Notes for lectures by K. Kajantie, Jyväskylä 18-22.8.08 1-

1. QCD & $\mathcal{N}=4$ SYM

Known to everybody:

broken by m_q $=2,3$

flavor $SU(N_f)$

$\nearrow \psi_{fis}$

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_f \bar{\psi}_f \, i\gamma^\mu(\partial_\mu + igA_\mu)\psi_f$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$$

$$[T_a, T_b] = i f_{abc} T_c$$ exact local color $SU(N_c)$

$$\begin{cases} \text{fundamental rep} & \psi^i \quad i = 1,\dots,N_c & \text{spinor} \\ \text{adjoint rep} & a = 1,\dots,N_c^2-1 & \text{vector} \end{cases}$$

$$S = \int d^dx \, \mathcal{L}$$

$\uparrow$ flat space, $dx^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ $\quad \eta_{\mu\nu} = (-+++\dots)$

Classically: $S = \int d^dx \, (\partial A + gA^2)^2$ $\qquad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \Rightarrow \frac{e^2}{\hbar}$ is dim.

dim. $\hbar = x^d \left(\frac{1}{x^2}A^2 + g^2A^4 + \dots\right)$

$\hbar = x^{d-2}A^2 \qquad \hbar = g^2 x^d \cdot \frac{\hbar^2}{x^{2d-4}}$

$g^2\hbar = x^{d-4} = \left(\frac{\hbar}{p}\right)^{d-4} = 1$ for $d=4$

Quantize theory, physics is in expectation values

$$\langle \varphi(x_1) \dots \varphi(y_n) \rangle$$

$$Z[J(y)] = e^{iW[J(y)]} = \int \mathcal{D}\varphi(x) \, e^{iS[\varphi(x)] + i\int d^4x \, J(x)\varphi(x)}$$

$\uparrow$ set of fields $\quad x = (t, \bar{x})$

Thermal:

$$Z = e^{-\frac{F(T)}{T}} = e^{p\frac{V}{T}} = Z[J=0, \text{periodic}] = \int \mathcal{D}\varphi(\tau, x) \, e^{-\frac{S_E}{\hbar}[\varphi(\tau,\bar{x})]}$$

$$\varphi(0,\bar{x}) = \varphi(\beta\hbar, \bar{x})$$

Some conventions:

Minkowski $\leftrightarrow$ Euclidean:

$$\tau = i t \qquad x_\mu x^\mu = x_0^2 - \vec{x}^2 \to -[x_4^2 + \vec{x}^2]$$

$$x_4 = i x^0 \qquad x^0 = -i x_4$$

$$\partial_0 \to i\partial_4 \qquad D_0 \to i D_4 \qquad A_0 = i A_4$$

$$e^{iS[x(t)]} = e^{i\int dt\, \ldots} = e^{i\int dt\left[\frac{1}{2}\dot{x}^2 - \ldots\right]}$$

$$= e^{-\int d\tau\, \frac{1}{2} \ldots + \ldots} = e^{-S_E[x(\tau)]}$$

Mostly $-$ $\leftrightarrow$ Mostly $+$

$+---$ $\leftrightarrow$ $-+++$ $\qquad \eta_{\mu\nu} \leftrightarrow -\eta_{\mu\nu} \qquad g_{\mu\nu} \leftrightarrow -g_{\mu\nu}$

$$\mathcal{L}[\varphi] = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi) \longrightarrow -\frac{1}{2}\eta \ldots \varphi \ldots V(\varphi)$$

$$\Box + m^2\varphi = 0 \longrightarrow (-\Box + m^2)\varphi = 0$$

$$\mathcal{L}[A] = -\frac{1}{4}F_{\mu\nu}^2 = -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} \longrightarrow -\frac{1}{4}F_{\mu\nu}^2$$

$$x^2 = x_\mu x^\mu = x_0^2 - \vec{x}^2 \longrightarrow -x^2 = -x_0^2 + \vec{x}^2$$

$$\Gamma = g\,\partial g \longrightarrow \Gamma$$

$$R^\mu{}_{\nu\alpha\beta} = \partial\Gamma + \Gamma^2 \longrightarrow R^\mu{}_{\nu\alpha\beta}$$

$$R_{\mu\nu\alpha\beta} = \qquad \to -R_{\mu\nu\alpha\beta}$$

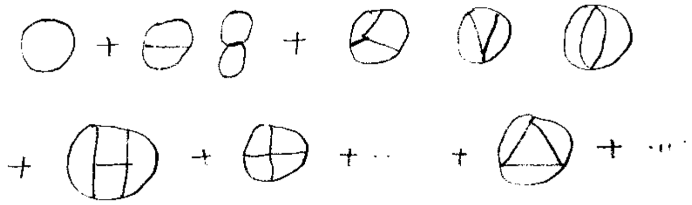
$$R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu} \to R_{\mu\nu}$$

$$R = g^{\mu\nu}R_{\mu\nu} \to -R$$

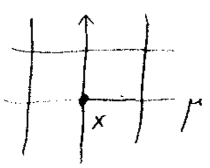
$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} \longrightarrow R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} - \Lambda g_{\mu\nu}$$

Regulate maintaining local GI:

— $\overline{MS}$ $\quad d = 4-2\varepsilon \quad$ expand in g $\qquad$ mass scale → $\mu^{2\varepsilon}\int d^{4-2\varepsilon}x$



— Lattice



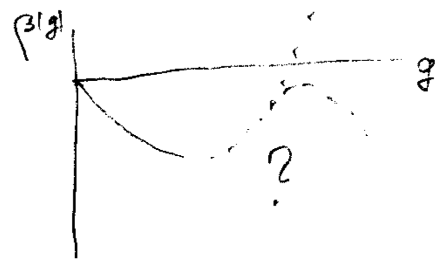
$\frac{1}{\text{mass}}$ scale

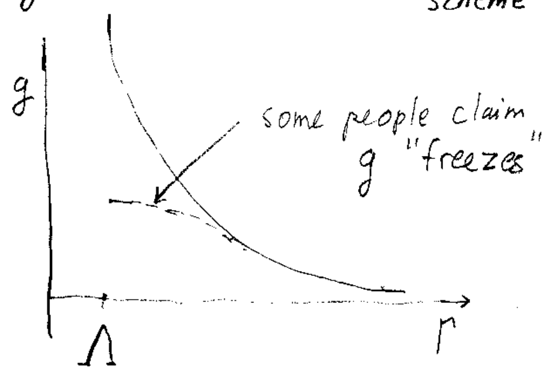
$$U_\mu(x) = e^{i g a A_\mu(x)}$$

$$\Rightarrow \quad \mu \frac{\partial g}{\partial \mu} \equiv \beta(g) = -\beta_0 g^3 - \beta_1 g^5 - \beta_2 g^7 - \beta_3 g^9 - \dots$$

Scheme independent, (β_0, β_1)
same for MS, $\overline{MS}$, latt,

$$\int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu} = \int_{g_1}^{g_2} \frac{dg}{\beta(g)} \quad \Rightarrow \text{ integration constant } \Lambda_{\text{scheme}}$$





running of g breaks <u>conformal invariance</u>

<u>Chiral symmetry</u> is trivially there in $\mathcal{L}_{QCD}$

$$\psi_L \to U_L \psi_L \qquad \psi_R \to U_R \psi_R$$

spontaneously broken to $SU_V(N_f)$ $\Rightarrow$ essential

$$\mathcal{L}_{\mathcal{N}=4\ SYM} = ?$$

A canonical way of deriving this (Divecchia, -th/9803026)

$$S = \int d^{10}x \left\{ -\frac{1}{4} F^{a}_{AB} F^{aAB} - \frac{1}{2} \bar{\lambda}^{a} i \Gamma_{A} \underbrace{(D^{A}\lambda)^{a}}_{\eta^{AB}(\partial_A \lambda^a - g f_{abc} A^b_A \lambda^c)} \right\}$$

$a = 1, \dots, N_c^2 - 1$; $A = 0, 1, \dots, 9$; Γ_A: 10d Dirac

SuSy generators Q^i_a, $\bar{Q}^i_{\dot{a}}$ $i = 1, \dots, \mathcal{N}$

$S \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$ $\mathcal{N}=1$ supersymm. theory in 10d

$8 = d-2$ bosonic dofs $= \frac{1}{4} 2^{\frac{d}{2}} = 8$ fermionic dofs $\quad / * d_A$

Inv. under
$$\begin{cases} \delta A^a_A = \frac{i}{2}(\bar{\lambda}^a \Gamma_A \xi - \bar{\xi} \Gamma_A \lambda^a) \\ \delta \lambda_a = \sigma_{AB} F^{AB}_a \xi \qquad \delta \bar{\lambda}_a = \dots \end{cases}$$

spinor reps. of $O(d)$: $(x_1^2 + \dots + x_d^2) = (\gamma_1 x_1 + \dots + \gamma_d x_d)^2$, $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. Find a $2^{\frac{d}{2}}$-dim rep for this = spinor

Do dimensional reduction:

$$x^A = (\underbrace{x^0, \vec{x}}_{x^\mu}, \underbrace{x^4, \dots, x^9}_{\text{fields do not depend on these } x^i})$$

$$\Rightarrow F^a_{ij\,=4,\dots,9} = g f_{abc} A^b_i A^c_j \rightarrow \text{get 6 adjoint scalars } \phi_i$$

$$F_{\mu i} = \partial_\mu A_i - \cancel{\partial_i A_\mu} - g A_\mu A_i \sim D_\mu A_i$$

$$\Rightarrow S[A^a_\mu, \phi^a_i, \psi^a, \bar{\psi}^a] = \int d^4x \left[-\frac{1}{4} F^2_{\mu\nu} + \frac{1}{2}(D_\mu \phi_i)^2 - \frac{1}{2} \bar{\psi}^a i \not{D} \psi^a - \bar{\psi} \phi \psi - \phi^4 \right]$$

S 1		1
S $\frac{1}{2}$		4
S 0	$\mathcal{N}=4$	6
S $-\frac{1}{2}$	susy	4
S -1		1
		8b+8f dofs

This theory is said to have identically $\beta(g) = 0 \Rightarrow \frac{\partial g(\mu)}{\partial \mu} = 0 \Rightarrow g$ = number

$$\begin{cases} g \ll 1 & \text{weakly coupled, pert theory} \\ g \gg 1 & \text{strongly " , AdS/CFT} \end{cases}$$

$(g^2 N_c \gg 1)$

No "hadrons"! No energy gap!

Conformal invariance

Invariance group of Maxwell

- Lorentz 1892: Lorentz $O(1,3)$ + Translations

$x^{\mu\prime} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ preserves $-t^2 + \bar{x}^2$

$$e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \qquad 6 = \frac{d(d-1)}{2} \quad + \quad 4 \quad = 10 \text{ param.}$$

(Poincaré group)

- Cunningham-Bateman 1909: also
  - Dilatations $D: x^{\mu} \to \lambda x^{\mu}$ 1
  - Special conf. transformations K_{μ}: "translation of inversion, $\frac{1}{x'} \sim \frac{1}{x} + b$"

$$x^{\mu} \to \frac{x^{\mu} + b^{\mu}x^2}{1 + 2x\cdot b + b^2 x^2} = x'^{\mu} \qquad 4$$

with indices $\Rightarrow$ take a square

$$\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} + b^{\mu} \qquad \frac{1}{x'^2} = \frac{1}{x^2} + 2\frac{b\cdot x}{x^2} + b^2$$

Total $10 + 5 = 15 = \frac{d(d-1)}{2}$ for $d = 6$ $\Rightarrow$ the group is $O(2,4)$

<u>Conformal algebra</u>: (cf ang. mom $[L_i, L_j] = i\varepsilon_{ijk}L_k$)

mass dimension		P_{α}	$M_{\alpha\beta}$	D	K_{α}
1	P_{μ}	0	$i(\eta_{\alpha\mu}P_{\beta} - \eta_{\beta\mu}P_{\alpha})$	$-iP_{\mu}$	$2i(\eta_{\mu\alpha}D + M_{\mu\alpha})$
0	$M_{\mu\nu}$		$i(\eta_{\mu\beta}M_{\nu\alpha} + \dots)$	0	$i(\eta_{\nu\alpha}K_{\mu} - \eta_{\mu\alpha}K_{\nu})$
0	D	antisymm		0	$-iK_{\alpha}$
−1	K_{μ}				0

Generalisations:

Have to know reps of $\Lambda(4) \sim SL(2,C)$

$M\psi_a \quad M^*\bar{\psi}_{\dot{a}} \quad M, M^*$ not equivalent

Braun et al 0806.2531

Poincare → SuSy algebra

Mass dim:			P_α	$M_{\alpha\beta}$	Q_b	$\bar{Q}_{\dot{b}}$
1	4	P_μ	0	$i\eta P$	0	0
0		$M_{\mu\nu}$	$-i\eta P$	$i\eta M$	$\sigma_{\mu\nu} Q$	$\bar{Q}\bar{\sigma}_{\mu\nu}$
$\frac{1}{2}$	2	Q_a	antis		0	$2\sigma^\mu_{a\dot{b}} P_\mu$
$\frac{1}{2}$	2	$\bar{Q}_{\dot{a}}$			symm	0

key relation!

can have diff # of bos & ferm generators: have to count states

Conformal → Superconformal algebra

superanalogue for D, K_μ = ?

$$\underset{8}{Q_a^i}, \underset{8}{\bar{Q}_{\dot{a}}^i} \qquad \underset{8}{\bar{S}_{\dot{a}}^i} \ \underset{8}{S_a^i} \qquad i = 1, \dots, 4 = N$$

mass dim: $+\frac{1}{2}$ (for Q, $\bar{Q}$), $-\frac{1}{2}$ (for $\bar{S}$, S)

$$[Q, K] \sim \sigma S$$

see d'Hoker , sect. 3.3
Lyng Petersen, " 3.6

$$\{Q_a^i, \bar{Q}_{\dot{a}}^j\} = \delta_{ij}\, \sigma^\mu_{a\dot{a}} P_\mu \Rightarrow U(N) \text{ invariance}$$

$SU(4) \sim SO(6)$

$4^2 - 1 = 15 \qquad \frac{6 \cdot 5}{2} = 15$

2. Master formula

4d QFT $\Longleftrightarrow$ 5d classical gravity in AdS

radius L

x $\qquad\qquad$ x, z

often $\quad z = \dfrac{L^2}{r}$

boundary $\qquad$ bulk

fields of N=4 SYM or related

$$Z[J] = e^{iW[J]} = \int \mathcal{D}\varphi \, e^{iS[\varphi(x)] + i\int d^4x \, J(x)\, \varphi(x)}$$

"single trace op" in SYM_4

For any operator $O(x)$ (combination of φ's, like $F^a_{\mu\nu}(x)$)

a generating functional of $\langle O(x_1) \dots O(x_n)\rangle$ is

$$\frac{1}{Z[0]} \int \mathcal{D}\varphi \; e^{iS[\varphi(x)] + i\int d^4x \, \varphi_0(x)\, O(x)} \quad (\Rightarrow \textstyle\sum_i \varphi_i O_i)$$

$\qquad -4 \quad 4-\Delta \quad \Delta$ $\quad$ ✓ have to fix dims by this!

$$\langle e^{i\int d^4x \varphi_0 O}\rangle = e^{iS_{Grav}[\varphi(x,z),\ \varphi(x,z\to 0) = z^{4-\Delta}\varphi_0(x)]}$$

$0 = \Delta - 4 + 4 - \Delta$

mostly written in Euclidian form!

in the $N_c \gg 1 \quad g^2 N_c \gg 1$ limit

Simplest for the thermal partition fn $\quad \langle e^{\int d^4x_E \varphi_0 O}\rangle$

$$Z[T] = \langle 1 \rangle = e^{-S^E_{Grav}} \qquad = e^{-S_{Grav}[\varphi(x,z)]}$$

before normalisation

Need class gravity

3. Classical gravity (or weak field quantum)

$m=0 \quad s=1 \qquad \overset{SU(N_c)\text{ matrix}}{A_\mu(x)} \qquad \int d^dx \left(\frac{-1}{4} F^{a2}_{\mu\nu}\right) \qquad$ gauge inv.

$\;\;\, =0 \quad s=2 \qquad g_{\mu\nu}(x) \qquad \int d^dx \sqrt{-g}\, R \qquad$ inv under $x^\mu \to x'^\mu(x)$

Given:

coordinates x: $(t\ x\ y\ z)$ $\quad (t\ r\ \theta\ \varphi)$

$(t\ x\ y\ \vartheta\ z) \quad (\tau\ \eta\ x^2\ x^3\ z)$

metric

$$\begin{array}{cc} & \begin{array}{cccc} t & r & \theta & \varphi \end{array} \\ g_{\mu\nu} = & \begin{pmatrix} -1 & & & 0 \\ & \frac{a(t)}{1-kr^2} & & \\ & & a(t)\, r^2 & \\ 0 & & & a(t)\, r^2 \sin^2\theta \end{pmatrix} \end{array} \qquad \text{FRW}$$

often diagonal

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \qquad g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu \qquad g = \det g_{\mu\nu}$$

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} V^\alpha \qquad V'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu} V_\alpha \qquad g'_{\mu\nu} = \cdots \quad g_{\alpha\beta}$$

Covariant derivative $\quad D_\mu = \partial_\mu + ig A_\mu$

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \underbrace{\Gamma^\mu_{\nu\alpha}}_{\text{Christoffel}} V^\alpha \;\to\; \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \nabla_\beta V^\alpha$$

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\alpha} \left[\partial_\nu g_{\lambda\alpha} + \partial_\lambda g_{\nu\alpha} - \partial_\alpha g_{\nu\lambda}\right]$$

$$\nabla_\nu V_\mu = \partial_\nu V_\mu - \Gamma^\alpha_{\nu\mu} V_\alpha$$

To find Γ I would find the eq. for a geodetic line by extremising $\quad L(x^\alpha, \dot{x}^\alpha) = g_{\mu\nu}(x^\alpha)\, \dot{x}^\mu \dot{x}^\nu \qquad \dot{x}^\mu \equiv \frac{dx^\mu}{ds}$

$$\Rightarrow \frac{\partial L}{\partial x^\alpha} - \partial_s \frac{\partial L}{\partial \dot{x}^\alpha} = 0 \;\Rightarrow\; \ddot{x}^\sigma + \underbrace{\Gamma^\sigma_{\mu\nu}}_{\text{read } \Gamma \text{ from the result}} \dot{x}^\mu \dot{x}^\nu = 0$$

$$\Gamma \sim g\,(\partial g + \partial g - \partial g)$$

Riemann $\qquad F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right]$

$$\left[\nabla_\mu, \nabla_\nu\right] V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta$$

$$R^\alpha{}_{\beta\mu\nu} = \underbrace{\partial\Gamma - \partial\Gamma}_{\text{contains } \partial^2 g} + \Gamma\Gamma - \underbrace{\Gamma\Gamma}_{\text{here } (\partial g)^2}$$

Ricci $\qquad R_{\mu\nu} = R_{\alpha\mu\alpha\nu} \equiv g^{\alpha\beta} R_{\alpha\mu\beta\nu} \qquad R = g^{\mu\nu} R_{\mu\nu}$

conventions! $\qquad R_{\mu\nu\alpha\beta} = 0 \quad R_{\mu\nu\alpha\beta} = 0 \qquad$ curvature scalar

$R_{\mu\nu\alpha\beta}$ has

d	$\frac{1}{12} d^2 (d^2-1)$
2	1
3	6
4	20
5	50

indep. comps

One can construct

d	$\frac{1}{12} d(d-1)(d-2)(d+3)$
2	$0 \to 1$
3	3
4	14
5	40 — Kretschmann scalar $R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}$

scalars from $R_{\mu\nu\alpha\beta}$, $g_{\mu\nu}$

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda \overset{V^\mu = \partial^\mu \varphi}{=} \frac{1}{\sqrt{g}}\, \partial_\mu [\text{illegible}]$$

[illegible]

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{g}}\, \partial_\mu \sqrt{g}\, T^{\mu\nu} + \underbrace{\Gamma^\nu_{\mu\lambda} T^{\mu\lambda}}_{=0 \text{ if } T^{\mu\nu} = -T^{\nu\mu}}$$

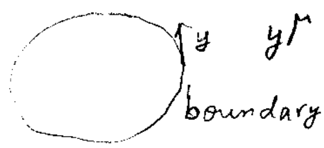
like $F^{\mu\nu}$

Einstein-Hilbert with Gibbons-Hawking boundary term + matter

lots of them in AdS !

$$S = \frac{1}{16\pi G_{d+1}} \left\{ \int_M d^{d+1}x \sqrt{-g}\,(R+2\Lambda) - \int_{\partial M} d^d y \sqrt{-\gamma}\, 2K \right\} + S_m$$

x^M



$$1 = \frac{1}{G_{d+1}}\, x^{d+1} \cdot \frac{1}{x^2}$$

$$\Rightarrow \frac{\mathcal{L}^{d-1}}{G_{d+1}} = \text{dimless}$$

will meet $\dfrac{\mathcal{L}^3}{4\pi G_5} = \dfrac{N_c^2}{2\pi^2}$

induced metric

$$\gamma_{\mu\nu} = g_{MN} \frac{\partial x^M}{\partial y^\mu} \frac{\partial x^N}{\partial y^\nu}$$

extrinsic curvature

$$K = g^{MN} K_{MN}$$

EOM from $\dfrac{\delta S}{\delta g^{\mu\nu}} = 0$: $\quad g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu}$

$$\Big[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$$

$\nabla^\mu T_{\mu\nu} = 0 \qquad \nabla^\alpha g_{\mu\nu} = 0$ since $\nabla^\alpha g_\mu{}^\nu = \nabla^\alpha \delta_\mu{}^\nu = 0$

and this is a tensor

$$AdS_{d+1} : \quad \Lambda = \frac{d(d-1)}{2\mathcal{L}^2} = \frac{6}{\mathcal{L}^2} \text{ for } AdS_5,\ d=4$$

$$R = -\frac{d(d+1)}{\mathcal{L}^2} \qquad R_{\mu\nu} = \frac{-d}{\mathcal{L}^2} g_{\mu\nu} \qquad R + 2\Lambda = \frac{-2d}{\mathcal{L}^2}$$

<u>Sign of Λ is a convention</u> : here $\Lambda_{AdS} > 0$

Sign of T_{00} matters, $S_m = 0 \Rightarrow 8\pi G T_{00}^{AdS} = \Lambda g_{00} = -\Lambda < 0$

$(-+++\,+)$

$$T_{00}^{dS} > 0$$

$$\int d^5x \sqrt{-g}\, e^{-2\Phi}\left[R + 4g^{MN}\partial_M\Phi\partial_N\Phi + \cdots\right] == \int d^5x \sqrt{-g}\left[R - \frac{1}{2}g^{MN}\partial_M\varphi\partial_N\varphi \cdots\right]$$

"String frame" $\quad g_{MN} \to e^{4\Phi/3} g_{MN}$ $\quad$ "Einstein frame"

More fields: $\quad$ + for $\eta = (+ - - - -)$

$$S_{EH} + \int d^dx\sqrt{-g}\left\{-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi) - \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \cdots\right\}$$

$$\Rightarrow \quad R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G\Big[\underbrace{\partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}L[\varphi]}_{T^\varphi_{\mu\nu}} + \underbrace{F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{2}g_{\mu\nu}F^2}_{T^A_{\mu\nu}}\Big]$$

$$\frac{\partial \mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} = 0 \quad\Rightarrow\quad \overbrace{\Box\varphi}^{\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\,g^{\mu\nu}\partial_\nu\varphi} - V'(\varphi) = 0$$

sign $(-\partial_0^2 + \vec\partial^2)\varphi - m^2\varphi$

$V = \frac{1}{2}m^2\varphi^2 \quad V' = m^2\varphi \quad = (\omega^2 - \vec k^2 - m^2)\varphi$

$$\frac{\partial\mathcal{L}}{\partial A^\alpha} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu A^\alpha} = 0 \Rightarrow g^{\mu\alpha}\nabla_\alpha F_{\mu\nu} = J_\nu = 0 \quad \text{Maxwell}$$

↓ if φ charged, e.g.

$= \cdots + \cdots = \cdots = \cdots\left[\cdots(\partial_\nu A_\rho - \partial_\rho A_\nu)\right]$

Strings & 3d gravity → p.10'

4d BH

solves $R_{\mu\nu} = 0$ for a spherically symm ansatz

static! Birkhoff

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$f(r) = g(r) = 1 - \frac{r_s}{r} \qquad r_s = 2G_4M$$

NOT in EOM $R_{\mu\nu} = 0$

$$T_H = \frac{\sqrt{f'(r_s)g'(r_s)}}{4\pi} = \frac{1}{4\pi r_s}$$

Charged BH: Reissner-Nordström

$$f(r) = 1 - \frac{2GM}{r} + \frac{G(q^2+p^2)}{r^2}$$

$$S = \frac{A}{4G_4} = \frac{r_s^2\overbrace{\Omega_2}^{4\pi}}{4G_4} = 4\pi G_4 M^2$$

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{12\, r_s^2}{r^6} \quad \text{true sing at } r = 0$$

(to p. 10!)

Strings & 2d gravity

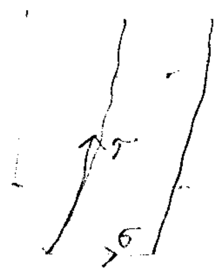
Relativistic ple:

$$S = -mc^2 \int_{\tau_1}^{\tau_2} d\tau = -mc \int_{t_1}^{t_2} dt \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \overset{NR}{=} \text{const} + \tfrac{1}{2} m v^2$$

$$1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2$$

"reparam. invariant" sign OK!

String:



$$\left\{ x^0 = X^0(\tau,\sigma) \right.$$

$$ds^2 = G_{\mu\nu}\, dx^\mu dx^\nu = \underbrace{G_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b}}_{\equiv h_{ab}} d\sigma^a d\sigma^b$$

$$S_{NG} = -T \int d^2\sigma \sqrt{-\det h_{ab}} \equiv S[\dot{X}^\mu, X'^\nu]$$

$$= -T \int d^2\sigma \sqrt{-h}\, \underbrace{\tfrac{1}{2} h^{ab} h_{ab}}_{=1}$$

$$= -\frac{T}{2} \int d^2\sigma \sqrt{-h}\, h^{ab} \left[\eta_{\mu\nu}\, \partial_a X^\mu \partial_b X^\nu - \bar{\Psi}^\mu i \rho^a \partial_a \Psi_\nu\right]_{\text{superstring}}$$

$\Rightarrow$ EOM $\quad T_{ab} = T\left[\partial_a X \cdot \partial_b X - \tfrac{1}{2} h_{ab} h^{cd} \partial_c X \partial_d X\right]$

classical $\quad T^a{}_a = 0$

Much of string theory is what happens
- in quantum theory
- when $\eta_{\mu\nu} \to G_{\mu\nu}(X)$; background indep

//preposterousuniverse.com/grnotes
Sean carroll

Other coords:

$$-f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega^2$$

factor out f by $\frac{dr}{f(r)} = dr^*$

$$\frac{dr^2}{f} = f\left(\frac{dr}{f}\right)^2$$

$$ds^2 = +f(r)\left[-dt^2 + dr_*^2\right] + r^2(r_*)\, d\Omega^2$$

transformation in one step

Further: $t_+ = t + r^*$ $\quad dt = dt_+ - dr_* = dt_+ - \frac{dr}{f(r)}$

$= dt_- + \frac{dr}{f(r)}$

$$ds^2 = f\left[-dt_+^2 + 2\,dt_+\,dr_* \underbrace{- dr_*^2 + dr_*^2}\right] + r^2 d\Omega^2$$

$$\boxed{ds^2 = -\left(1 - \frac{r_s}{r}\right) dt_+^2 + 2\,dt_+\,dr + r^2\, d\Omega_2^2}$$

Eddington-Finkelstein $\quad$ original r!

$ds^2 = 0$ t_s = const

$\left\{\begin{array}{l} dt_+ = 0 \\ \frac{2dr}{dt_+} = 1 - \frac{r_s}{r} \begin{cases} <0 & r<r_s \\ >0 & r>r_s \end{cases}\end{array}\right.$

Pure AdS_5

$\begin{cases} x^\mu \to \lambda x^\mu \\ z \to \lambda z \end{cases}$

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right)$$

pull out one coord^{-2}!
need a scale $\mathcal{L}$ (length)

$$= \frac{r^2}{\mathcal{L}^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{\mathcal{L}^2}{r^2}\, dr^2$$

$r = \frac{\mathcal{L}^2}{z}$

$\begin{cases} x^\mu \to \lambda x^\mu \\ r \to \frac{1}{\lambda} r \end{cases}$

"Pure" means maximal symmetry; AdS_{d+1}:

$$\Lambda = \frac{R}{d(d+1)}$$

$$R_{\mu\nu\alpha\beta} = \#\,(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) = -\frac{1}{\mathcal{L}^2}(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha})$$

$$R_{\nu\beta} = g^{\mu\alpha} R_{\mu\nu\alpha\beta} = \#\,((d+1) g_{\nu\beta} - g_{\nu\beta}) = \#\cdot d\, g_{\nu\beta} = \frac{-d}{\mathcal{L}^2}\, g_{\nu\beta}$$

AdS_5 is the surface

$$-t_1^2 - t_2^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = -\mathcal{L}^2$$

can be parametrised by $t_1 = \mathcal{L}\frac{t}{z}$, $x_1 = \mathcal{L}\frac{y_1}{z}$, $x_2 = \mathcal{L}\frac{y_2}{z}$, $x_3 = \mathcal{L}\frac{y_3}{z}$, $t_2 = -\frac{-\mathcal{L}^2 + \vec{y}^2 + z^2 - t^2}{2z}$

with

$-\infty < t < \infty$ $\quad 0 < z < \infty$

$-\infty < y_i < \infty$

$$\begin{cases} t_2 + x_4 = -\frac{\mathcal{L}^2}{z} \\ t_2 - x_4 = -\frac{\vec{y}^2 + z^2 - t^2}{z} \end{cases}$$

$$\frac{\mathcal{L}^2}{z^2}(-t^2 + \vec{y}^2) - \frac{1}{4z^2}\underbrace{\left[(\mathcal{L}^2 + \vec{y}^2 + z^2 - t^2)^2 - (-\mathcal{L}^2 + \vec{y}^2 + z^2 - t^2)^2\right]}_{4\mathcal{L}^2(\vec{y}^2 + z^2 - t^2)}$$

$$= \frac{\mathcal{L}^2}{z^2}\left[-t^2 + \vec{y}^2 - (\vec{y}^2 + z^2 - t^2)\right] = -\mathcal{L}^2 \quad !!$$

in the space with flat metric

$$ds^2 = -dt_1^2 - dt_2^2 + dx_1^2 + \dots + dx_4^2$$
$$= \frac{\mathcal{L}^2}{z^2}\left(-dt^2 + d\vec{y}^2 + dz^2\right)$$

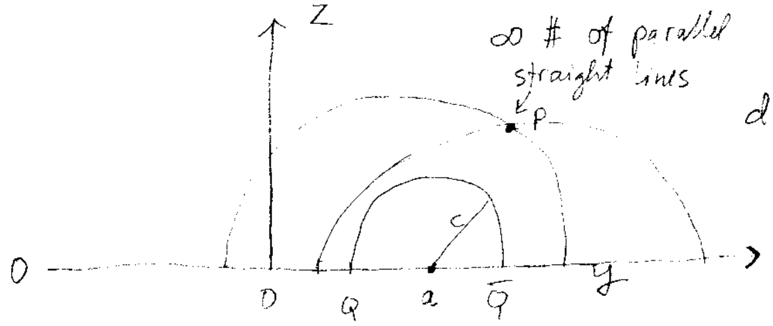
after some work

The surface and thus AdS_5 has the symmetry $O(2,4)$!
= conformal group in 4d !

BG really is $AdS_5 \otimes S_5$

The $N=4$ SUSY's $SO(6)$ rotational symmetry is mapped to rotating S_5 in 6d space!

Poincare plane; non-Eucl. geometry:



Why this?

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dy^2)$$

Geodesics: $S = \int ds = L \int \sqrt{\frac{dz^2 + dy^2}{z^2}} = L \int dz \frac{1}{z}\sqrt{1+y'(z)^2}$

$$= L \int \frac{dy}{z}\sqrt{1+z'(y)^2} \qquad \int dz\, L(y'(z))$$

$$= \int dy\, L(z(y), z'(y)) \qquad \text{Simpler!}$$

$$\overset{=0}{\frac{\partial L}{\partial y}} - \partial_z \frac{\partial L}{\partial y'(z)} = 0 \qquad \frac{\partial L}{\partial y'} = \frac{y'(z)}{z\sqrt{1+y'^2}} = \text{const} = \frac{1}{c} = \frac{1}{z\sqrt{z'(y)^2+1}}$$

$$z^2\left[1 + \left(\frac{dz}{dy}\right)^2\right] = c^2 \qquad \left(\frac{dz}{dy}\right)^2 = \frac{c^2}{z^2} - 1$$

$$\frac{dz}{dy} = \frac{1}{z}\sqrt{c^2 - z^2} \qquad \frac{d\frac{1}{2}z^2}{\sqrt{c^2-z^2}} = dy \Rightarrow \boxed{(y-a)^2 + z^2 = c^2}$$

- In complex coordinates:

$\uparrow z$

$\cdot\ z = x + iy$

$\to x$

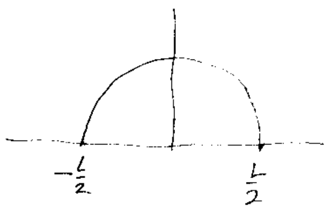
$$\text{th}\frac{\rho}{2}(z_1, z_2) = \left|\frac{z_1 - z_2}{z_1 - \bar{z}_2}\right|$$

$$\rho(z_1, z_2) = \log \frac{|z_1 - \bar{z}_2| + |z_1 - z_2|}{|z_1 - \bar{z}_2| - |z_1 - z_2|}$$

distance between z_1, z_2 in metric $\frac{1}{y^2}(dx^2 + dy^2)$

- $Q\bar{Q}$ energy in [illegible] of Wilson line (Exercise):

$$S = \underbrace{T}_{=\frac{1}{2\pi\alpha'}} (\Delta \text{time}) \cdot \int_{-L/2}^{L/2} dx\, \frac{1}{z^2}\sqrt{1 - \frac{z^4}{z_0^4} + z'(x)^2}$$



4. Master formula, correlators

$4-\Delta \quad \Delta \leftarrow$ mass dims

$$\frac{1}{Z}\int \mathcal{D}\Phi\, e^{-\int d^4x \left[\frac{1}{2} g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi + V(\Phi) + \varphi_0(x)\Phi(x) + \varphi_0(x)\underbrace{\Phi^\Delta}_{\equiv \mathcal{O}(x)} + \right.}$$

quantum field

$$= e^{-\frac{1}{16\pi G_5}\int d^5x \sqrt{-g}\,\frac{1}{2} g^{MN}\partial_M\varphi\,\partial_N\varphi + \frac{1}{2}m^2\varphi^2}$$

pure AdS_5 — needed to fix dimensions! Find $m^2 < 0$!

does NOT fix normalization

$(\Box - m^2)\varphi = 0$

$$\varphi(x, z\to 0) = z^{4-\Delta}\varphi_0(x) \qquad m^2 = \Delta(\Delta-4) \;!$$

For p-form bulk fields $(\Delta-p)(\Delta+p-4) = m^2$

Need mom. space,

$$\varphi(x,z) = \int \frac{d^4k}{(2\pi)^4} e^{ikx}\varphi(k,z)\varphi_0(k) \to z^{4-\Delta}\varphi_0(x)$$

$$\varphi(k, z\to 0) = z^{4-\Delta}(1 + \mathcal{O}(z^2))$$

$$\int d^4x\, \varphi_0(x)\Phi^\Delta(x) = \int \frac{d^4k}{(2\pi)^4}\varphi_0(k)\,\Phi(-k) = \int dk_1\, dk_2\, (2\pi)^4\delta(k_1+k_2)\,\varphi_0(k_1)\varphi_0(k_2)$$

Fact 1: $\int d^5x \sqrt{-g}\left[\frac{1}{2}g^{MN}\partial_M\varphi\partial_N\varphi + \frac{1}{2}m^2\varphi^2\right]$

$$= \int d^5x \frac{1}{2}\sqrt{-g}\left[(-\Box + m^2)\varphi\right]\varphi + \frac{1}{2}\int d^5x\, \partial_N\left[\sqrt{-g}\,\partial^N\varphi\cdot\varphi\right]$$

$= 0$ for soln's of EOM / "on shell" — Only this remains

dissipation, boundary contribution: holography

Fact 2: $\varphi(k,z) \equiv z^2 u(z)$

$$\Rightarrow (\Box - m^2)\varphi = \frac{z^2}{L^2}\left\{z^2 u'' + z u' - [k^2 z^2 + \overbrace{4 + (mL)^2}^{\equiv \nu^2}]\right\}$$

$-k_0^2 + \vec{k}^2$ Standard Bessel eq.

$$\Rightarrow \varphi(k,z) \sim z^2 I_{-\nu}(kz) \quad \text{or} \quad z^2 K_\nu(kz) \qquad \begin{cases} K_\nu(z) = K_{-\nu}(z) \\ I_m = I_{-m} \end{cases}$$

$$k \equiv \sqrt{k^2} \qquad \nu = \sqrt{4 + (mL)^2}$$

Euclidian or spacelike

on shell action

Witten, hep-th/9802150 3700 citations

Basically Gubser-Klebanov-Polyakov, hep-th/9802109, 3200 citations 2008

$$\frac{1}{16\pi G_5}\int d^5x\ \partial_N\Big[\underbrace{\sqrt{-g}}_{\frac{L^5}{z^5}}\ \underbrace{g^{NM}}_{g^{zz}=\frac{z^2}{L^2}}\partial_M \varphi\cdot\varphi\Big]$$

φ: normalised soln of $(\Box - m^2)\varphi = 0$

$$= \frac{1}{16\pi G_5}\int\frac{d^4k}{(2\pi)^4}\left\{\Big|_{\varepsilon}^{\infty\to z_0}\frac{L^3}{z^3}\varphi'_z(k,z)\underbrace{\varphi(-k,z)}_{=\varphi^*(k,z)} + \int_\varepsilon^\infty dz\,\frac{L^3}{z^3}(-k^2)\varphi(k,z)\varphi(-k,z)\right\}$$

- need convergence for $z\to\infty$!
- for finite T there will be upper limit $z_0 \sim \frac{1}{\pi T}$

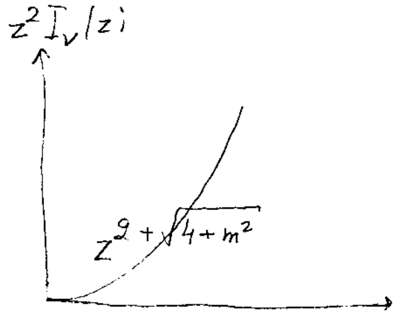
"compactify μ-directions so that there is no boundary there" !

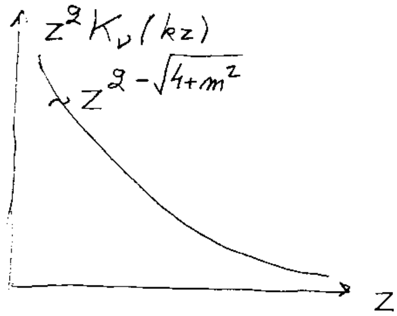
Two solutions:

Compare J^μ: Noether current in scalar ED: $j_\mu = \frac{e}{2i}(\varphi^*\partial_\mu\varphi - \partial_\mu\varphi^*\varphi)$

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu\frac{1}{\Gamma(1+\nu)}\left[1+\frac{1}{1+\nu}\left(\frac{z}{2}\right)^2+\dots\right.$$

$$K_\nu(z) = \frac{1}{2}\left(\frac{2}{z}\right)^\nu\left[\Gamma(\nu) - \Gamma(\nu-1)\frac{z^2}{4}+\dots\right] + (-)^{1+\nu}\log\frac{z}{2}\cdot I_\nu(z) + \dots$$



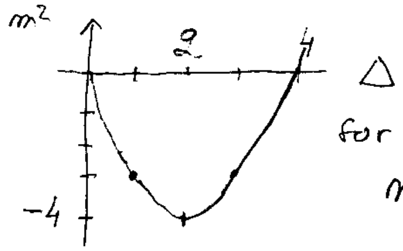


Have to choose $z^2K_\nu(kz)$ $(\to 0$ at $z\to\infty)$ $\quad \nu = |\Delta - 2|$

with normalisation $\quad C = \frac{2}{\Gamma(\nu)}\left(\frac{k}{2}\right)^\nu$

$$\varphi(k, z\to 0) = 1\cdot z^{4-\Delta} = C\cdot\frac{\Gamma(\nu)}{2}\left(\frac{kz}{2}\right)^{-\nu}\cdot z^2 = C\,z^2K_\nu(kz)$$

$$\Rightarrow 4-\Delta = 2-\sqrt{4+m^2} \Rightarrow \sqrt{4+m^2} = \Delta - 2 \Rightarrow m^2 = \Delta(\Delta-4)$$



doubly valued! $m^2\to\Delta_+,\Delta_-$

for $0<\Delta<4$

$m^2<0$!

$$S_{Grav} = \frac{L^3}{16\pi G_5} \int \frac{d^4k_1}{(\ )^4} \frac{d^4k_2}{(\ )^4} (2\pi)^4 \delta(k_1+k_2)\, \varphi_0(k_1)\varphi_0(k_2) \cdot \overset{\swarrow ?}{\frac{1}{2}} \left. \frac{\varphi(k_1,z)\,\varphi_z'(k_2, z)}{z^3} \right|_{z=\varepsilon}$$

$$\Rightarrow \langle \mathcal{O}(k_1)\,\mathcal{O}(k_2)\rangle = \frac{L^3}{16\pi G_5} \left. \frac{\varphi(k_1,z)\,\varphi_z'(k_2, z)}{z^3}\right|_{z=\varepsilon} \cdot (2\pi)^4 \delta^4(k_1+k_2)$$

$$\varphi(k, z) = \frac{2}{\Gamma(\nu)} \left(\frac{k}{2}\right)^\nu z^2 K_\nu(kz)$$

$\Delta \quad 4 \quad \Delta-4$

$\mathcal{O}(x) = \int d^4k \ \mathcal{O}(k)$

Ex $\quad \Delta = 3 \quad \nu = 1 \quad \varphi(k,z) = k z^2 K_1(kz)$

$$\begin{cases} \varphi(k,z) = z + \frac{k^2}{4}\left(2\log\frac{kz}{2} + 2\gamma - 1\right) z^3 + \mathcal{O}(z^5) \\ \frac{d}{dz}\varphi(k,z) = 1 + \frac{k^2}{4}\left(6\log\frac{kz}{2} + 6\gamma - 1\right) z^2 + \mathcal{O}(z^4) \end{cases}$$

$$\varphi_k \cdot \varphi_{-k}' = z + z^3 \cdot 2k^2\left(\log\frac{kz}{2} + \gamma - \frac{1}{4}\right) + \cdots \quad + i \ast (\cdots)$$

$$\langle \mathcal{O}(k_1)\,\mathcal{O}(k_2)\rangle = \left[\frac{\sim 1}{\varepsilon^2} + \frac{N_c^2}{8\pi^2}\, 2k^2\left(\log\frac{k\varepsilon}{2} + \gamma - \frac{1}{4}\right)\right] \cdot (2\pi)^4 \delta^4(k_1+k_2)$$

Son-Starinets th/0205051 : $\frac{N_c^2}{8\pi^2} \frac{1}{\varepsilon^2} \quad k^2 \log k + \cdots$

$\Rightarrow$ p. 16' for finite T & Minkowski correlators

$$\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle \sim \int d^4k_1 d^4k_2\, \delta^4(k_1+k_2)\, k^2 e^{ik_1x_1 + ik_2x_2}$$

$$\sim \int d^4k\, k^2 e^{ik(x_1-x_2)} \sim \frac{1}{|x_1-x_2|^6}$$

Dimensionally

A particularly interesting $\Delta = 3$ operator is $\bar{q}q$:

$$\int d^4x \; m_q \, \bar{q}q(x)$$

$$\phi_0(x) = m_q \qquad \phi_0(x) = m_q$$

$$\varphi(x_0, z) = z\, m_q + \langle \bar{q}q \rangle \, z^3 + \dots$$

the full gravity dual would determine $\langle \bar{q}q \rangle$

"non-normalisable mode" has less powers of z when $z \to 0$ if $\Delta > 2$

$$= z^{4-\Delta} \varphi_0^{\sigma} + z^{\Delta} \langle \mathcal{O} \rangle \frac{2\Delta - 4}{2\sqrt{4+m^2}}$$

"normalisable mode"

Klebanov – Witten

$\Rightarrow$ Holographic model for hadrons EKSS ph/0501128

$$\int d^4x \; \bar{\psi} \gamma^\mu \dots A_\mu$$

corresponding conserved current (some global symmetry)

$A_\mu^a(x, z)$ $\int d^5x \; \dots$

gauge inv in bulk theory

$$\int d^4x \; T_{\mu\nu} \cdot \delta g^{\mu\nu}$$

$$\partial_\mu T^{\mu\nu} = 0$$

5. 5d flat black hole (in many forms)

$$ds^2 = \frac{L^2}{z^2}\left[-\left(1-\frac{z^4}{z_0^4}\right)dt^2 + d\vec{x}^2 + \frac{dz^2}{1-\frac{z^4}{z_0^4}}\right]$$

inside BH

$0 \quad z_* \quad z_0$

$$R^2_{\mu\nu\rho\beta} = \frac{1}{L^4}\left(40 + 72\frac{z^8}{z_0^8}\right) \qquad T_H = \frac{1}{\pi z_0} \qquad S = \frac{A}{4G_5} = V_3 \frac{L^3}{4G_5 z_0^3}$$

Extrinsic curvature of a surface $z = z_*$:

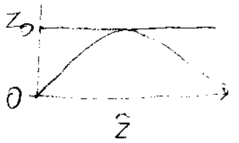
$$K = -\frac{2}{L}\left[\frac{1}{\sqrt{1-\frac{z_*^4}{z_0^4}}} + \sqrt{1-\frac{z_*^4}{z_0^4}}\right]$$

$$= \frac{L^2}{\tilde{z}^2}\left\{-\frac{\left(1-\frac{\tilde{z}^4}{4z_0^4}\right)^2}{1+\frac{\tilde{z}^4}{4z_0^4}}dt^2 + \left(1+\frac{\tilde{z}^4}{4z_0^4}\right)d\vec{x}^2 + dz^2\right\}$$

only outside of BH!

$$z^2 = \frac{\tilde{z}^2}{1+\frac{\tilde{z}^4}{4z_0^4}}$$

FG form $\frac{L^2}{z^2}\, g_{\mu\nu}\, dx^\mu dx^\nu + dz^2$

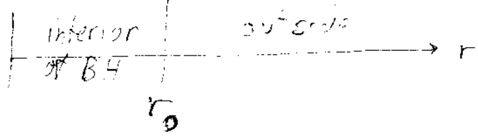


$$g_{\mu\nu}(x, z=0) = g^{(0)}_{\mu\nu}(x) = \text{boundary metric} = (-1,+++)$$

$$= \frac{L^2}{z^2}\left[-\left(1-\frac{z^4}{z_0^4}\right)dt_+^2 + 2\,dt_+\,dz + d\vec{x}^2\right]$$

Eddington-Finkelstein-type

$$dt = dt_+ + \frac{dz}{1-\frac{z^4}{z_0^4}}$$



$$z = \frac{L^2}{r} \qquad z_0 = \frac{L^2}{r_0}$$

$$= -\frac{r^2}{L^2}\left(1-\frac{r_0^4}{r^4}\right)dt_+^2 + 2\,dt_+\,dr + \frac{r^2}{L^2}d\vec{x}^2$$

$$= -\frac{r^2}{L^2}\left(1-\frac{r_0^4}{r^4}\right)u_\mu u_\nu\, dx^\mu dx^\nu - 2u_\mu dx^\mu + \frac{r^2}{L^2}(\eta_{\mu\nu}+u_\mu u_\nu)dx^\mu dx^\nu$$

$$x^\mu = (t_+, x^i) \qquad u^\mu = (\gamma, \gamma\vec{v})$$

Make $r_0 \to r_0(x^\mu)$, $u^\mu \to u^\mu(x^\mu)$ and solve $R_{\mu\nu} + \frac{4}{L^2}g_{\mu\nu} = 0$ expanding in $\partial_\mu r_0, \partial_\mu u^\nu$

interesting recent work! "Boosted black brane" Minwalla et al 0712.2456 + many related papers

$$\underset{L=1}{=} -r^2 a(t_+,r)\,dt_+^2 + 2\,dt_+\,dr + r^2 b(t_+,r)\,\tau_+^2 d\eta^2 + r^2 c(t_+,r)\,dx_T^2$$

t-dep. soln. with in- and outside of horizon

Nakamura et al 0807.3797

Solving $R_{MN} = -\frac{4}{L^2} g_{MN}$

Use Mathematica or Maple!

[illegible]

[illegible] coord: $\tau \ \eta \ x^2 \ x^3 \ z$

$$ds^2 = \frac{1}{z^2}\left[-\frac{\left(1-\frac{e_0}{3}\frac{z^4}{\tau^{4/3}}\right)^2}{1+\frac{e_0}{3}\frac{z^4}{\tau^{4/3}}} d\tau^2 + \left(1+\frac{e_0}{3}\frac{z^4}{\tau^{4/3}}\right)\left(\tau^2 d\eta^2 + dx_\perp^2\right) + dz^2 \right]$$

solves $R_{\mu\nu} + \frac{4}{L^2} g_{\mu\nu} = 0$ at large τ

"time dependent $z_0 = \frac{1}{\pi T} = \left(\frac{3}{e_0}\right)^{\frac{1}{4}} \tau^{\frac{1}{3}}$"

corrections $\sim \underbrace{\frac{1}{T(\tau)\tau}}_{\text{dimless}} \sim \frac{1}{\tau^{2/3}} \Rightarrow$ read viscosity η & other 2nd order hydro coefficients

Same in EF type coordinates (Nakamura...)

$$z_0 = \frac{L^2}{r_0} = \left(\frac{3}{e_0}\right)^{\frac{1}{4}} \tau^{\frac{1}{3}}$$

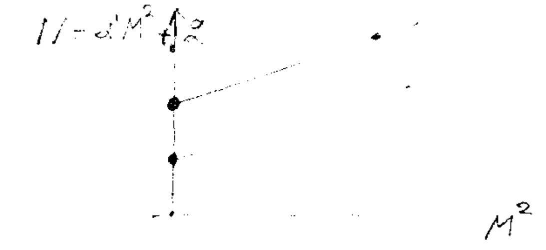
also needed!

$$ds^2 = -\frac{r^2}{L^2}\left[1-\left(\frac{w}{r\tau_+^{1/3}}\right)^4\right] d\tau_+^2 + 2\, d\tau_+\, dr + r^2\left(1+\frac{1}{r\tau_+}\right)^2 \tau_+^2 d\eta^2 + r^2 dx_\perp^2$$

(qualitative)
Some steps on string th → class gr

ST: $S = -\frac{T}{2}\int d^2\sigma \sqrt{-h}\, h^{\alpha\beta} g_{\mu\nu}(X)\,\partial X^\mu \partial X^\nu + \dots$ + fermionic!

— Has excitations of $M^2 \simeq \frac{1}{\alpha'}(N-2)$, $N = 0, 1, \dots$



— For $\alpha' \to 0$ $M^2 \gg \frac{1}{\alpha'}$ unless $M = 0$

⇒ strings → point ⇒ field theory,
$T\to\infty$ supergravity

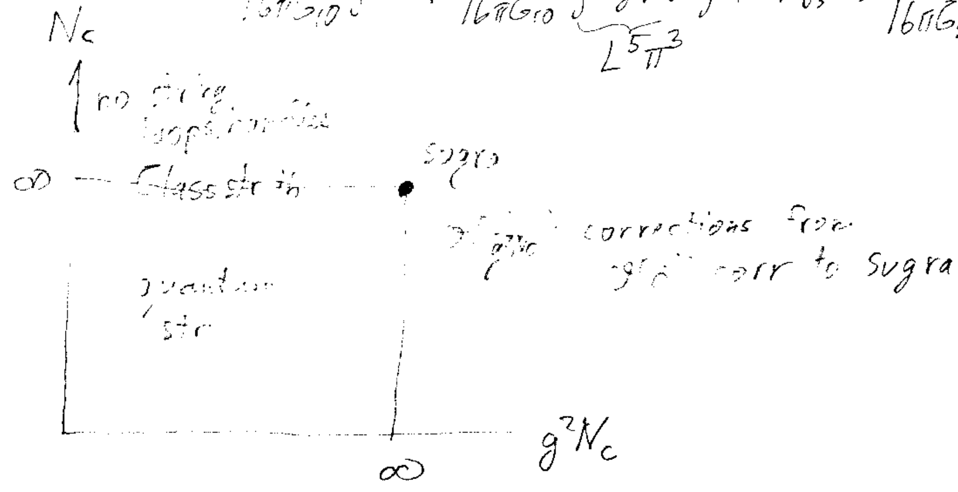
$$S = \frac{1}{16\pi G_{10}}\int d^{10}x \sqrt{-g}\, e^{-2\phi}\left[R + 4(\nabla\phi)^2 - \frac{1}{2\cdot 5!}F^2_{\mu\ldots} + \dots\right]$$

ST ⇒ match Sugra: $16\pi G_{10} = (2\pi)^7 \alpha'^4 g_s^2$

$g_s = \frac{g^2}{4\pi}$

$L^2 = \alpha'\sqrt{g^2 N_c}$ $\Rightarrow \frac{L^3}{4\pi G_5} = \frac{N_c^2}{2\pi^2}$

$\left(\frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{g} = \frac{1}{16\pi G_{10}}\underbrace{\int d^5y\sqrt{-g}}_{L^5\pi^3}\int d^5x\sqrt{-g_5} = \frac{1}{16\pi G_5}\right)$



6. Finite T matter on the boundary

(1) $$Z_{CFT} = e^{-\frac{E}{T}} = e^{p(T)\frac{V}{T}} = \int \mathcal{D}\varphi \; e^{-S_E[\varphi,\, g^{(0)}_{\mu\nu}(x)]}$$ no ext current!

Master formula! $\varphi(\beta,\bar{x}) = \varphi(0,\bar{x})$

$$= e^{-S_{Grav}[G_{MN}(x, z; z_0)]}$$ the G_{MN} giving the smallest S_{Grav} !

$z_0 \uparrow T$; has to be regularised!!

(2) Determine full $T_{\mu\nu}(x)$

—

(1) For flat 5d BH: $a = 1 - \frac{z^4}{z_0^4}$

$$ds^2 = \frac{L^2}{z^2}\left\{-a(z)\,dt^2 + d\bar{x}^2 + \frac{dz^2}{a(z)}\right\} \qquad \sqrt{-g} = \frac{L^5}{z^5} \quad \sqrt{-\gamma} = \frac{L^4}{z^4}\sqrt{a}$$

$$S_{Grav} = \frac{1}{16\pi G_5}\left[\int d^5x\, \frac{L^5}{z^5}\frac{-8}{L^2} - \int d^4x\, \frac{L^4}{z^4}\sqrt{a}\left[\underbrace{-\frac{4}{L}\left(\frac{1}{\sqrt{a}} + \sqrt{a}\right)}_{=2K} + \frac{6}{L} + \frac{L}{2}R(\gamma)\right]\right]$$

$$\boxed{\int d^5x \sqrt{g}\left(R + \frac{12}{L^2}\right) - \int d^4x \sqrt{-\gamma}\left[2K + \frac{6}{L} + \frac{L}{2}R(\gamma)\right]}$$

$\varepsilon \to 0$ —— $z_0 = \frac{1}{\pi T}$ —— z $\qquad \int d^4x = \int_0^\beta d\tau \int d^3x = \frac{V}{T}$

$$= \frac{L^3}{16\pi G_5}\frac{V}{T}\left\{\underbrace{\int_\varepsilon^{z_0} dz\, \frac{-8}{z^5}}_{2z^{-4}} + \frac{4}{\varepsilon^4}\left(1 + 1 - \frac{\varepsilon^4}{z_0^4}\right) - \frac{6}{\varepsilon^4}\sqrt{1 - \frac{\varepsilon^4}{z_0^4}} + \underbrace{\frac{1}{2}L^2 R(\gamma)}_{=0}\right\}$$

Note: also bulk action contributes, EOM does not

intrinsic curv. of $z = \varepsilon$

$$\left\{\frac{2}{z_0^4} - \frac{2}{\varepsilon^4} + \frac{8}{\varepsilon^4} - \frac{4}{z_0^4} - \frac{6}{\varepsilon^4} + \frac{3}{z_0^4}\right\}$$

sign has to be adjusted! CT tuned to cancel $\frac{1}{\varepsilon^4}$

$$\Rightarrow p(T) = \frac{L^3}{16\pi G_5}\cdot\frac{2-4+3}{z_0^4} = \frac{\pi^2 N_c^2}{8}\,T^{-4}$$

$$\frac{L^3}{4\pi G_5} = \frac{N_c^2}{2\pi^2}$$

very subtle !!
best to do with diffgeo5.m

Do a coord. transf. in bulk

$\rightarrow$ may change $g^{(0)}_{\mu\nu}(x)$ by a conf. transf.

Does not matter if bound. theory is conf. invariant!

Ex: Work out $p(T)$ from

$$g_{MN} = \frac{L^2}{z^2}\begin{pmatrix} -\frac{a^2}{b} & & & & \\ & b & & & \\ & & b & & \\ & & & b & \\ & & & & 1 \end{pmatrix} \qquad a = 1 - \frac{z^4}{4z_0^4} \qquad b = 1 + \frac{z^4}{4z_0^4} \qquad z_0 = \frac{1}{\pi T}$$

$$\sqrt{-G} = \frac{L^5}{z^5}\,ab \qquad \int d^4x \mathrel{\hat{=}} \frac{V}{T} \qquad \sqrt{-\gamma} = \frac{L^4}{z^4}\,ab \qquad K = \frac{1}{L}\left(-4 + \frac{za'}{a} + \frac{zb'}{b}\right)$$

$$S_{gr} = \frac{L^3}{16\pi G_5}\cdot\frac{V}{T}\left\{\int_\varepsilon^{z_0\sqrt{2}} \frac{dz(-8)}{z^5}\left(1 - \frac{z^8}{16z_0^8}\right) - \frac{1}{L^3}\sqrt{-\gamma}\left[2K + \frac{6}{L} + \frac{L}{2}R(\gamma)\right]\right\}$$

$$\int_\varepsilon^{z_0\sqrt{2}} \left(2z^{-4} + \frac{2z^4}{16z_0^8}\right) - \underbrace{\frac{1}{\varepsilon^4}\left[-8 + 6\right]}_{\text{cancel}} + O(\varepsilon^4)$$

$$\frac{2}{4z_0^4} - \frac{2}{\varepsilon^4} + \frac{4}{8z_0^4}$$

$$= \frac{L^3}{16\pi G_5}\,\frac{1}{z_0^4}\cdot\frac{V}{T}$$

just as for standard flat BH, but now all comes from bulk action!

$$\frac{1}{L^3}\sqrt{-\gamma}\left(2K + \frac{6}{L}\right) = \frac{1}{z^4}\left[-8(a+b) + 2ab\right]$$

7. Comparing actions: Hawking-Page transition

Herzog hep-th/0608151 toy model

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left(R+\frac{12}{L^2}\right) = -\frac{1}{16\pi G_5 L^2}\int d^5x\sqrt{-g}\cdot 8$$

$R = -\frac{20}{L^2}$ [Boundary term is neglected in this toy model!] "hard wall model"

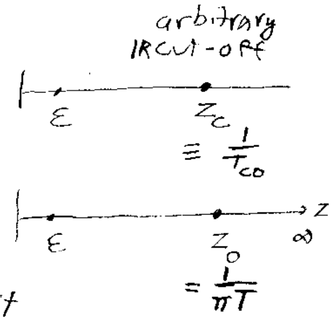
Two metrics:

pure AdS

(1) $$ds^2 = \frac{L^2}{z^2}\left(d\tau^2 + d\vec{x}^2 + dz^2\right) \qquad \sqrt{-g} = \left(\frac{L}{z}\right)^5$$

↑ Euclidian time

(2) $$ds^2 = \frac{L^2}{z^2}\left[a(z)\,d\tau^2 + d\vec{x}^2 + \frac{dz^2}{a(z)}\right] \qquad \sqrt{-g} = \left(\frac{L}{z}\right)^5$$



$$a(z) = 1 - \frac{z^4}{z_0^4} \qquad \text{Flat AdS BH, } \tau = it$$

$$\boxed{Z_{CFT} = \langle 1\rangle = e^{-f\frac{V}{T}} = e^{+p\frac{V}{T}} = e^{S_{grav}}}$$

$$\int d^5x = \int d\tau \underbrace{\int d^3x}_{\Rightarrow V_3} \int_{\varepsilon} dz$$

ε ← value diverges at $z\to 0$; difference will be finite

Limits are obvious for (2) $\int dz\, 8z^{-5} = -2z^{-4}$

$$S_{(2)}/V_3 = -\frac{L^3}{16\pi G_5}\int_0^{\beta} d\tau \int_{\varepsilon}^{\min(z_0,z_c)} \frac{8\,dz}{z^5} = -\frac{L^3}{16\pi G_5 T}\left(\frac{2}{\varepsilon^4} - \frac{2}{(\min z_0, z_c)^4}\right)$$

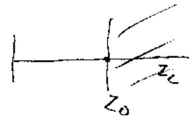
No bdry term!?

! [To compare actions at same T demand same physical period

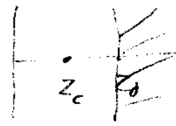
$$\Delta\tau_{(1)} = \sqrt{a(\varepsilon)}\,\Delta\tau_{(2)}$$

$$S_{(1)}/V_3 = -\frac{L^3}{8\pi G_5}\frac{1}{T}\underbrace{\sqrt{1-\frac{\varepsilon^4}{z_0^4}}\left(\frac{1}{\varepsilon^4} - \frac{1}{z_c^4}\right)}_{\frac{1}{\varepsilon^4} - \frac{1}{z_c^4} - \frac{1}{2z_0^4}}$$

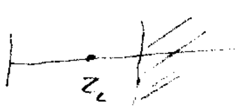
$$\left[S_{(2)} - S_{(1)}\right]/V_3 = + \frac{L^3}{8\pi G_5 T}\left[\frac{1}{z_0^4} - \frac{1}{z_c^4} - \frac{1}{2z_0^4}\right] \qquad z_0 < z_c$$



$$= \frac{p_{(2)} - p_{(1)}}{T} \qquad \left[\frac{1}{z_c^4} - \frac{1}{z_c^4} - \frac{1}{2z_0^4}\right] \qquad z_0 > z_c$$



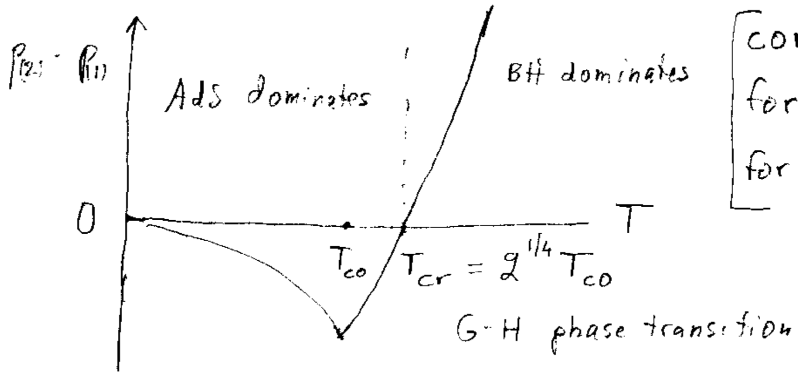
$$\begin{cases} \dfrac{L^3}{8\pi G_5 T}\left(\dfrac{1}{2z_0^4} - \dfrac{1}{z_c^4}\right) \\ \dfrac{L^3}{8\pi G_5 T}\left(-\dfrac{1}{2z_0^4}\right) \end{cases}$$



one can also soften the cut-off by $\theta(z-z_c) \Rightarrow e^{-cz^2}$ ~~"soft model"~~

Put here, for concreteness $\boxed{\dfrac{L^3}{4\pi G_5} = \dfrac{N_c^2}{2\pi^2}}$ $\qquad \dfrac{1}{z_0} = \pi T$

$$\Rightarrow p_{(2)} - p_{(1)} = \begin{cases} \dfrac{\pi^2 N_c^2}{4}\left(\dfrac{T^4}{2} - T_{co}^4\right) & T > T_{co} \\ \quad \times \left(-\dfrac{T^4}{2}\right) & T < T_{co} \end{cases}$$



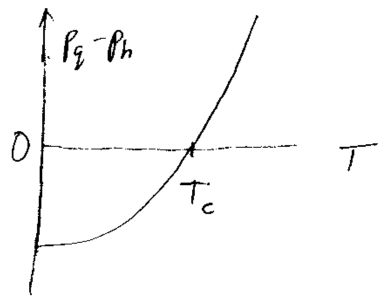
correct coefficient for $p(T) = aT^4$ for $N=4$ hot SYM!

5d bulk transition!

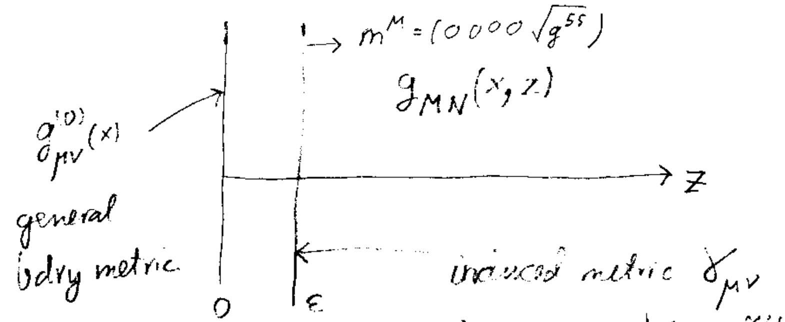
compare bag model for hot QCD:

$$\begin{cases} p_q = a_q T^4 - B \\ p_h = a_h T^4 \end{cases}$$

$$a_q > a_h$$



2. $T_{\mu\nu}(x)$



$K_{MN} \Leftrightarrow K_{\mu\nu}$ $\quad$ $K = g^{MN}K_{MN}$ ← extrinsic curvature

"2nd fund form" = "extrinsic curvature tensor"

$n^M K_{MN} = 0$

$$S = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\left(R + \frac{12}{L^2}\right) - \int d^4x\sqrt{-\gamma}\,2K + S_{CT}$$

$$\Rightarrow \langle T_{\mu\nu}\rangle = \frac{-2\delta S}{\sqrt{-\gamma}\,\delta\gamma^{\mu\nu}} \overset{L=1}{=} \frac{1}{8\pi G_5}\Big(K_{\mu\nu} - \overset{\sim\frac{1}{L}}{K}\gamma_{\mu\nu} \underbrace{- \overset{d-1}{3}\gamma_{\mu\nu} + \overset{\frac{1}{d-2}\sim\frac{L}{2}}{\frac{1}{2}}\left[R_{\mu\nu}(\gamma) - \frac{1}{2}R(\gamma)\gamma_{\mu\nu}\right]}_{CT's}\Big)_{z\to 0}$$

variation of $\int d^5x$ just gives EOM ⇒ no contrib.!

"simply" has to be evaluated!

K_{MN} = extrinsic

$K = g^{MN}K_{MN}$ = tr[Inverse[metric] · extrinsic]

$\sqrt{-\gamma} = (-\mathrm{Det}(\mathrm{induced}))^{1/2}$

Some generalities:

- Write $g_{MN} = \frac{L^2}{z^2}\begin{pmatrix} g_{\mu\nu}(x,z) & 0 \\ 0 & 1 \end{pmatrix}$ "gauge choice" FG form

- EOM $R_{MN} + \frac{4}{L^2} g_{MN}$ becomes $\left(' = \frac{\partial}{\partial z^2} \right)$
Skenderis -th/0009230 eq 2.5

$$
\begin{cases}
\mu\nu & R_{\mu\nu}(g) - 2 g'_{\mu\nu} - g'^{\alpha\beta} g_{\alpha\beta}\, g_{\mu\nu} + z^2\left[2 g''_{\mu\nu} - 2 g'_{\mu\alpha} g'^{\alpha}{}_{\nu} + g'^{\alpha}{}_{\alpha}\, g'_{\mu\nu}\right] = 0 \\
\mu z & \nabla^{\mu}(g)\left(g'_{\mu\nu} - g'^{\alpha\beta} g_{\alpha\beta}\, g_{\mu\nu}\right) = 0 \\
zz & g^{\mu\nu} g''_{\mu\nu} - \frac{1}{2} g'^{\alpha\beta} g'_{\alpha\beta} = 0
\end{cases}
$$

Try to solve: $g_{\mu\nu}(x,z) = g^{(0)}_{\mu\nu}(x) + g^{(2)}_{\mu\nu}(x) z^2 + g^{(4)}_{\mu\nu}(x) z^4 + \dots$

$g'_{\mu\nu}(x,z) = g^{(2)}_{\mu\nu}(x) + 2 g^{(4)}_{\mu\nu} \cdot z^2$

$g''_{\mu\nu}(x,z) = 2 g^{(4)}_{\mu\nu}(x) + \dots$

$z^2 = 0: \quad g^{\mu\nu}_{(0)} \Big| \; R_{\mu\nu}(g^{(0)}_{\mu\nu}) - 2 g^{(2)}_{\mu\nu} - g^{(2)\alpha\beta} g^{(0)}_{\alpha\beta}\, g^{(0)}_{\mu\nu} = 0$

$R(g^{(0)}_{\mu\nu}) - 2\,\mathrm{Tr}\, g^{(2)} - 4\,\mathrm{Tr}\, g^{(2)} = 0 \qquad \mathrm{Tr}\, g^{(2)} = \frac{1}{6} R(g^{(0)}_{\mu\nu})$

$\mu\nu$: $\quad g^{(2)}_{\mu\nu} = \frac{1}{2}\left[R_{\mu\nu}(g^{(0)}) - \frac{1}{6} R(g^{(0)})\, g^{(0)}_{\mu\nu}\right]$

For flat bdry metric $g^{(2)}_{\mu\nu} = 0$!

zz: $\quad 2\,\mathrm{Tr}\, g^{(4)} - \frac{1}{2}\mathrm{Tr}(g_{(2)})^2 \Rightarrow \mathrm{Tr}\, g^{(4)} = \frac{1}{4}\mathrm{Tr}\, g_{(2)}^2$

$$
\begin{cases}
g^{(2)}_{\mu}{}^{\alpha}\, g^{(2)}_{\alpha\nu} = \frac{1}{4}\left(R_{\mu\alpha} R^{\alpha}{}_{\nu} - \frac{1}{3} R R_{\mu\nu} + \frac{1}{36} R^2 g^{(0)}_{\mu\nu}\right) \\
\mathrm{Tr}\, g_{(2)}^2 = \frac{1}{4} R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{18} R^2
\end{cases}
$$

Main point is that the vector μz equation to leading order (put $z^2=0$) gives

$$\nabla^\mu(g^{(0)})\Big[g^{(2)}_{\mu\nu} - \mathrm{Tr}\,g_2\, g^{(0)}_{\mu\nu}\Big] = \nabla^\mu(g^0)\,\tfrac{1}{2}\Big[R_{\mu\nu} - \tfrac{1}{2}R\,g^{(0)}_{\mu\nu}\Big] = 0$$

put in $g^{(2)}_{\mu\nu}$ from $\mu\nu$-eq

Einstein tensor is conserved!

but to next order in z^2 one finds that it can be written in the form

$$\nabla^\mu(g^0)\Big[g^{(4)}_{\mu\nu} - \frac{1}{8}\underbrace{\Big(R_{\mu\alpha}R^\alpha{}_\nu - \frac{1}{2}RR_{\mu\nu} + \big(\tfrac{5}{36}R^2 - \tfrac{1}{4}R^{\alpha\beta}R_{\alpha\beta}\big)g^{(0)}_{\mu\nu}\Big)}_{\text{curvature tensors of boundary metric}}\Big] = 0$$

$$\Rightarrow g^{(4)}_{\mu\nu} = \overbrace{\frac{1}{8}(\qquad\qquad)}^{\equiv A^{(4)}_{\mu\nu}} + \underbrace{t_{\mu\nu}}_{\text{some conserved tensor}} \qquad \mathrm{Tr}A^{(4)} = \frac{1}{8\cdot 18}R^2 = \frac{1}{4}(\mathrm{Tr}\,g_2)^2$$

$$\boxed{\langle T_{\mu\nu}\rangle = \frac{4L^3}{16\pi G_5}\big(g^{(4)}_{\mu\nu} - A^{(4)}_{\mu\nu}\big)}$$

Some conserved tensor $\nabla^\mu(g^0)\, t_{\mu\nu} = 0$

Initial condition, not specified by Einstein!

Energy momentum tensor!

From zz eq

$$t^\mu{}_\mu = \mathrm{Tr}\,g^{(4)} - \mathrm{Tr}A^{(4)} = \frac{1}{4}\mathrm{Tr}\,g_2^2 - \frac{1}{4}(\mathrm{Tr}\,g_2)^2$$

$$= \frac{1}{4}\Big[\mathrm{Tr}\,g_2^2 - (\mathrm{Tr}\,g_2)^2\Big] = \frac{1}{4}\Big[\frac{1}{4}R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{18}R^2 - \frac{1}{36}R^2\Big] \qquad \frac{3}{2\cdot 18} = \frac{1}{12}$$

$$= \frac{1}{4}\Big(\frac{1}{4}R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{12}R^2\Big)$$

Trace anomaly $\frac{N_c^2}{16\pi^2} \cdot 2 \quad \frac{1}{4} R_{\alpha\beta}^2 \quad \frac{1}{12} R^2$ from direct computation

$$\langle T^\mu{}_\mu \rangle = \frac{L^3}{16\pi G_5}\left(\frac{1}{4} R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{12} R^2\right) \qquad T_{\mu\nu} = \begin{pmatrix} \varepsilon & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \quad T^\mu{}_\mu = -\varepsilon + 3p = 0$$

$\varepsilon = T\dot{s} - p = Tp' - p = (4-1)p$

AdS_5/CFT_4 encodes remarkably the trace anomaly of CFT_4 in curved space (computed with the field content of $\mathcal{N}=4$ SYM & $L^3/(4\pi G_5) = N_c^2/2\pi^2$

$\overbrace{\qquad\qquad}^{\approx g_{\mu\nu}''''(x,0)}$

$g^{(0)}_{\mu\nu}(x)$ $\quad T_{\mu\nu}(x)$ are initial conditions for integrating $g_{MN}(x,z)$ from $R_{MN} + \frac{4}{L^2} g_{MN} = 0$!!
For second order diff eq. you need both $f(0)$ and $f'(0)$!

Ex: $\langle T^\mu{}_\mu \rangle$ if bdry metric is $\begin{pmatrix} -1 & & & \\ & r^2(t) & & \\ & & r^2(t) & \\ & & & r^2(t) \end{pmatrix}$ = FRW

$$g_{MN} = \frac{L^2}{z^2}\begin{pmatrix} -a(t,z) & & & & 0 \\ & b(t,z) & & & \\ & & b(t,z) & & \\ & & & b(t,z) & \\ 0 & & & & 1 \end{pmatrix} \qquad \left.\begin{matrix} a(t,z) \\ b(t,z) \end{matrix}\right\} \text{ given explicitly in th/0612226}$$

$$\Rightarrow \langle T^\mu{}_\mu \rangle = -\varepsilon + 3T^1{}_1 = \frac{3N_c^2}{8\pi^2}\,\frac{-r'^2 r''}{r^3}$$

so you have gravity dual of FRW, but AdS of course does not provide $r(t)$! Need Einstein in bdry theory

Ex. Now it is trivial to redo flat BH (p.18):

Transform it 1st to FG form

$$\frac{L^2}{z^2}\left\{-\frac{\left(1-\frac{z^4}{4z_0^4}\right)^2}{1+\frac{z^4}{4z_0^4}}\,dt^2+\left(1+\frac{z^4}{4z_0^4}\right)d\vec{x}^2+dz^2\right\} \qquad \pi T=\frac{1}{z_0}$$

$$\left(\eta_{\mu\nu}+g^{(4)}_{\mu\nu}\,z^4+\ldots\right)dx^\mu dx^\nu$$

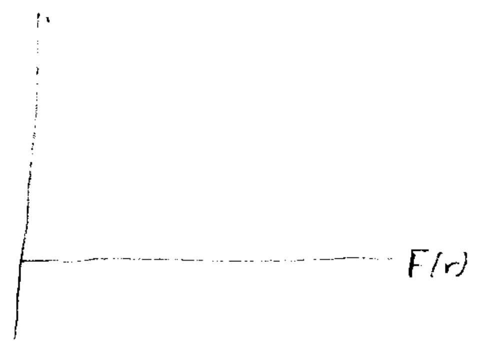
$$g^{(4)}_{\mu\nu}=\begin{pmatrix}3&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}\frac{1}{4z_0^4} \qquad T_{\mu\nu}=\frac{N_c^2}{2\pi^2}\cdot\frac{1}{4}\,\pi^4T^4\begin{pmatrix}3&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$$

$$T_{\mu\nu}=\begin{pmatrix}\varepsilon&&&\\&p&&\\&&p&\\&&&p\end{pmatrix} \qquad p=\frac{\pi^2N_c^2}{8}T^4$$

Ex

$$ds^2=-F(r)\,dt^2+\frac{dr^2}{F(r)}+r^2\,d\Omega^2$$

$$F(r)=\hat{r}^2+k-\frac{\mu}{\hat{r}^2}$$



Application 4: Expectation values of Wilson loops

$$P\exp\left[ig\int_C A^\mu dx_\mu\right] \quad C = \text{closed loop}, \quad \text{Tr is gauge invariant}$$

Expectation value of a Wilson loop [21] in the boundary field theory = Action of the string hanging from the loop in the 5th dimension.

Take Q at $x = -L/2$, $\bar{Q}$ at $L/2$. How deep does the string connecting them hang in the z direction, i.e., what is $z = z(x,t) = z(x)$ for the extremal configuration (expected to be static, no t)?

Particle action $= -m\int d\tau \Rightarrow$ String action $= -T\int dA$. $T = \frac{1}{2\pi\alpha'} =$ **Tension.**
String $X^\mu(\tau,\sigma)$ moving in a space with metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ has the action:

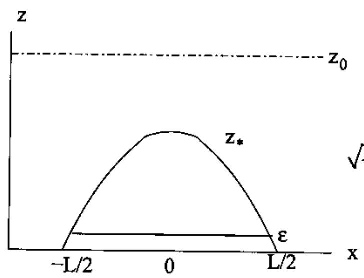
$$S = -T\int d\tau d\sigma\sqrt{-\det h_{ab}}, \qquad h_{ab} = g_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}, \qquad \sigma^a = (\tau,\sigma).$$

Nambu-Goto: $g_{\mu\nu} = \eta_{\mu\nu}$

$$h_{ab} = \begin{pmatrix} \dot{X}\cdot\dot{X} & \dot{X}\cdot X' \\ X'\cdot\dot{X} & X'\cdot X' \end{pmatrix} \qquad X\cdot Y \equiv \eta_{\mu\nu}X^\mu Y^\nu \qquad {}^{\cdot} = \partial/\partial\tau,\ {}' = \partial/\partial\sigma.$$

[21] For a discussion of a Wilson loop and its relation to a $Q-\bar{Q}$ (free)energy at $T=0$ and $T\neq 0$, and real $t \Leftrightarrow$ imaginary time $\tau = it$, see H.J. Rothe, Lattice gauge theories, Ch. 7 and 8 and Sect. 20.2.

String hanging [22] from a Q and a $\bar{Q}$ at $z=0$ in the metric $ds^2 = g_{tt}(z)dt^2 + g_{xx}(z)dx^2 + g_{zz}(z)dz^2$. 35



$$\left.\begin{aligned} g_{tt} &= -g(z) \\ g_{xx} &= 1 \\ g_{zz} &= \frac{1}{g(z)} \end{aligned}\right\} * \frac{L^2}{z^2}$$

$$\sqrt{-h} = \sqrt{g(z) + z'^2 - \frac{1}{g}\dot{z}^2}\ \frac{L^2}{z^2}$$

$$\int dt\,dz\sqrt{g(z) + z'^2 - \frac{1}{g}\dot{z}^2}\cdot\frac{L^2}{z^2}$$

$$\sigma^1 = t, \quad \sigma^2 = x, \qquad X^\mu = (t,x,0,0,z(t,x) \to z(x))$$

$$h_{ab} = \begin{pmatrix} g_{tt} + g_{zz}\dot{z}^2 & g_{zz}\dot{z}z' \\ g_{zz}z'\dot{z} & g_{xx} + g_{zz}z'^2 \end{pmatrix} \qquad \sqrt{-h} = \sqrt{-g_{tt}g_{xx} - g_{tt}g_{zz}z'^2 - g_{xx}g_{zz}\dot{z}^2}$$

Extremize ($T = 1/(2\pi\alpha')$ = tension, $\dot{z} = 0$, $g(z) = 1 - z^4/z_0^4$ specialising to the 5d AdS BH)

$$S = T\Delta t\int_{-L/2}^{L/2} dx\sqrt{-h} = T\Delta t\,2\int_\epsilon^{z_*} dz\,x'(z)\sqrt{-h} = \Delta t\frac{\mathcal{L}^2}{\pi\alpha'}\int_\epsilon^{z_*}\frac{dz}{z^2}\sqrt{1 + g(z)x'(z)^2}$$

Key technical point: $dx\,L = dx\,L(z(x),z'(x)) = dz\,x'(z)L = dz\,L_{\text{new}}(x'(z))$

[22] Rey-Yee, hep-th/9803001; Sonnenschein, hep-th/9910089

Equation of motion is

$$\frac{\partial L}{\partial x(z)} - \frac{d}{dz}\frac{\partial L}{\partial x'(z)} = 0 \Rightarrow \frac{\partial L}{\partial x'(z)} = \frac{-g_{tt}g_{xx}x'}{\sqrt{-g_{tt}g_{zz} - g_{tt}g_{xx}x'^2}} = \text{constant.} \tag{5}$$

The constant is fixed neatly so that the maximum value of z is z_*, $z(x=0) = z_*$, $z'(z_*) = 0$. This gives $z'^2 = (dz/dx)^2$ from which by integration (Exercise)

$$L = 2\int_{\epsilon\to 0}^{z_*} \frac{dz}{\sqrt{\frac{g_{xx}}{g_{zz}}\left(\frac{g_{tt}g_{xx}}{g^*_{tt}g^*_{xx}} - 1\right)}} = 2\int_0^{z_*} \frac{dz}{\sqrt{g(z)\left[\frac{g(z)}{z^4}\frac{z_*^4}{g(z_*)} - 1\right]}} \qquad g^*_{tt} \equiv g_{tt}(z_*), \text{etc.}$$

Inserting this z'^2 to the action gives the extremal action (Exercise)

$$S = T\Delta t\, 2\int_\epsilon^{z_*} dz \sqrt{\frac{-g_{tt}g_{zz}}{1 - \frac{g^*_{tt}g^*_{xx}}{g_{tt}g_{xx}}}} = T\Delta t\, 2\int_\epsilon^{z_*} dz \frac{\mathcal{L}^2}{z^2}\frac{1}{\sqrt{1 - \frac{z^4}{g(z)}\frac{g(z_*)}{z_*^4}}}$$

Separate divergence of the 5d AdS BH at $z \to 0$:

$$\int_\epsilon^{z_*} dz\,\frac{1}{z^2}f(z) = \int_\epsilon^{z_*} dz\,\frac{1}{z^2}[f(z) - 1 + 1] = \frac{1}{\epsilon} + \int_0^{z_*} dz\,\frac{1}{z^2}[f(z) - 1] - \frac{1}{z_*};$$

throw away $1/\epsilon$. For Euclidian Wilson loop $\langle \Delta t \times R - \text{loop}\rangle \sim \exp[-\Delta t V(R)]$ so write here $V(L, z_0) = S/\Delta t$:

$$V(L, z_0) = 2T\mathcal{L}^2\left[\int_0^{z_*}\frac{dz}{z^2}\left(\frac{1}{\sqrt{1 - \frac{z^4}{g(z)}\frac{g(z_*)}{z_*^4}}} - 1\right) - \frac{1}{z_*}\right]$$

Scaling $z = yz_0$, $z_* = y_m z_0$: [23]

$$L = 2z_0\int_0^{y_m}\frac{dy}{\sqrt{(1-y^4)[q(y_m)/q(y) - 1]}} \qquad q(y) \equiv y^4/(1-y^4) \tag{6}$$

$$V(L, z_0) = \frac{\mathcal{L}^2}{\alpha'\pi z_0}\left\{\int_0^{y_m}\frac{dy}{y^2}\left[\frac{1}{\sqrt{1 - q(y)/q(y_m)}} - 1\right] - \frac{1}{y_m}\right\} \tag{7}$$

Units of length and energy given by

$$\pi z_0 = \frac{1}{T_H}, \qquad -V_{Q\bar{Q}} \equiv \frac{\mathcal{L}^2}{\alpha'\pi z_0} = \sqrt{g^2 N_c}\, T_H.$$

Small L (and $y_m \to 0$, $q(y) \approx y^4$):

$$V(L) = -\frac{0.2285\sqrt{g^2N_c}}{L}$$

Intermediate $0 < L < L_{\max}$: can fit to

$$V(L) = -\frac{4}{3}\frac{\alpha_s}{L} + \sigma L$$

Small L again for $y_m \to 1$:

$$V(L, z_0) \to V_{Q\bar{Q}} = \frac{\mathcal{L}^2}{\pi\alpha'}\int_\epsilon^{z_0}\frac{dz}{z^2} \Rightarrow -\frac{\mathcal{L}^2}{\alpha'\pi z_0}$$

After some y_m, for $L >$ some value (see Fig) the dominant config is that with separate $Q\bar{Q}$. This is also a solution of (5) with $x' = 0$, $z_* = z_0$.

[23] Plotted with Mathematica using ParametricPlot[{L[ym], V[ym]}, {ym, 0.15, 0.994}] in Fig.1

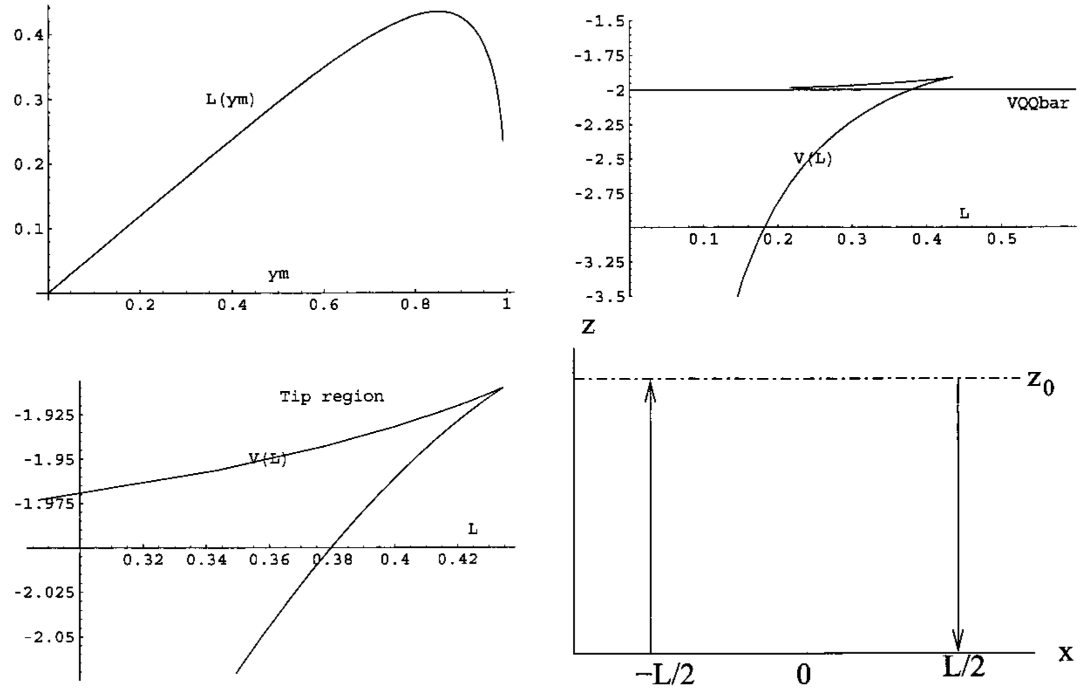


Figure 1: The distance L as function of y_m evaluated from (6), the extremal action and its tip region from (7) (all scaled by the factors outside the integral) and the $Q\bar{Q}$ configuration.

Picture from AdS/CFT:

- at small distances conformally invariant form $V \sim 1/L$, $\sqrt{..}$ dependence on g^2N_c is typical of strong coupling.
- at some distance interaction is screened and $Q\bar{Q}$ separate.
- one can put in numbers [24]
- mathematics of the curves is pretty: the independent $Q\bar{Q}$ solution is obtained also from the string-connected solution when $z_* \to z_0$ and the two branches approach each other

This was just an example of numerous applications of AdS/CFT to Wilson loop computations. Works even for gluonic scattering amplitudes!! [25]

[24] Large number of papers, mine is Kajantie-Tahkokallio-Yee, hep-ph/0609254
[25] Alday-Maldacena, arXiv:0705.0303