5d black holes and 4d thermodynamics

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28 June 2008

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Setup:

- Bulk gravity in $d+1$ dim/boundary CFT in d dim duality, AdS$_{d+1}$/CFT$_d$ duality

- Black holes in 5d have temperature T and entropy

$$S = \frac{A}{4G_5}$$

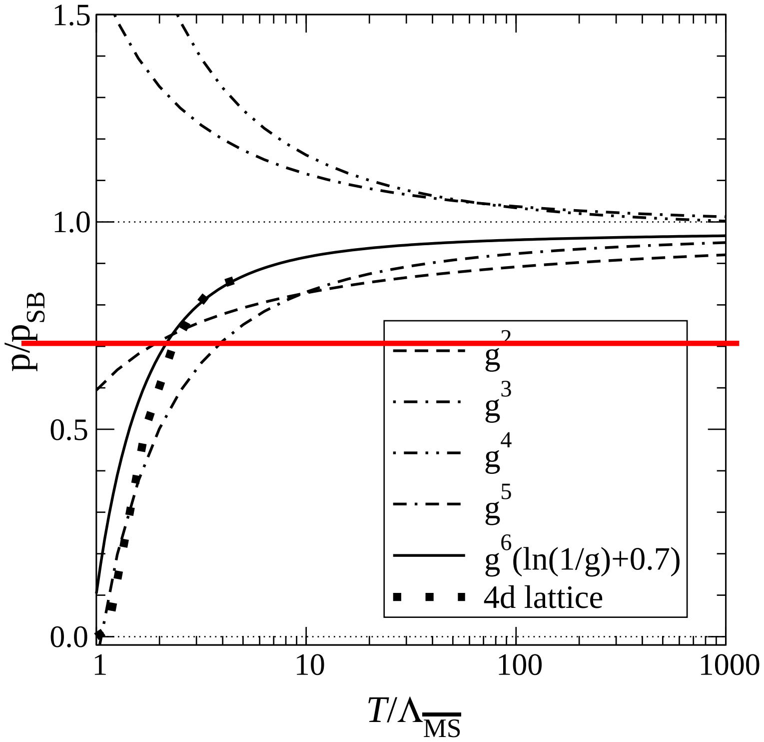
  which are also those of boundary CFT

- Can one in the boundary theory define pressure satisfying $p'(T) = s(T)$?

$$e^{p(T)V/T} = \int \mathcal{D}\phi \, e^{-\int_0^\beta d\tau d^3x L} = e^{-S_\text{grav}} = e^{\frac{-1}{2\pi G_5 \mathcal{L}^2} \int_0^\beta d\tau d^3x \int_\epsilon^{z_0} dz \sqrt{-g(z_0)}}$$

The prototype is AdS$_5$/CFT$_4$ with CFT$_4 = (\mathcal{N} = 4$ SYM), with some experimental support:

$\mathcal{N}=4$ SYM prediction "compared with hot QCD":



Expectation:

Wrong for $T \lesssim 3T_c$ (not conformal) and $T \gtrsim 100T_c$ (not strongly coupled)

Black Holes in Classical Gravity

Einstein-Hilbert for AdS$_{d+1}$ (some coordinates $x^\mu = x^0, x^1, ..., x^d$):

$$S[g_{\mu\nu}] = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x\sqrt{-g}\left(R + \frac{d(d-1)}{\mathcal{L}^2}\right) = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x\sqrt{-g}\frac{-2d}{\mathcal{L}^2},$$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu,$$

EOM from $\delta S/\delta g_{\mu\nu} = 0$:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = 0\left(= 8\pi G T_{\mu\nu},\ T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}\right).$$

For AdS$_{d+1}$ of radius $\mathcal{L}$:

$$\Lambda = \frac{d(d-1)}{2\mathcal{L}^2} \Rightarrow R_{\mu\nu} = -\frac{d}{\mathcal{L}^2}g_{\mu\nu}$$

There is a length scale $\mathcal{L}$ associated with AdS!

1. Black hole in our world, $d = 4$, solution of

$$R_{\mu\nu} = 0 \qquad \text{or of } R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

which is asymptotically flat ($\eta_{\mu\nu}$) and regular on and outside an event horizon (coordinates t, r, θ, ϕ):

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 \underbrace{(d\theta^2 + \sin^2\theta d\phi^2)}_{\equiv d\Omega_2^2}$$

$$F(r) = 1 - \frac{r_s}{r} \qquad r_s = 2MG_4$$

$$T_{\text{Hawk}} = \frac{F'(r_s)}{4\pi} = \frac{1}{4\pi r_s} = \frac{1}{8\pi M G_4},$$

$$S_{\text{BH}} = \frac{A}{4G_4} = 4\pi G_4 M^2 \qquad A = r_s^2 \Omega_2.$$

$$dM = TdS, \quad \text{but} \quad M = 2TS!$$

Charged BH, Reissner-Nordström:

$$F(r) = 1 - \frac{2GM}{r} + \frac{G(q^2 + p^2)}{r^2}$$

2. Pure AdS$_5$, a solution of

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{6}{\mathcal{L}^2} g_{\mu\nu} \quad \text{with} \quad R_{\mu\nu\alpha\beta} = \frac{R}{(d-1)(d-2)} (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha})$$

With coordinates t, x^1, x^2, x^3, z

$$ds^2 = \frac{\mathcal{L}^2}{z^2}(-dt^2 + d\mathbf{x}^2 + dz^2) \quad z = 0 \,\text{is}\, \textbf{boundary},\ z > 0 \,\text{is}\, \textbf{bulk}$$

AdS$_5$ black hole:

$$ds^2 = \frac{\mathcal{L}^2}{\tilde{z}^2} \left[-\left(1 - \frac{\tilde{z}^4}{z_0^4}\right) dt^2 + d\mathbf{x}^2 + \frac{d\tilde{z}^2}{1 - \tilde{z}^4/z_0^4} \right]$$

with temperature

$$T_{\text{Hawk}} = \frac{1}{\pi z_0}$$

and entropy

$$S = \frac{A}{4G_5} = V_3 \cdot \frac{\pi^2 N_c^2}{2} T^3 \quad A = \int d^3x \sqrt{-\gamma} = V_3 \frac{\mathcal{L}^3}{z_0^3}, \quad \frac{\mathcal{L}^3}{4\pi G_5} = \frac{N_c^2}{2\pi^2}$$

Now you have entropy, what about $p(T)$, even better $T_{\mu\nu}$?

Expect $T_{\mu\nu}(x)$ to be related to $g_{MN}(x,z)$.

Method: write the 5d metric in the form

$$g_{MN} = \frac{\mathcal{L}^2}{z^2}\begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}$$

and expand near $z=0$:

$$g_{\mu\nu}(x,z) = g^{(0)}_{\mu\nu}(x) + g^{(2)}_{\mu\nu}(x)z^2 + g^{(4)}_{\mu\nu}(x)z^4 + \ldots.$$

Then

$$T_{\mu\nu} = \frac{\mathcal{L}^3}{4\pi G_5}\left[g^{(4)}_{\mu\nu} + \ldots\right]$$

Bulk black hole metric was

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-\frac{(1-z^4/(4z_0^4))^2}{1+z^4/(4z_0^4)}dt^2 + \left(1+\frac{z^4}{4z_0^4}\right)(dx_1^2+dx_2^2+dx_3^2) + dz^2\right]$$

$$\equiv \frac{\mathcal{L}^2}{z^2}\left\{\left[g_{\mu\nu}(x,0) + \underbrace{g^{(4)}_{\mu\nu}(x)}_{\sim T_{\mu\nu}} z^4 + ...\right] dx^\mu dx^\nu + dz^2\right\}$$

$$\Rightarrow g^{(4)}_{\mu\nu} = \mathrm{diag}(3,1,1,1)\frac{1}{z_0^4}, \qquad \frac{1}{z_0} = \pi T.$$

Magnitude: Relating string theory $\longrightarrow$ supergravity

$$16\pi G_{10} = 16\pi G_5 \mathcal{L}^5 \pi^3 = (2\pi)^7 \alpha'^4 g_s^2, \quad g_s = g^2/4\pi \qquad \text{nontrivial!!}$$

$g_s =$ closed string coupling, one handle costs g_s^2. $\mathcal{L}^4 = g^2 N_c \alpha'^2$

$$\Rightarrow \frac{\mathcal{L}^3}{G_5} = \frac{2N_c^2}{\pi}$$

$$\Rightarrow T_{\mu\nu} = \frac{\mathcal{L}^3}{4\pi G_5} g^{(4)}_{\mu\nu} = \frac{N_c^2}{2\pi^2} g^{(4)}_{\mu\nu} = \begin{pmatrix} 3aT^4 & 0 & 0 & 0 \\ 0 & aT^4 & 0 & 0 \\ 0 & 0 & aT^4 & 0 \\ 0 & 0 & 0 & aT^4 \end{pmatrix} \qquad a = \frac{\pi^2 N_c^2}{8}$$

Black holes with other symmetries:

$$ds^2 = -\left(\hat{r}^2 + k - \frac{\mu}{\hat{r}^2}\right) dt^2 + \frac{dr^2}{\hat{r}^2 + k - \mu/\hat{r}^2} + r^2 d\tilde{\Omega}^2_{3,k} \qquad \hat{r} \equiv \frac{r}{\mathcal{L}}$$

$$\begin{aligned} d\tilde{\Omega}^2_{3,k} &= d\eta^2 + \sin^2\eta\,(d\theta^2 + \sin^2\theta d\phi^2) & k = +1 \quad S^3 \\ &= dy_1^2 + dy_2^2 + dy_3^2 & k = 0 \quad R^3 \\ &= d\eta^2 + \sinh^2\eta\,(d\theta^2 + \sin^2\theta d\phi^2) & k = -1 \quad H^3 \quad \Leftarrow \text{we} \end{aligned}$$

$$T = \frac{F'(r_+)}{4\pi} = \frac{1}{\pi\mathcal{L}}\left(\hat{r}_+ - \frac{1}{2\hat{r}_+}\right) \Rightarrow \hat{r}_+ = \tfrac{1}{2}\left(\pi T\mathcal{L} + \sqrt{(\pi T\mathcal{L})^2 + 2}\right)$$

r_+ is the larger root of $\hat{r}^2 - 1 - \frac{\mu}{\hat{r}^2} = 0$

$$T \geq 0 \text{ means } \hat{r}_+ \geq \frac{1}{\sqrt{2}}, \quad \mu \geq -\frac{1}{4}$$

$$s = \frac{S}{\mathcal{L}^3\tilde{\Omega}_3} = \frac{1}{4G_5}\hat{r}^3_+,$$

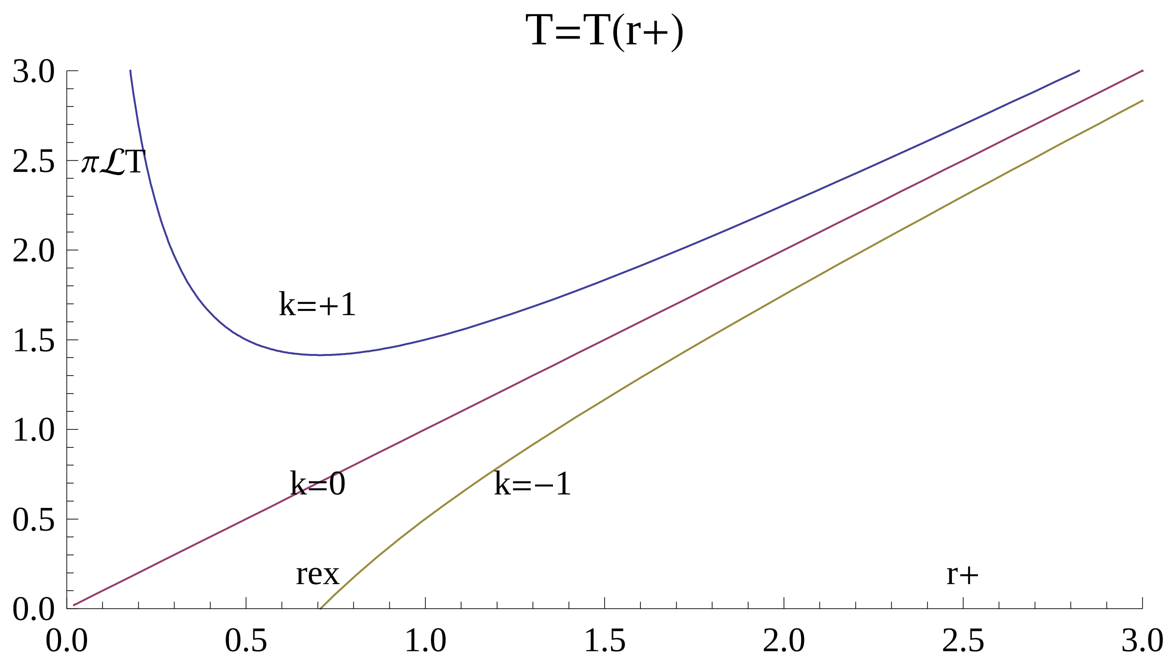


Figure 1: The dependence of temperature on r_+ for transverse metrics S^3,R^3,H^3 $(k = +1, 0, -1)$; $\pi\mathcal{L}T$ is plotted vs. $r_+/\mathcal{L}$.

For $k = -1$ $s(T = 0) = \frac{N_c^2\sqrt{2}}{8\pi\mathcal{L}^3}$ is non-zero!

Multiply degenerate ground state!

Boundary thermo of 5d AdS BH with hyperbolic horizon, $k = -1$

Transform metric to the form:

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left\{-\frac{\left[1-\left(\frac{\mu+\frac{1}{4}}{4}\right)\hat{z}^4\right]^2}{\left[1+\frac{\hat{z}^2}{2}+\left(\frac{\mu+\frac{1}{4}}{4}\right)\hat{z}^4\right]}dt^2 + \left[1+\frac{\hat{z}^2}{2}+\left(\frac{\mu+\frac{1}{4}}{4}\right)\hat{z}^4\right]\mathcal{L}^2\, d\tilde{\Omega}_3^2 + dz^2\right\}$$

Boundary metric:

$$g^{(0)}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \mathcal{L}^2 & 0 & 0 \\ 0 & 0 & \mathcal{L}^2 \sinh^2\eta & 0 \\ 0 & 0 & 0 & \mathcal{L}^2 \sinh^2\eta \sin^2\theta \end{pmatrix}, \qquad R_\mu{}^\nu = -\tfrac{2}{\mathcal{L}^2}\,\mathrm{diag}(0,1,1,1)$$

$=$ hyperbolic Robertson-Walker with constant scale factor!

$\langle T_\mu{}^\nu\rangle_{\mathrm{vacuum}} = 0$, Bunch 1978.

Energy-momentum tensor is all fluid, no vacuum:

$$T_\mu{}^\nu = \frac{\mathcal{L}^3}{4\pi G_5}\frac{\mu+\frac{1}{4}}{4\mathcal{L}^4}\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \frac{\mathcal{L}^3}{4\pi G_5} = \frac{N_c^2}{2\pi^2}.$$

$$\mu + \tfrac{1}{4} = \tfrac{1}{4}(\pi T\mathcal{L})^2\left(\pi T\mathcal{L} + \sqrt{(\pi T\mathcal{L})^2+2}\right)^2,$$

$$\Rightarrow \quad p(T) = \frac{1}{3}\epsilon(T) = \frac{\pi^2 N_c^2}{8}T^4\,\frac{1}{4}\left(1+\sqrt{1+\frac{2}{(\pi T\mathcal{L})^2}}\right)^2, \quad p(0)=0$$

$$\text{From } \frac{A}{4G_5}: \quad s(T) = \frac{\pi^2 N_c^2}{2}T^3\,\frac{1}{8}\left(1+\sqrt{1+\frac{2}{(\pi T\mathcal{L})^2}}\right)^3, \qquad s(0) = \frac{N_c^2\sqrt{2}}{8\pi\mathcal{L}^3}$$

$$d\epsilon = T ds$$

but

$$\frac{dp}{dT} = s(T)\,\frac{1}{\sqrt{1+2/(\pi T\mathcal{L})^2}} \neq s(T) \ \text{ unless } T \gg 1/\mathcal{L} \qquad \epsilon \neq Ts - p$$

This strongly coupled fluid in curved space is never thermalised:

Estimate collision time by

$$\eta = \frac{1}{4\pi} s = p\tau_c \Rightarrow$$

$$\tau_c = \lambda_{\text{free}} = \frac{s}{4\pi p} = \mathcal{L} \frac{\hat{r}_+^3}{\hat{r}_+^4 - \hat{r}_+^2 + \frac{1}{4}} > \mathcal{L} \cdot \frac{\mathcal{L}}{r_+} \approx \frac{1}{\pi T}$$

- $\tau_c \ll \mathcal{L}$ if $r_+ \gg \mathcal{L}$, big black holes, $\pi T \gg 1/\mathcal{L}$

- even then collision time is $\sim$ microscopic time $1/\pi T$

But there is a further mystery (for us):

The boundary partition function was to be evaluated from

$$Z_{\rm CFT} = e^{p(T)V/T} = e^{\frac{-1}{2\pi G_5 \mathcal{L}^2}\int_0^\beta d\tau d^3x \int_\epsilon^{z_0} dz \sqrt{-g(z_0)}}$$

where in full glory the regulated gravity action in d dim is

$$S = \frac{1}{16\pi G_{d+1}}\left\{\int d^{d+1}x\sqrt{-g}\,\frac{-2d}{\mathcal{L}^2} - \int d^dx\sqrt{-\gamma}\left[2K + \frac{2d-2}{\mathcal{L}} + \frac{\mathcal{L}}{d-2}R(\gamma)\right]_{z=\epsilon}\right\}$$

$\gamma_{\mu\nu}$ = induced metric on the surface $z=\epsilon$, K = its extrinsic curvature.

The z-integral above is:

$$\int_\epsilon^{\hat{z}_0} \frac{d\hat{z}}{\hat{z}^5}\left(1-\frac{\hat{z}^4}{\hat{z}_0^4}\right)\left(1+\frac{\hat{z}^2}{2}+\frac{\hat{z}^4}{\hat{z}_0^4}\right) = \frac{1}{4\epsilon^4} + \frac{1}{4\epsilon^2} - \frac{1+\hat{z}_0^2}{2\hat{z}_0^4} + \mathcal{O}(\epsilon^2), \quad \hat{z}_0^4 = \frac{4}{\mu+\frac{1}{4}}.$$

and the $p(T)$ coming from here satisfies $dp/dT = s(T)$, $\epsilon = Ts - p$ is unchanged but

$$\epsilon - 3p = -\frac{3N_c^2T^2}{8\mathcal{L}^2}\left(1+\sqrt{1+\frac{2}{(\pi T\mathcal{L})^2}}\right)$$

What happens if a coordinate transformation is carried out in the bulk?

The boundary metric is changed by a conformal (Weyl) transformation:

$$g^{(0)}(x) \to \omega^2_{\text{Weyl}}(x) g^{(0)}(x)$$

A coordinate trafo $t, r \Rightarrow \tau, z$ in the AdS BH ($d = 4$):

$$ds^2 = \frac{\mathcal{L}^2}{z^2} \left\{ - \frac{\left[1 - \frac{\mu}{4}\left(\frac{z}{\tau}\right)^d\right]^2}{\left[1 + \frac{\mu}{4}\left(\frac{z}{\tau}\right)^d\right]^{\frac{2(d-2)}{d}}} \, d\tau^2 + \left[1 + \frac{\mu}{4}\left(\frac{z}{\tau}\right)^d\right]^{\frac{4}{d}} \tau^2 \, d\tilde{\Omega}^2_{d-1} + dz^2 \right\}.$$

The boundary metric, $\omega_{\text{Weyl}} = \tau/\mathcal{L}$,

$$g^{(0)}_{\mu\nu} = -d\tau^2 + \tau^2 (d\eta^2 + \sinh^2\eta \, d\Omega^2_{d-2})$$

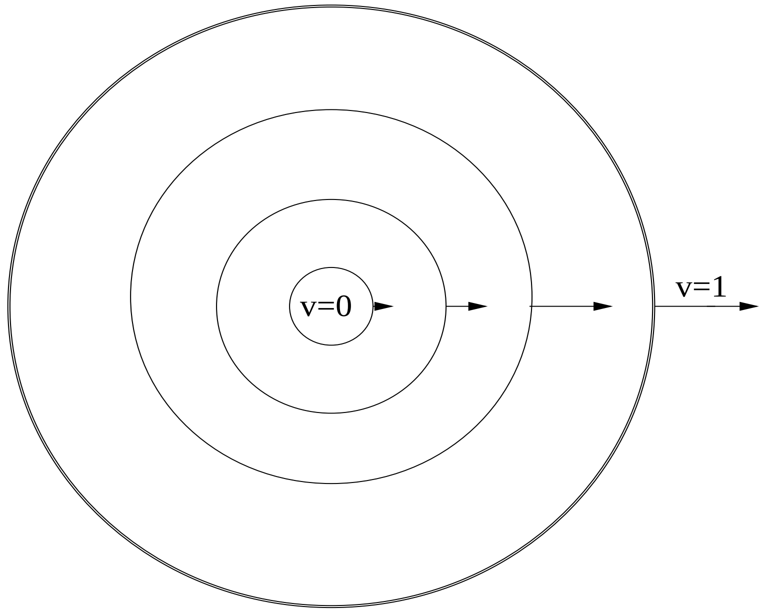
is flat; generalises Bjorken expansion to spherical expansion!

Where did the horizon at $r = r_+$ disappear? Now a zero at $z = (4/\mu)^{1/d} \, \tau$.

Spherical similarity expansion in 1+(d-1)

$T_{\mu\nu} = (\epsilon + p)\frac{x_\mu x_\nu}{\tau^2} + p g_{\mu\nu}$

Fixed time t:



$$\mathbf{v} = \frac{\mathbf{x}}{t}\theta(t - |\mathbf{x}|), \quad u^\mu = (\gamma, \gamma \mathbf{v}) = \frac{x^\mu}{\tau}, \quad \tau = \sqrt{t^2 - \mathbf{x}^2}$$

Natural coordinates:

$$\begin{aligned} t &= \tau \cosh \eta \\ x^i &= \tau \sinh \eta \, \omega^i, \quad i = 1, d-1 \\ d\Omega_{d-2}^2 &= \sum_{i=1}^{d-1} d\omega_i^2 \end{aligned}$$

$$ds^2 = -d\tau^2 + \tau^2 \left(d\eta^2 + \sinh^2 \eta d\Omega_{d-2}^2\right) \equiv -d\tau^2 + \tau^2 d\tilde{\Omega}_{d-1}^2$$

For $d = 2$ all is clear (0705.1791):

$$F(r) = \hat{r}^2 - 1 - \frac{\mu}{\hat{r}^{d-2}} = \hat{r}^2 - 1 - \mu = 0 \;\Rightarrow\; \hat{r}_+ = \sqrt{\mu + 1}$$

$$T_\mu{}^\nu = \frac{\mathcal{L}}{16\pi G_3}\frac{\mu + 1 - 1}{\tau^2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\epsilon(\tau) & 0 \\ 0 & p(\tau) \end{pmatrix} + \underbrace{\frac{\mathcal{L}}{16\pi G_3}\frac{-1}{\tau^2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{vacuum}}$$

$$s(\tau) = \frac{\pi\mathcal{L}}{2G_3}T(\tau) = p'(\tau), \quad T(\tau) = \frac{\sqrt{\mu + 1}}{2\pi\tau}$$

For $d = 4$ in the coordinates $(\tau, \eta, \theta, \phi)$:

$$T_\mu{}^\nu = \frac{\mathcal{L}^3}{4\pi G_5}\frac{\mu}{4\tau^4}\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \frac{\mathcal{L}^3}{4\pi G_5} = \frac{N_c^2}{2\pi^2}$$

$\langle T_\mu{}^\nu \rangle_{\text{vacuum}}$ is unknown in these coordinates! Maybe in $\mu + \frac{1}{4} - \frac{1}{4}$ the $-\frac{1}{4}$ is vacuum?

For large $\hat{r}_+$, large BH, again consistent thermo with

$$T(\tau) = \frac{1}{4\pi\tau}\left(d\hat{r}_+ - \frac{d-2}{\hat{r}_+}\right) \approx \frac{d\hat{r}_+}{4\pi\tau}, \text{ etc}$$

Conclusions

- The famous result $p_{\text{strong coupling}} = \frac{3}{4} p_{\text{ideal}}$ comes from an AdS BH with a flat horizon

- For a $k = +1$ spherical horizon there is a Hawking-Page phase transition at $T = 1/\mathcal{L}$

- For a $k = -1$ hyperbolic horizon T can go all the way to zero. T, s and ϵ, p can be defined but the system approaches thermalisation only when $T \gg 1/\mathcal{L}$

- The AdS hyperbolic BH is the gravity dual of a spherically expanding system, which is the closer to thermal equilibrium the larger $T\tau$ is. Needs still computation of vacuum energy in relevant coordinates

- Inclusion of three chemical potentials associated with SU(4) symmetry could sharpen the analysis of thermodynamic (non)equilibrium. Much work has been done with a multitude of rotating and charged 5d BHs.