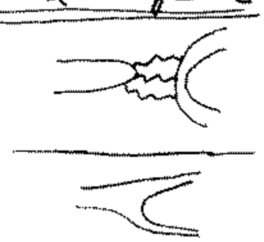
$\left(\frac{\alpha}{4\pi} \ll 1 !\right)$

<u>QCD & large N_c</u> $\quad e = 0.3\,,\ \frac{1}{N_c=3} = 0.3$ also!



$\rightarrow 0$ in large N_c limit! :

$$N_c \cdot 2_g \;\Big|\; \mu \frac{\partial g}{\partial \mu} = -\beta_0 g^3 - \beta_1 g^5 \qquad \tilde{g}^2 \equiv g^2 N \overset{\nearrow C_A}{} \Leftarrow \quad f_{acd} f_{bcd} = N_c \delta_{ab}$$

$$\mu \frac{\partial \tilde{g}^2}{\partial \mu} = -2\beta_0 N_c \frac{(\tilde{g}^2)^2}{N_c^2} - 2\beta_1 N_c \frac{(\tilde{g}^2)^3}{N_c^3}$$

$$\left[\mu \frac{\partial \tilde{g}^2}{\partial \mu} = -\left(\frac{11}{3} - \frac{2}{3}\frac{N_f}{N_c}\right)\tilde{g}^4 \frac{1}{(4\pi)^2} - \left(\frac{34}{3} - \frac{N_F}{N_c}\left(\frac{13}{3} - \frac{1}{N^2}\right)\right)\frac{1}{(4\pi)^4}\tilde{g}^6 + \cdots\right.$$

Has a "sensible" large N_c limit

$$\frac{1}{\tilde{g}^2(\mu)} = 2\beta_0 \log \frac{\mu}{\underbrace{\Lambda_{QCD}}_{\text{kept fixed when } N_c \to \infty}}$$

So take $N_c \to \infty$ with $\tilde{g}^2 = g^2 N_c$ fixed; Λ_{QCD}, meson masses fixed

$$\begin{cases} D_\mu = \partial_\mu + i \frac{\tilde{g}}{\sqrt{N}} A_\mu \\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i \frac{\tilde{g}}{\sqrt{N}}[A_\mu, A_\nu] \end{cases} \qquad A_\mu = A_\mu^a \underbrace{T^a_{ij}}_{N\times N \text{ hermitian}}$$

$$\langle A_\mu A_\nu \rangle = \langle A_\mu^a T^a A_\nu^b T^b \rangle = D_{\mu\nu}(x-y)\, T^a T^a$$

$$\sim\!\sim\!\sim = D_{\mu\nu}(x-y)\, T^a_{ij} T^a_{kl} = D_{\mu\nu} \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$$

Rescale $\hat{A} = \frac{\tilde{g}}{\sqrt{N}} A_\mu \quad \psi = \sqrt{N}\hat{\psi}$

$F_{\mu\nu} = \frac{\sqrt{N}}{\tilde{g}} \hat{F}_{\mu\nu}$

$$^a_{ij}\sim\!\sim\!\sim^a{}_{kl} = \left({}^i_j \rightrightarrows {}^l_k - \frac{1}{N_c} \supset\;\subset \right)$$

$$\Rightarrow \mathcal{L} = N\left[-\frac{1}{2\tilde{g}^2}\mathrm{Tr}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \sum_1^{N_F} \bar{\hat{\psi}}_f (i\not{D} - m_f)\hat{\psi}_f\right]$$

N counting; $\quad \sim\!\sim \;\; - \;\sim \frac{1}{N} \qquad \text{vertices} \;\; N \qquad \bigcirc \sim N$

1. <u>each propag</u> $\sim \frac{1}{N}$ 2. <u>each vertex</u> $\sim N$ 3. <u>each color index loop</u> $\sim N$

<u>Example</u> 3 loop vacuum diags 6 propag, 4 vertices

(looks just like a string amplitude!)

$(F_a)_{bc} = -i f_{abc}$

$\underbrace{\mathrm{Tr}\, F_a F_b F_c}_{\frac{1}{2} N_c\, i f_{abc}}\ f_{abc} \sim N_c^4 \cdot \frac{\tilde{g}^4}{N_c^2} \sim N_c^2 \tilde{g}^4$

$\underbrace{\frac{1}{2} N_c\, i f_{abc}}_{N_c d_A}$

maximum!

$P_{QCD} \sim d_A T^4 \sim N_c^2$

4 color index loops

$\frac{1}{N_c^6} N_c^4 \cdot N_c^4 = N_c^2$

topology = sphere

$\mathrm{Tr}\, T_a T_b T_c\ f_{abc} \sim N_c^3 \frac{\tilde{g}^4}{N_c^2} \sim N_c \tilde{g}^4$

$\frac{1}{2} \underbrace{T_F}_{\frac{1}{2}}\ i\ \underbrace{f_{abc} f_{abc}}_{N_c^3}$

topology = sphere with a hole

$\frac{1}{N_c^6} N_c^4 N_c^3 = N_c$

$\underbrace{\mathrm{Tr}\, T_a T_a T_b T_b}_{C_F^2 N_c} = N_c^3 \cdot \frac{\tilde{g}^4}{N_c^2} \sim N_c \tilde{g}^4$

$C_F = \frac{N_c^2 - 1}{2 N_c}$

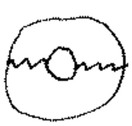
$\frac{1}{N_c^2} N_c^3 = N_c$

$\underbrace{\mathrm{Tr}\, T_a T_b T_a T_b}_{-\frac{1}{2} C_F} = N_c \frac{\tilde{g}^4}{N_c^2} \sim \frac{1}{N_c} \tilde{g}^4$

one loop only!

$\frac{1}{N_c^2} \cdot N_c = \frac{1}{N_c}$!

torus with 1 hole?



$\underbrace{\mathrm{Tr}\, T_a T_b}_{T_F \delta_{ab} = \frac{1}{2}\delta_{ab}}\ \mathrm{Tr}\, T_a T_b = \frac{1}{4} d_A \sim N_c^2 \frac{\tilde{g}^4}{N_c^2} \sim \tilde{g}^4$

$\frac{1}{N_c^2}\ N_c^2 = 1$

2 loops

general:

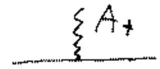
$$\boxed{N^{2 - 2 * \text{handles} - \text{holes}}}$$

Leading:  just glue

Leading with one q loop  q loop filled with glue $\Rightarrow$ quark propagator is 

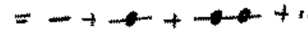
QCD in 1+1 d (effectively Abelian, $F_{+-} = -\partial_- A_+$ in $A^+ = A_- = 0$ gauge) can be solved exactly for $N_c \gg 1$ (only planar diags of type +  +  since no self interactions)

$\dim g^2 = (\text{energy})^{4-d}$

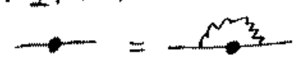
$\bar\psi(i\not{D} - m)\psi =$  inter.

$\not{D} = \gamma^+ D_+ + \gamma^- D_- = \gamma^+(\partial_+ + igA_+) + \gamma^-\partial_-$

$\gamma^0 = \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \gamma^1 = \begin{pmatrix}0&-1\\1&0\end{pmatrix} \quad \gamma^+ = \begin{pmatrix}0&0\\\sqrt2&0\end{pmatrix} \quad \gamma^- = \begin{pmatrix}0&\sqrt2\\0&0\end{pmatrix}$

Full = 

Eq for 1PI piece:

$\Rightarrow$ exact quark propagator from 

is $\dfrac{1}{\not{p} - M} \quad M^2 = m^2 - \dfrac{g^2}{2\pi}$

This kind of gap equation is often written as an uncontrollable approximation!

Meson spectrum:

$q\bar q$

$q\bar q$ 

$\bar q q$ production?

Bethe-Salpeter  for meson wave function

$\Phi(y_q)$ $\quad y_q$ = rapidity of q in meson rest frame

2d ED; just U(1) invariance $\Rightarrow$ "confinement"

$\frac{1}{r} \to \log r \to r = |x|$
3d $\quad$ 2d $\quad$ 1d

$\mathcal{L} = -\frac12 F_{01}F^{01} + \bar\psi[i(\not\partial + ie\not{A}) - m]\psi$

$= \frac12(\partial_1 A_0)^2 + \bar\psi[i(\gamma^0\partial_0 + \gamma^1\partial_1) - e\gamma^0 A_0 - m]\psi$ $\qquad A_1 \to A_1 + \frac{1}{e}\partial_1\chi = 0$

$\partial_x\chi(t,x) = -eA_1(t,x)$

transforms A_1 away

can still do x-indep. transf.

Maxwell: $\quad \vec\nabla\cdot\vec E = \rho \qquad \dfrac{\partial\mathcal{L}}{\partial A_0} = \partial_1 \dfrac{\partial\mathcal{L}}{\partial\partial_1 A_0}$

$\partial_x^2 A_0(t,x) = -e\bar\psi\gamma^0\psi \equiv -eJ^0$

$\begin{cases} A_0(t,x) \to A_0(t,x) + \frac{1}{e}\partial_0\chi(t) = 0 & \text{at some } x_0 \\ A_1 = 0 \to \frac{1}{e}\partial_1\chi(t) = 0 \end{cases}$

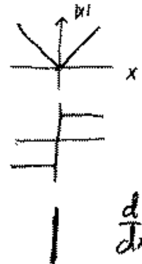
$\Rightarrow A_0(t,x) = \displaystyle\int_{-\infty}^{\infty} dy\, \underbrace{\tfrac12|x-y|}\,(-e)J^0(t,y) + C_1 x + C_2$

$\Rightarrow A_0 = A_0(t,x)$

$A_0(t,x_0) = 0$

but how do you get this from

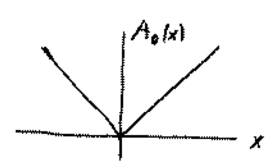
$\displaystyle\lim_{m\to0}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ikx}}{k^2+m^2} = \frac{1}{2m}e^{-m|x|}$ $\quad = \frac{1}{m^2}\delta(x)$



$\frac{d}{dx}|x| = 2\theta(x) - 1$

$\frac{d^2}{dx^2}|x| = 2\theta'(x) = 2\delta(x)$

Ex: $J^0 = \delta(x) \Rightarrow A_0(x) = -\frac{e}{2}|x|$



linear confinement

eliminate A_0 $\Rightarrow \boxed{\mathcal{L} = \bar\psi(i\not\partial - m)\psi + \dfrac{e^2}{4}\displaystyle\int_{-\infty}^{\infty} dy\, \psi^\dagger\psi(x)\,|x-y|\,\psi^\dagger\psi(y)}$

While we are here at 2d ED, consider it as a prototype example of fermion spectral flow

$$\boxed{\begin{array}{l} \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^1 \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma^2 \\ \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3 \\ \gamma^\mu\gamma^5 = -\varepsilon^{\mu\nu}\gamma_\nu \quad \varepsilon^{01} = +1 \end{array}}$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi$$

U(1) invariance:
$$\begin{cases} D_\mu = \partial_\mu + igA_\mu & G = e^{ig\alpha(x)} \\ A_\mu \to A_\mu' = A_\mu + \frac{i}{g}\partial_\mu G \cdot G^{-1} = A_\mu - \partial_\mu \alpha \\ \psi \to \psi' = G\psi \\ D_\mu\psi \to G\, D_\mu \psi \end{cases}$$

1+1 d Maxwell $\partial_\mu F^{\mu\nu} = J^\nu$, $E = F^{01} = -F^{10}$

$$\begin{cases} \partial_1 E = J^0 \\ \partial_0 E = -J^1 \end{cases} \Rightarrow \begin{cases} \partial_0\partial_1 A_1 - \partial_1^2 A_0 = J^0 \\ \partial_0\partial_1 A_0 - \partial_0^2 A_1 = J^1 \end{cases} \quad \partial_0 J^0 + \partial_1 J^1 = 0$$

$$= \frac{1}{2}\left(\partial_0 A_1 - \partial_1 A_0\right)^2 + \bar{\psi}\left\{i\left[\gamma^0(\partial_0 + igA_0) + \gamma^1(\partial_1 + igA_1)\right] - m\right\}\psi$$

q's: $\quad A_0 \quad A_1 \qquad\qquad \psi$

$$\Pi_0 = 0 \quad \Pi_1 = \frac{\partial\mathcal{L}}{\partial\partial_0 A_1} = \partial_0 A_1 \qquad \Pi_\psi = \frac{\partial\mathcal{L}}{\partial\partial_0\psi} = i\gamma^+$$

$\Downarrow$

Gauss

$$\partial_1\left(\partial_0 A_1 - \partial_1 A_0\right) = g\psi^+\psi$$

$$\left[\alpha(t,x) = \int_{t_0}^{t} dt\, A_0(t,x)\right]$$

Now take $\boxed{A_0 = 0}$: $\left[A_0(t,x) \to A_0'(t,x) = A_0(t,x) - \partial_0\alpha(t,x) = 0\right.$

$$\mathcal{H} = p\dot{q} - \mathcal{L} = \frac{1}{2}\left(\partial_0 A_1\right)^2 + \psi^+ \mathcal{H}_f\, \psi$$

$$\mathcal{H}_f = -i\gamma^0\gamma^1\big(\underbrace{\partial_1}_{+ip_1 \text{ on } e^{ipx}} + igA_1\big) + m\gamma^0 = \overbrace{\gamma^0\gamma^1}^{\equiv\gamma^5}\left(p_1 + gA_1\right) + m\gamma^0$$

$$\left(\equiv \vec{\alpha}\cdot(\vec{p} + g\vec{A}) + m\gamma^0\right)$$

Nothing special so far. Now introduce topology by finite volume and (anti) periodicity $\Rightarrow$ [string compactification!]

0 $\qquad\qquad$ L

$$\begin{cases} A_1(x+L) = A_1(x) \\ \psi(x+L) = \psi(x) \end{cases} \quad \left(\begin{array}{l}\text{could also} \\ \text{take } = -\psi(x)\end{array}\right)$$

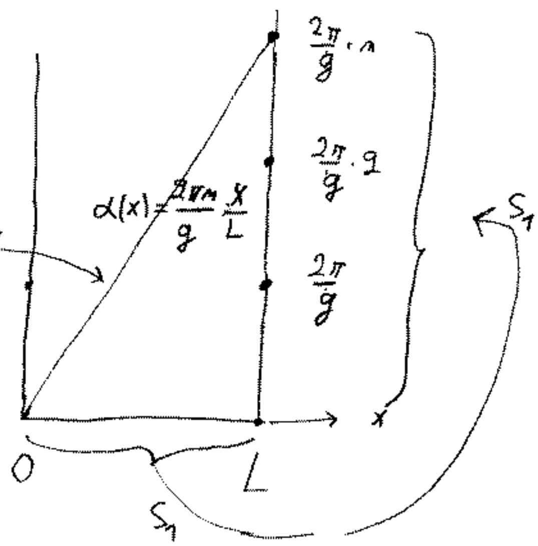
$$G(x+L) = G(x) \Rightarrow \alpha(x+L) = \alpha(x) + \frac{2\pi}{g} m$$

$$e^{ig\alpha(x+L)} = e^{ig\alpha(x) + i2\pi m}$$

After fixing $A_0 = 0$ one can still do t-independent gauge transformations on $A_1(t,x)$, $E = \partial_0 A_1(t,x)$

$$A_1 \to A_1' = A_1 - \partial_1 \alpha(x)$$

Now $\alpha(x+L) = \alpha(x) + \frac{2\pi n}{g}$



one representative of a large gauge transf.
$G(x) = e^{i 2\pi n \frac{x}{L}}$

For this

$$A_1 \to A_1' = A_1 - \frac{2\pi n}{gL}$$

$$0 \to A_1' = -\frac{2\pi n}{gL}$$ another vacuum configuration,

$$A_1^{vac} = +\frac{i}{g}\partial_1 G \cdot G^{-1} = -\partial_1 \alpha(x) = -\frac{2\pi n}{gL}$$

The gauge transf. maps $S_1 \to S_1$:

$$N_{CS} = N_{winding} = -\frac{g}{2\pi}\int_0^L dx \underbrace{A_1^{vac}(x)}_{-\frac{2\pi}{gL}n} = n \equiv \frac{1}{2\pi i}\int_0^L dx \underbrace{\partial_1 G \cdot G^{-1}}_{\equiv L_1}$$

$\boxed{\text{Wilson line: } L = e^{-ig\int_0^L dx A_1(x)}}$

Cf. 1+3d SU(2) $\quad N_{CS} = \frac{g^2}{16\pi^2}\int d^3x\, \varepsilon_{ijk}\, Tr\left(A_i F_{jk} - \frac{2}{3} i g A_i A_j A_k\right)$ can be calculated for any config; integer for A_i^{vac}

$$A_i \to G A_i G^{-1} + \frac{i}{g} L_i \qquad \to N_{CS} + \text{surface term} + \frac{1}{12\pi^2}\int d^3x\, \varepsilon_{ijk}\, Tr(L_i L_j \underbrace{L_k}_{\partial_k G\cdot G^{-1} \sim A_k^{vac}})$$

K^0

$$\partial_\mu \bar\psi \gamma^\mu \gamma^5 \psi = \underbrace{\frac{g^2 N_f}{16\pi^2} Tr F_{\mu\nu}\tilde F^{\mu\nu}}_{} = 2g^2 N_f\, \partial_\mu K^\mu$$

axial anomaly, breaks $U_A(1)$ in QCD

$F^a_{\mu\nu} F^a_{\alpha\beta}\varepsilon^{\mu\nu\alpha\beta}$ $\overset{U(1)}{A_\mu F_{\alpha\beta}\varepsilon^{\mu\alpha\beta}}$ $F_{\alpha\beta}\varepsilon^{\alpha\beta}$

1+3   1+2   1+1

Anomaly

In 1+1 the anomaly is

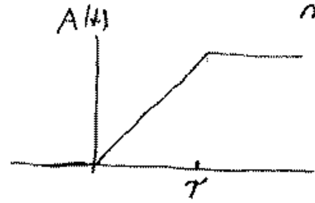
$$\partial_\mu J_5^\mu = \frac{e}{2\pi} \underbrace{\varepsilon^{\mu\nu} F_{\mu\nu}}_{= \partial_\mu (2\varepsilon^{\mu\nu} A_\nu)} + 2mi\, \bar\psi \gamma^5 \psi \overset{?}{=} -\varepsilon^{\mu\nu} \partial_\mu J_\nu$$

some anomalous current like

$$J_5^\mu = \bar\psi \gamma^\mu \gamma^5 \psi = -\varepsilon^{\mu\nu} J_\nu$$

$$i\partial_\mu J_5^\mu = -\bar\psi \gamma^5 \underbrace{i\gamma^\mu \partial_\mu \psi}_{m\psi} + \dots = -2m \bar\psi \gamma^5 \psi \quad (\text{Dirac})$$

If



then $\frac{e}{2\pi} \int_0^\tau dt \int dx\, \varepsilon_{\mu\nu} F^{\mu\nu} = \#\left\{\begin{array}{l}\text{pairs} \\ \text{particles}\end{array}\right.$ produced, $\tau \ll \frac{1}{m}$

For $\tau \gg \frac{1}{m}$ no pairs are created!

Fermion energy levels:

$$i\partial_0 \psi = H_f \psi = E\psi$$

$$\Big[\underbrace{\gamma^0\gamma^1}_{\equiv \gamma^5} (\underbrace{p_1}_{-i\partial_1} + gA_1) + m\gamma^0\Big] \psi = E\psi$$

Write: $\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} = \psi_+ \begin{pmatrix}1\\0\end{pmatrix} + \psi_- \begin{pmatrix}0\\1\end{pmatrix} = \overbrace{\frac{1+\gamma_5}{2}}^{\begin{pmatrix}1&0\\0&0\end{pmatrix}} \psi + \overbrace{\frac{1-\gamma_5}{2}}^{\begin{pmatrix}0&0\\0&1\end{pmatrix}} \psi = \psi_R + \psi_L$

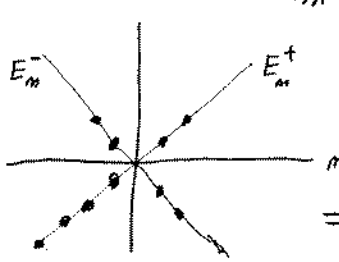
$\gamma^5 \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \quad \gamma^5 \begin{pmatrix}0\\1\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix}$ — right-moving, left-moving ($\vec\alpha = \gamma^0 \vec\gamma$ = velocity!)

$\boxed{m = 0}$

ψ_+: $\quad [-i\partial_1 + gA_1(x)]\, \psi_m^{(+)}(x) = E_m^+ \psi_m^+(x) \qquad \psi_m^+(x+L) = \psi_m^+(x)$

$$\Rightarrow \psi_m^+(x) = e^{i p_{1m} x - ig\int_0^x dy\, A_1(y)} = e^{i p_{1m}(x+L) - ig\left(\int_0^x + \int_x^{L+x}\right) dy\, A_1(y)}$$

$\int_x^{L+x} = \int_0^L$ since $A_1(y+L) = A_1(y)$



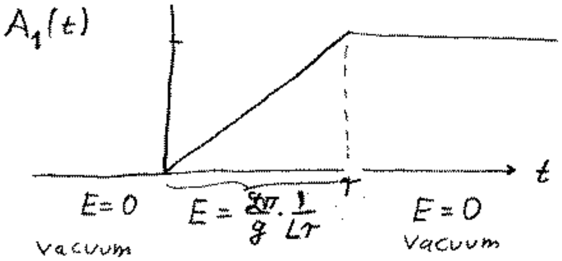
$$= \text{same} * \underbrace{e^{i p_{1m} L - ig\int_0^L dy\, A_1(y)}}_{= i2\pi m}$$

$$\Rightarrow p_{1m} = \frac{2\pi m}{L} + \frac{g}{L}\int_0^L dy\, A_1(y) \equiv \frac{2\pi}{L}(m - N_{cs})$$

$$\Rightarrow \boxed{E_m^\pm(N_{cs}) = \pm \frac{2\pi}{L}(m - N_{cs})} = \text{Eigenvalues of } D_1$$

Now <u>argue</u> adiabatic change of $A_1(x) \to A_1(t,x)$, $E = \partial_0 A_1 \neq 0$

$\sqrt{}$ $A_1(t,x) = A_1(t)$



what sets the scale here? L or $\frac{1}{m}$ or $\frac{1}{g}$?

$\left(\dim g^2 = (\text{GeV})^{4-d} = \text{GeV}^2 \quad \dim A_\mu = 1 \quad \dim F_{\mu\nu} = \dim E = \text{GeV}\right)$

slow must mean $\tau \gg \frac{1}{m}$ or $\tau \gg \frac{1}{g}$ or $\tau \approx L$

However:
Peskin-Schröder (19.13) $\Pi_{\mu\nu}(p) = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\frac{g^2}{\pi}$ from

$$D^{-1}_{\mu\nu} = D^{-1}_{\mu\nu 0} + \Pi_{\mu\nu}$$

$$= -p^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)p_\mu p_\nu + \frac{g^2}{\pi}g_{\mu\nu} - \frac{g^2}{\pi}\frac{p^\mu p^\nu}{p^2}$$

$$= \left(-p^2 + \frac{g^2}{\pi}\right)g_{\mu\nu} + p_\mu p_\nu\left[1 - \frac{1}{\xi} - \frac{g^2}{\pi p^2}\right]$$

"photon mass²" but also pole at $p^2 = 0$! Exact! (Schwinger 1962)

cf for quarks $M^2 = m^2 - \frac{g^2}{2\pi}$

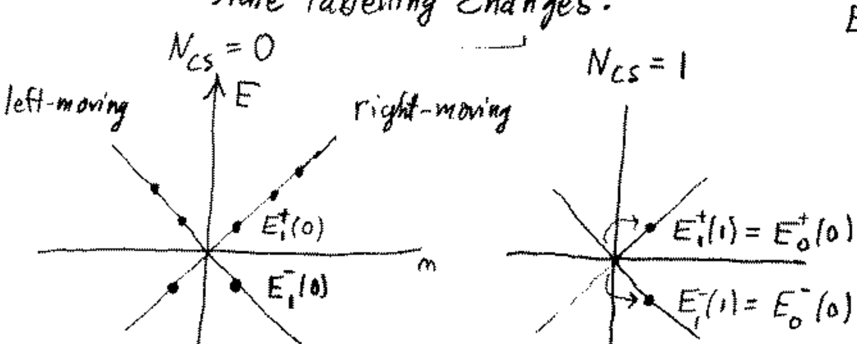
$$\overset{?}{\Rightarrow} \quad D_{\mu\nu} = \frac{1}{p^2 - \frac{g^2}{\pi}}\left\{g_{\mu\nu} - \left(1 - \xi + \frac{g^2}{\pi p^2}\xi\right)\frac{p_\mu p_\nu}{p^2}\right\}$$

We only change N_{CS}, $E_m(N_{CS}) = \frac{2\pi}{L}(m - N_{CS})$ or $\boxed{E_m(N_{CS}) = E_{m-N_{CS}}(0)}$

$\partial_\mu J^\mu_5 = \frac{e}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu}$ $\quad \int d^2x\, \partial_\mu J^\mu_5 = \frac{g}{\pi}\underbrace{\int_0^\tau dt\int_0^L dx}_{\tau\cdot L}\, \underbrace{F_{01}}_{= \text{const} = \frac{2\pi}{g}\frac{1}{L\tau}} = 2 = N_R - N_L$

state labelling changes:

$E_m(1) = E_{m-1}(0)$



so if originally only $E<0$ was occupied we now have $\begin{cases}\text{one more } \psi^+ & N_R = 1 \\ \text{one less } \psi^- & N_L = -1\end{cases}$

$N_R - N_L = 2$