Expanding systems in gauge theory/gravity duality

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23 May 2007

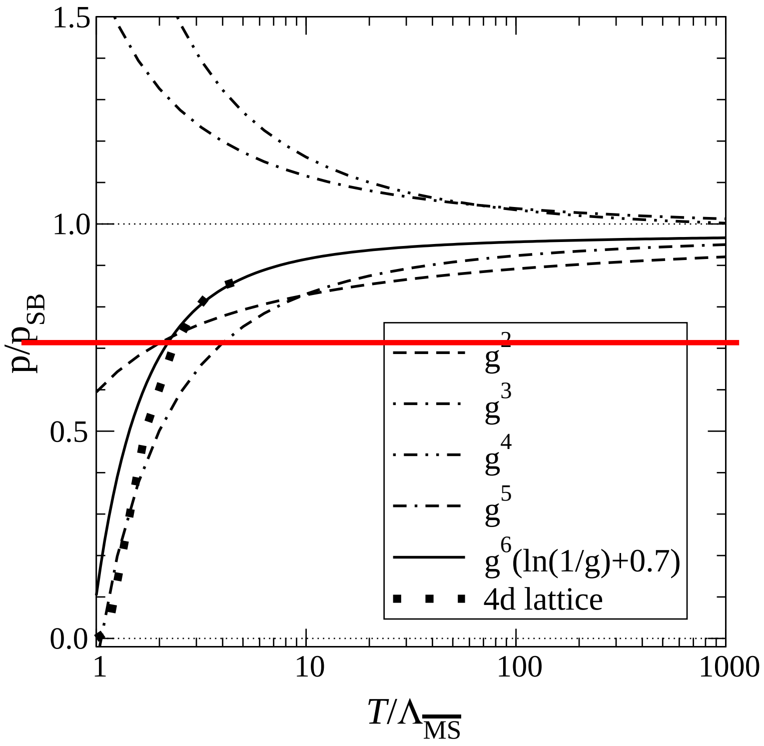
Work with Jorma Louko (Nottingham), T. Tahkokallio (Helsinki)

Janik, Peschanski; Nakamura, Sin, Kim; Kovchegov, Taliotis

Nastase; Shuryak, Sin, Zahed

The celebrated 3/4 and $\hbar/4\pi$

$$\frac{p_{\text{QCD}}}{aT^4}$$



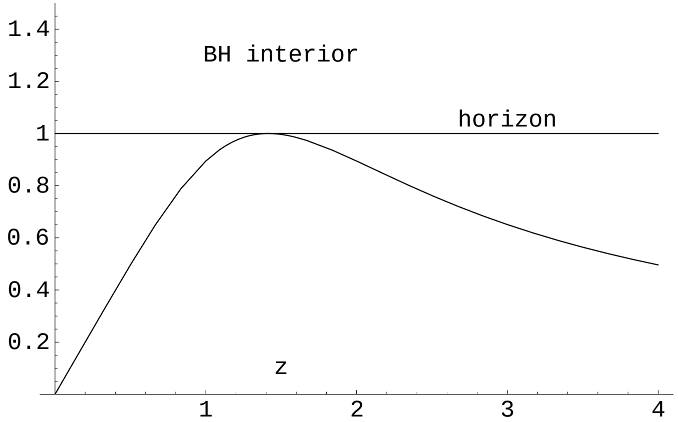
Points: Lattice Monte Carlo, Curves: Perturbation theory

$$\frac{\eta}{s} = \frac{\hbar}{4\pi}\left[1 + \frac{135\zeta(3)}{16\sqrt{2}\lambda^{3/2}} + \ldots\right] \qquad \frac{\eta}{s} \approx \frac{p\tau_c}{s} \approx \frac{T^4\tau_c}{T^3} = T\tau_c \gtrsim \hbar$$

These are static results, derived in AdS$_5$/CFT duality from

$$ds^2 = \frac{\mathcal{L}^2}{\tilde{z}^2}\left[-(1-\frac{\tilde{z}^4}{z_0^4})dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + \frac{1}{1-\tilde{z}^4/z_0^4}d\tilde{z}^2\right] \qquad T_{\text{Hawk}} = \frac{1}{\pi z_0}$$

Transform $\tilde{z}^2 = z^2/(1+z^4/4z_0^4)$:



Polchinski, cosmicvariance.com /2006/12/07/:
Physicists have found that some of the properties of this plasma are better modeled (via duality) as a tiny black hole in a space with extra dimensions than as the expected clump of elementary particles in the usual four dimensions of spacetime.

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-\frac{(1-z^4/(4z_0^4))^2}{1+z^4/(4z_0^4)}dt^2 + \left(1+\frac{z^4}{4z_0^4}\right)(dx_1^2+dx_2^2+dx_3^2) + dz^2\right]$$

$$\equiv \frac{\mathcal{L}^2}{z^2}\left\{\left[g^{(0)}_{\mu\nu}(x,0) + \underbrace{g^{(4)}_{\mu\nu}(x)}_{\sim T_{\mu\nu}}z^4 + \ldots\right]dx^\mu dx^\nu + dz^2\right\}$$

$$T_{\mu\nu} = \frac{\mathcal{L}^3}{4\pi G_5} g^{(4)}_{\mu\nu} = \begin{pmatrix} 3aT^4 & 0 & 0 & 0 \\ 0 & aT^4 & 0 & 0 \\ 0 & 0 & aT^4 & 0 \\ 0 & 0 & 0 & aT^4 \end{pmatrix}$$

$$a = \frac{\pi^2 N_c^2}{6} \left[\frac{3}{4} + \frac{45\zeta(3)}{64\sqrt{2}} \frac{1}{\lambda^{3/2}} + ... \right]$$

For small $\lambda \equiv g^2 N_c$, counting $2 + 6 + 7/8 \times (4+4) = 15 \times$ color massless dofs:

$$a = N_c^2 \frac{\pi^2}{6} \left[1 - \frac{3}{2\pi^2}\lambda + \frac{3+\sqrt{2}}{\pi^3}\lambda^{3/2} + a\lambda^2 \log\lambda + b\lambda^2 + c\lambda^{5/2} + d\lambda^3 \log\lambda + ... \right]$$

(Nieto computed a, b, c, Laine d 2 years ago, unpublished)

$$\begin{aligned} f(\lambda) &= 1 - \frac{3}{2\pi^2}\lambda + \frac{3+\sqrt{2}}{\pi^3}\lambda^{3/2} \\ &- \frac{1}{(2\pi)^4}\lambda^2 \left[45 + 18\sqrt{2} - 24\gamma + 50\log 2 + 48\log\pi - 24\frac{\zeta'(-1)}{\zeta(-1)} - 24\log\lambda \right] \\ &- \frac{1}{(2\pi)^5}\lambda^{5/2} \left[123 + \frac{11}{8}\sqrt{2} + \left(3 + \frac{\sqrt{2}}{2}\right)\pi^2 + \left(\frac{53}{2}\sqrt{2} - 264\right)\log 2 \right. \\ &\left. - \left(21 + \frac{75}{4}\sqrt{2}\right)\log\left(1+\sqrt{2}\right) \right] . \end{aligned}$$

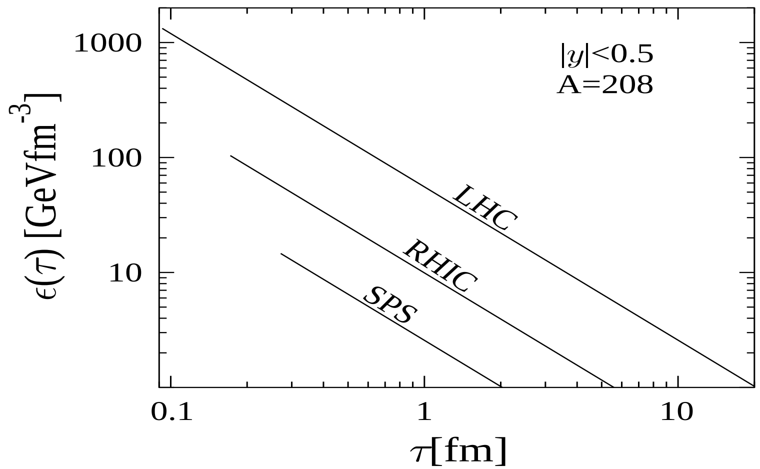
Expanding matter?

1+1+2d Bjorken similarity flow:

$\epsilon(T) = 3p(T) = 3aT^4$, $\eta = p'(T)/(4\pi) = aT^3/\pi$, $a = \pi^2 N_c^2/8$, $\zeta = 0$

$$v(t,x) = \frac{x}{t} \equiv \tanh\Theta(\tau,\eta), \qquad \Theta(\tau,\eta) = \eta, \qquad u^\mu = \frac{x^\mu}{\tau},$$

$$T(t) = \left(T_i + \frac{1}{6\pi\tau_i}\right)\left(\frac{\tau_i}{\tau}\right)^{1/3} - \frac{1}{6\pi\tau}.$$



$$1/\tau^{1/3}, 6\pi \text{ simple}, \quad T_i, \tau_i \text{ (very) hard}, \ T_i\tau_i \sim \hbar$$

Shuryak-Sin-Zahed

Search for time-dependent solutions of AdS_{d+1}:

$$R_{MN} - \frac{1}{2} R g_{MN} - \frac{d(d-1)}{2\mathcal{L}^2} g_{MN} = 0$$

for $d = 4$, $\mathrm{AdS}_5 \times \mathrm{S}_5$, boundary theory: $\mathcal{N} = 4$ SYM

$$\mathcal{L}^4 = 4\pi g_s N_c \alpha'^2 = g^2_{\text{YM}} N_c \alpha'^2, \qquad \frac{\mathcal{L}^3}{G_5} = \frac{2N_c^2}{\pi}$$

$$ds^2 = \frac{\mathcal{L}^2}{z^2} \left[-a(\tau, z) d\tau^2 + \tau^2 b(\tau, z) d\eta^2 + c(\tau, z)(dx_2^2 + dx_3^2) + dz^2 \right]$$

$$ds^2 = \frac{\mathcal{L}^2}{z^2} [-a(t, z) dt^2 + b(t, z) d\mathbf{x}^2 + dz^2]$$

for $d = 2$, $\mathrm{AdS}_3 \times \mathrm{S}_3 \times T_4$, boundary theory: a 2d CFT

$$\mathcal{L}^4 = g_s^2 \frac{16\pi^4 \alpha'^2}{V_4} Q_1 Q_5 \alpha'^2, \qquad \frac{\mathcal{L}}{G_3} = 4 Q_1 Q_5.$$

$$ds^2 = \frac{\mathcal{L}^2}{z^2} \left[-a(\tau, z) d\tau^2 + b(\tau, z) \tau^2 d\eta^2 + dz^2 \right].$$

$d = 2$

General solution with boundary $\sim$ Minkowski scales with z/τ:

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-(1-\frac{z^2}{v^2\tau^2})^2 d\tau^2 + (1+\frac{z^2}{v^2\tau^2})^2\tau^2 d\eta^2 + dz^2\right]$$

Suggests a horizon at $z = v\tau$ moving with velocity v. However, structure of AdS$_3$ is completely known (BTZ)! Transform $\tau, z \rightarrow V, U \rightarrow t, r$

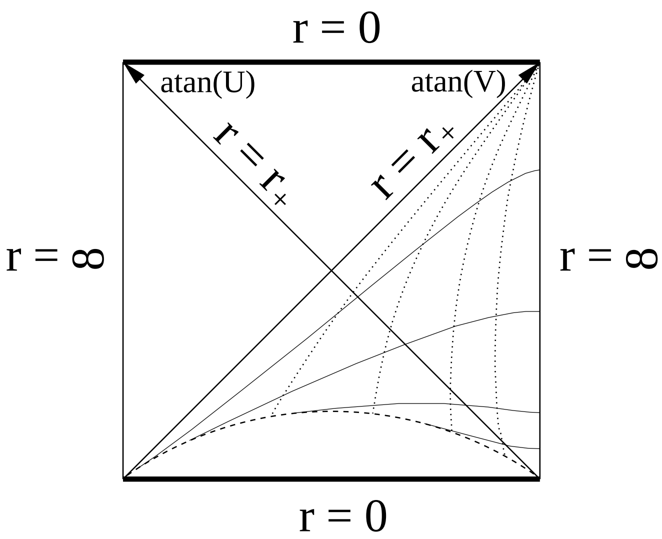
$$V = \left(\frac{2\tau - \left(\sqrt{M}+1\right)z}{2\tau + \left(\sqrt{M}-1\right)z}\right)\left(\frac{\tau}{\mathcal{L}}\right)^{\sqrt{M}}, \quad r = \mathcal{L}\sqrt{M}\left(\frac{1-UV}{1+UV}\right), \quad M \equiv 1 + \frac{4}{v^2} = M_{\mathsf{BH}} \cdot 8G_3$$

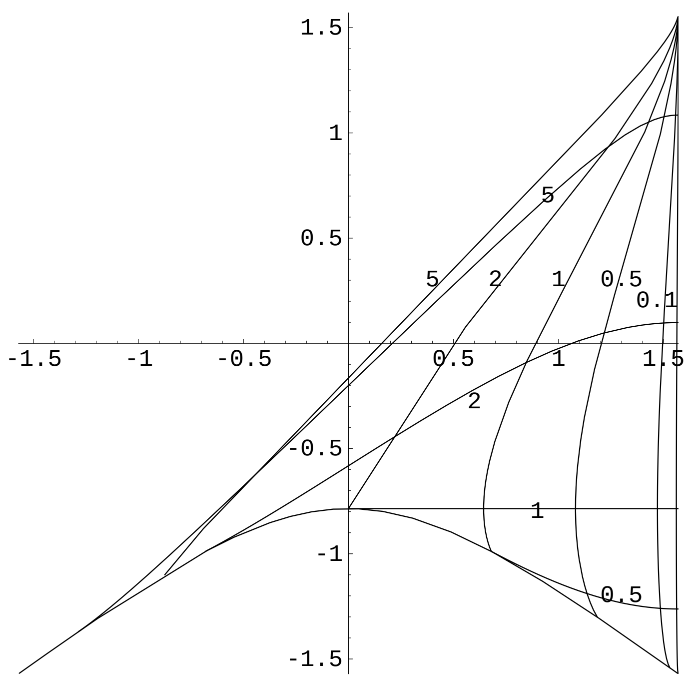
$$U = -\left(\frac{2\tau - \left(\sqrt{M}-1\right)z}{2\tau + \left(\sqrt{M}+1\right)z}\right)\left(\frac{\tau}{\mathcal{L}}\right)^{-\sqrt{M}}, \qquad t = \frac{\mathcal{L}}{2\sqrt{M}}\ln\left|\frac{V}{U}\right|.$$

$$\Rightarrow \; ds^2 = \mathcal{L}^2\left[-\frac{4}{(1-UV)^2}dV\,dU + M\left(\frac{1-UV}{1+UV}\right)^2 d\eta^2\right]$$

$$ds^2 = -\left(\frac{r^2}{\mathcal{L}^2} - M\right)dt^2 + \frac{dr^2}{r^2/\mathcal{L}^2 - M} + r^2 d\eta^2$$

Kajantie-Louko-Tahkokallio, arXiv:0705.1791[hep-th]





Boundary is $r = \infty$, $z = 0$!

The region $0 < z < v\tau =$

part of interior of while hole + exterior of black hole.

$$r_m = \frac{2\mathcal{L}}{v} = \mathcal{L}\sqrt{M-1} < r < r_+ = \mathcal{L}\sqrt{M} < r < \infty$$

Matter comes out of a white hole!

We will use $r_m \int d\eta$ to give the "area" of BH!

Energy-momentum tensor in boundary CFT

$$ds^2 = \frac{\mathcal{L}^2}{z^2}[g_{\mu\nu}dx^\mu dx^\nu + dz^2]$$

$$g_{\mu\nu} = g^{(0)}_{\mu\nu}(\tau) + g^{(2)}_{\mu\nu}(\tau)z^2 + .. \qquad g^{(0)}_{\mu\nu} = \mathrm{diag}(-1, \tau^2)$$

$$T_{\mu\nu} = \frac{\mathcal{L}}{8\pi G_3}[g^{(2)}_{\mu\nu} - g^{(0)}_{\mu\nu}\mathrm{Tr}(g^{(2)}_{\mu\nu})].$$

$$T^\mu_\nu = \begin{pmatrix} -\epsilon(\tau) & 0 \\ 0 & p(\tau) \end{pmatrix}, \qquad \epsilon(\tau) = p(\tau) = \frac{\mathcal{L}}{4\pi G_3}\frac{1}{v^2\tau^2} = \pi Q_1 Q_5 \left(\frac{1}{\pi v \tau}\right)^2$$

Unproblematic; but what is entropy density and T: $S = s(T)V = p'(T)V$?

$$\mathrm{Try}: \; S = \frac{A}{4G_3} = \frac{r_m \int d\eta}{4G_3} = \frac{\mathcal{L}}{4G_3}\frac{2}{v\tau} \times \int \tau d\eta, \quad V = \int \tau d\eta$$

$$\Rightarrow s(T) = \frac{\mathcal{L}}{2G_3}\frac{1}{v\tau} \qquad T(\tau) = \frac{1}{\pi v \tau} = \frac{\sqrt{M-1}}{2\pi\tau}$$

The τ dependent coordinate singularity at $r_m = 2\mathcal{L}/v$ was used to get the expected T!

Effectively: scale static T_H, s by $\mathcal{L}/\tau$, $T_{BTZ} = \frac{\sqrt{M}}{2\pi\mathcal{L}} \longrightarrow \frac{\sqrt{M}}{2\pi\tau}$

Thermodynamics:

$$\epsilon(T) = p(T) = \pi Q_1 Q_5 T^2(\tau), \qquad s(T) = 2\pi Q_1 Q_5 T(\tau)$$

$$T = \frac{1}{\pi v \tau}, \quad \pi T \tau = \frac{1}{v} \gtrsim \hbar \to v \lesssim 1,\ M \gtrsim 4.$$

Compare with ideal BE-FD 1+1d gas with $N_b = N_f = 4Q_1Q_5$:

$$\epsilon(T) = p(T) = (N_b + \frac{1}{2}N_f)\frac{\pi}{6}T^2 = \pi Q_1 Q_5 T^2, \qquad s(T) = 2\pi Q_1 Q_5 T$$

Just the same, no 3/4

Thermalisation: smallest τ_i for which $T = T_i \frac{\tau_i}{\tau}$

Rotating metric

The time dependent form of the standard rotating BTZ metric is

$$
\begin{aligned}
ds^2 \;=\; \frac{\mathcal{L}^2}{z^2}\Bigg(&-\left\{1-\frac{(M-1)z^2}{2\tau^2}+\frac{1}{16}\left[(M-1)^2-(J/\mathcal{L})^2\right]\frac{z^4}{\tau^4}\right\}d\tau^2-\frac{Jz^2}{\mathcal{L}\tau}d\tau\,d\eta\\
&+\left\{1+\frac{(M-1)z^2}{2\tau^2}+\frac{1}{16}\left[(M-1)^2-(J/\mathcal{L})^2\right]\frac{z^4}{\tau^4}\right\}\tau^2 d\eta^2+dz^2\Bigg).
\end{aligned}
$$

$$
T_{\mu\nu}=\frac{\mathcal{L}}{16\pi G_3\tau^2}\begin{pmatrix} M-1 & -(J/\mathcal{L})\tau \\ -(J/\mathcal{L})\tau & (M-1)\tau^2 \end{pmatrix}=(\epsilon+p)u_\mu u_\nu+pg^{(0)}_{\mu\nu}.
$$

$$
\epsilon=p=\frac{\mathcal{L}}{16\pi G_3\tau^2}\sqrt{(M-1)^2-(J/\mathcal{L})^2},\quad u^\mu=(\cosh(\eta+\phi),\,\sinh(\eta+\phi))
$$

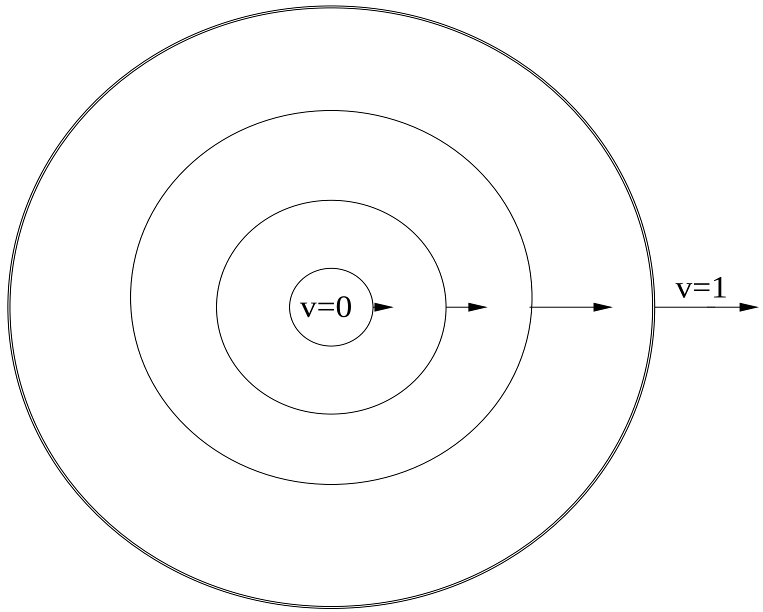
Metric with $J\neq 0$ describes a flow boosted with the velocity

$$
v_{\text{boost}}=\tanh\phi=\frac{J/\mathcal{L}}{M-1+\sqrt{(M-1)^2-(J/\mathcal{L})^2}}
$$

Similarity expansion in 1+3d

$$T_{\mu\nu} = (\epsilon + p)\frac{x_\mu x_\nu}{\tau^2} + p g_{\mu\nu}$$

Fixed time t:



$$\mathbf{v} = \frac{\mathbf{x}}{t}\theta(t - |\mathbf{x}|), \quad u^\mu = (\gamma, \gamma\mathbf{v}) = \frac{x^\mu}{\tau}, \quad \tau = \sqrt{t^2 - \mathbf{x}^2}$$

$$\epsilon'(\tau) + \frac{3}{\tau}(\epsilon + p) = 0 \underset{p=\epsilon/3}{\Rightarrow} \quad \epsilon(\tau) = \frac{\epsilon_0}{\tau^4} = \frac{\epsilon_0}{(t^2 - \mathbf{x}^2)^2}$$

Gravity dual of spherical similarity expansion would be a time dependent solution of

$$R_{MN} + 4g_{MN} = 0$$

with the symmetries (coordinates $= t, r, \theta, \phi, z$)

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-a(t,r,z)dt^2 + b(t,r,z)dt\,dr + c(t,r,z)dr^2 + g(t,r,z)d\Omega_2^2 + dz^2\right],$$

which expanded near boundary

$$g_{\mu\nu}(x,z) = \eta_{\mu\nu} + ... + g^{(4)}_{\mu\nu}(x)z^4 + ...$$

leads to $T_{\mu\nu} = (\epsilon + p)x_\mu x_\nu/\tau^2 + pg_{\mu\nu}$, $\epsilon = 3p = 3aT^4(t)$:

$$T_{\mu\nu} = \frac{\mathcal{L}^3}{4\pi G_5} g^{(4)}_{\mu\nu} = \begin{pmatrix} 4p\frac{t^2}{\tau^2} - p & 4p\frac{tr}{\tau^2} & 0 & 0 \\ 4p\frac{tr}{\tau^2} & 4p\frac{r^2}{\tau^2} + p & 0 & 0 \\ 0 & 0 & r^2 p & 0 \\ 0 & 0 & 0 & r^2 p \sin^2\theta \end{pmatrix}$$

Can "only" do 1+1 dim or $r = 0$ in 1+3d, 2 functions, 2 variables:

$$ds^2 = \frac{\mathcal{L}^2}{z^2}[-a(t,z)dt^2 + b(t,z)d\mathbf{x}^2 + dz^2]$$

Solution in 1+3 dimensions: $r = r(t)$

$$ds^2 = [-a(t,z)dt^2 + b(t,z)d\mathbf{x}^2 + dz^2]/z^2, \quad R_{MN} + 4g_{MN} = 0$$

$$a(t,z) = \frac{\left[\left(1 - \frac{r''}{4r}z^2\right)^2 - \left(\frac{r''}{4r} - \frac{r'^2}{4r^2}\right)^2 z^4 - \frac{1}{4r^4 z_0^4} z^4\right]^2}{\left[\left(1 - \frac{r'^2}{4r^2}\, z^2\right)^2 + \frac{1}{4r^4 z_0^4} z^4\right]} \xrightarrow[r(t)=1]{} \frac{(1 - z^4/(4z_0^4))^2}{1 + z^4/(4z_0^4)}$$

$$b(t,z) = r^2 \left[\left(1 - \frac{r'^2}{4r^2}\, z^2\right)^2 + \frac{1}{4r^4 z_0^4} z^4\right] \xrightarrow[r(t)=1]{} 1 + \frac{z^4}{4z_0^4}$$

Again a time dependent solution with a "horizon" at $a(t,z) = 0$:

$$z_{H\pm}^2 = \frac{4r^2}{rr'' \pm \sqrt{4/z_0^4 + (r'^2 - rr'')^2}} \xrightarrow[r(t)=1]{} 2z_0^2, \quad \pi T_H = \frac{1}{z_0}$$

Is there a coordinate transformation transforming away the t-dependence?

Boundary metric now is RW:

$$g_{\mu\nu}(x,0) = (-1, r^2(t), r^2(t), r^2(t))$$

Brane gravity adds a brane and Einstein with G_4 to determine $r(t)$.

$T_{\mu\nu}$:

$$g_{\mu\nu}(t,z) = g^{(0)}_{\mu\nu}(t) + g^{(2)}_{\mu\nu}(t)z^2 + g^{(4)}_{\mu\nu}(t)z^4 + ...,$$

$$T_{\mu\nu} = \frac{\mathcal{L}^3}{4\pi G_5}\left[g^{(4)}_{\mu\nu} - \frac{1}{8}g^{(0)}_{\mu\nu}[(\mathrm{Tr}\, g_{(2)})^2 - \mathrm{Tr}\, g^2_{(2)}] - \frac{1}{2}(g_{(2)}g^{-1}_{(0)}g_{(2)})_{\mu\nu} + \frac{1}{4}\mathrm{Tr}\, g_{(2)}\cdot g_{(2)\mu\nu}\right]$$

$$T_{tt} = \epsilon(t) = \frac{3N_c^2}{8\pi^2}\left(\frac{1}{z_0^4 r^4} + \frac{r'^4}{4r^4}\right) \sim \frac{T_0^4}{r^4(t)} + \frac{1}{t^4} \sim \underbrace{T^4(t)}_{\text{radiation}} + \underbrace{t^{-4}}_{\text{curvature}},$$

$$T^1_1(t) = \frac{1}{3}\epsilon(t) - \underbrace{\frac{N_c^2}{8\pi^2}\frac{r'^2 r''}{r^3}}_{\text{trace anomaly}/3}$$

$\epsilon = 3p \sim T^4$ are known, $T(t) = ?$, $s = p'(T) = ?$, $r(t) = ?$. Obvious that $r(t) = t/t_0$ works nicely:

$$\epsilon(t) = \frac{3\pi^2 N_c^2}{8}\frac{1}{t^4}\left(\frac{t_0^4}{\pi^4 z_0^4} + \frac{1}{4}\right) \Rightarrow T(t) = \frac{1}{\pi z_0}\frac{t_0}{t} \quad \text{if } t_0 \gg z_0$$

$T \sim 1/\tau^{1/3}$ follows if expansion is in 1d, thermalisation in 3d.

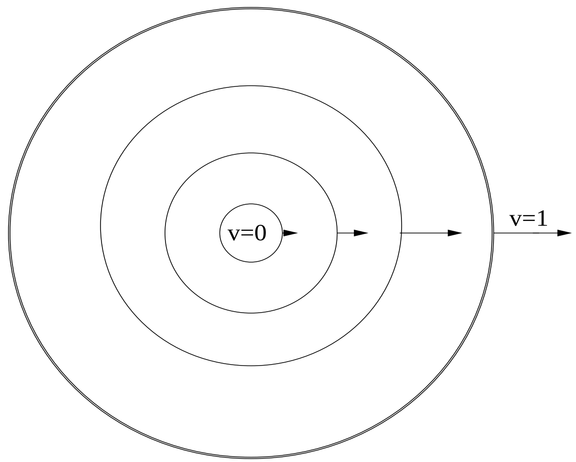
The JP argument for fixing $r \sim t^p$: $R = -20/\mathcal{L}^2$, $R^{\mu\nu}R_{\mu\nu} = 80/\mathcal{L}^4$,

$$R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = \frac{1}{\mathcal{L}^4}\left\{40 + 72\left[\frac{z^2}{z_0^2 b(t,z)}\right]^4\right\}$$

This is $112/\mathcal{L}^4$ at the horizon if $r(t) = t/t_0$. If $r(t) \sim t^{p>1}$, this is $\sim t^{8(p-1)}$ for $t \gg z_0$ (similarly $p < 1$) , thus

$$r(t) = \frac{t}{t_0}$$

We thus have gravity dual of matter in the center of spherical bang:



Thermalisation condition: temperature can be defined for

$$t_0/z_0 = \pi T t_0 > 1 = \hbar$$

Conclusions

1. Lattice QCD can determine purely Euclidian quantities, no real time.

2. AdS/CFT can compute real time correlators, but using time independent gravitational backgrounds (linear response)

$$\langle e^{\int d^4x \mathcal{O}(x)\phi_0(x,0)}\rangle_{\mathrm{FT}} = e^{-\int d^4x dz L(\phi(x,z))}$$

$\Rightarrow$ correlators between states at $t=-\infty$ and $t=+\infty$.

3. How does one handle processes starting at some $t=0$? "Gauge invariant" formulation $=$? Can one derive $\mathbf{new}$ results for t dependent processes from AdS/CFT?