is on the curve $y = 3/(8\pi^2 x)(1 + 3x/2 + \mathcal{O}(x^2))$.

The $g^6 \log g$ terms are

$$\frac{f}{g^6} = \underbrace{a \log \frac{T}{\Lambda_E}}_{\text{4d theory}} + \underbrace{a \log \frac{\Lambda_E}{m_E} + b \log \frac{m_E}{\Lambda_M} + b \log \frac{\Lambda_M}{g_M^2}}_{\text{3d theory}}$$

Adding:

$$\frac{f}{g^6} = a \log \frac{T}{m_E} + b \log \frac{m_E}{g_M^2} = (a+b) \log \frac{1}{g}$$

We will discuss 3d theories in MSbar or lattice regularisation with ONE regularisation scale, g_E^2 or $1/a$ (relating the two schemes correctly is crucial). In the 3d sector the log term is $(\Lambda_E \sim \Lambda_M \sim g_E^2)$

$$a \log \frac{g_E^2}{m_E} + b \log \frac{m_E}{g_E^2} = (-a+b) \log \frac{1}{g}$$

The (colossal) 4-loop perturbative computation of a, b is under way.

To get b you must find the UV log div of 3d SU(3) gauge theory. In msbar the result is ZERO since no scale whatsoever in the problem: UV + IR = 0. The IR singularities have to be shielded by a gluon mass, say.

In the 3d sector

$$
\begin{aligned}
&(f_M + f_G)/T^3 = \\
&g^3 + g^4 \log\frac{1}{g} + g^5 + \qquad \leftarrow \text{known} \\
&+(-a+b)g^6 \log\frac{1}{g} + \qquad \leftarrow \text{find numerically} \\
&+cg^6] \qquad\qquad \leftarrow \text{numerically small}
\end{aligned}
$$

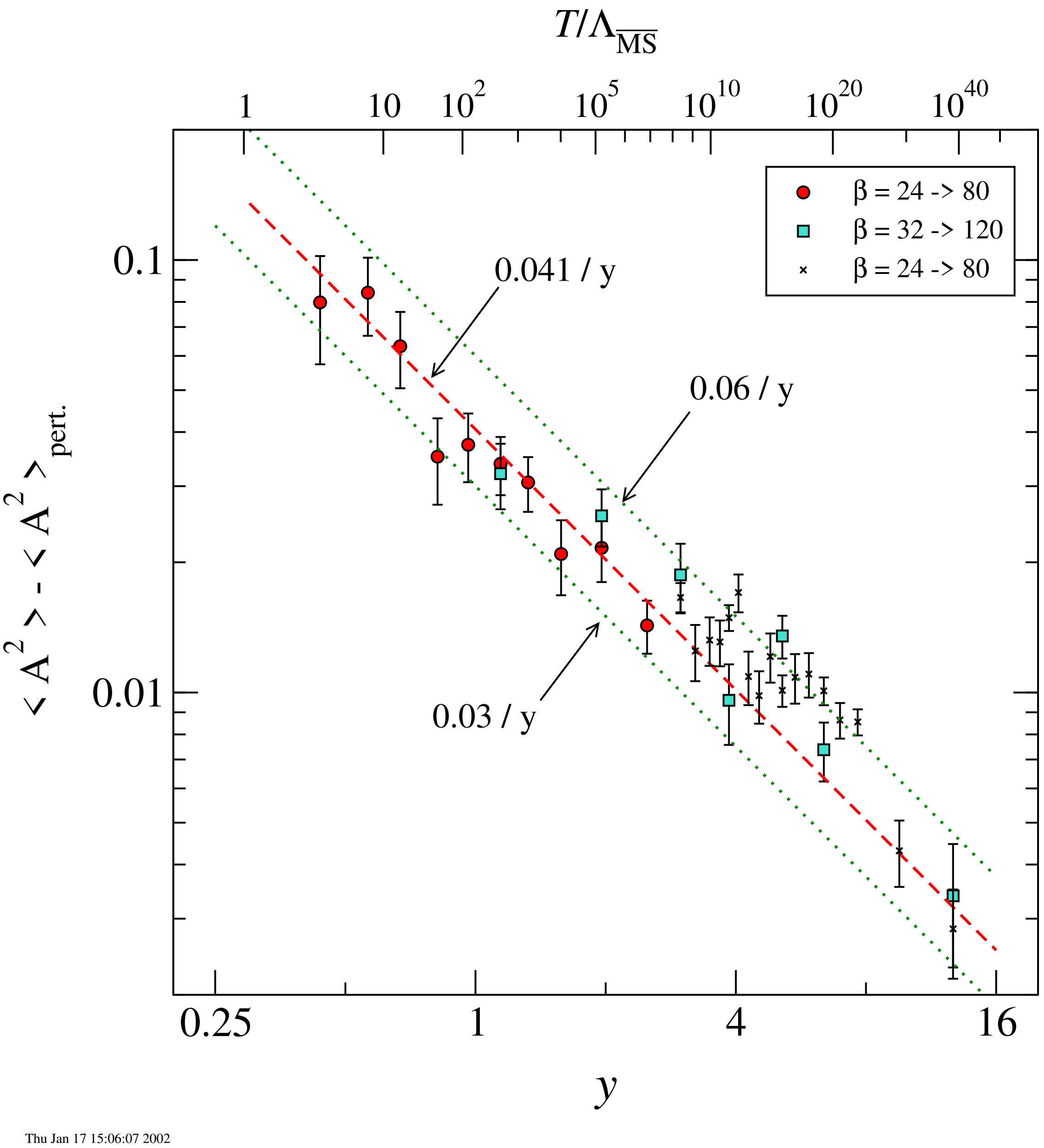
Determine $-a+b$ numerically by

$$
\frac{\partial((f_M + f_G)/g^6)}{\partial(1/g^2)} = \frac{1}{g} + \log\frac{1}{g} + 1 + g + (-a+b)\frac{g^2}{2} \sim \langle A_0^2 \rangle
$$

$$
e^{-f} = \int \mathcal{D}A_0 \, e^{-yA_0^2 + \ldots}
$$

So measure on the lattice $\langle A_0^2 \rangle$, relate lattice and MSbar schemes, take the large V small a (continuum) limits, subtract the known 1+2+3-loop perturbative part and find the $g^2 \sim 1/y \sim 1/\log(T/\Lambda_{\overline{\rm MS}})$ dependence:

Result $(y \sim 1/g^2 \sim \log(T/\Lambda_{\overline{\mathrm{MS}}}))$; after $6 \cdot 10^{15}$ flops:

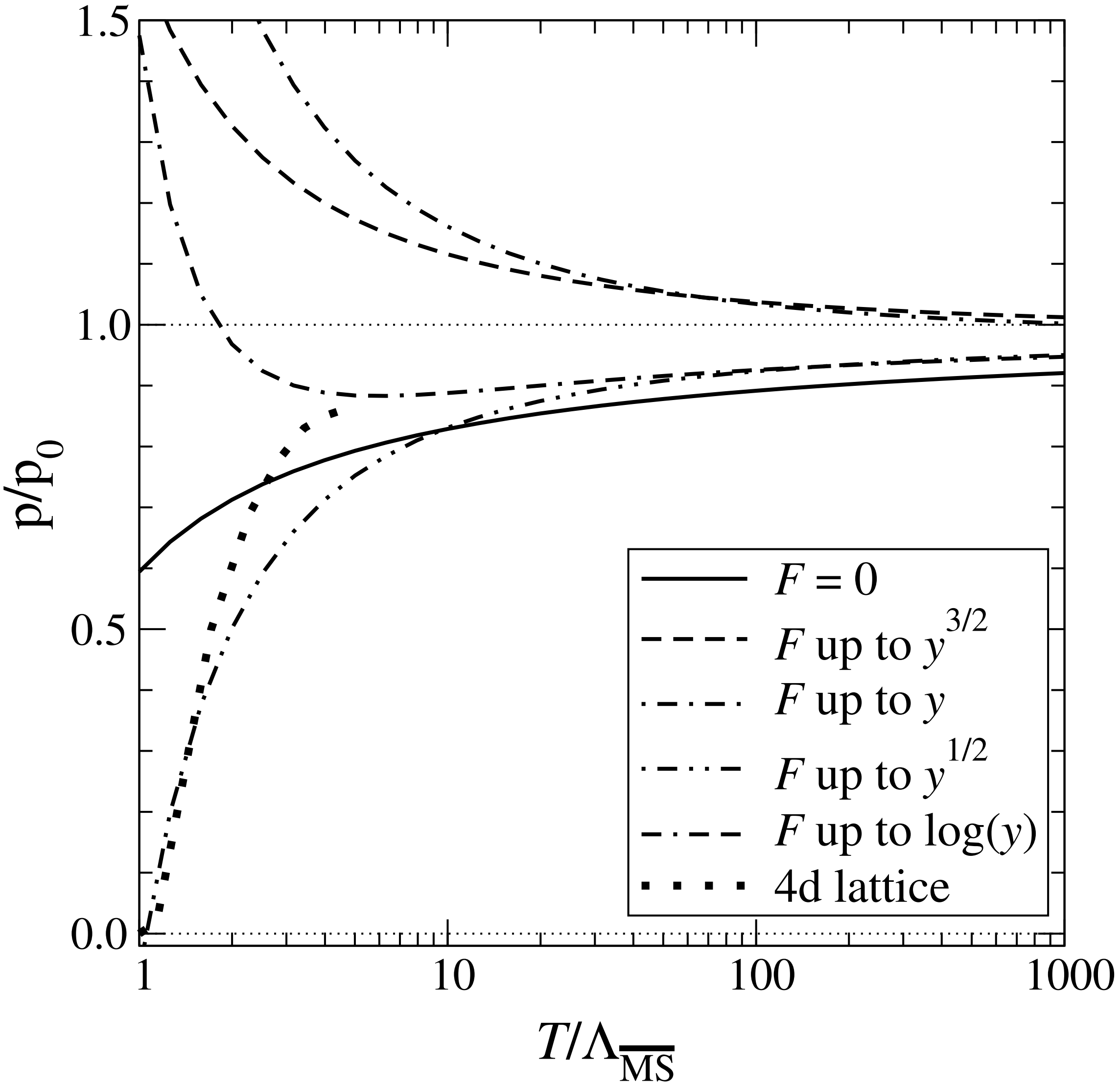


$$\Rightarrow \frac{d_A C_A^3}{2(4\pi)^4}(-a+b) = 0.041(2) \rightarrow -a+b = 9.5$$

$$d_A = 8,\ C_A = 3$$

Note that $0.04/y$ saturates data over all y!

The physical pressure $p/p_{\rm SB} = 1 + ... + (a+b)g^6 \log g + cg^6$ and we know $-a+b$. Let's assume b dominates and see what we get:



Excellent connection to 4d lattice data at about $T = 5T_c$!

Note that there are no further terms; everything is in the numerics.

Suggests that a (UV log of 3d) and c (the entirely nonperturbative constant) are small!

Conclusions

- We are now in hot QCD progressing like in jet physics or DIS: there is an "infinite" perturbative series (LO,NLO,NNLO,..) series leading to logarithmic scale dependence and something entirely nonperturbative (pdf at some scale, the g^6 constant)

- At the 4loop NNNLO level in hot QCD there are two log-constants a, b; the combination $-a+b$ has been numerically determined

- The analytic computation is under way: the 4loop computation of $g^4\lambda, g^2\lambda^2, \lambda^3$ terms is done

- A good connection to 4d lattice date at $T = T_c$ using just b (the UV divergence of 3d SU(3) gauge theory) suggests that the other constants are numerically small