QCD matter at finite temperature and density: lattice Monte Carlo, perturbation theory and holography

K. Kajantie

Helsinki Institute of Physics

Odense 20-21 November 2013

QCD thermodynamics

$$e^{p(T,\mu;m_q)\frac{V}{T}} =$$

$$\int \mathcal{D}A\mathcal{D}q \; e^{-\int^{1/T} d\tau d^3x \left[\frac{1}{g^2}F^2 + \bar{q}(\partial + A)q + m_q \bar{q}q + \mu q^\dagger q\right]}$$

Color N_c, Flavor N_f, QCD scale Λ_{QCD}

$m_q = 0$ to have chiral symmetry

$QCD_{physical}$: m_u, m_d, m_s, m_c, m_b, m_t

Reminder: coupling and mass run, are scheme dependent:

$$\lambda(\mu) \equiv \frac{N_c g^2(\mu)}{8\pi^2} \qquad x_f = \frac{N_f}{N_c}$$

We include **all** $\lambda(\mu)$

$$\mu\frac{d\lambda}{d\mu} = \beta(\lambda) = -\underbrace{\frac{11-2x_f}{3}}_{b_0}\lambda^2 - \underbrace{\frac{34-13x_f}{6}}_{b_1}\lambda^3 - ..$$

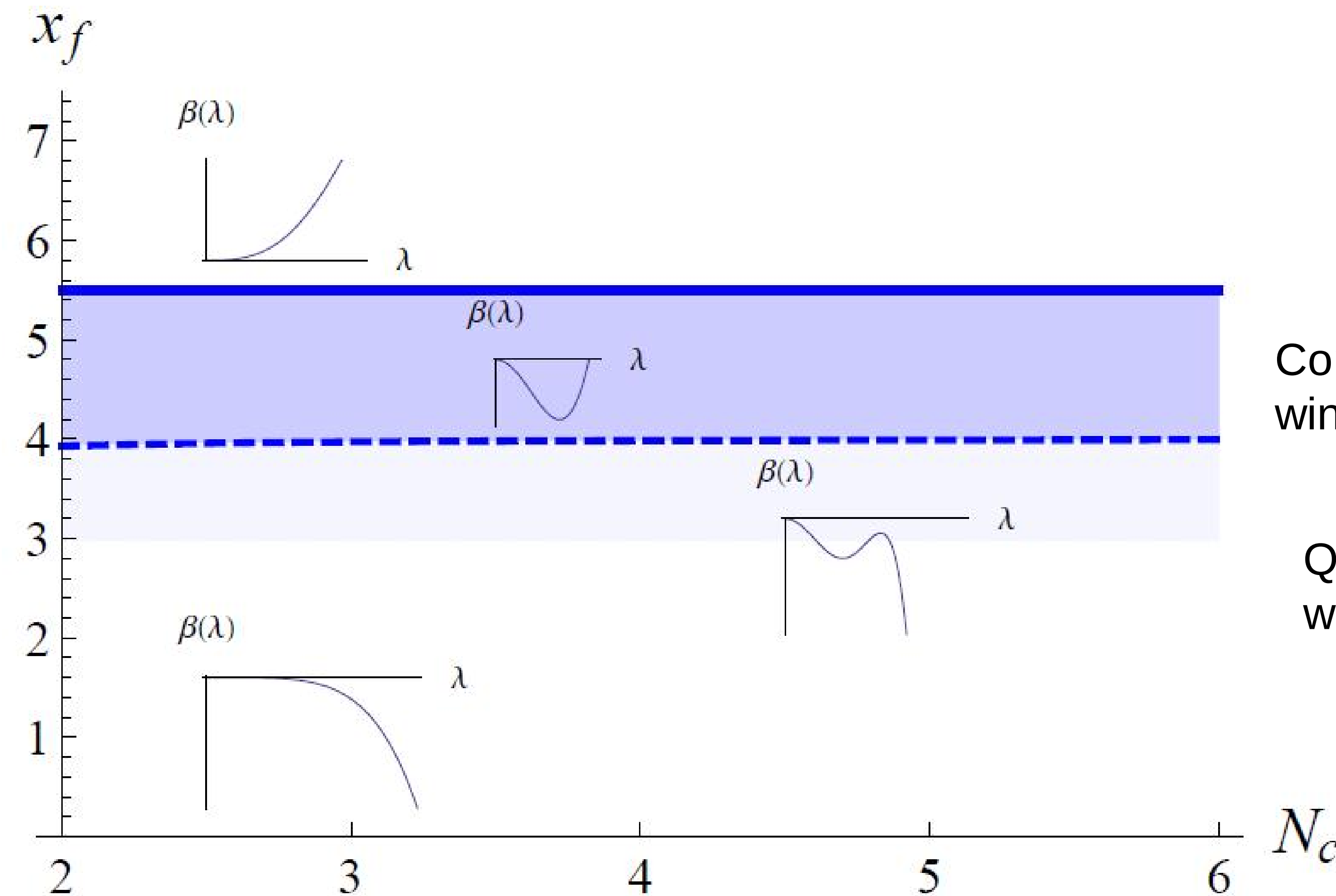
$$\mu\frac{d\log m}{d\mu} = \gamma_m(\lambda) = -\underbrace{\frac{3}{2}}_{\gamma_1}\lambda - ..$$

For $x_f > 5.5$ loose asymptotic freedom!

$$\log\mu = \int_{\lambda_0}^{\lambda(\mu)} \frac{d\lambda}{\beta(\lambda)}$$

$$m(\mu) = m_0(\log\mu)^{-\frac{9}{2(11-2x_f)}}$$

Beta function in various ranges of x_f:



Conformal window

Quasiconformal walking

Expected thermo:

$$2N_c^2 + \frac{7}{2} N_c N_f$$

At large T,μ quark-gluon plasma with chiral symmetry if m_q=0

in between a chiral transition at $T_\chi(\mu)$

At small T,μ a quark-gluon system with chiral symmetry broken

A deconfinement transition to a hadronic phase

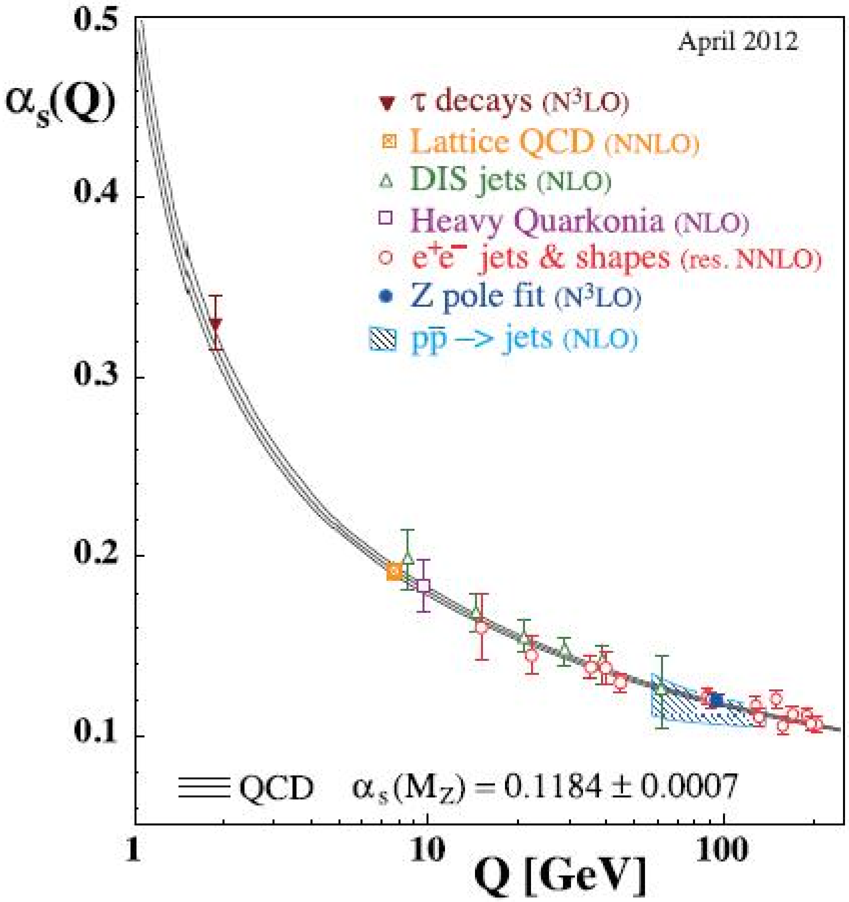
$$N_f^2$$

Chiral transition has an order parameter: condensate

No order parameter, symmetry, associated with confinement!

Perturbative if

$$\lambda \equiv \frac{N_c g^2}{8\pi^2} = \frac{3\alpha_s}{2\pi} \gtrsim 0.15$$



1. Lattice Monte Carlo

2. Perturbation theory

3. Holography – Gauge/gravity duality

4. Chiral effective theories

1. Lattice and finite T, $\mu = 0$

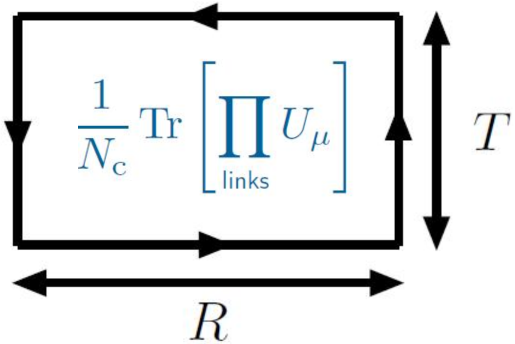
1. For the equation of state, evaluate the integral

You always have the confining magnetic sector!

$$Z(T,V) = e^{p(T)\frac{V}{T}} = \int \mathcal{D}[A\bar{\psi}\psi]e^{-\int_0^{1/T} d\tau \overbrace{d^3x\, \mathcal{L}_{\text{QCD}}}}$$

2. Spatial string tension(T)

Expectation value of

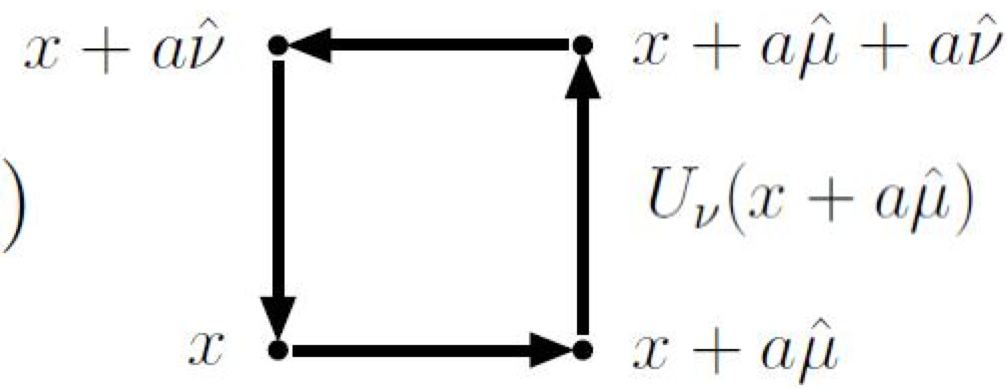


with path in spatial directions

3. Polyakov line $\frac{1}{N_c}\mathrm{Tr}\left[\prod_{\text{links}} U_0\right]$ with path in τ direction

Lattice: $N_t \cdot N_s^3$ $\quad \frac{1}{T} = N_t a \ll N_s a = V^{1/3}$

$P_{\mu\nu}(x)$



$$U_\mu(x) = e^{igaA_\mu(x)}$$

G(x), ψ(x) on sites, gauge fields on links!

Gauge transfo $\quad U'_\mu(x) = G(x)U_\mu(x)G^{-1}(x + a\hat{\mu})$

Tr[loop] = gauge invariant

$$S_E \equiv \frac{1}{g_0^2} \sum_x \sum_{\mu,\nu=0}^{3} \mathrm{Tr}\left[\mathbb{1} - P_{\mu\nu}(x)\right] \approx a^4 \sum_x \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu}$$

On the lattice one Monte Carloes expectation values = derivatives of logZ

$$\langle I \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U \, I[U] e^{-S_E[U]}$$

$4N_t N_s^3 (N_c^2 - 1) \sim 10^7$ dim integral

Normalisation cancels!

$$\langle \text{gauge noninvariant} \rangle = 0$$

Fermions

$$a^4 \sum_{x,y} \bar{\psi}(x) \left[D(x,y) + M\delta_{x,y} \right] \psi(y) \quad -\frac{r}{2} \sum_x a^5 \bar{\psi}(x) \Delta_\mu \Delta_\mu^* \psi(x)$$

Wilson term

Grassman variables integrated over:

$$\mathcal{Z} = \int \mathcal{D}U_\mu \ \text{Det}[D + M] \exp\left\{ -S_E^{(\text{gluons})} \right\}$$

$10^7 \cdot 10^7$ sparse matrix

$$\langle \psi(x) \bar{\psi}(y) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U_\mu \ \text{Det}[D + M][D + M]^{-1}(x,y) e^{-S_E^{(\text{gluons})}}$$

Long non-ending story, chiral symmetry, overlap fermions, domain wall fermions, connection to analytic formulas of chiral perturbation theory

Lattice and p(T)

What expectation value gives the EoS? Since

$$\log Z = \frac{p(T)}{T} V = \log \int \mathcal{D}U e^{-\beta(a) S_\Box(U)} \quad \frac{1}{T} \sim a \quad V \sim a^3$$

$$\frac{-1}{VT^3} a \frac{d \log Z}{da} = \frac{\epsilon - 3p}{T^4} = T \frac{\partial}{\partial T} \frac{p(T)}{T^4}$$

$$= \frac{N_t^3}{N_s^3} a\beta'(a) \langle S_\Box \rangle_{(T)-(T=0)} \qquad a \frac{d(ma)}{da} \sum_x \langle \bar{\psi}\psi \rangle$$

So "just" determine the expectation value of the plaquette action times lattice beta function!!

$$c_s^2 = \frac{dp}{d\epsilon} = \frac{s}{T s'(T)} \qquad T \frac{\partial}{\partial T} \frac{s}{T^3} = \frac{s}{T^3} \left(\frac{1}{c_s^2} - 3 \right)$$

Physics is in decimals:

$$\frac{\epsilon - 3p}{T^4} = N_t^4 \, a\frac{d}{da}\frac{2N_c}{g^2(a)}\left[\langle\frac{S_\Box}{N_t N_s^3}\rangle_{N_t N_s^3} - \langle\frac{S_\Box}{N_s^4}\rangle_{N_s^4}\right]$$

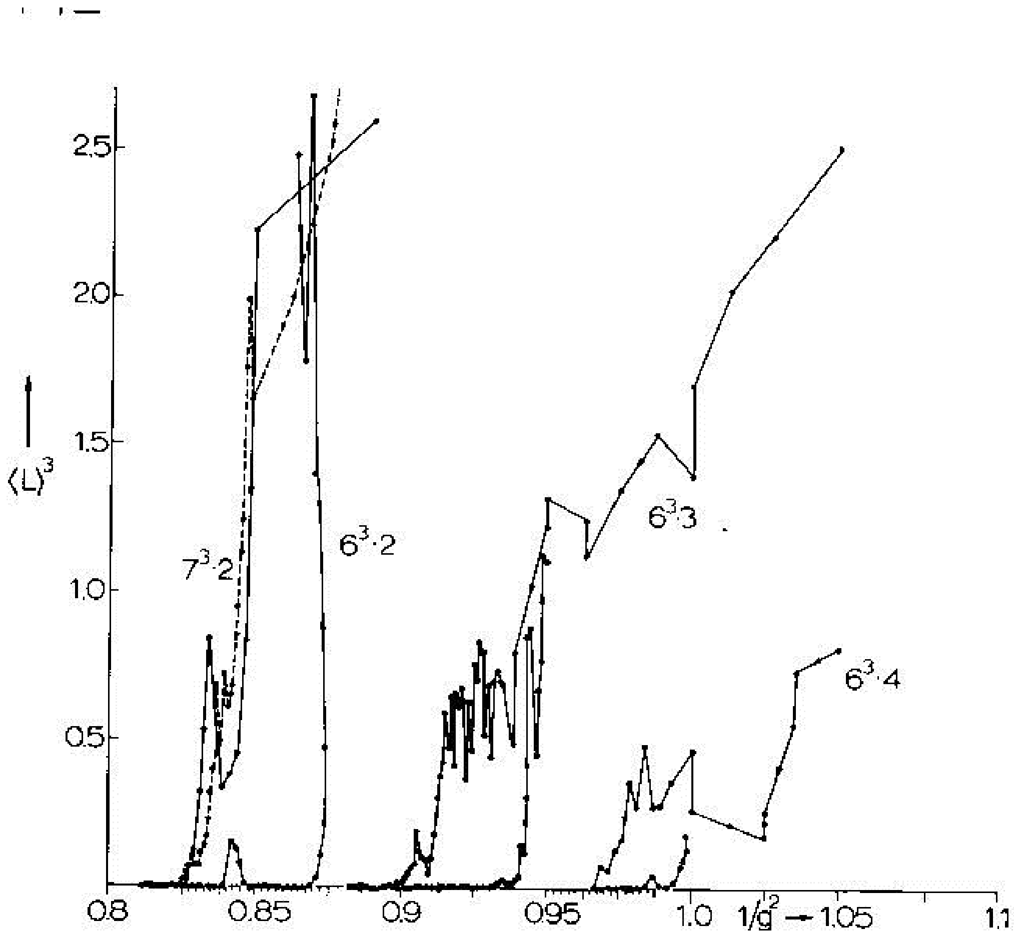
$\mathcal{O}(1)$ $\mathcal{O}(1)$ Action per point

$$\approx N_t^4\left(0.6 + \frac{1}{N_t^4} - 0.6\right)$$

The bigger and better the lattice, the deeper is physics buried!

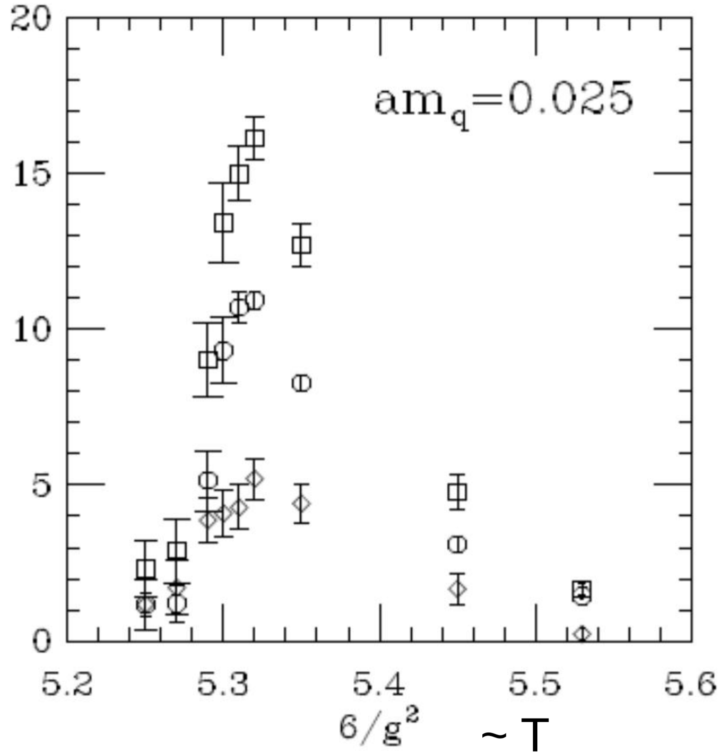
Some history:

1982, SU(3) : Kajantie-Montonen-Pietarinen



1994, $N_f = 2$: Blum- Gottlieb-Kärkkäinen-Toussaint

$$\frac{\epsilon - 3p}{T^4}$$

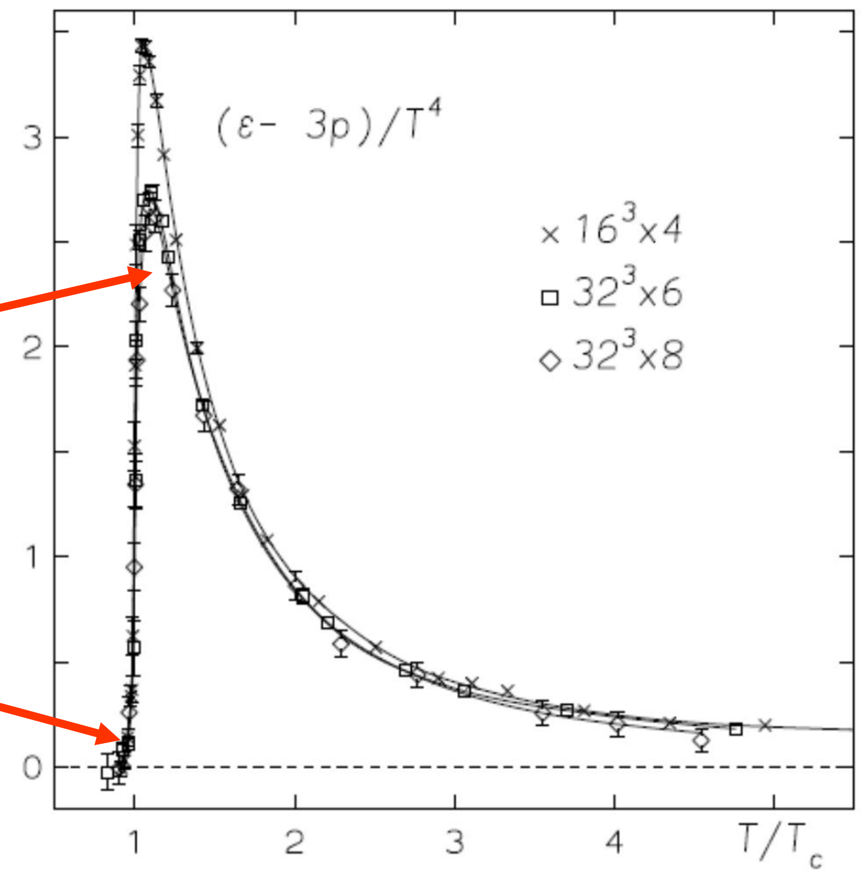


~ T

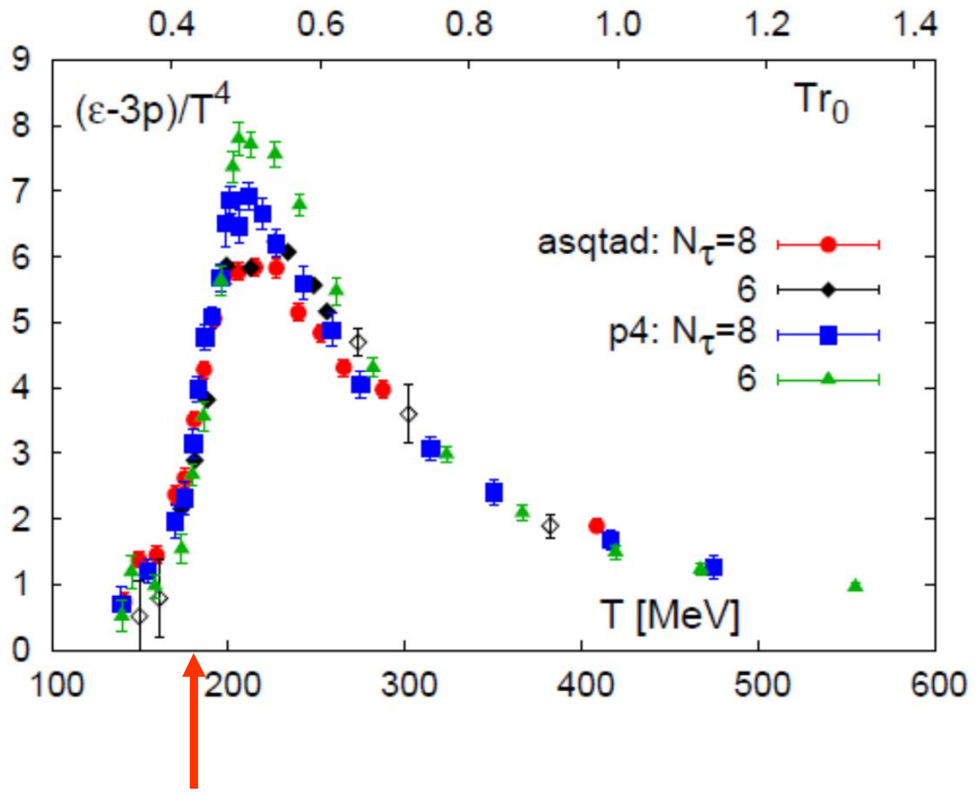
1996, pure SU(3): Boyd-Engels-Karsch-Laermann…

1st order transition

$$\frac{L}{T_c^4 N_c^2} \approx 0.3$$



2009: N_f = 2+1 0903.4379,   23 authors

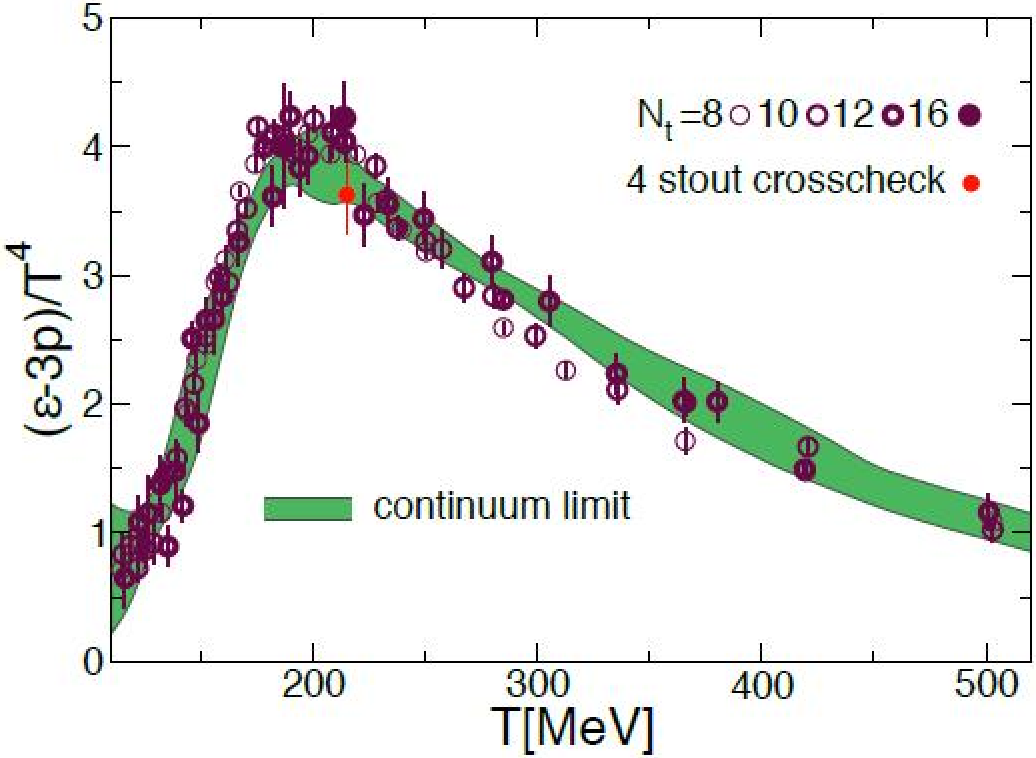


T_c

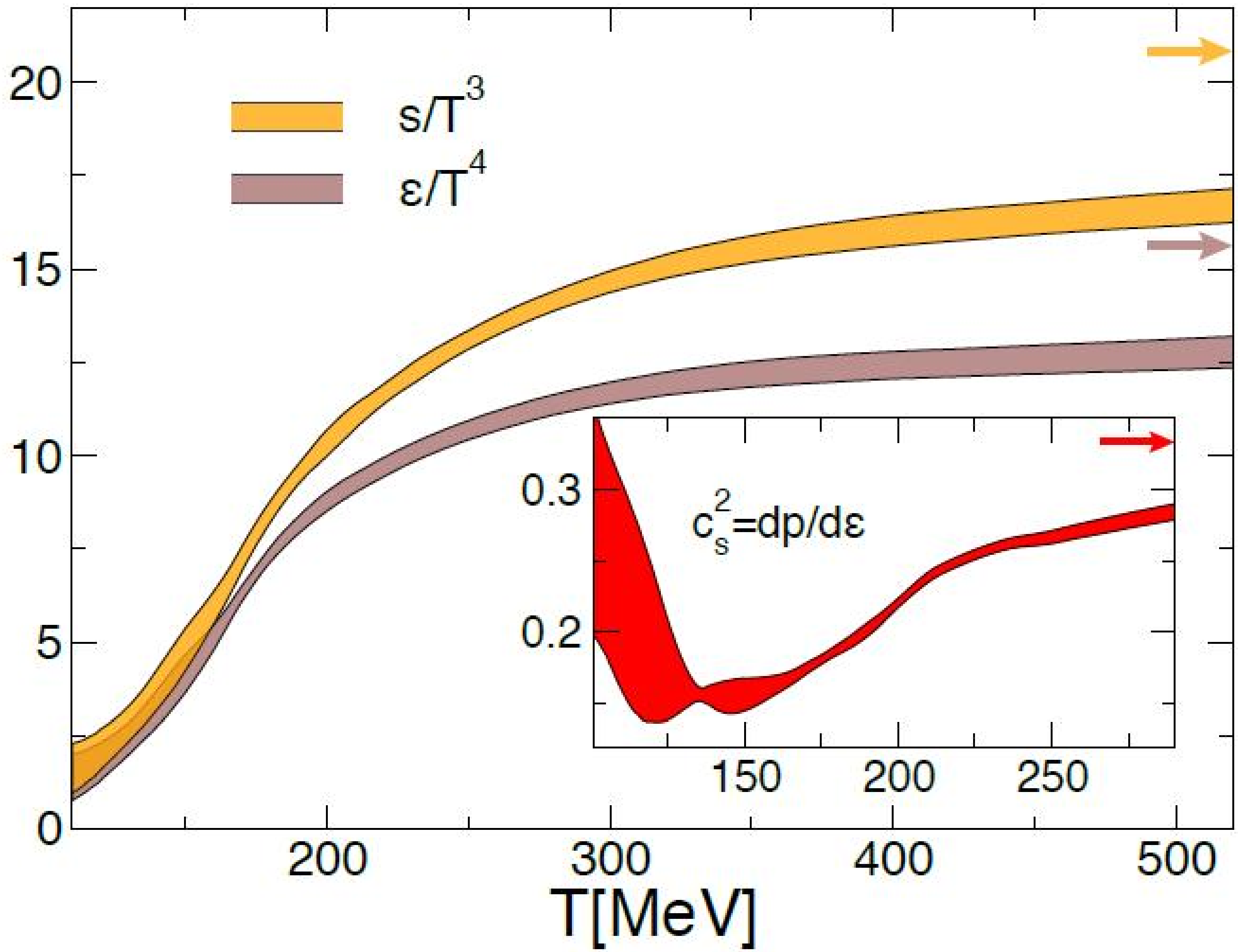
Continuous transition!!

2013: QCD_{uds}

Wuppertal-Budapest 1309.5258



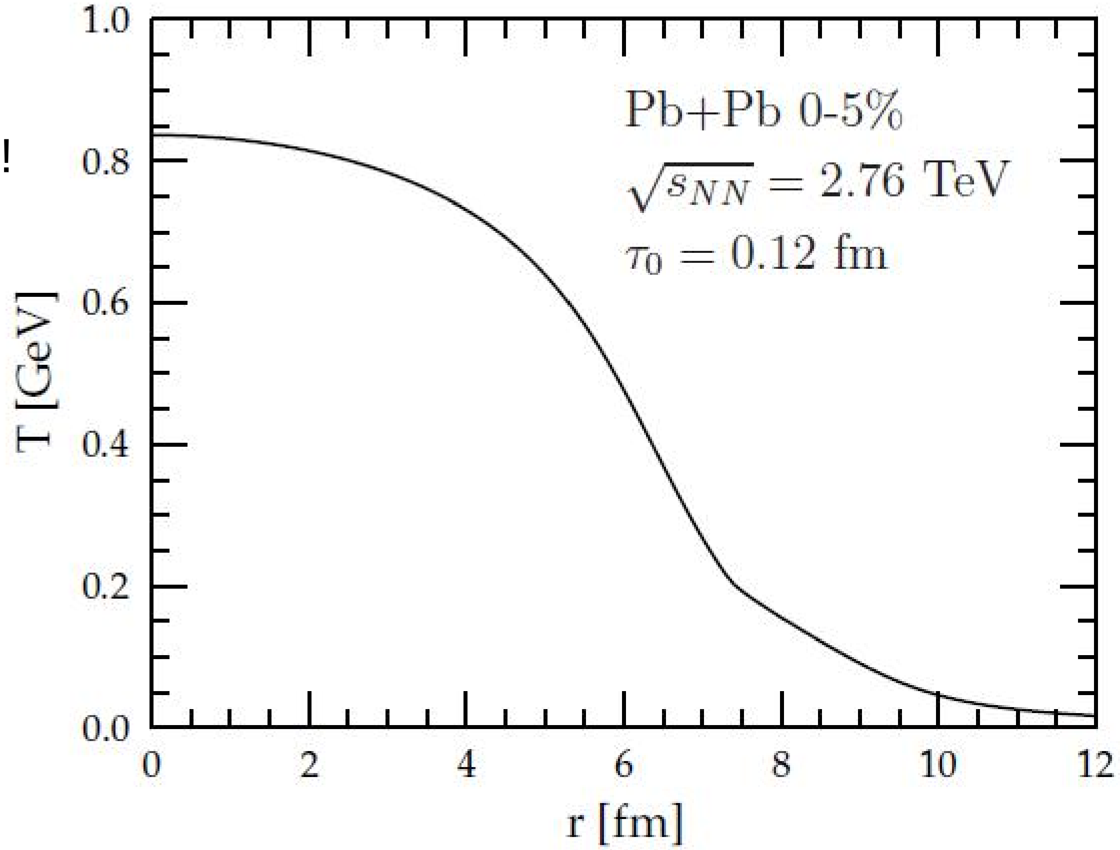
Wuppertal-Budapest 1309.5258



What temperatures can be reached at LHC?

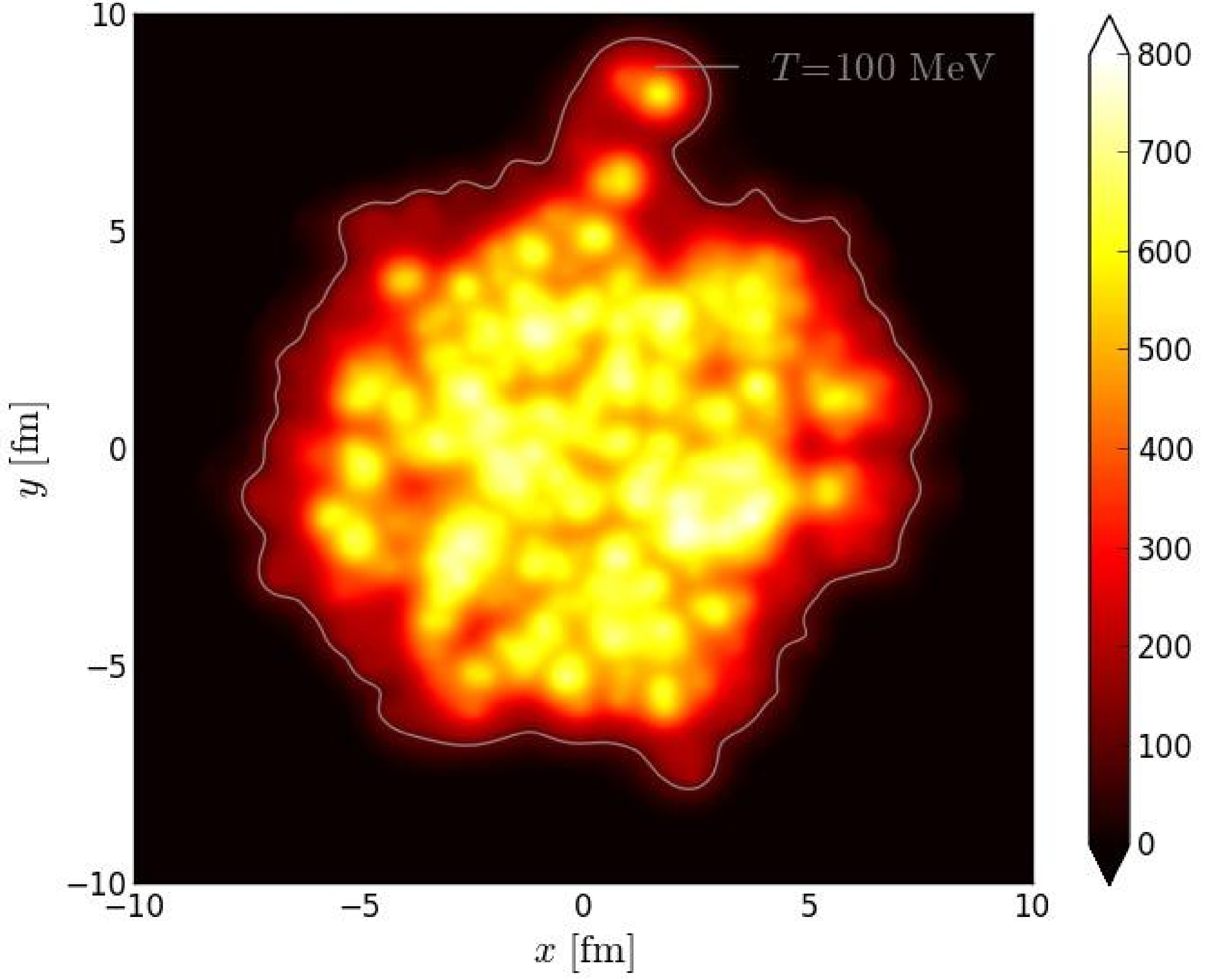
Jyväskylä hydro group, Eskola, Niemi,…:

Up to 800 MeV!



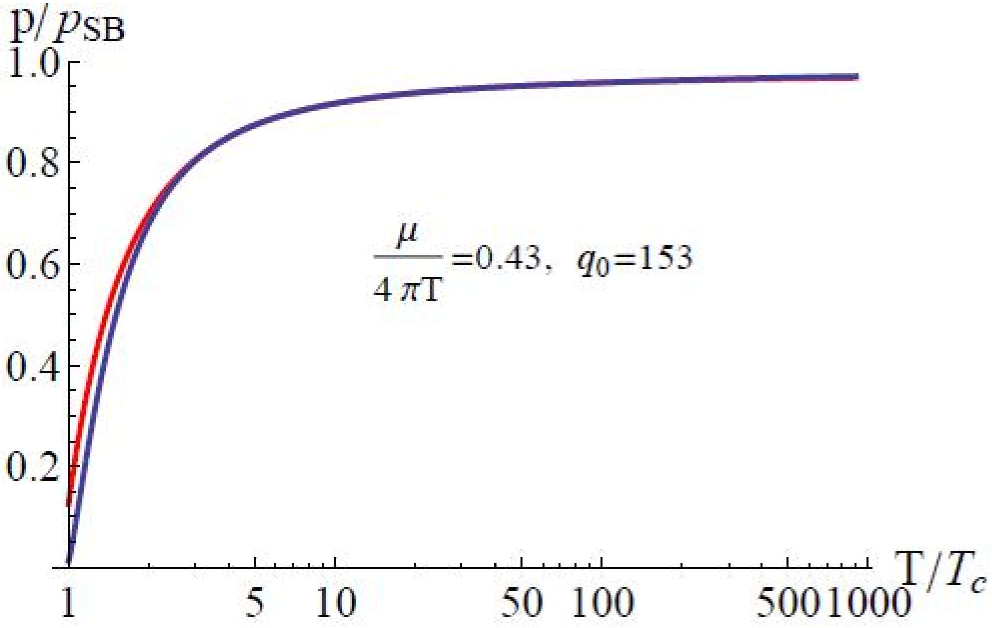
Fluctuations important! Transverse plane, proper time = 0.2 fm:

Jyväskylä hydro group, Eskola, Niemi,...:



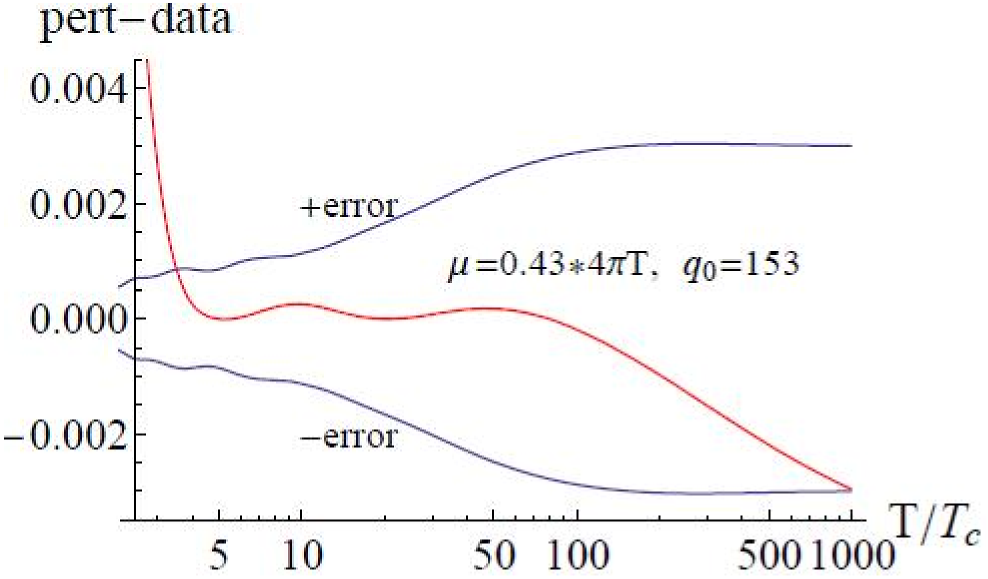
Pure SU(3)

Budapest-Wuppertal 1204.6184



Continuum data! (blue)

Pert (red)



pert < lattice data

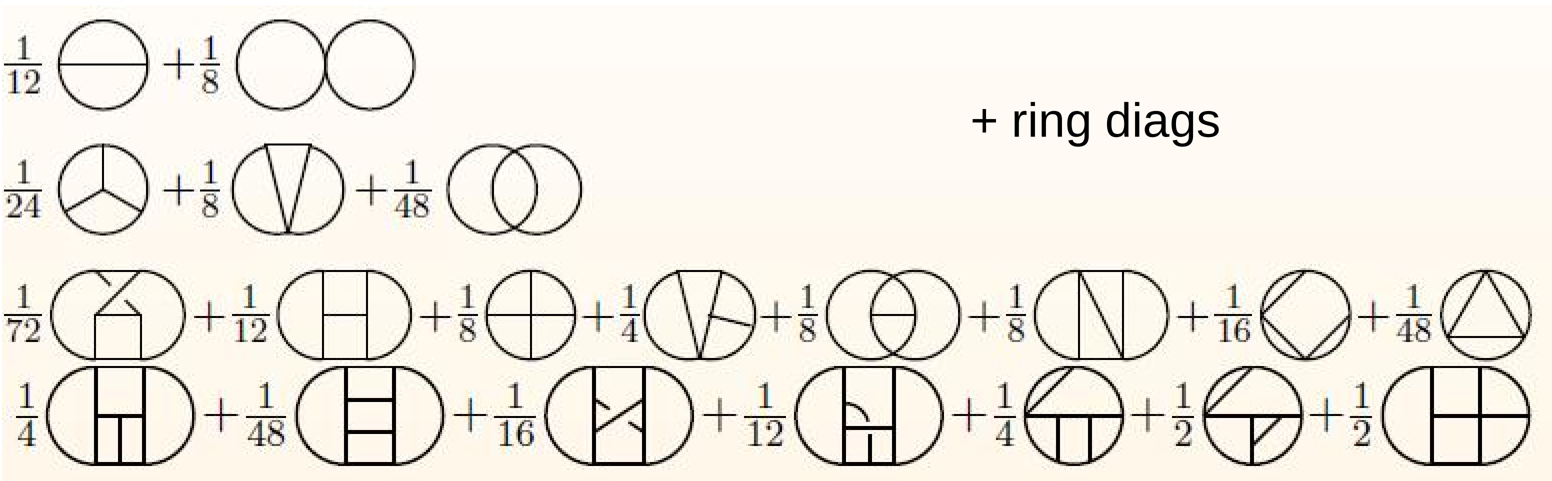
Perhaps lattice data
should be corrected
by a tiny amount down

2. Perturbation theory for p(T)

$$e^{p(T)V/T} = \int \mathcal{D}A\, e^{-(\partial A + gA^2)^2}$$

$$= \int \mathcal{D}A\, e^{A\partial^2 A}\left[1 + \sum_n \frac{1}{n!}(2g\partial A \cdot A^2 + g^2 A^4)^n\right]$$

Generate vacuum diagrams:



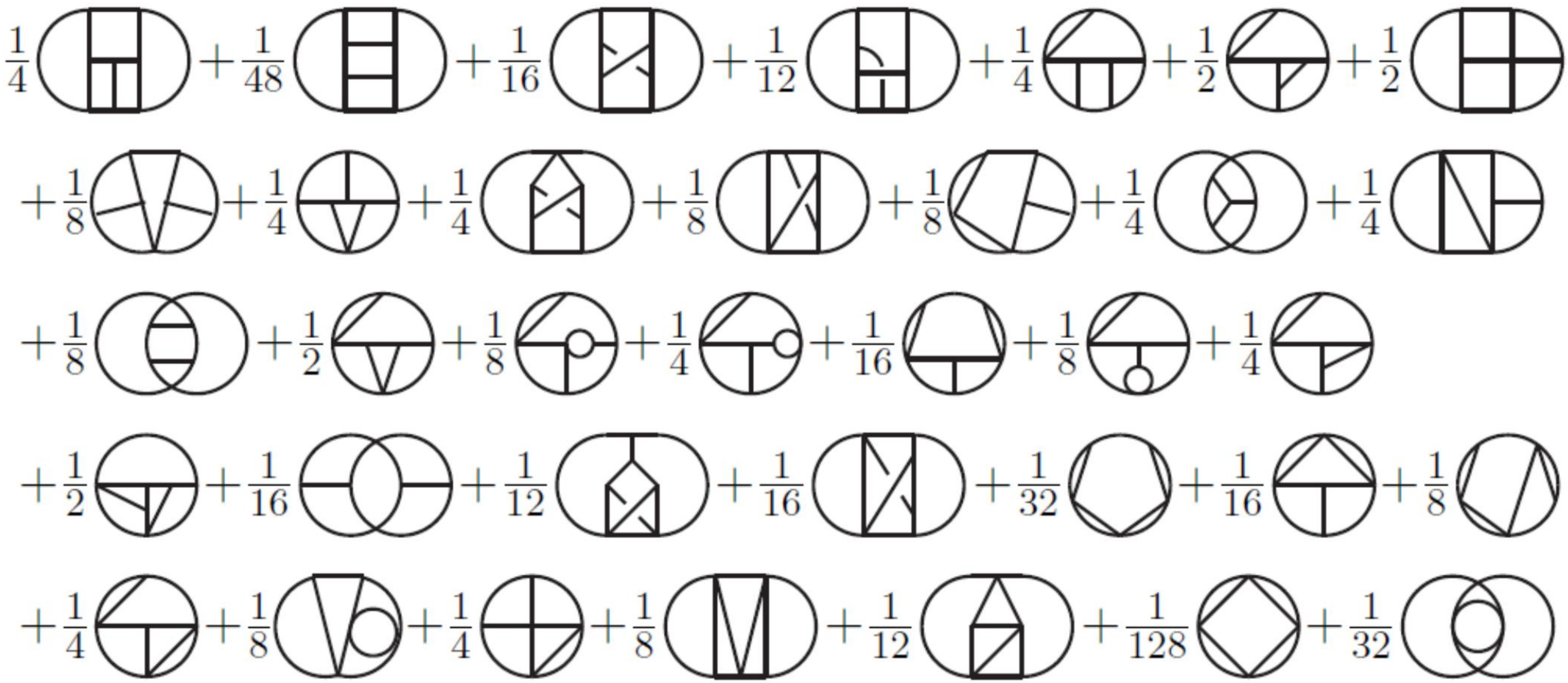
$$T\sum_n \int \frac{d^{3-2\epsilon}k}{(2\pi n T)^2 + \mathbf{k}^2}$$

IR divs at k=0; physics is electric screening and magnetic sector confinement

$$g^{n\geq 6}: \quad \infty \text{ number of loops}$$

All topologically distinct 5-loop vacuum diags;

Kajantie-Laine-Schröder
hep-ph/0109100



Exercise in futility (mathematics): generalise to n loops

No wonder QCD matter becomes strongly interacting!

$$c_{\rm SB}+c_2g^2+c_3g^3+(c_4'\log g+c_4)g^4+c_5g^5+(c_6'\log g+c_6)g^6+c_7g^7+..$$

c_2 Shuryak 78, c_3 Kapusta 79, c_4' Toimela 83, c_4 Arnold-Zhai 94,
c_5 Zhai-Kastening, Braaten-Nieto 95, c_6' Kajantie-Laine-Rummukainen-Schröder 03

$$p/p_{\rm SB} =$$

$$1-\frac{5}{2}\lambda(\bar\mu)+\frac{20}{\sqrt{3}}\lambda^{3/2}+\left[30\log(\frac{2}{3}\lambda)+p_2b_0\log\frac{\bar\mu}{4\pi T}+99.0784\right]\lambda^2$$

$$+[\frac{3}{2}p_3b_0\log\frac{\bar\mu}{4\pi T}-227.746]\lambda^{5/2}+$$

$$\Big\{(-42.8187+60b_0\log\frac{\bar\mu}{4\pi T})\log(\frac{2}{3}\lambda)-140.915\log\lambda+p_2b_0^2\log^2\frac{\bar\mu}{4\pi T}$$

$$+(p_2b_1+p_4b_0+2b_0\,99.0784)\log\frac{\bar\mu}{4\pi T}+q_0\Big\}\lambda^3+\mathcal{O}(\lambda^{7/2})$$

$$\frac{1}{\lambda(\mu)}=b_0\log\frac{\mu}{\Lambda}+\frac{b_1}{b_0}\log(\log\frac{\mu}{\Lambda})$$

3. Holography, AdS/CFT

The prototype:

$\mathcal{N}=4$ SuSy in full glory (1 vector, 4 fermions, 6 scalars, all adjoint) [9]

$$S[A^a_\mu, \phi^a_i, \psi^a, \bar{\psi}^a] = \frac{1}{2g^2}\int d^4x \left\{ \frac{1}{2} F^{a\,2}_{\mu\nu} + (\partial_\mu \phi^a_i + f_{abc} A^b_\mu \phi^c_i)^2 + \bar{\psi}^a i\gamma^\mu (\partial_\mu \psi^a + f_{abc} A^b_\mu \psi^c) \right.$$

$$\left. + i f_{abc} \bar{\psi}^a \Gamma^i \phi^b_i \psi^c - \sum_{i<j} f_{abc} f_{ade} \phi^b_i \phi^c_j \phi^d_i \phi^e_j + \partial_\mu \bar{c}^a (\partial_\mu c^a + f_{abc} A^b_\mu c^c) + \xi (\partial_\mu A^a_\mu)^2 \right\}$$

NOT QCD! Conformally invariant on quantum level: coupling does not run!

$\mathcal{N}=4$ SYM has the symmetry O(2,4), just like AdS_5

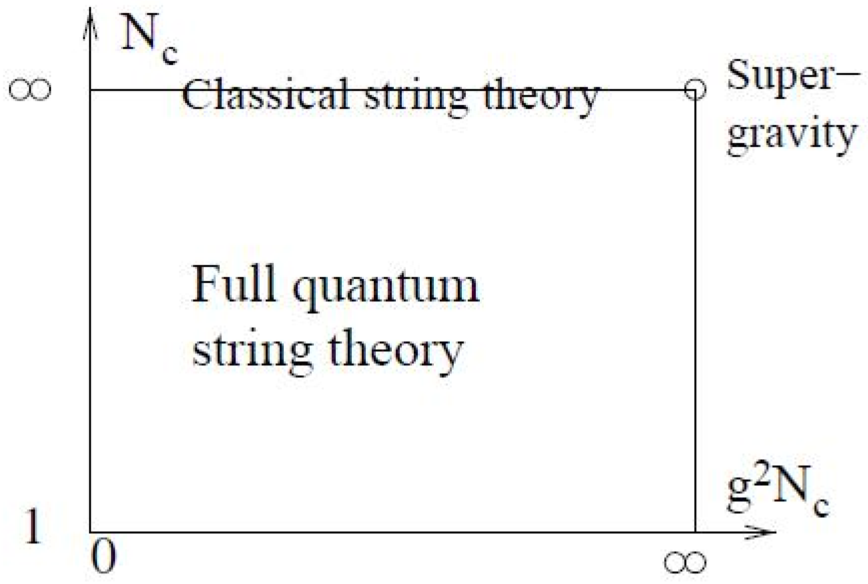
$$S = \frac{1}{16\pi G_5}\int d^5x \sqrt{-g}\left(R + \frac{12}{\mathcal{L}^2}\right) \quad ds^2 = \frac{\mathcal{L}^2}{z^2}\left(-dt^2 + d\mathbf{x}^2 + dz^2\right)$$

AdS/CFT: Quantum string theory = Quantum N=4 SuSy

String theory becomes classical gravity (calculable!) if

N_c>>1: no loops

g^2N_c>>1: strings become points



Thermo of N=4 SuSy:

BH in 5d asymptotically ($z \to 0$) AdS_5

$$ds^2 = \frac{\mathcal{L}^2}{\tilde{z}^2}\left[-\left(1-\frac{\tilde{z}^4}{z_0^4}\right)dt^2 + d\mathbf{x}^2 + \frac{d\tilde{z}^2}{1-\tilde{z}^4/z_0^4}\right]$$

$$T_{\text{Hawk}} = \frac{1}{\pi z_0} \qquad S = \frac{A}{4G_5} \qquad = V_3 \cdot \frac{\pi^2 N_c^2}{2} T^3$$

strong coupling!

$$\frac{\mathcal{L}^3}{16\pi G_5} = \frac{N_c^2}{2\pi^2}$$

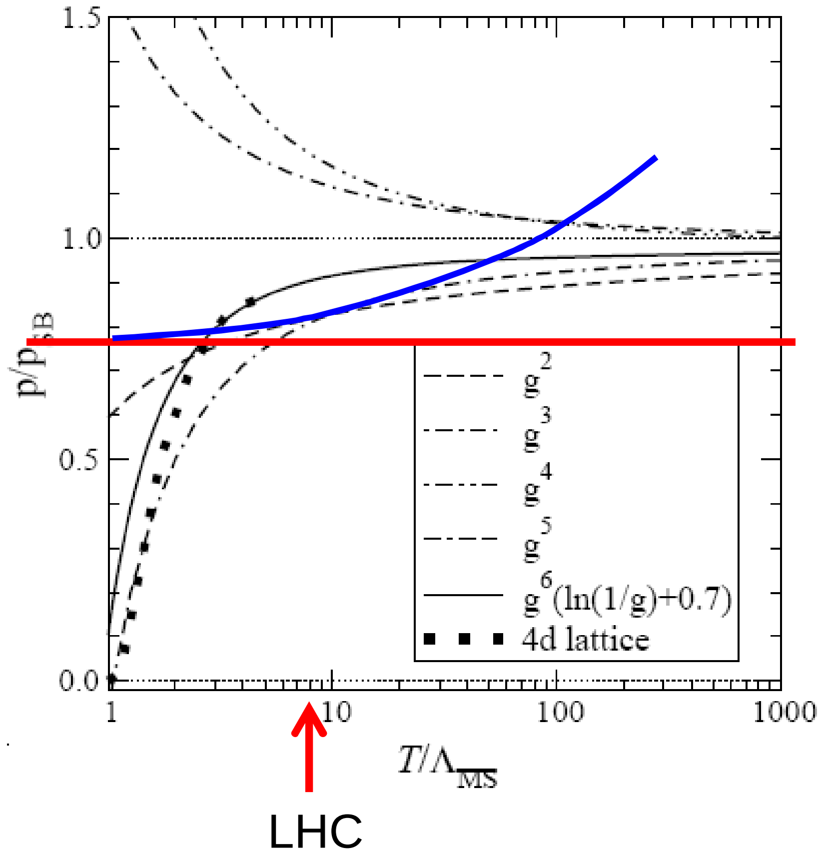
string theory

⟶ The famous 3/4 :

p(T): lattice, perturbation theory, AdS/CFT

$$(g_B + \frac{7}{8} g_F)\frac{\pi^2}{90}T^4 = (8+7)d_A\frac{\pi^2}{90}T^4 = \frac{\pi^2(N_c^2-1)}{6}T^4$$

weak coupling



4. Holography: AdS/QCD

Alho, Järvinen, Kajantie, Kiritsis, Tuominen 1210.4516+

To make this more QCD-like we add more structure to metric:

$$ds^2 = b^2(z)\left[-f(z)dt^2 + d\mathbf{x}^2 + \frac{dz^2}{f(z)}\right]$$

$$b(z) \to \frac{\mathcal{L}}{z}, \quad f(0) = 1, \quad f(z_h) = 0, \quad 4\pi T = -\dot{f}(z_h)$$

temperature
entropy

and add three scalars to describe essential QCD dynamics:

a dilaton for confinement: $\frac{N_c g^2(\mu)}{8\pi^2} \to \lambda(z)$ μ=1/z

a tachyon for chiral symmetry: $m \to \tau(z)$

a potential for quark number: $\mu \to A_0(z)$

$$q^\dagger q = \bar{q}\gamma^0 q$$

$$\int \mathcal{D}A\mathcal{D}q\ e^{-\int^{1/T} d\tau d^3x\left[\frac{1}{g^2}F^2+\bar{q}(\partial+A)q+m_q\bar{q}q+\mu q^\dagger q\right]}$$

Confinement
Asymptotic freedom
Dilaton

Chiral symmetry
Quark mass = 0
Tachyon

Quark density
$A_0(z)$

$$S = \frac{1}{16\pi G_5} \int d^5x \, \mathcal{L},$$

$$\mathcal{L} \;=\; \sqrt{-g}\Big[R + [-\tfrac{4}{3}\, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V_g(\lambda)]$$

Usual scalar action for dilaton

$$-\; V_f(\lambda,\tau)\sqrt{-\det\left[g_{ab} + \kappa(\lambda)\partial_a\tau\partial_b\tau + \omega(\lambda,\tau)F_{ab}\right]}\Big]$$

$$\sqrt{1 + ..\,\dot{\tau}^2 + ..\,\dot{A}_0^2}$$

DBI action for tachyon and potential

A_0 is cyclic $\frac{\partial L_f}{\partial \dot{A}_0} = \tilde{n}$

$$V_g(\lambda) = \frac{12}{\mathcal{L}_0^2}\left[1 + \frac{88\lambda}{27} + \frac{4619\lambda^2}{729}\frac{\sqrt{1+\ln(1+\lambda)}}{(1+\lambda)^{2/3}}\right]$$

Tachyon action is particularly interesting; **string theory** enters

When string tension grows, strings become points

Closed string theory, containing gravity, becomes supergravity

Open string theory, containing gauge theory, goes to DBI:

Dirac-Born-Infeld

$$\mathcal{L}_{\mathrm{DBI}} = -\frac{1}{\ell^4}\sqrt{-\det(g_{\mu\nu} + D_\mu\tau D_\nu\tau + \ell^2 F_{\mu\nu})}$$

$$\underset{\tau\equiv 0}{=} -\frac{1}{\ell^4}\sqrt{1-\ell^4(E^2-B^2)-\ell^8(E\cdot B)^2} = \frac{1}{2}(E^2-B^2)+\frac{1}{2}\ell^4(E\cdot B)^2+\ldots$$

$$\ell^2 = 1/T = 2\pi\alpha'$$

Physics in these functions:

Z<<1 UV asymptotic freedom, small λ

Z>>1 IR confinement

$$\lambda(z) = \frac{1}{b_0 \log(1/\Lambda z)} + \cdots$$

grows towards IR

$$\lambda_h = \lambda(z_h)$$

parameter!

$$\tau(z) = m\,(\log\frac{1}{\Lambda z})^{-\frac{3}{2b_0}}\,z + \langle\bar{q}q\rangle\,(\log\frac{1}{\Lambda z})^{\frac{3}{2b_0}}\,z^3 + ..$$

$$\tau_h = \tau(z_h)$$

fixes m_q=0

Solve $\dot{A}_0$ from $\frac{\partial L_f}{\partial \dot{A}_0} = \tilde{n}$

$$A_0(z) = \mu + \int_0^z dz\,\dot{A}_0(z) = \mu - nz^2 + ..$$

chemical potential
number density

$$n = \frac{\tilde{n}}{4\pi}s = \tilde{n}\frac{b_h^3}{16\pi G_5}$$

The five functions b(z), f(z), λ(z), τ(z), A_0(z) are obtained as solutions of Einstein's equations shooting from the horizon

Two types of tachyon solutions:
$\tau = 0$: chirally symmetric, no condensate
τ nonzero: chirality broken, nonzero condensate

As in lattice Monte Carlo, particularly time consuming is fixing $m_q = 0$. One has to choose $\tau(z_h)$ properly:

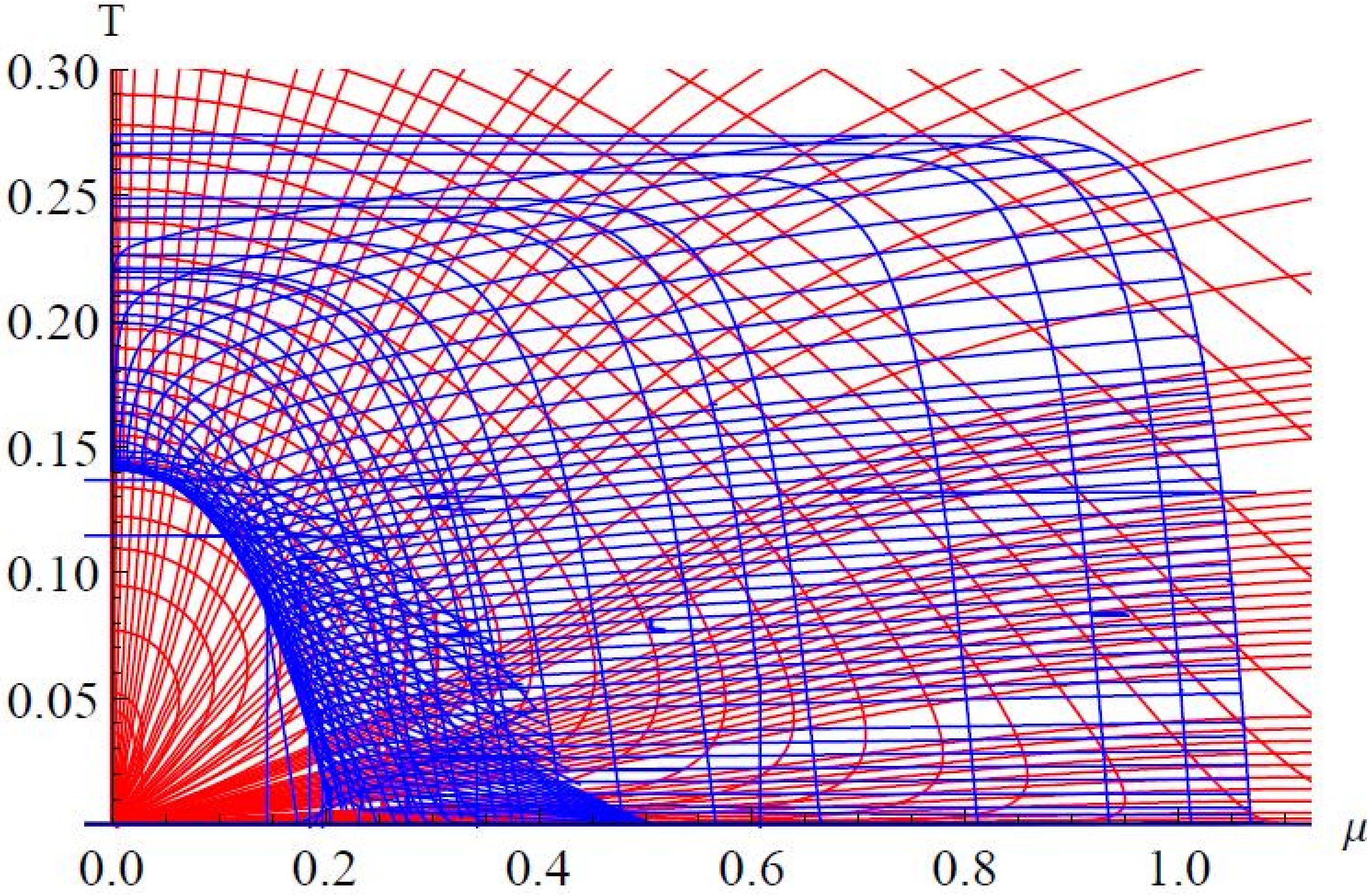
Two parameters: $\lambda_h, \ \tilde{n}$

$$T(\lambda_h, \tilde{n}), \ \ \mu(\lambda_h, \tilde{n})$$

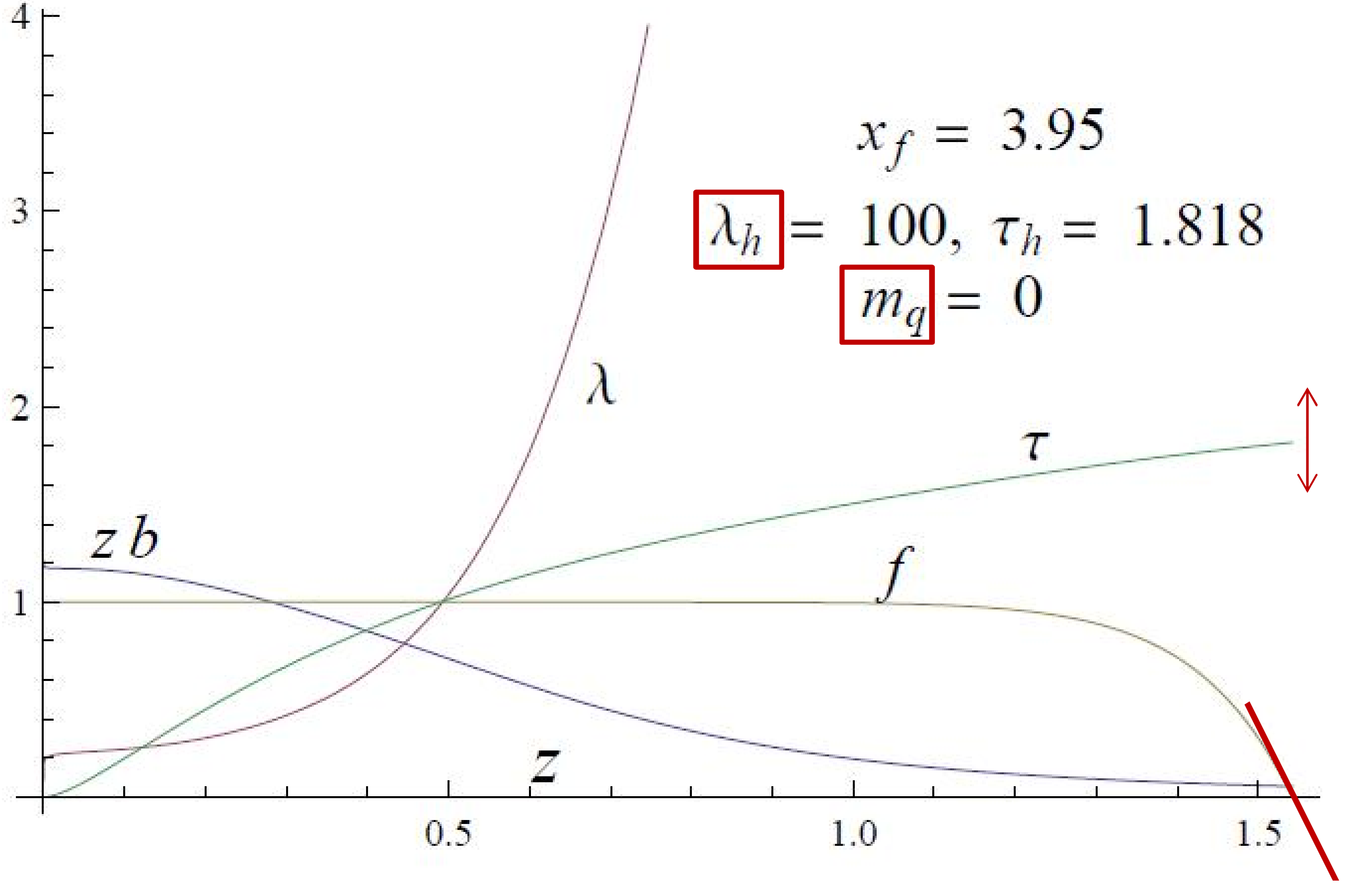
$$dp = sdT + nd\mu \Rightarrow p_s(\lambda_h, \tilde{n}), \, p_b(\lambda_h, \tilde{n})$$

3. Results

$$p(\lambda_h, \tilde{n}) \Rightarrow p(T, \mu)$$



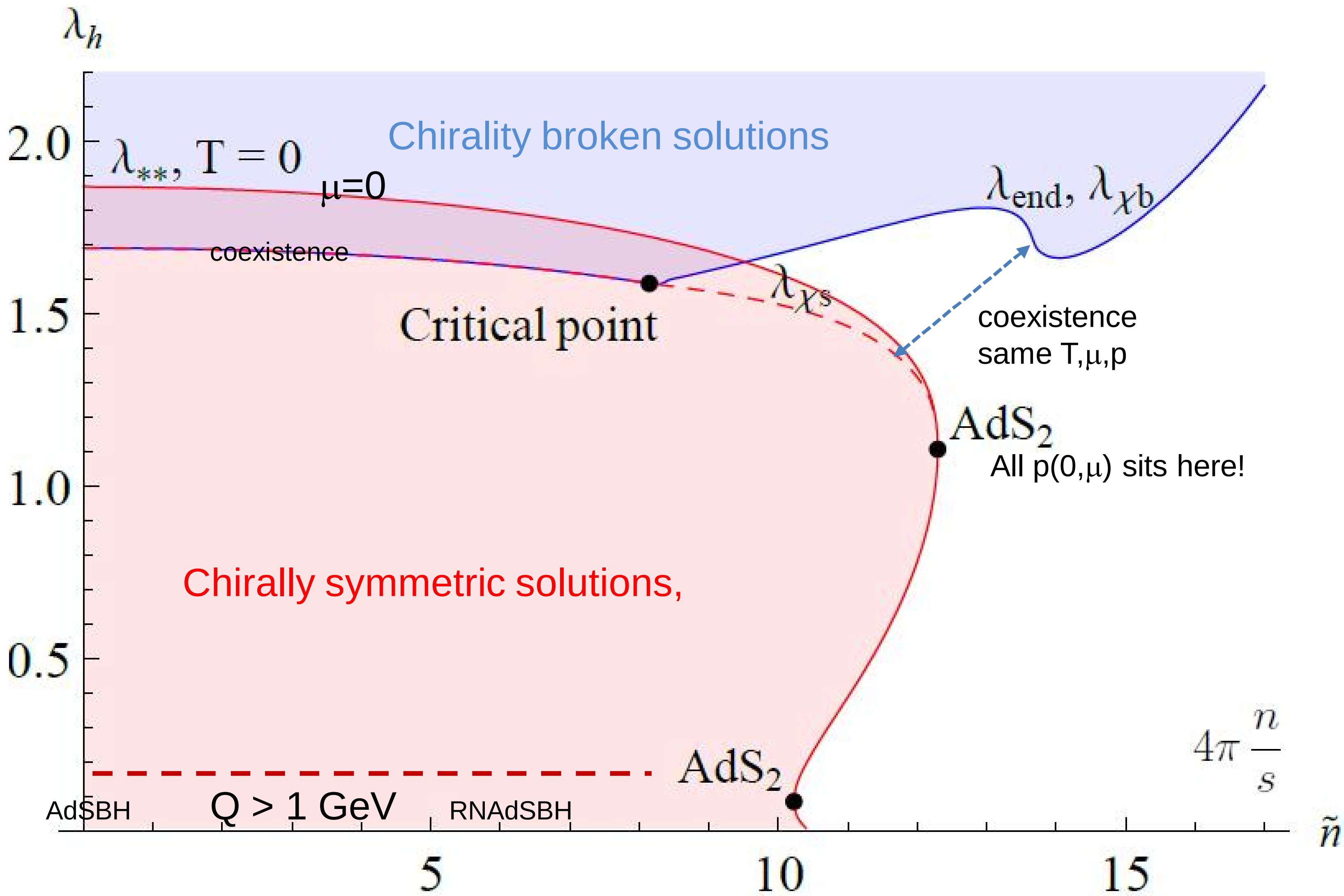
Typical bulk field configuration:



Tune $\tau(z_h)$ so that near $z=0$ $\tau(z) \sim z^3$

$$4\pi T = -f'(z_h)$$

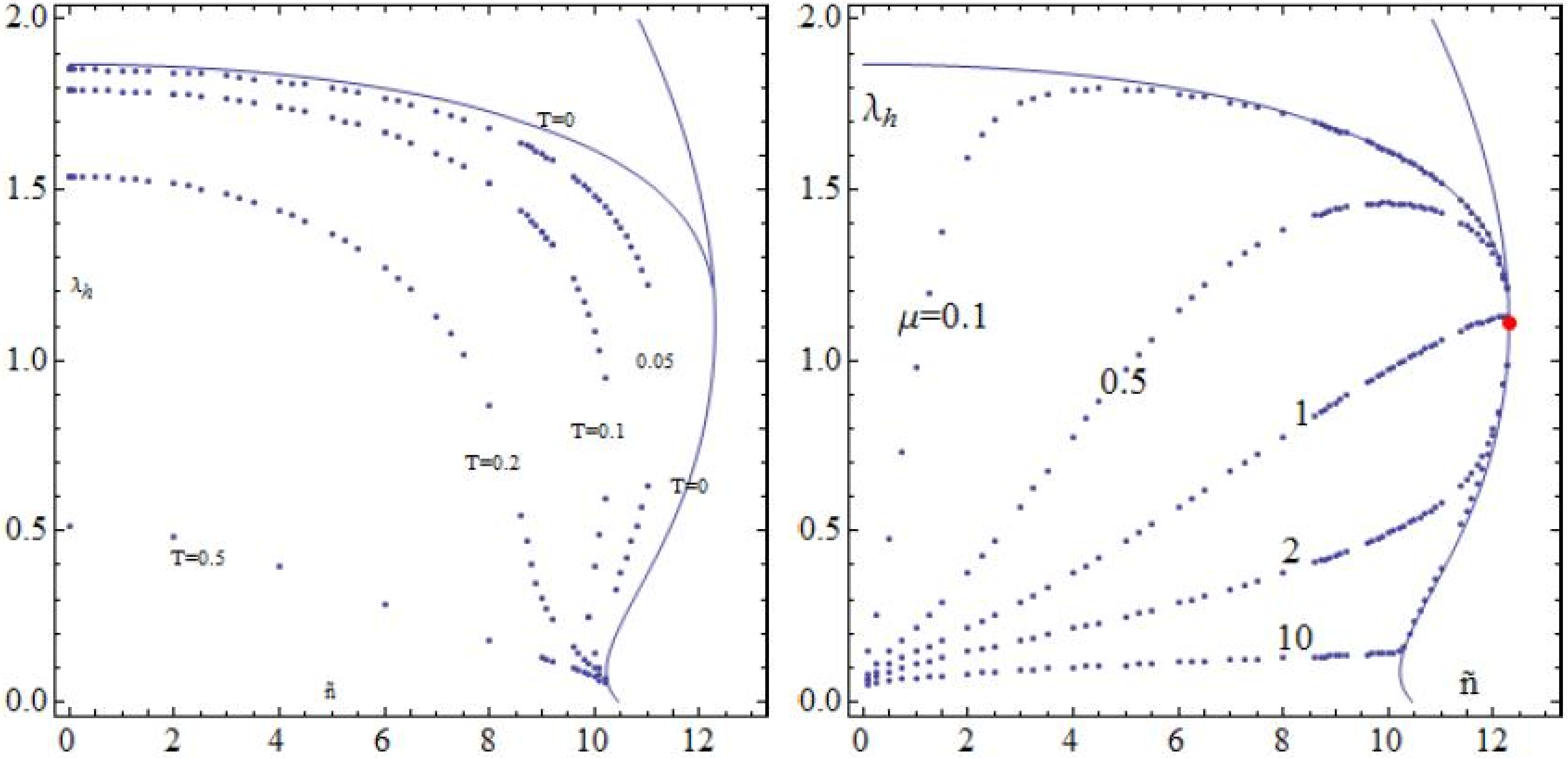
Physical region



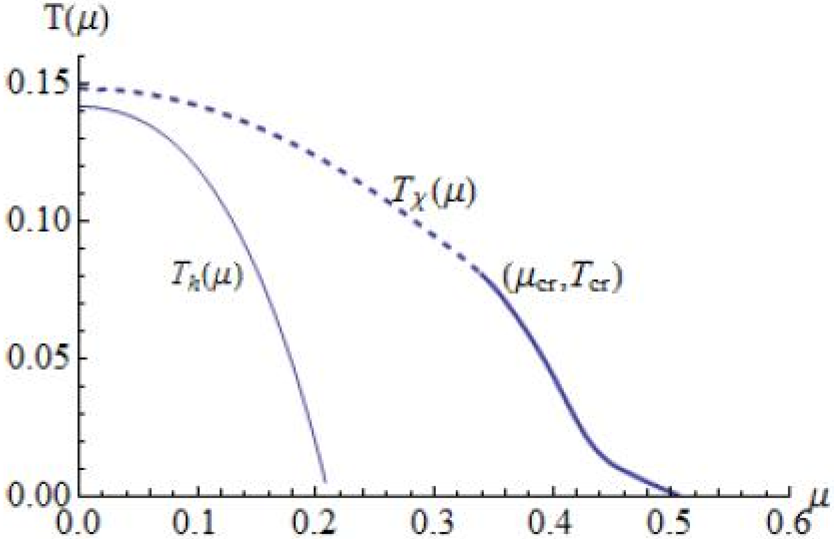
~ charged Reissner-Nordström BHs

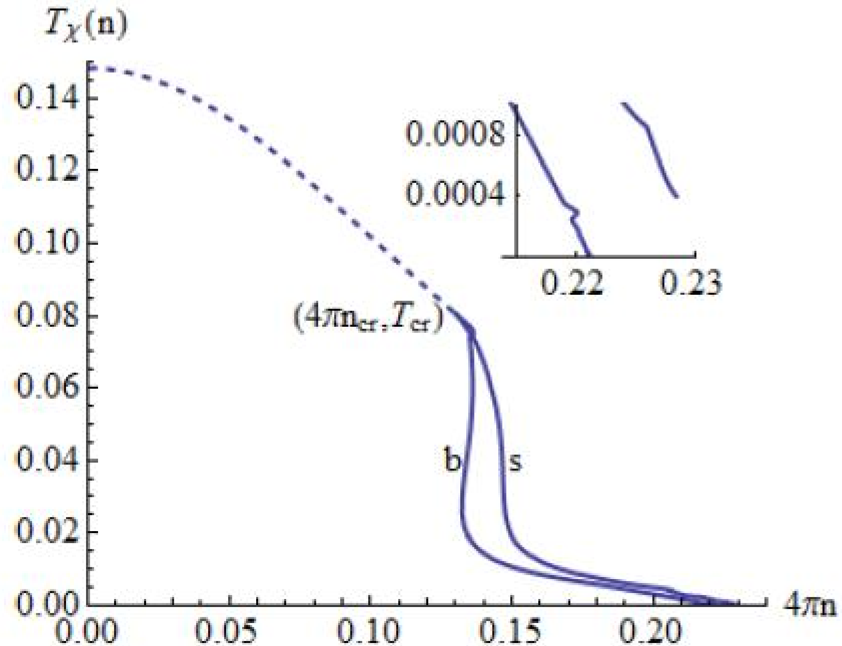
Constant T,μ on $\lambda_h, \tilde{n}$ plane

(chirally symmetric sols)



$$T_\chi(\mu) \Rightarrow T_\chi(n)$$



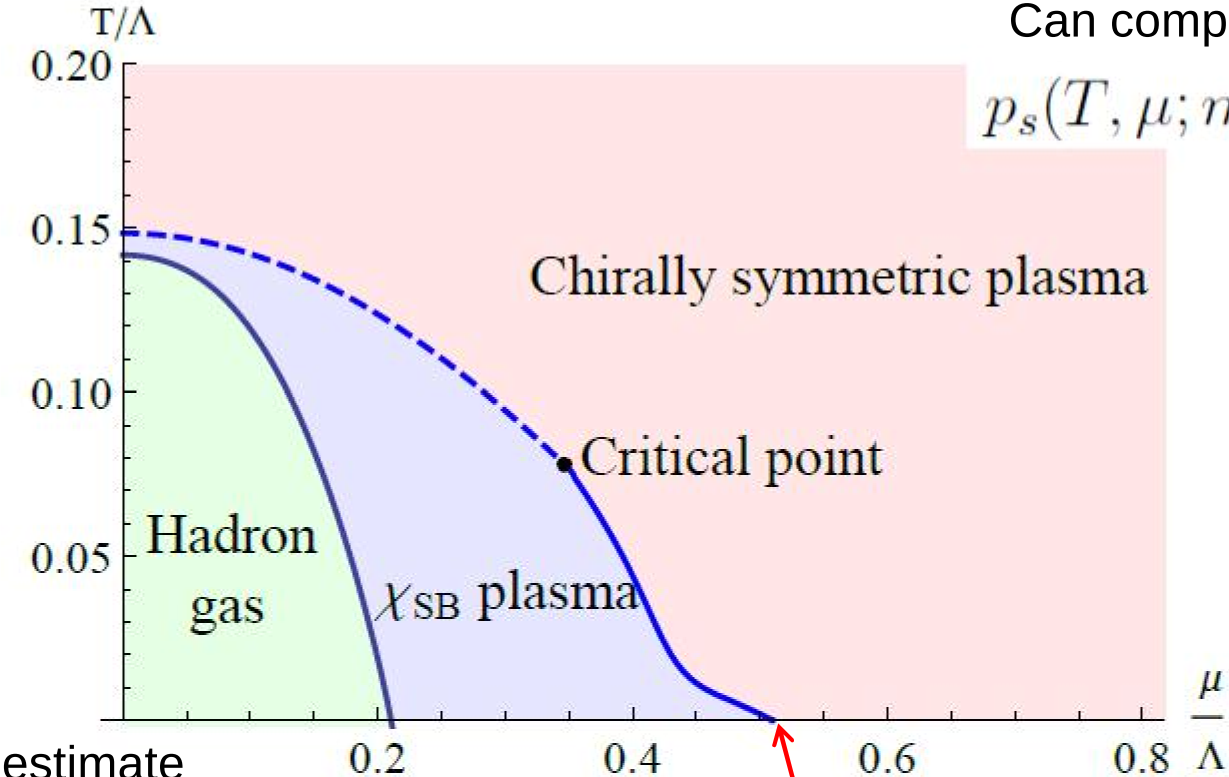


Entropy finite at T=0!!

On coexistence line T, μ, p(T,μ) equal

Can compute

$p_s(T, \mu; m_q = 0)$

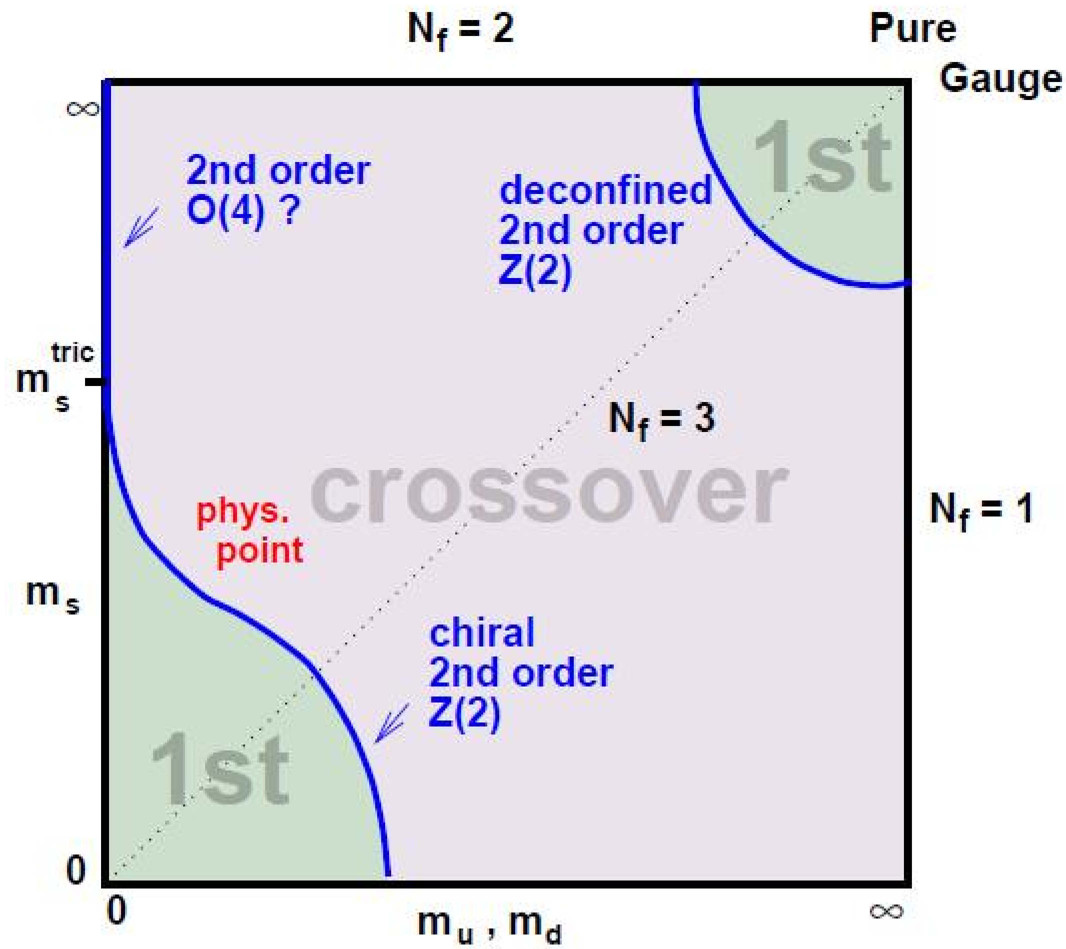


Can only estimate
$p_h \sim 0$

Can compute $p_b(T, \mu; m_q = 0)$

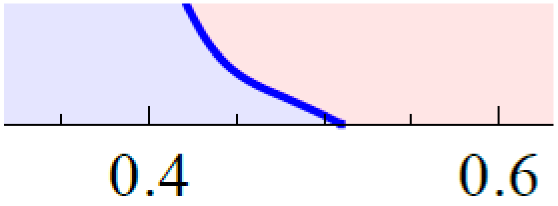
T=0 quantum phase transition!

Order of transition?



N_c=N_f=3 is here!

T=0



chirality broken state at λ_h=infty

T=0
μ = 0.506
p = 0.0016

symmetric state at AdS_2 point

$$T = 0 \Rightarrow f'(z_h) = 0 \qquad ds^2 = -z^2 dt^2 + \frac{dz^2}{z^2} + d\bar{x}^2 \quad \mathrm{AdS}_2 \times R^3$$

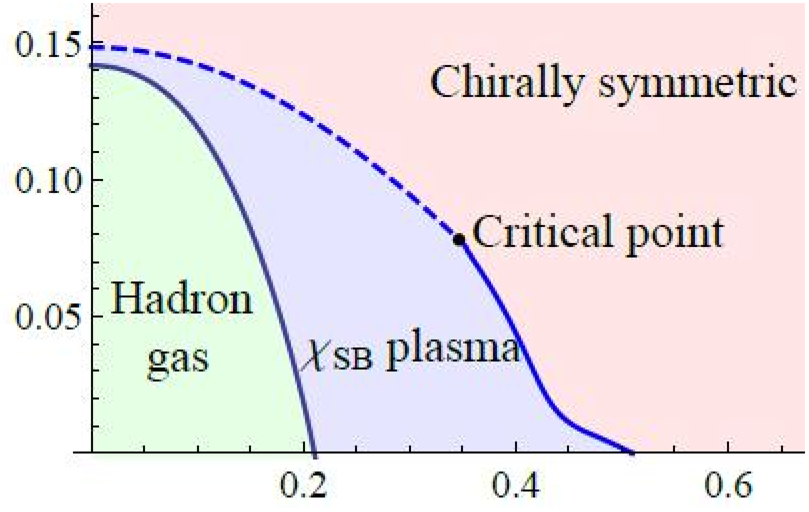
Actually the expansion for getting all T=0 solutions is more complicated:

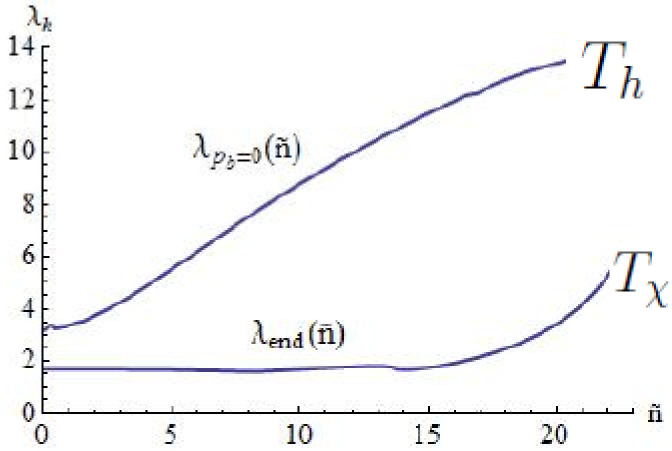
$$f(z) = z^2 + z^{2+\mathrm{noninteger}} + \dots$$

why is entropy finite? No baryon operator, nuclear matter..
No qq operator, color-flavor locking, etc

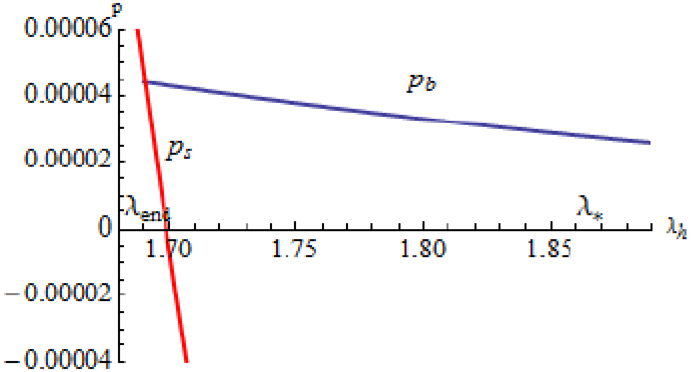
4. Deconfinement

The condition $p_b(\lambda_h, \tilde{n}) = p_{\text{hadr}} \approx 0$



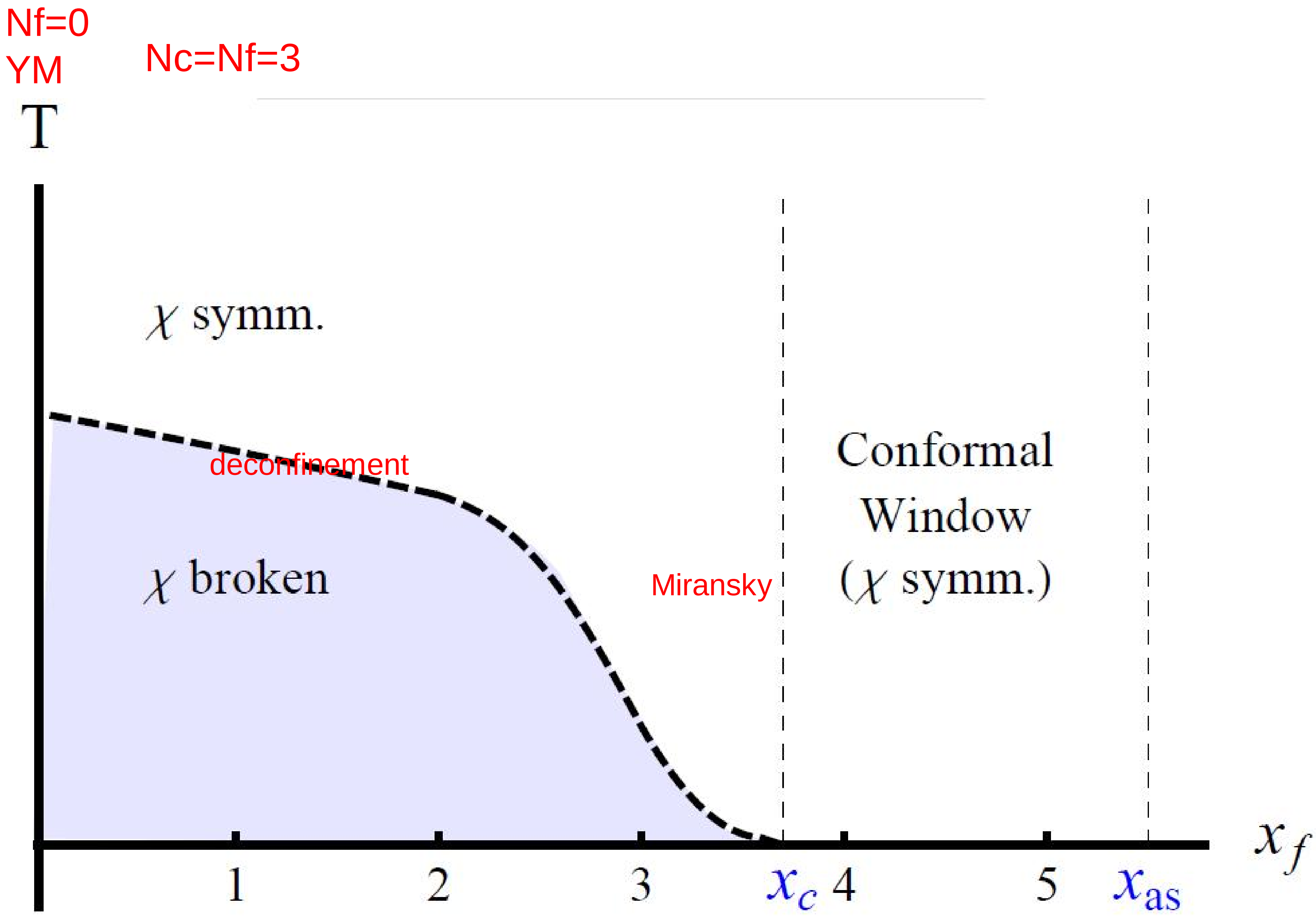


Gives us T_h



$2N_c^2 + \frac{7}{8}4N_fN_c$ vs N_f^2 dofs

Phase diagram for larger N_f: 1210.4516



Now put here a perpendicular μ axis :

5. Conclusions

- This model is an effective theory connecting strong coupling holography to the weak coupling region

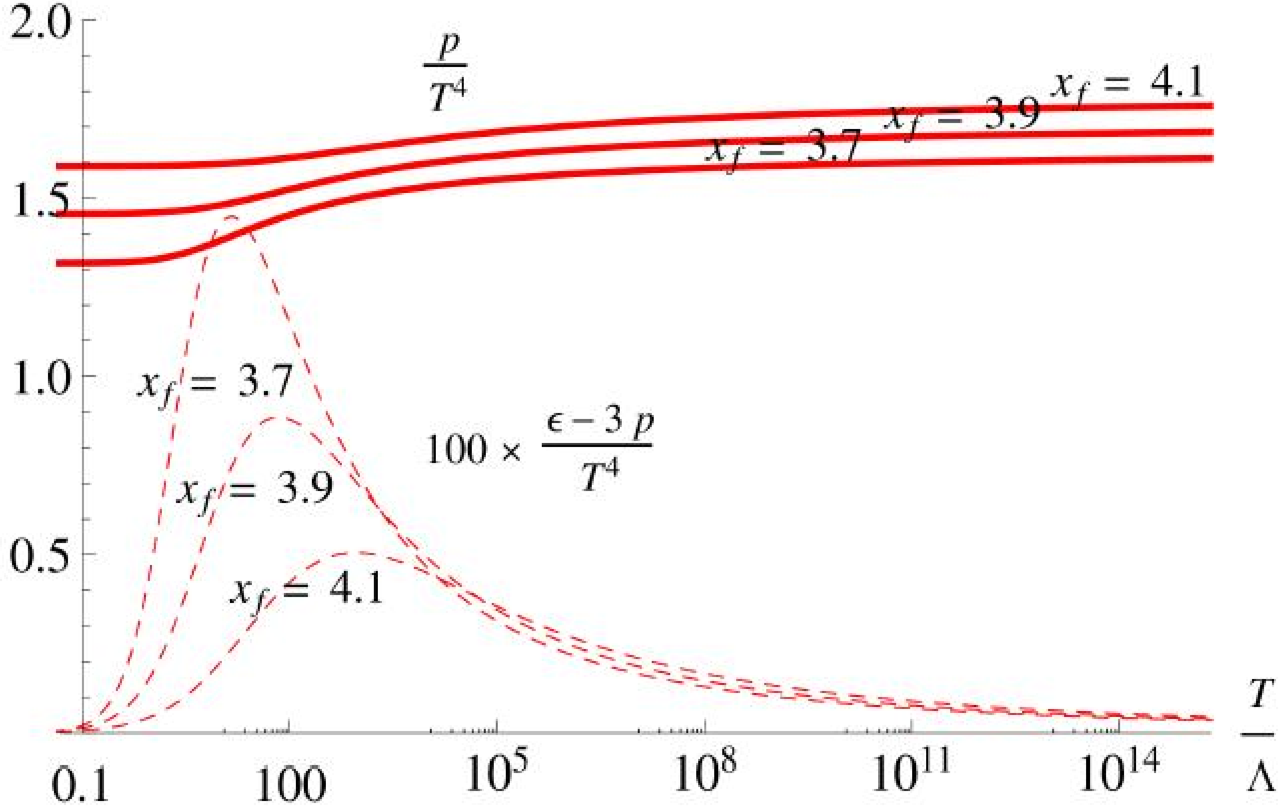
- The subtle interplay between confinement/chiral symmetry and charged black holes with and without tachyons produces a coexistence line with a critical point. Quite impressive

- The potentials $V_g(\lambda)$, $V_f(\lambda)$ are constrained but not completely: predictive power is limited. Offers a **framework, alternatives**

- Not a cheap simple way to solve QCD!

- Much to do: more and better numbers, other potentials, larger Nf, more on T=0, other BSM theories (technicolor!), correlators, magnetic fields, theta vacua, baryons....

Overflow



Normalised to SB at T=infty

Gauge/gravity duality

-4 $\quad\Delta_-\quad$ $4-\Delta_-$

$$\langle \exp\left[i \int d^4x\, \phi_0(x)\, \mathcal{O}(x)\right] \rangle$$

$$\exp\left[i \int d^4x \int dz \sqrt{-g}\, \mathcal{L}_{\mathrm{grav}}[g_{\mu\nu}, \ldots, \phi(x,z)]\right]$$

$$\phi(x,z) = \phi_0(x) z^{\Delta_-} + \langle \mathcal{O} \rangle z^{4-\Delta_-} + \ldots$$

Dofs of gravity ~ area, not volume!

AdS_5 has boundary at z=0 and scale L

N_c, $g^2 N_c$ large