Expanding conformal matter in gauge theory/gravity duality

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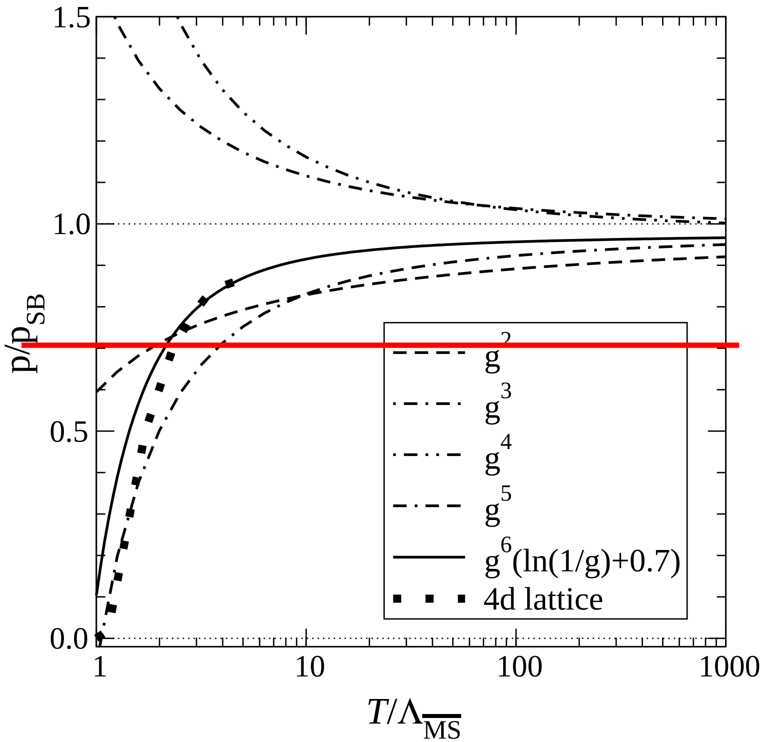
1. Background

$\mathcal{N}=4$ SYM prediction "applied to hot QCD":

$$p(T) = p_{\text{SB}}(T)\ [0.75 + (0.15\,\mathrm{Log}\frac{T}{T_c})^{1.5} + ..]$$



Wrong for $T\lesssim 3T_c$ (not conformal) and $T\gtrsim 100T_c$ (not strongly coupled)

Now make this conformal matter expand, $\infty > T > 0$:

Bjorken expansion (1983) of massless conformal fluid, $\epsilon = 3aT^4$ in 1+1+2 dim:

$$v(t,x) = \frac{x}{t}$$

$$\epsilon(\tau) \sim T^4(\tau) \sim \frac{1}{\tau^{4/3}}, \qquad T(\tau) \sim \frac{1}{\tau^{1/3}}.$$

Same with viscosity $\eta \sim T^3$, $\zeta = 0$ (Hosoya-KK 1985)

$$T(\tau) = T_f \left(\frac{\tau_f}{\tau}\right)^{1/3} - \frac{c}{\tau}$$

Dimensions of dissipative coefficients in 4d:

$$T_{\mu\nu} = \epsilon u_\mu u_\nu + \eta(\partial u)_{\mu\nu} + \lambda_2(\partial u)^2_{\mu\nu} + \lambda_3(\partial u)^3_{\mu\nu} + \lambda_4(\partial u)^4_{\mu\nu}$$

$$\Rightarrow \qquad \epsilon, \eta, \lambda_2, \lambda_3, \lambda_4 \qquad \sim T^4, T^3, T^2, T, 1$$

Dimensionless large-τ expansion parameter:

$$\frac{1}{\tau T(\tau)} = \frac{1}{T_f \tau_f^{1/3} \tau^{2/3}} \sim \frac{1}{\tau^{(d-2)/(d-1)}}$$

Results from AdS/CFT [1]

$$T(\tau) = T_f \left(\frac{\tau_f}{\tau}\right)^{1/3} + \frac{\eta_0}{\tau} - \frac{\eta_0^2(1-\log 2)}{T_f \tau_f^{1/3}} \frac{1}{\tau^{5/3}} + \frac{\eta_0^3 A}{(T_f \tau_f^{1/3})^2 \tau^{7/3}} + \frac{\eta_0^4 B}{(T_f \tau_f^{1/3})^3 \tau^3} + ..$$

$$\eta_0 = -\frac{1}{6\pi}$$

$$\epsilon(\tau) = \frac{3\pi^2}{8} N_c^2 \left(\frac{(T_f \tau_f^{1/3})^4}{\tau^{4/3}} + \frac{4\eta_0 (T_f \tau_f^{1/3})^3}{\tau^2} + \frac{2\eta_0^2 (1+\log 4)(T_f \tau_f^{1/3})^2}{\tau^{8/3}} + \right.$$
$$\left. + \frac{4\eta_0^3 T_f \tau_f^{1/3}(-2+\log 8 + A)}{\tau^{10/3}} + \frac{\eta_0^4(-5+6\log^2 2 + 12A + 4B)}{\tau^4} + .. \right)$$

For $d = 2$ only one term in expansion[2]:

$$\epsilon(\tau) = p(\tau) = \frac{\mathcal{L}}{16\pi G_3} \frac{M}{\tau^2} = \frac{\pi \mathcal{L}}{4G_3} T^2(\tau) \qquad T(\tau) = \frac{\sqrt{M}}{2\pi\tau}.$$

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[1] Janik-Peschanski, Baier-Romatschke-Son-Starinets-Stephanov,...

[2] Kajantie-Louko-Tahkokallio

2. The AdS$_{d+1}$/CFT$_d$ setup for boost inv conf flow, $d = 4$

$(t, x, x_2, x_3) \Rightarrow (\tau = \sqrt{t^2 - x^2}, \eta = \frac{1}{2}\log[(t+x)/(t-x)], x_2, x_3), \quad ds^2 = -d\tau^2 + \tau^2 d\eta^2 + dx_T^2$

Metric ansatz:

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[a(\tau,z)d\tau^2 + \tau^2 b(\tau,z)d\eta^2 + c(\tau,z)(dx_2^2 + dx_3^2) + dz^2\right]$$

Solve from

$$R_{MN} = -\frac{4}{\mathcal{L}^2} g_{MN},$$

expand near the boundary $z = 0$:

$$a(\tau,z) = -[1 + a_0(\tau)z^4 + a_1(\tau)z^6 + \mathcal{O}(z^8)]$$

to get

$$\epsilon(\tau) = T_{\tau\tau} = \frac{\mathcal{L}^3}{4\pi G_5} g_{\tau\tau}^{(4)} = -\frac{N_c^2}{2\pi^2} a_0(\tau)$$

Solving 5d classical gravity get flow of conf $\mathcal{N} = 4$ SYM matter in 4d !?

How do you solve 5 nonlin PDOs, $\tau\tau$, $\eta\eta$, TT, zz, τz comps of Einstein?

Conformal boost invariant flow in general

$T^{\mu}_{\mu} = 0,\ \nabla_\mu T^{\mu\nu} = 0\ \Rightarrow$ in the local rest frame

$$T^{\mu}_{\ \nu} = \begin{pmatrix} -\epsilon(\tau) & 0 & 0 & 0 \\ 0 & -\epsilon(\tau) - \tau\epsilon'(\tau) & 0 & 0 \\ 0 & 0 & \epsilon(\tau) + \frac{1}{2}\tau\epsilon'(\tau) & 0 \\ 0 & 0 & 0 & \epsilon(\tau) + \frac{1}{2}\tau\epsilon'(\tau) \end{pmatrix}.$$

$$\text{positivity condition}: \quad -4\epsilon(\tau) \leq \tau\epsilon'(\tau) \leq 0$$

If $\epsilon \sim \tau^{-p}$, $0 \leq p \leq 4$ and

$$T^{\mu}_{\ \nu} = \epsilon(\tau) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & p-1 & 0 & 0 \\ 0 & 0 & 1 - \frac{1}{2}p & 0 \\ 0 & 0 & 0 & 1 - \frac{1}{2}p \end{pmatrix}$$

$$p = 0, \quad \text{constant}, \quad p_L = -\epsilon,\ p_T = \epsilon$$

$$p = \tfrac{4}{3}, \quad \text{thermal}, \quad p_L = p_T = \tfrac{1}{3}\epsilon$$

$$p = 4, \quad \text{"vacuum"},\ \text{Casimir}, \quad p_T = 3\epsilon,\ p_T = -\epsilon$$

Not true vacuum in the sense $T_{\mu\nu} = \text{const} \times g^{(0)}_{\mu\nu}$!

3. AdS$_3$/CFT$_2$

(x^+, x^-, z), (τ, η, z), (t, x, z), (light cone, Milne, Minkowski)

$$x^\pm = \frac{x^0 \pm x^1}{\sqrt{2}} = \frac{\tau}{\sqrt{2}} e^{\pm\eta}, \qquad t = \tau\cosh\eta,\ x = \tau\sinh\eta,$$
$$ds^2 = -2dx^+dx^- = -d\tau^2 + \tau^2 d\eta^2 = -dt^2 + dx^2.$$

General solution of

$$R_{MN} - \tfrac{1}{2} R\, g_{MN} - \tfrac{1}{\mathcal{L}^2} g_{MN} = 0, \quad g_{MN} = \tfrac{\mathcal{L}^2}{z^2}\begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}$$

is

$$g_{MN} = \frac{\mathcal{L}^2}{z^2}\begin{pmatrix} g(x^+)z^2 & -1 - \frac{z^4}{4} g(x^+) f(x^-) & 0 \\ -1 - \frac{z^4}{4} g(x^+) f(x^-) & f(x^-) z^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$g(x^+) = 0,\ f(x^-) = \delta(x^-)$ gives an "Aichelburg-Sexl shock wave", grav field of a particle moving with $x = +t$.

Now you have two clouds of particles colliding!

Compute $T_{\mu\nu}$

Expand

$$g_{\mu\nu}(x^{\pm}, z) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} g(x^+) & 0 \\ 0 & f(x^-) \end{pmatrix} z^2 + (\dots) z^4$$

and read from general results

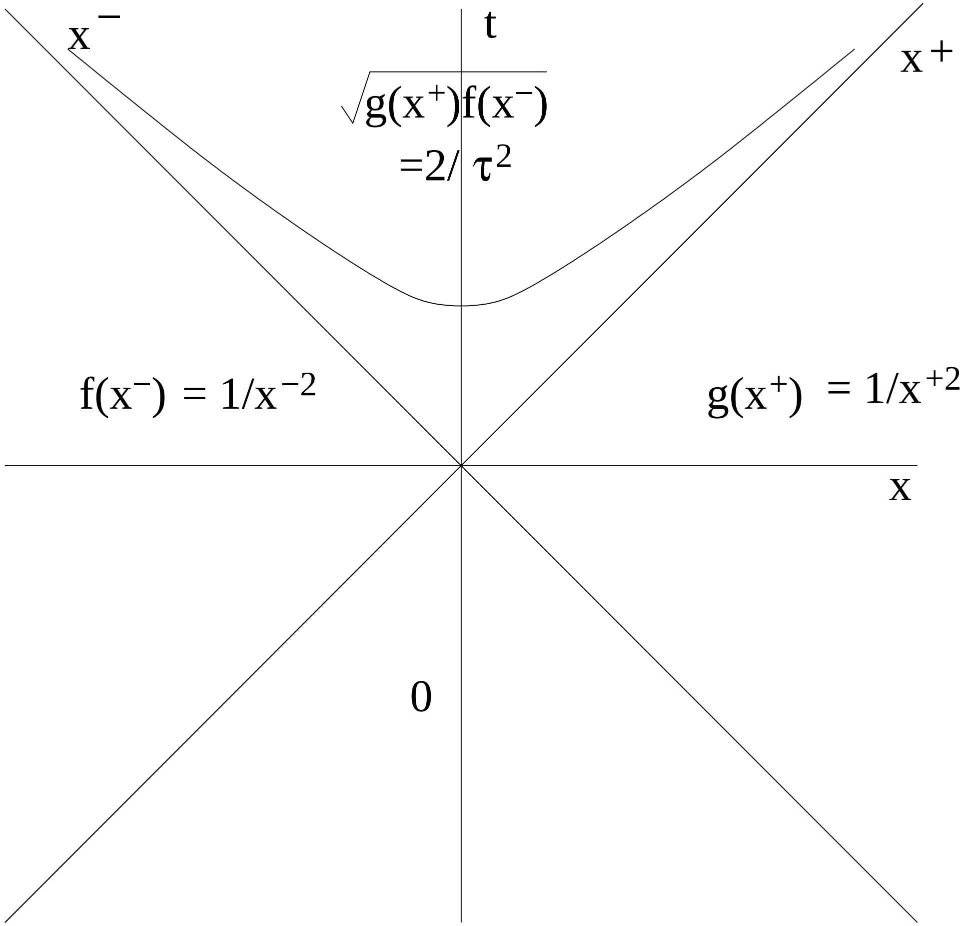
$$T_{\mu\nu} = \frac{\mathcal{L}}{8\pi G_3} \begin{pmatrix} g(x^+) & 0 \\ 0 & f(x^-) \end{pmatrix} = (\epsilon + p) u_\mu u_\nu + p g^{(0)}_{\mu\nu}$$

Solve $2u^+u^- = u^2 = -1$:

$$u_\mu = \left(-\left(\frac{g(x^+)}{4f(x^-)} \right)^{1/4}, -\left(\frac{f(x^-)}{4g(x^+)} \right)^{1/4} \right)$$

$$\epsilon = p = \frac{\mathcal{L}}{8\pi G_3} \sqrt{g(x^+) f(x^-)}$$

Special case 1: $g(x^+) = \frac{M-1}{4x^{+2}}$, $f(x^-) = \frac{M-1}{4x^{-2}}$



$$\epsilon = p \sim \sqrt{g(x^+)f(x^-)} = \frac{M-1}{4x^+x^-} = \frac{M-1}{2\tau^2}$$

Diverges for $\tau \to 0$!

Same metric in other coordinates:

(τ, η, z),

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-\left(1 - \frac{(M-1)z^2}{4\tau^2}\right)^2 d\tau^2 + \left(1 + \frac{(M-1)z^2}{4\tau^2}\right)^2 \tau^2 d\eta^2 + dz^2\right]$$

Horizon at $\tau = \frac{1}{2}\sqrt{M-1}z$?

(t, r, η)

By explicit coordinate transformations[3]:

$$ds^2 = -\left(\frac{r^2}{\mathcal{L}^2} - M\right) dt^2 + \frac{dr^2}{r^2/\mathcal{L}^2 - M} + r^2 d\eta^2.$$

Completely static: the nonrotating BTZ black hole!

$$T = \frac{\sqrt{M}}{2\pi\mathcal{L}}, \qquad \frac{S}{\text{"Vol"}} = \frac{S}{\mathcal{L}\Delta\eta} = \frac{\sqrt{M}}{4G_3}. \qquad M, \text{ not } M-1\ !$$

Where is time dependent $s(\tau) \sim 1/\tau$?

[3] Kajantie-Louko-Tahkokallio

Defining temperature:

Take $M = 1$:

$$ds^2 = \frac{1}{z^2}\left(-d\tau^2 + \tau^2 d\eta^2 + dz^2\right) \qquad (\star)$$

seems AdS, no T – for inertial observers!

$\tau = e^t r/\sqrt{r^2-1},\; z = e^t/\sqrt{r^2-1} \;\Rightarrow$

$$ds^2 = -(r^2-1)dt^2 + \frac{dr^2}{r^2-1} + r^2 d\eta^2.$$

Has timelike Killing ∂_t with horizon, etc.[4]

AdS $(\star)$ has many Killing vectors. Choose physically correct one: timelike, commutes with ∂_η (boost invariance!) $\Rightarrow$ ∂_t!!

Noninertial observers!

[4] BHTZ, gr-qc/9302012

Fluid+Casimir/vacuum, time dependent entropy

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-\left(1 - \frac{(M-1)z^2}{4\tau^2}\right)^2 d\tau^2 + \left(1 + \frac{(M-1)z^2}{4\tau^2}\right)^2 \tau^2 d\eta^2 + dz^2\right]$$

$$\Rightarrow\ T_{\mu\nu} = \frac{\mathcal{L}(M-1)}{16\pi G_3}\begin{pmatrix} \tau^{-2} & 0 \\ 0 & 1 \end{pmatrix} = \frac{\mathcal{L}M}{16\pi G_3}\begin{pmatrix} \tau^{-2} & 0 \\ 0 & 1 \end{pmatrix} - \frac{\mathcal{L}}{16\pi G_3}\begin{pmatrix} \tau^{-2} & 0 \\ 0 & 1 \end{pmatrix}$$

$T_{\mu\nu}$ = sum of fluid ($M > 0$) and a Casimir/vacuum contribution - renormalised $T_{\mu\nu}$ in Milne coordinates[5].

Gauge/gravity duality gives all there is in field theory (most strikingly anomalies of T^{μ}_{μ} in curved boundary)

$$\text{Same as}\ \ ds^2 = -\left(\frac{r^2}{\mathcal{L}^2} - M\right)dt^2 + \frac{dr^2}{r^2/\mathcal{L}^2 - M} + r^2 d\eta^2.$$

[5] Birrell-Davies, Eq. (7.24)

The equivalent metric is the well-understood completely static nonrotating BTZ black hole with entropy (density)

$$T = \frac{\sqrt{M}}{2\pi\mathcal{L}}, \qquad \frac{S}{\text{"Vol"}} = \frac{S}{\mathcal{L}\Delta\eta} = \frac{\sqrt{M}}{4G_3}. \qquad M, \text{ not } M-1\ !$$

For an expanding system one should measure entropy not with $\mathcal{L}\Delta\eta$ but $\tau d\eta$ as longitudinal volume element:

$$s(\tau) = \frac{\Delta S}{\tau\Delta\eta} = \frac{\sqrt{M}}{4G_3}\frac{\mathcal{L}}{\tau}, \qquad T(\tau) = \frac{\sqrt{M}}{2\pi\tau}$$

Consistent!

Moral: understanding the global structure is important!

For the record, here are the coordinate transformations from the time dependent

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-(1-\frac{(M-1)z^2}{4\tau^2})^2 d\tau^2 + (1+\frac{(M-1)z^2}{4\tau^2})^2 \tau^2 d\eta^2 + dz^2\right]$$

to a static form for any M:

Transform stepwise $\tau, z \;\rightarrow\; V, U \;\rightarrow\; t, r$

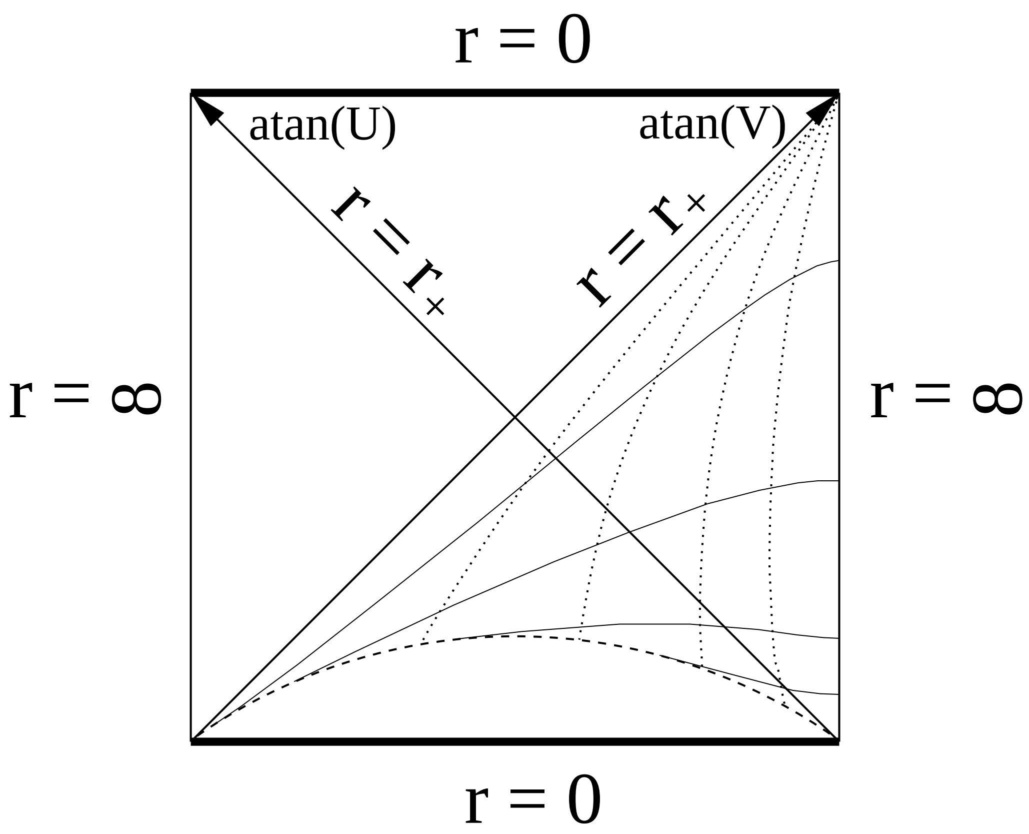
$$V \;=\; \left(\frac{2\tau - \left(\sqrt{M}+1\right)z}{2\tau + \left(\sqrt{M}-1\right)z}\right)\left(\frac{\tau}{\mathcal{L}}\right)^{\sqrt{M}}, \quad r = \mathcal{L}\sqrt{M}\,\left(\frac{1-UV}{1+UV}\right), \quad M = M_{\mathsf{BH}}\cdot 8G_3$$

$$U \;=\; -\left(\frac{2\tau - \left(\sqrt{M}-1\right)z}{2\tau + \left(\sqrt{M}+1\right)z}\right)\left(\frac{\tau}{\mathcal{L}}\right)^{-\sqrt{M}}, \qquad t = \frac{\mathcal{L}}{2\sqrt{M}}\ln\left|\frac{V}{U}\right|.$$

$$\Rightarrow\; ds^2 = \mathcal{L}^2\left[-\frac{4}{(1-UV)^2}dV\,dU + M\left(\frac{1-UV}{1+UV}\right)^2 d\eta^2\right]$$

$$ds^2 = -\left(\frac{r^2}{\mathcal{L}^2} - M\right)dt^2 + \frac{dr^2}{r^2/\mathcal{L}^2 - M} + r^2 d\eta^2$$

For the record, here is also the Penrose diagram:

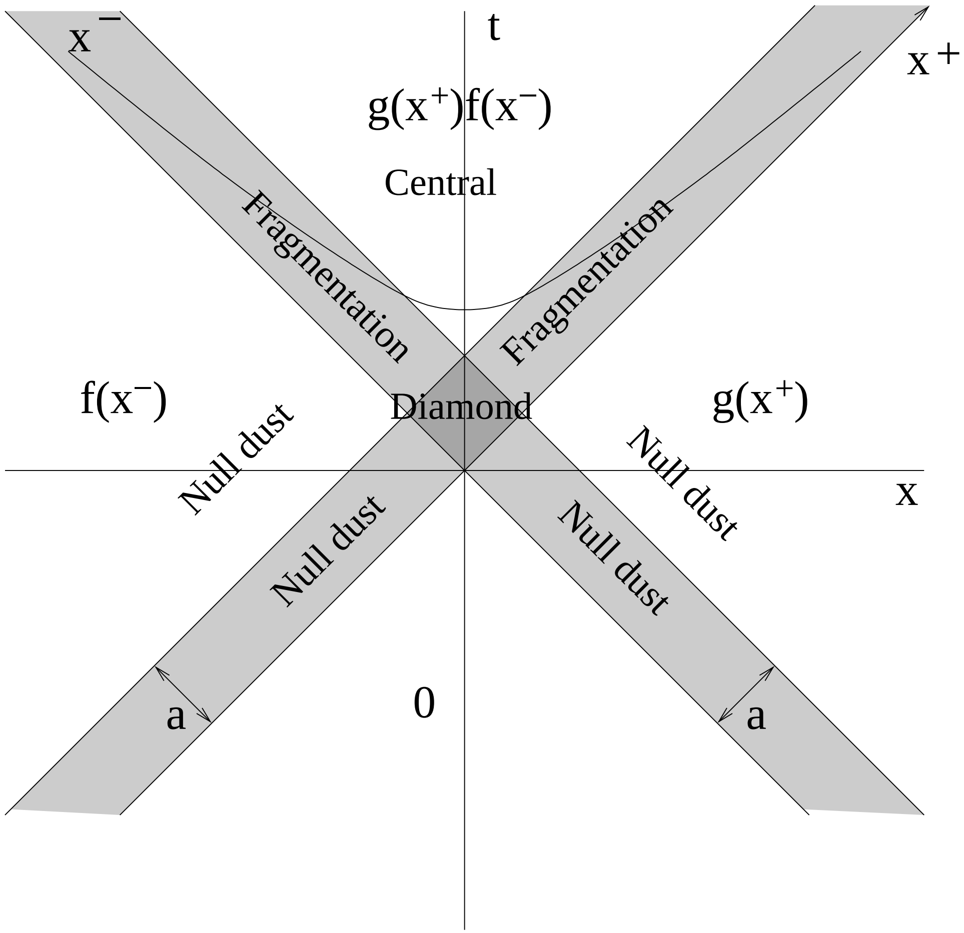


The region $0 < z < 2\tau/\sqrt{M-1}$ is part of interior of while hole + exterior of black hole.

The naive horizon $\tau = \frac{1}{2}\sqrt{M-1}z$ (dotted) is behind the true horizon $r = r_+,\ \tau = \frac{1}{2}(\sqrt{M}+1)z$.

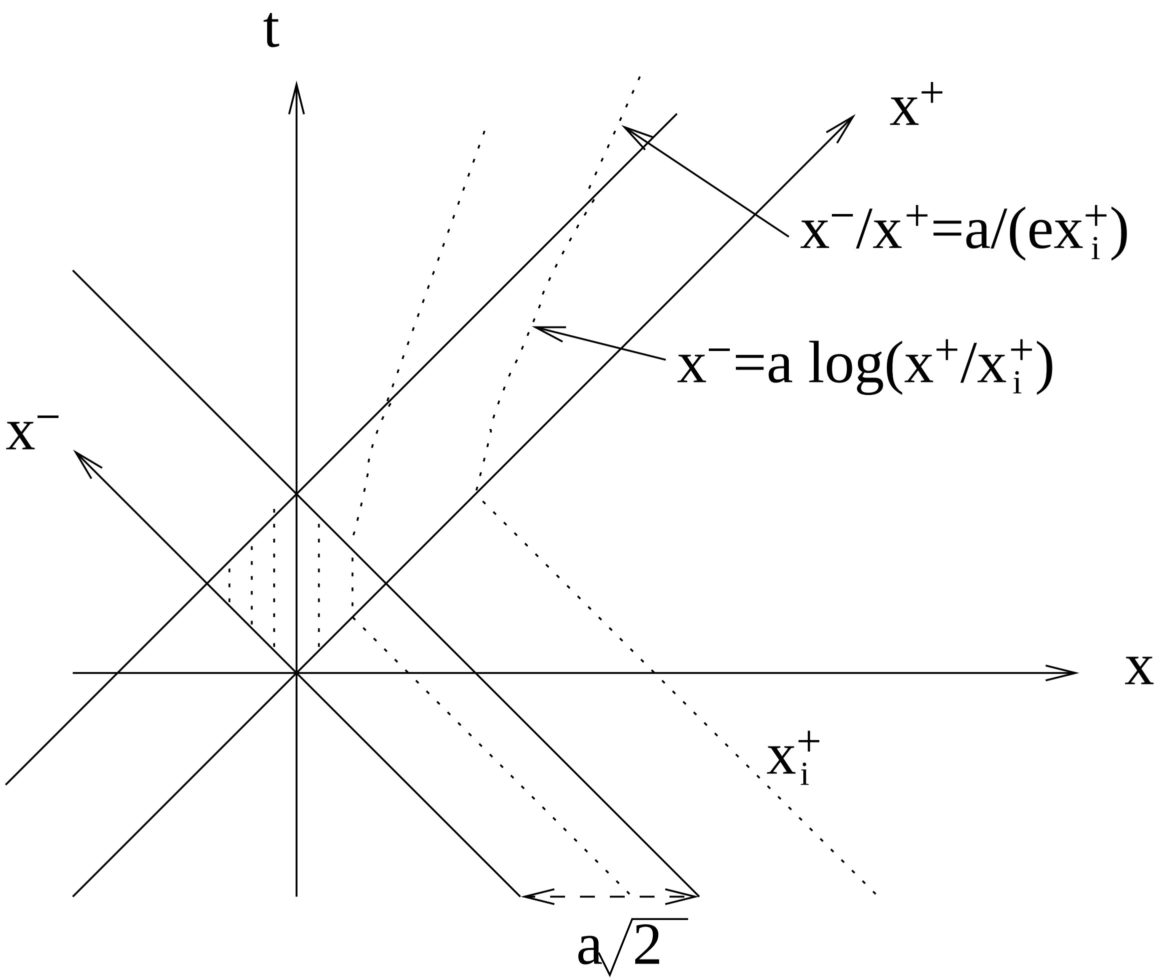
Special case 2: core + tail, some phenomenology

$$f(x) = g(x) = \frac{M-1}{4}\left[\frac{1}{x^2}\Theta(x-a) + \frac{1}{a^2}\Theta(x)\Theta(a-x)\right] \qquad a \sim \frac{1}{Q_s}$$



Energy density per unit rapidity is finite due to imposed boost noninvariance!

Conformal matter is opaque:



Dotted: particle paths. Matter recoils!

4. Back to AdS_5/CFT_4

Metric ansatz:

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[a(\tau,z)d\tau^2 + \tau^2 b(\tau,z)d\eta^2 + c(\tau,z)(dx_2^2+dx_3^2)+dz^2\right]$$

Large-τ solutions obtained by expanding

$$a(\tau,z) = a_0(v) + a_1(v)\frac{1}{\tau^{2/3}} + a_2(v)\frac{1}{\tau^{4/3}} + ..., \qquad v \equiv \frac{z}{\tau^{1/3}},$$

solving $a_i(v)$ exactly and determining constants by regularity.

Could it be that the solution just is a time dependent coordinate transformation of the AdS_5 black hole

$$ds^2 = \frac{\mathcal{L}^2}{\tilde{z}^2}\left[-(1-\frac{\tilde{z}^4}{z_0^4})dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + \frac{1}{1-\tilde{z}^4/z_0^4}d\tilde{z}^2\right]$$

or

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-\frac{(1-z^4/(4z_0^4))^2}{1+z^4/(4z_0^4)}dt^2 + \left(1+\frac{z^4}{4z_0^4}\right)(dx_1^2+dx_2^2+dx_3^2)+dz^2\right]$$

or even

$$ds^2 = -\left(\frac{r^2}{\mathcal{L}^2} + 1 - \frac{\mu\mathcal{L}^2}{r^2}\right)dt^2 + \frac{dr^2}{(....)} + r^2 d\Omega_3^2 \qquad ??$$

Solutions depending on z/τ

Assume a, b, c only depend on z/τ:

$$a(\tau, z) = -g^2(s), \qquad s \equiv \frac{z^2}{\tau^2} \qquad g(0) = 1,\ g'(0) = 0$$

Then $g(s) = 1 + \frac{1}{2}\, g''(0)s^2 + ...,$

$$g^{(4)}_{\tau\tau} = \frac{-g''(0)}{\tau^4}, \quad \epsilon(\tau) = \frac{N_c^2}{2\pi^2}\ \frac{-g''(0)}{\tau^4}$$

and tensor structure is that of Casimir/vacuum, $T^{\mu}_{\ \nu} \sim \mathrm{diag}(1, -3, 1, 1)/\tau^4$.

The ODE for $g(s)$

$$g(s)g'(s)[g(s) - sg'(s)] = s[g^2(s) - s]g''(s)$$

can be solved analytically $\Rightarrow a, b, c$.

So one has a family (parameter: $g''(0)$) of AdS$_5$ solutions leading to a maximally τ dependent energy density $\sim 1/\tau^4$ in the boundary flow! Can this be split in fluid + Casimir?

A surprise is waiting:

By a change of variables the AdS_5 scaling solution becomes ("bubble of nothing"[6])

$$ds^2 = \frac{\mathcal{L}^2}{\zeta^2}\left\{\left[1-\frac{\zeta^2}{2\mathcal{L}^2}+\frac{(\mu+\frac{1}{4})\zeta^4}{4\mathcal{L}^4}\right]\mathcal{L}^2\left[-d\gamma^2+e^{-2\gamma}\mathcal{L}^{-2}(dx_2^2+dx_3^2)\right] + \frac{\left[1-\frac{(\mu+\frac{1}{4})\zeta^4}{4\mathcal{L}^4}\right]^2}{\left[1-\frac{\zeta^2}{2\mathcal{L}^2}+\frac{(\mu+\frac{1}{4})\zeta^4}{4\mathcal{L}^4}\right]}\mathcal{L}^2 d\eta^2 + d\zeta^2\right\}$$

Coordinates $(\gamma, x^2, x^3, \eta, \zeta)$, $\mu = 4g''(0)$.

A new time γ and transverse coordinates form a 3d De Sitter space!

Expanding around $\zeta = 0$, (coordinates γ, x^2, x^3, η):

$$T^{\mu}{}_{\nu} = \frac{N_c^2}{2\pi^2}\frac{1+4\mu}{16\mathcal{L}^4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

The above is an example of what may appear if one studies exact solutions and their global properties.

[6] Balasubramanian-Ross, Aharony-Fabinger-Horowitz-Silverstein

Conclusions

Lacking exact solutions of AdS_5 gravity equations with symmetries appropriate for boost invariant longitudinal flow in 1+1+2d we have studied cases where exact solutions can be obtained.

In 1+1d boundary the fluid and Casimir/vacuum parts can be correctly identified since the global structure is known. The Casimir/vacuum part necessarily appears.

In 1+1d an exact non-boostinvariant solution simulates heavy ion collisions with central and fragmentation regions and an analogue of saturation scale.

In 1+1+2d an exact solution with z/τ scaling leads to an energy density $\sim 1/\tau^4$. Are there fluid + Casimir components in this?