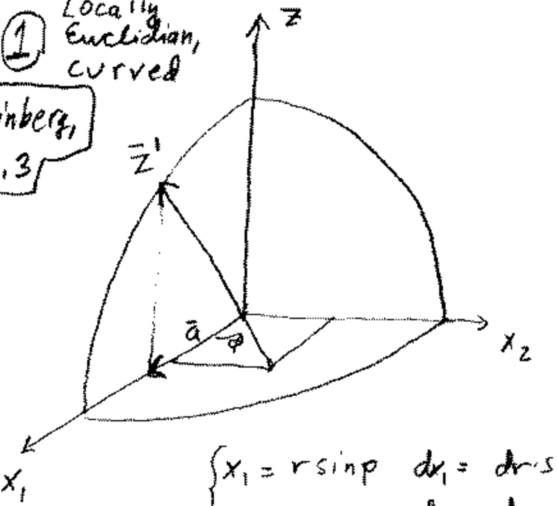
Metrics by imbedding

(1) Locally Euclidian, curved

Weinberg, 13.3



$$z^2 + \bar{x}^2 = R^2 \qquad \Big| \; z^2 - \bar{x}^2 = R^2$$

$$\Rightarrow \; z\,dz + \bar{x}\cdot d\bar{x} = 0$$

Metric

$$ds^2 = dz^2 + dx_1^2 + dx_2^2$$

$$= \frac{(x_1\,dx_1 + x_2\cdot dx_2)^2}{R^2 - \bar{x}^2} + dx_1^2 + dx_2^2$$

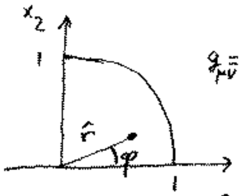
$$\begin{cases} x_1 = r\sin\varphi & dx_1 = dr\,s\varphi + r c\varphi\, d\varphi \\ x_2 = r\cos\varphi & dx_2 = dr\,c\varphi - r s\varphi\, d\varphi \end{cases}$$

$$x_1\,dx_1 + x_2\,dx_2 = r\,dr$$

$$dx_1^2 + dx_2^2 = dr^2 + r^2 d\varphi^2$$

$$= \frac{r^2 dr^2}{R^2 - r^2} + dr^2 + r^2 d\varphi^2$$

$$= \frac{dr^2}{R^2 - r^2} + r^2 d\varphi^2$$



$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{1-\hat{r}^2} & \\ & \hat{r}^2 \end{pmatrix} (R^2)^2$$

metric in x^1, x^2

$$\boxed{ds^2 = R^2\Big[\underbrace{\frac{d\hat{r}^2}{1-\hat{r}^2} + \hat{r}^2 d\varphi^2}_{d\Omega^2}\Big]}$$

$$V = \int d^2x\sqrt{g} = \int dr\,d\varphi\,\frac{rR^2}{\sqrt{1-r^2}} = \pi R\int_0^1 \frac{dy}{\sqrt{1-y}} = 2\pi R^2 = \frac{1}{2}\,(\text{Area of sphere})$$

It seems the origin is a special point. However by rotating $z \to \bar{z}'$ you can bring the origin to any point $\bar{a}$ On the sphere $(\bar{0}, 1) \to (\bar{a}, \sqrt{1-\bar{a}^2})$

$$\begin{cases} \bar{x}' = \bar{x} + \bar{a}\,(z - b\,\bar{x}\cdot\bar{a}) \\ z' = \sqrt{1-\bar{a}^2}\,z - \bar{a}\cdot\bar{x} \end{cases} \qquad \bar{x}'^2 + z'^2 = \bar{x}^2 + z^2 \quad \text{if } b = \frac{1-\sqrt{1-\bar{a}^2}}{\bar{a}^2}$$

We will generalize this to

$$\begin{cases} z^2 - y_1^2 - \;\; - y_m^2 + y_{m+1}^2 = R^2 \Rightarrow AdS_{m+1} & (\text{in the space of } y_1 \cdots y_{m+1}) \\ + z^2 + y_1^2 + \cdots + y_m^2 - y_{m+1}^2 = R^2 \Rightarrow dS_{m+1} & (\;\; - '' - \;\;) \end{cases}$$

but start from known RW (where the above is the spatial part!)

$$m = 1 \quad \begin{cases} z^2 - y_1^2 + y_2^2 = R^2 & AdS_2 \\ z^2 + y_1^2 - y_2^2 = R^2 & dS_2 \end{cases} \text{ just the same: } R = R\Gamma_\mu = 0 \text{ for } \overset{d=2}{m=1}$$

p. 34

hep-th/0212396

de Sitter, anti de Sitter from RW

Start from RW:

$$d\tau^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

$= R^2(\eta)(d\eta^2 - d\chi^2 - f^2(\chi)\, d\Omega^2)$

$d\eta = \frac{dt}{R(t)}$ $\quad d\chi = \frac{dr}{\sqrt{1-kr^2}}$

$f^2(\chi) = \operatorname{sh}\chi \quad \chi \quad \sin\chi$

$k = \ -1 \quad\ 0 \quad\ 1$

with
$$\begin{cases} \ddot{R} = -\frac{4\pi G}{3}(\varepsilon + 3p)\, R & \text{"acceleration"} \\ \dot{R}^2 + k = \frac{8\pi G}{3}\varepsilon R^2 & \text{"energy"} \end{cases}$$

and only cosmological constant $= \varepsilon_{vac} = -p_{vac}$

$R = R^{\mu\nu} g_{\mu\nu}$
$= -8\pi G(\varepsilon - 3p)$

Conventions: $\quad T^{\mu\nu} = \varepsilon_{vac}\, g^{\mu\nu}$

$$\Lambda = 8\pi G\, \varepsilon_{vac} \qquad \dim \Lambda = \frac{1}{s^2}$$

$$\Rightarrow \boxed{\dot{R}^2 + k = \frac{\Lambda}{3} R^2}$$

$k = -1$ open $\quad \varepsilon < \varepsilon_{cr}$
$k = +1$ closed $\quad \varepsilon > \varepsilon_{cr}$

$\operatorname{ch}^2 - \operatorname{sh}^2 = 1$

$\Lambda > 0$

$k=0 \qquad \dot{R}^2/R^2 = \frac{\Lambda}{3}$ $\quad$ must be $\Lambda > 0 \qquad R(t) = e^{\pm\sqrt{\frac{\Lambda}{3}}\, t}$

$\Lambda > 0 \qquad R = \sqrt{\frac{3}{\Lambda}}\ \operatorname{sh}\sqrt{\frac{\Lambda}{3}}\, t \qquad \dot{R} = \operatorname{ch}\sqrt{\frac{\Lambda}{3}}\, t \qquad \dot{R}^2 = \operatorname{sh}^2\sqrt{\frac{\Lambda}{3}}\, t + 1 = \frac{\Lambda}{3}R^2 + 1 \quad k=-1$

$\qquad = \sqrt{\frac{3}{\Lambda}}\ \operatorname{ch}\sqrt{\frac{\Lambda}{3}}\, t \qquad \dot{R} = \operatorname{sh}\sqrt{\frac{\Lambda}{3}}\, t \qquad \dot{R}^2 = \operatorname{ch}^2\sqrt{\ }\, t - 1 = \frac{\Lambda}{3}R^2 - 1 \quad k=+1$

Claim: All these $\Lambda > 0$ sol's are de Sitter space but in different coordinates!

$\Lambda < 0 \qquad \dot{R}^2 + k = -\frac{|\Lambda|}{3}R^2 \Rightarrow k = -1 \qquad R(t) = \sqrt{\frac{3}{-\Lambda}}\ \sin\sqrt{\frac{-\Lambda}{3}}\, t$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu} \quad (+8\pi G T_{\mu\nu})$$

same as $G_{\mu\nu} = 0$ with $T_{\mu\nu} = \frac{\Lambda}{8\pi G}g_{\mu\nu}$

$$\Rightarrow R - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = R\left(1 - \frac{1}{2}d\right) = -\Lambda d \Rightarrow R = \Lambda\frac{-d}{1-\frac{1}{2}d} = \Lambda\frac{2d}{d-2}$$

$\Lambda > 0$ for de Sitter

$$R_{\mu\nu} = \left(\frac{1}{2}R - \Lambda\right)g_{\mu\nu} = \left(\frac{d}{d-2} - 1\right)\Lambda g_{\mu\nu} = \frac{2}{d-2}\Lambda g_{\mu\nu}$$

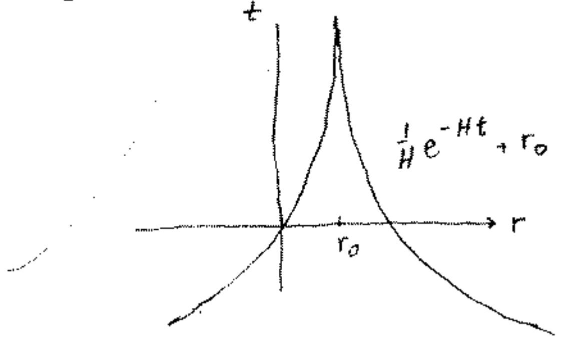
$$R_{\mu\nu\alpha\beta} = \Lambda\frac{2}{(d-1)(d-2)}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})$$

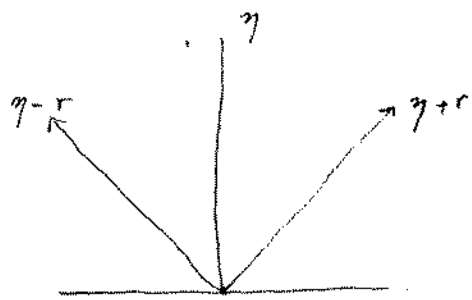
These are valid for all "maximally symmetric spaces", characterised by a "curvature" $\sim\Lambda$ & signature of metric

Other forms of de Sitter:

1. $$\boxed{ds^2 = dt^2 - e^{2Ht}(dr^2 + r^2 d\Omega^2)} \qquad H^2 = \frac{\Lambda}{3}$$

Light: $\pm dt\, e^{-Ht} = dr \qquad \pm\frac{1}{H} e^{-Ht} = r - r_0$





$dt = e^{Ht} d\eta$

2. $\eta = -\frac{1}{H} e^{-Ht} \qquad d\eta = e^{-Ht} dt$

$$\boxed{ds^2 = \frac{1}{(H\eta)^2}\left[d\eta^2 - dr^2 - r^2 d\Omega^2\right]}$$

3. Static coord's
$(t, r) \to (\tau, \rho)$
SW, (13.3.42)

$$\begin{cases} r = e^{-H\tau} \dfrac{\rho}{\sqrt{1-H^2\rho^2}} \\ t = \tau + \dfrac{1}{2H} \ln(1-H^2\rho^2) \end{cases} \qquad \begin{cases} dr = r\left[-H d\tau + \dfrac{1}{1-H^2\rho^2}\dfrac{d\rho}{\rho}\right] \\ dt = d\tau - \dfrac{H\rho}{1-H^2\rho^2}\, d\rho \end{cases}$$

$$e^{2Ht} = e^{2H\tau}(1-H^2\rho^2)$$

$$\boxed{ds^2 = (1-H^2\rho^2)\, d\tau^2 - \frac{1}{1-H^2\rho^2} d\rho^2 - \rho^2 d\Omega^2} \qquad \text{Seems static!}$$

Light $\quad d\tau = \dfrac{1}{1-H^2\rho^2} d\rho \quad \Rightarrow \quad \boxed{H\rho = \tanh H\tau} \qquad H d\rho = \dfrac{1}{ch^2 H\tau} H d\tau$

$= (1 - th^2 H\tau) H d\tau$

$= (1 - H^2\rho^2) H d\tau$

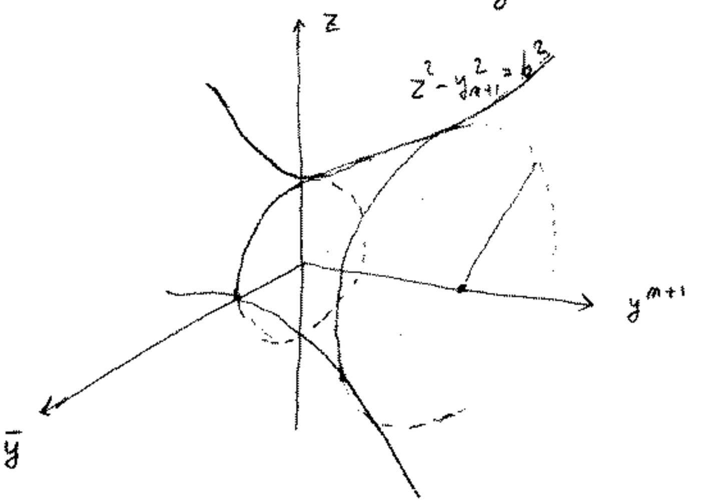
Classical geometry of de Sitter
Kim - Oh - Park hep-th/0212326

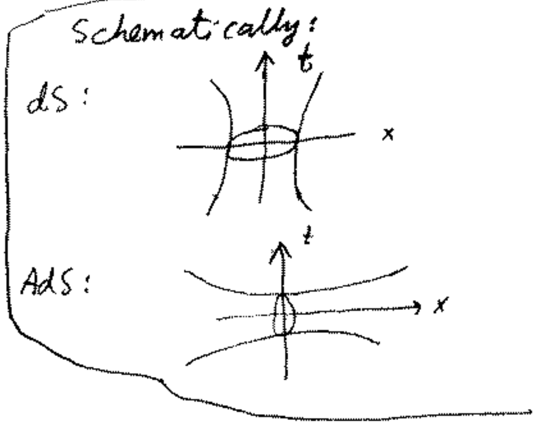
dS_{m+1} with these conventions:

$$Z_4^2 + Z_1^2 + Z_2^2 + Z_3^2 \quad - Z_0^2$$

← notation in Birrel-Davies (5.49) for dS_4

Constraint $\quad Z^2 + \underbrace{y_1^2 + \cdots + y_m^2}_{\equiv \bar{y}^2} - y_{m+1}^2 = b^2$





(1) $\begin{cases} Z = b\, x^0 \operatorname{ch}\frac{t}{b} \\ y^i = b\, x^i \operatorname{ch}\frac{t}{b} \quad i=1,\dots,m \\ y^{m+1} = b \operatorname{sh}\frac{t}{b} \end{cases}$ $\quad ds^2 = dz^2 + d\bar{y}^2 - (dy^{m+1})^2$

$$\underset{b=1}{=} (\operatorname{ch}\tau\, dx^\mu + x^\mu \operatorname{sh}\tau\, dt)^2 - \operatorname{ch}^2\tau\, dt^2$$

$$= \operatorname{ch}^2\tau\, (dx^\mu)^2 + x_\mu x_\mu \operatorname{sh}^2\tau\, dt^2 - \operatorname{ch}^2\tau\, dt^2$$

$\sum_0^m x_\mu x_\mu = 1$

$\Rightarrow x_\mu dx_\mu = 0$

$$\boxed{ds^2 = -dt^2 + b^2 \operatorname{ch}^2\frac{t}{b}\, d\Omega_m^2} \underset{(m=3)}{=} -dt^2 + b^2\operatorname{ch}^2\frac{t}{b}\left[d\theta_0^2 + \sin^2\theta_0 \cdot (d\theta_1^2 + \sin^2\theta_1 d\theta_2^2)\right]$$

$r = \sin\theta_0$

$d\theta_0 = \dfrac{dr}{\sqrt{1-r^2}}$

p. 30: $k = +1$ RW

$R(t) = b\operatorname{ch}\frac{t}{b}$ $\quad b = \sqrt{\frac{3}{\Lambda}}$

$$d\Omega_m^2 = d\theta_0^2 + \sin^2\theta_0\, d\theta_1^2 + \cdots + \sin^2\theta_0 \cdots \sin^2\theta_{m-2}\, d\theta_{m-1}^2$$

$\begin{cases} x^0 = \cos\theta_0 \\ x^1 = \sin\theta_0 \cos\theta_1 \\ x^2 = \sin\theta_0 \sin\theta_1 \cos\theta_2 \\ x^{m-1} = \sin\theta_0 \cdots \cos\theta_{m-1} \\ x^m = \sin\theta_0 \cdots \sin\theta_{m-1} \end{cases}$

$0 \le \theta_{0,1} \le \pi$

$0 \le \theta_2 \le 2\pi$

(2) $\quad m=3$ $\quad \begin{cases} z = \sqrt{b^2 - r^2}\, \operatorname{ch}\frac{t}{b} \\ y_{m+1} = \sqrt{b^2 - r^2}\, \operatorname{sh}\frac{t}{b} \\ \left.\begin{array}{l} y_1 = \\ y_2 = \\ y_3 = \end{array}\right\} \text{spherical } r\ \theta\ \varphi \end{cases}$

$$z^2 - y_{m+1}^2 = b^2 - r^2$$

$$\Rightarrow z^2 + \bar{y}^2 - y_{m+1}^2 = b^2$$

$$\bar{y}^2 = r^2$$

$z + y_{m+1} = \sqrt{b^2 - r^2}\, e^{t/b} > 0$

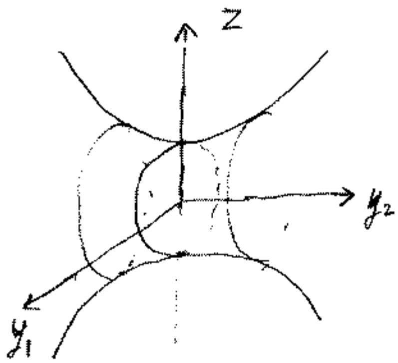
Coordinate singularity at $r = b$

$$\begin{cases} dz = \dfrac{-r}{\sqrt{b^2-r^2}} \operatorname{ch}\frac{t}{b}\, dr + \sqrt{b^2-r^2}\, \frac{1}{b} \operatorname{sh}\frac{t}{b}\, dt \\ dy_{m+1} = \dfrac{-r}{\sqrt{b^2-r^2}} \operatorname{sh}\frac{t}{b}\, dr + \sqrt{b^2-r^2}\, \frac{1}{b} \operatorname{ch}\frac{t}{b}\, dt \end{cases}$$

$$ds^2 = -\left(1 - \frac{r^2}{b^2}\right) dt^2 + \frac{r^2}{b^2 - r^2} dr^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$\boxed{ds^2 = -\left(1 - \frac{r^2}{b^2}\right) dt^2 + \frac{1}{1 - \frac{r^2}{b^2}} dr^2 + r^2\, d\Omega_2^2} = \text{"static" form}$$

<u>dS by Embedding:</u> see also p.33 as before, but now



$$x_1 \; x_2 \Rightarrow \underset{x^1\ x^2\ x^3}{- \; - \; -} \quad \underset{x^{n+1}}{+}$$

dSitter choose $\boxed{z^2 + y_1^2 + \dots + y_n^2 - y_{n+1}^2 = R^2}$

(Weinberg 13.3.39 has $+z^2 + \bar{x}^2 - t^2 = \frac{1}{K} = R^2$)

$\eta = (+ \quad +++ \quad -)$

$$\begin{cases} d\tau^2 = dt^2 - d\bar{x}^2 - dz^2 \\ z\,dz + \bar{x}\cdot d\bar{x} - t\,dt = 0 \end{cases}$$

$$d\tau^2 = -dt^2 + d\bar{x}^2 + \frac{(t\,dt - \bar{x}\cdot d\bar{x})^2}{R^2 + t^2 - \bar{x}^2} \qquad (13.3.39) \quad R^2 = \frac{1}{K}$$

$$\begin{cases} t = R\left[\frac{1}{2}\hat{r}'^2 \,\mathrm{ch}\, \hat{t}' + (1 + \frac{1}{2}\hat{r}'^2)\,\mathrm{sh}\,\hat{t}'\right] \\ \bar{x} = \bar{x}' e^{\hat{t}'} \end{cases} \qquad \hat{r} = \frac{r}{R} \text{ etc}$$

$$\Rightarrow \boxed{d\tau^2 = dt'^2 - e^{2\hat{t}'} d\bar{x}'^2} = \text{flat RW metric}$$

<u>AdS by embedding</u>

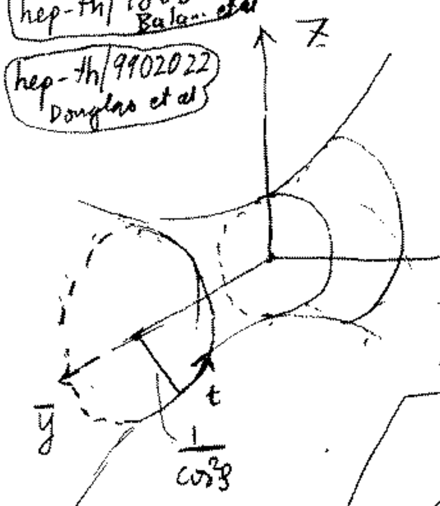
Now start from $\left(+ \overset{1\ 2\ \dots\ n}{- - - -} \overset{n+1}{+}\right) = \eta_{ab}$

$\quad z \quad \bar{y} \quad t$

or: $y^0 \quad y^i \quad y^{n+1}$

See hep-th/9805171 Bala.. et al

hep-th/9902022 Douglas et al

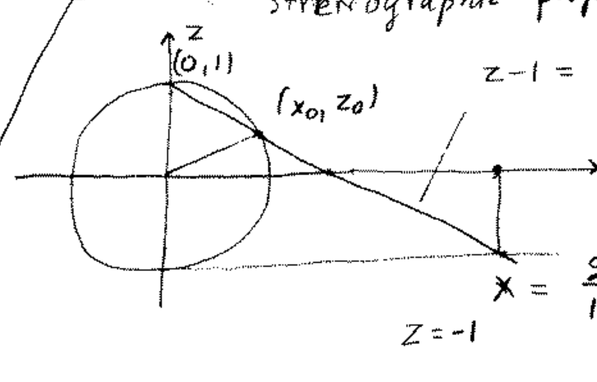


$$\boxed{z^2 - \bar{y}^2 + (y^{n+1})^2 = b^2}$$

One could compute $ds^2 = g_{\mu\nu}\, dy^\mu dy^\nu$ as above but it is more convenient to directly introduce new coordinates satisfying this constraint.

– induces AdS_{n+1} in $\bar{y}, y^{n+1}$

Sterеographic projection $S^1 \to R^1$



$$z - 1 = \frac{z_0 - 1}{x_0}\, x$$

mapping $S^1 \to R^1$

$$(x_0, z_0) \to X = \frac{2x^0}{1 - z_0}$$

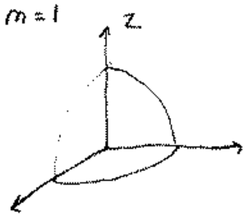
$x_0^2 + z_0^2 = 1$

$$X = \frac{2x^0}{1 - z_0}$$

$Z = -1$

① Implement $z^2 - \bar{y}^2 + (y^{m+1})^2 = b^2\ (=1)$ $\quad z^2 + (y^{m+1})^2 = b^2 + \bar{y}^2 \ge b^2$

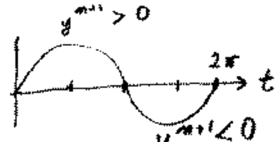
$m=1$

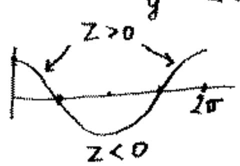


by $\boxed{z = \frac{\cos t}{\cos\rho},\quad y^i = z_i \tan\rho,\quad y^{m+1} = \frac{\sin t}{\cos\rho},\quad \sum_1^m z_i^2 = 1}$
$i = 1, \dots, m$

$\tan\rho > 0 \quad \cos\rho > 0$

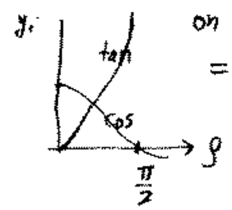
$$\frac{\cos^2 t + \sin^2 t}{\cos^2\rho} - \sum_{i=1}^m z_i^2 \tan^2\rho = \frac{1}{\cos^2\rho} - \tan^2\rho = 1$$

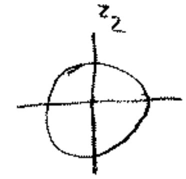




$\boxed{0 \le t < 2\pi \quad 0 \le \rho \le \frac{\pi}{2} \quad z_i \in S^{m-1}}$

~~angle around~~ circle $\Rightarrow$ radius of the circle on z, y^{m+1} plane $= \frac{1}{\cos^2\rho}$





$$ds^2 = (dz)^2 - d\bar{y}^2 + (dy^{m+1})^2$$

$$= \left(-\frac{\sin t}{\cos\rho} dt + \cos t \frac{\sin\rho}{\cos^2\rho} d\rho\right)^2 - \sum_i \left(\tan\rho\, dz_i + z_i \frac{1}{\cos^2\rho} d\rho\right)^2$$
$$+ \left(\frac{\cos t}{\cos\rho} dt + \sin t \frac{\sin\rho}{\cos^2\rho} d\rho\right)^2$$

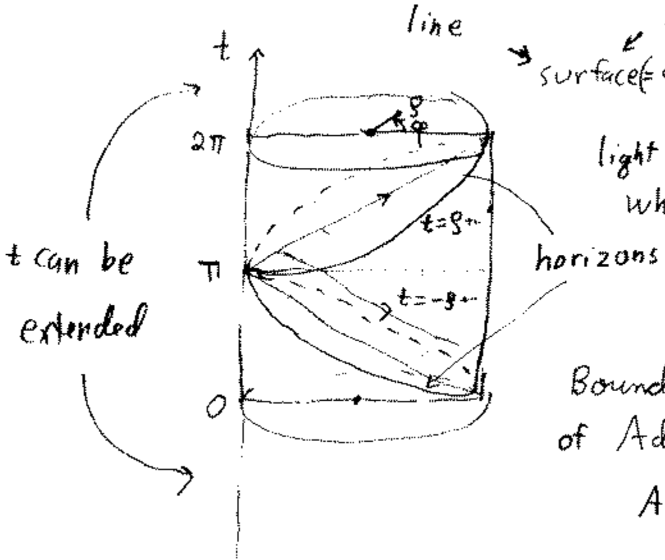
$\boxed{AdS_{m+1} \text{ is}}$

$$\Rightarrow \boxed{-ds^2/R^2 = -\frac{1}{\cos^2\rho} dt^2 + \frac{1}{\cos^2\rho} d\rho^2 + \tan^2\rho\, d\Omega_{m-1}^2}$$

$d\Omega_{m-1}$ surf. element of S^{m-1}

$m=2$:
AdS_3 $\quad -\frac{ds^2}{R^2} = \underbrace{-\frac{1}{\cos^2\rho} dt^2}_{\text{line}} + \underbrace{\frac{1}{\cos^2\rho} d\rho^2 + \tan^2\rho\, d\phi^2}_{\text{disc}}$

$\rightarrow$ surface(=cylinder) is a boundary at spatial infinity



light: $dt = \pm d\rho \Rightarrow t = \pm\rho + C$
What happens at $\rho = 0$?

CFT on cylinder = quantum gravity in bulk

$\nearrow R^{m-1}$ by addings points at ∞

Boundary of AdS_{m+1} is $t,\ S^{m-1}$ = cylinder
$AdS_5 \Rightarrow t,\ S^3$

(2) Here is a 2nd way of enforcing

$$z^2 - y_1^2 - \cdots - y_m^2 + y_{m+1}^2 = R^2 \ :$$

$$\boxed{Z = \rho\,\frac{1+x^2}{1-x^2} \qquad y^i = \rho\,\frac{2x^i}{1-x^2} \qquad y^{m+1} = \rho\,\frac{2x^{m+1}}{1-x^2}}$$

$$\rho^2\,\frac{1}{(1-x^2)^2}\Big[(1+x^2)^2 - 4\,\bar{x}^2 + 4(x^{m+1})^2\Big] = \rho^2$$

$$\text{if} \quad x^2 \equiv \bar{x}^2 - (x^{m+1})^2 \equiv \eta_{\mu\nu}\,x^\mu x^\nu$$

$$\begin{cases} dz = d\rho\,\dfrac{1+x^2}{1-x^2} + \rho\,\dfrac{2}{(1-x^2)^2}\,2x_\mu\,dx^\mu \\[2ex] dy^\mu = d\rho\,\dfrac{2x^\mu}{1-x^2} + \underbrace{2\rho\left[\dfrac{1}{1-x^2}\,dx^\mu + x^\mu\,\dfrac{2x_\nu\,dx^\nu}{(1-x^2)^2}\right]}_{+\,2\rho\,\frac{1}{(1-x^2)^2}\left[\delta^\mu{}_\nu\,(1-x^2) + 2x^\mu x_\nu\right]dx^\nu} \end{cases} \qquad d\,\frac{x^\mu}{1-x^2} \qquad dx^2 = 2x_\mu\,dx^\mu$$

$$\Rightarrow \quad dz^2 - \eta_{\mu\nu}\,dy^\mu dy^\nu = d\rho^2 - \frac{4\rho^2}{(1-x^2)^2}\,dx^2 \qquad \text{Take } \rho^2 = b^2$$

$$\Rightarrow \boxed{AdS_{m+1} \text{ is } \quad ds^2 = b^2\,\frac{4}{(1-x^2)^2}\,\eta_{\mu\nu}\,dx^\mu dx^\nu \quad (\underbrace{+\,+\cdots+}_{m}\,-)}$$

This metric:

$$g_{\mu\nu} = \frac{4b^2}{(1-x^2)^2}\,\eta_{\mu\nu} \qquad x^2 = \eta_{\mu\nu}\,x^\mu x^\nu$$

should be the extremum of

$$\int d^{m+1}x\,\sqrt{g}\left[R + \frac{m(m-1)}{b^2}\right] \quad , \text{ i.e., be the solution of}$$

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\left(R + \frac{m(m-1)}{b^2}\right) = 0$$

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R - \frac{m(m-1)}{2b^2}\,g_{\mu\nu} = 0 \qquad \Lambda = -\frac{m(m-1)}{2b^2}$$

This looks like a good exercise in $g_{\mu\nu}$, $\Gamma^\lambda_{\mu\nu}$, $R^\mu{}_{\nu\lambda\rho}$:

$$g^{\mu\nu} = e^{-\phi(x)}\eta^{\mu\nu}$$

$$g_{\mu\nu} = \frac{4b^2}{(1-x^2)^2}\eta_{\mu\nu} = e^{\log 4b^2 - 2\log(1-x^2)}\eta_{\mu\nu} \equiv e^{\phi(x)}\eta_{\mu\nu}$$

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\lambda}\big(\underbrace{\partial_\nu g_{\rho\lambda}}_{\partial_\nu\phi\cdot e^\phi\eta_{\rho\lambda}} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho}\big)$$

$$\partial_\rho \Big| \quad = \frac{1}{2}\left[\partial_\nu\phi\,\delta^\mu_\rho + \partial_\rho\phi\,\delta^\mu_\nu - \partial^\mu\phi\,\eta_{\nu\rho}\right]$$

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu$$
$$= \eta_{ab}\,de^a\,de^b$$
$$= g_{\mu\nu}\frac{dx^\mu}{de^a}\frac{dx^\nu}{de^b}de^a de^b$$
$$= \eta_{ab}\frac{de^a}{dx^\mu}\frac{de^b}{dx^\nu}dx^\mu dx^\nu$$

$$g_{\mu\nu} = \eta_{ab}\frac{\partial e^a}{\partial x^\mu}\frac{\partial e^b}{\partial x^\nu}$$

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial e^a}\frac{\partial^2 e^a}{\partial x^\mu\partial x^\nu}$$

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\lambda\rho}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma}\Gamma^\lambda_{\nu\rho}$$

$$[\nabla_\rho\nabla_\sigma - \nabla_\sigma\nabla_\rho]V_\nu = +V_\mu R^\mu{}_{\nu\rho\sigma}$$

$$\partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} = \frac{1}{2}\left[\partial_\rho\partial_\nu\phi\,\delta^\mu_\sigma + \cancel{\partial_\rho\partial_\sigma\phi}\,\delta^\mu_\nu - \eta_{\nu\sigma}\partial_\rho\partial^\mu\phi - (\rho\leftrightarrow\sigma)\right]$$

$$\Gamma^\mu_{\lambda\rho}\Gamma^\lambda_{\nu\sigma} - (\rho\leftrightarrow\sigma) = \frac{1}{4}\left(\partial_\lambda\phi\,\delta^\mu_\rho + \partial_\rho\phi\,\delta^\mu_\lambda - \partial^\mu\phi\,\eta_{\lambda\rho}\right)\left(\partial_\nu\phi\,\delta^\lambda_\sigma + \partial_\sigma\phi\,\delta^\lambda_\nu - \partial^\lambda\phi\,\eta_{\nu\sigma}\right) - (\cdot)$$

$$= \frac{1}{4}\Big[\partial_\sigma\phi\,\partial_\nu\phi\,\delta^\mu_\rho + \partial_\nu\phi\,\partial_\sigma\phi\,\delta^\mu_\rho - \partial_\lambda\phi\,\partial^\lambda\phi\,\delta^\mu_\rho\,\eta_{\nu\sigma}$$
$$+ \partial_\rho\phi\,\partial_\nu\phi\,\delta^\mu_\sigma + \cancel{\partial_\rho\phi\,\partial_\sigma\phi}\,\delta^\mu_\nu - \partial_\rho\phi\,\partial^\mu\phi\,\eta_{\nu\sigma}$$
$$- \partial^\mu\phi\,\partial_\nu\phi\,\eta_{\sigma\rho} - \partial^\mu\phi\,\partial_\sigma\phi\,\eta_{\nu\rho} + \partial^\mu\phi\,\partial_\rho\phi\,\eta_{\nu\sigma} - (\sigma\leftrightarrow\rho)$$

$$= \frac{1}{4}\Big[2\partial_\nu\phi\,\partial_\sigma\phi\,\delta^\mu_\rho - \delta^\mu_\rho(\partial\phi)^2\eta_{\nu\sigma} + \partial_\rho\phi\,\partial_\nu\phi\,\delta^\mu_\sigma - \cancel{\partial_\rho\phi\,\partial^\mu\phi}\,\eta_{\nu\sigma} - \cancel{\partial^\mu\phi\,\partial_\sigma\phi}\,\eta_{\nu\rho} + \boxed{\partial^\mu\phi\,\partial_\rho\phi\,\eta_{\nu\sigma}}$$
$$- 2\partial_\nu\phi\,\partial_\rho\phi\,\delta^\mu_\sigma + \delta^\mu_\sigma(\partial\phi)^2\eta_{\nu\rho} - \partial_\sigma\phi\,\partial_\nu\phi\,\delta^\mu_\rho + \cancel{\partial_\sigma\phi\,\partial^\mu\phi}\,\eta_{\nu\rho} + \cancel{\partial^\mu\phi\,\partial_\rho\phi}\,\eta_{\nu\sigma} - \partial^\mu\phi\,\partial_\sigma\phi\,\eta_{\nu\rho}\Big]$$

$$= \frac{1}{4}\Big[\partial_\nu\phi\,\partial_\sigma\phi\,\delta^\mu_\rho - \partial_\nu\phi\,\partial_\rho\phi\,\delta^\mu_\sigma + \partial^\mu\phi\,\partial_\rho\phi\,\eta_{\nu\sigma} - \partial^\mu\phi\,\partial_\sigma\phi\,\eta_{\nu\rho} + \delta^\mu_\sigma(\partial\phi)^2\eta_{\nu\rho}$$
$$- \delta^\mu_\rho(\partial\phi)^2\eta_{\nu\sigma}\Big]$$

For general $\phi(x)$, $g_{\mu\nu} = e^{\phi(x)}\eta_{\mu\nu}$, from the above

$$\boxed{R_{\mu\nu} = \left(1-\frac{d}{2}\right)\left(\partial_\mu\partial_\nu\phi - \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\right) + \frac{1}{2}\eta_{\mu\nu}\left[\left(1-\frac{d}{2}\right)\partial_\alpha\phi\,\partial^\alpha\phi - \partial_\alpha\partial^\alpha\phi\right]}$$

$$= R_{\mu\underbrace{\alpha\nu\rho}}$$

For the present $\phi(x) = \text{const} - 2\log(1-x^2)$:

$$\begin{cases} \partial_\mu\phi\,\partial_\nu\phi = \dfrac{16\, x_\mu x_\nu}{A^2} & \qquad A \equiv 1-x^2 \\[2ex] \partial_\mu\partial_\nu\phi = \dfrac{4}{A}\eta_{\mu\nu} + \dfrac{8}{A^2}x_\mu x_\nu \end{cases}$$

$$R^\mu{}_{\nu\rho\sigma} = \frac{1}{2}\Big[\Big(\frac{4}{A}\eta_{\rho\nu} + \frac{8}{A^2}x_\rho x_\nu\Big)\delta^\mu_\sigma - \eta_{\nu\sigma}\Big(\frac{4}{A}\delta^\mu_\rho + \frac{8}{A^2}x_\rho x^\mu\Big)$$
$$- \Big(\frac{4}{A}\eta_{\sigma\nu} + \frac{8}{A^2}x_\sigma x_\nu\Big)\delta^\mu_\rho + \eta_{\nu\rho}\Big(\frac{4}{A}\delta^\mu_\sigma + \frac{8}{A^2}x_\sigma x^\mu\Big)\Big] + \Gamma\Gamma - \Gamma\Gamma$$

$$= \frac{1}{2}\Big[\delta^\mu_\sigma\Big(\frac{8}{A}\eta_{\nu\rho} + \boxed{\frac{8}{A^2}x_\nu x_\rho}\Big) - \delta^\mu_\rho\Big(\frac{8}{A}\eta_{\nu\sigma} + \cancel{\frac{8}{A^2}x_\sigma x_\nu}\Big)$$
$$- \frac{8}{A^2}\eta_{\nu\sigma}\, x_\rho x^\mu + \frac{8}{A^2}\eta_{\nu\rho}\, x_\sigma x^\mu\Big]$$

$$+ \frac{1}{4}\Big\{\cancel{\frac{16}{A^2}x_\nu x_\sigma}\,\delta^\mu_\rho\ \boxed{-\frac{16}{A^2}x_\nu x_\rho\,\delta^\mu_\sigma} + \frac{16}{A^2}x^\mu x_\rho\,\eta_{\nu\sigma} - \frac{16}{A^2}x^\mu x_\sigma\,\eta_{\nu\rho}$$
$$+ \delta^\mu_\sigma\,\eta_{\nu\rho}\frac{16x^2}{A^2} - \delta^\mu_\rho\,\eta_{\nu\sigma}\frac{16x^2}{A^2}\Big\}$$

$$= \frac{4}{A}\left(\delta^\mu_\sigma\eta_{\nu\rho} - \delta^\mu_\rho\eta_{\nu\sigma}\right) + \frac{4x^2}{A^2}\left(\delta^\mu_\sigma\eta_{\nu\rho} - \delta^\mu_\rho\eta_{\nu\sigma}\right)$$

$$= \frac{4}{A}\Big(1 + \frac{x^2}{A}\Big)\left(\delta^\mu_\sigma\eta_{\nu\rho} - \delta^\mu_\rho\eta_{\nu\sigma}\right) = + \frac{4}{(1-x^2)^2}\left(\delta^\mu_\sigma\eta_{\nu\rho} - \delta^\mu_\rho\eta_{\nu\sigma}\right)$$

$$\boxed{R^\mu{}_{\nu\rho\sigma} = \frac{4}{(1-x^2)^2}\left(\delta^\mu_\sigma\eta_{\nu\rho} - \delta^\mu_\rho\eta_{\nu\sigma}\right) = \frac{1}{b^2}\left(\delta^\mu_\sigma g_{\nu\rho} - \delta^\mu_\rho g_{\nu\sigma}\right)}$$

$$\boxed{\frac{1}{b^2}g_{\mu\nu} = \frac{4}{(1-x^2)^2}\eta_{\mu\nu}}$$

$D \equiv m+1$

$$\Rightarrow R_{\mu\rho} = R^{\sigma}{}_{\mu\sigma\rho} = \frac{1}{b^2}\Big(g_{\mu\sigma}\delta^\sigma_\rho - g_{\mu\rho}\underbrace{\delta_\sigma{}^\sigma}_{=D}\Big) = \frac{1}{b^2}g_{\mu\rho}\big(1-(m+1)\big) = -\frac{m}{b^2}g_{\mu\rho}$$

$$R = -\frac{1}{b^2}m(m+1) = \frac{2(m+1)}{m-1}\Lambda \equiv \frac{2d}{d-2}\Lambda < 0$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{m}{b^2}g_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\frac{1}{b^2}m(m+1) = \frac{1}{b^2}g_{\mu\nu}\Big(\underbrace{-m + \tfrac{1}{2}(m^2+m)}_{\frac{1}{2}(m^2-m)}\Big) = \frac{m(m-1)}{2b^2}g_{\mu\nu}$$

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{m(m-1)}{2b^2}g_{\mu\nu} = 0} \Rightarrow \boxed{\Lambda = \text{"}8\pi G\varepsilon_{\text{vac}}\text{"} = -\frac{m(m-1)}{2b^2} < 0}$$

③ 3rd way of implementing ("Poincaré coordinates")

$$Z^2 - y_1^2 - y_2^2 - \cdots - y_{m-1}^2 - y_m^2 + y_{m+1}^2 = b^2$$

$$b\frac{t}{r} \qquad b\frac{x_1}{r} \qquad b\frac{x_2}{r} \qquad b\frac{x_{m-1}}{r} \qquad \frac{1}{2r}(-b^2+\bar{x}^2+r^2-t^2) \qquad y_m + y_{m+1} = -\frac{b^2}{r} \le 0$$

$$\frac{-1}{2r}(b^2+\bar{x}^2+r^2-t^2)$$

$$\boxed{-\infty \le t \le \infty \quad -\infty < x_i < \infty \quad 0 \le r \le \infty}$$

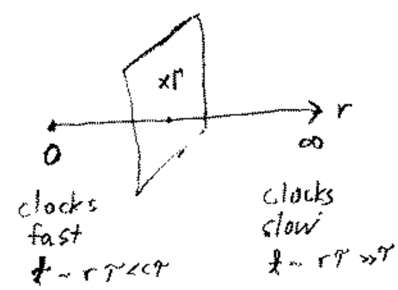
$$\frac{b^2}{r^2}(t^2-\bar{x}^2) - \frac{1}{4r^2}\Big[\underbrace{(-b^2+\bar{x}^2+r^2-t^2)^2 - (b^2+\bar{x}^2+r^2-t^2)^2}_{-4b^2(\bar{x}^2+r^2-t^2)}\Big]$$

$$= \frac{b^2}{r^2}\left[t^2-\bar{x}^2+(\bar{x}^2+r^2-t^2)\right] = b^2 \quad !$$

$\boxed{AdS_{m+1} \text{ is}}$

$$\Rightarrow \boxed{ds^2 = \underbrace{\frac{b^2}{r^2}}_{\text{warp factor}}(-dt^2 + dx_1^2 + \cdots + dx_{m-1}^2 + dr^2)}$$

inv. under $x^M = (x^\mu, r) \to \lambda x^M$



Again conformally flat!

Exercise: $ds^2 = e^{\phi(x^m)}\eta_{\mu\nu}dx^\mu dx^\nu \qquad x^0, x^1, \ldots, x^{m-1}, x^m$

$\phi(x) = \log b^2 - 2\log(x^m) \qquad g_{\mu\nu} = \frac{b^2}{x_m^2}\eta_{\mu\nu} \qquad \partial_\mu\phi = -\frac{2}{x^m}\delta_\mu^m$

bottom of p. 35: $\qquad \partial_\mu\partial_\nu\phi = \frac{2}{(x^m)^2}\delta_\mu^m\delta_\nu^m$

$$R_{\mu\nu} = \left(1-\frac{d}{2}\right)\underbrace{\left(\frac{2}{(x^m)^2}\delta_\mu^m\delta_\nu^m - \frac{1}{2}\cdot\frac{4}{(x^m)^2}\delta_\mu^m\delta_\nu^m\right)}_{=0} + \frac{1}{2}\eta_{\mu\nu}\underbrace{\left[\left(1-\frac{d}{2}\right)\frac{4}{(x^m)^2}\delta_\mu^m\delta_\mu^m - \frac{2}{(x^m)^2}\delta_\mu^m\delta_\mu^m\right]}_{\eta_{\mu\nu}\cdot\frac{1}{(x^m)^2}(1-d)}$$

$$\boxed{R_{\mu\nu} = -\frac{d-1}{(x^m)^2}\eta_{\mu\nu} = -\frac{d-1}{b^2}g_{\mu\nu}}$$

as should for AdS_{m+1} $\quad d = m+1$

Compare (p. 15):

$$ds^2 = \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2$$

$$= \frac{r^2}{R^2}\Big[\eta_{\mu\nu}dx^\mu dx^\nu + \underbrace{\left(\frac{R^2}{r^2}dr\right)^2}_{=d\rho^2} + \frac{R^4}{r^2}d\Omega_5^2\Big] \qquad \frac{R}{\rho} = -\frac{r}{R}$$

$$= \frac{R}{\rho^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + d\rho^2\right) + R^2 d\Omega_5^2$$

back to the above!