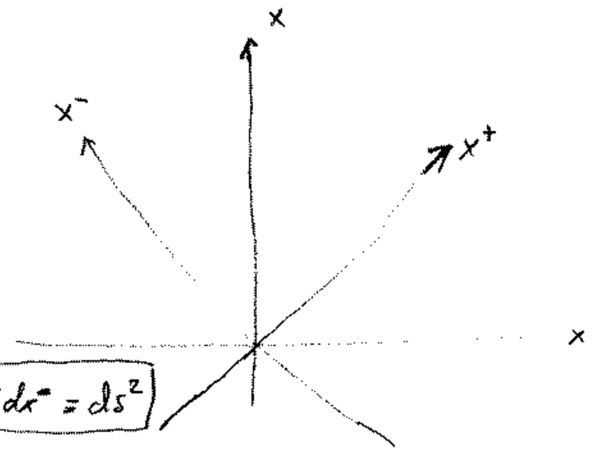
Local → global structure

Ex. 1 Flat space

$$\boxed{ds^2 = dt^2 - dx^2} + r^2 d\Omega_3^2$$



1) $-\infty < x^{\pm} = \dfrac{t \pm x}{\sqrt{2}} < \infty$

$t = \dfrac{x^+ + x^-}{\sqrt{2}} \qquad x = \dfrac{x^+ - x^-}{\sqrt{2}}$

$ds^2 = \frac{1}{2}(dx^+ + dx^-)^2 - \frac{1}{2}(dx^+ - dx^-)^2 = \boxed{2dx^+ dx^- = ds^2}$

2) $t = \tau \,\mathrm{ch}\,\eta \qquad \tau = \sqrt{t^2 - x^2} \qquad \eta = \frac{1}{2}\log\frac{t+x}{t-x} \qquad \tau^2 = 2x^+x^- = 2\tan u^+ \tan u^-$

$x = \tau \,\mathrm{sh}\,\eta$

$ds^2 = (\mathrm{ch}\eta\, d\tau + \tau\,\mathrm{sh}\eta\, d\eta)^2 - (\mathrm{sh}\eta\, d\tau + \tau\,\mathrm{ch}\eta\, d\eta)^2$

$\boxed{ds^2 = d\tau^2 - \tau^2 d\eta^2}$ = "Milne universe"

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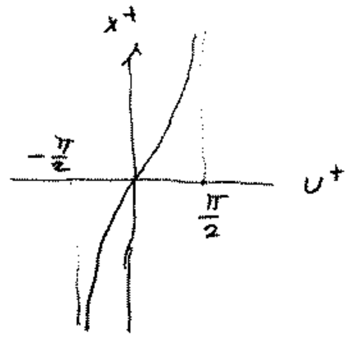
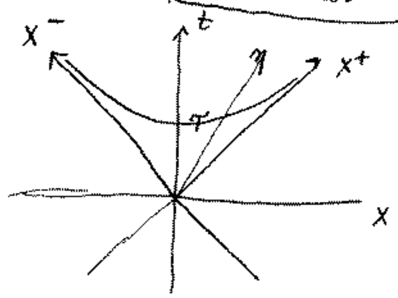
3) Further map to a finite range by

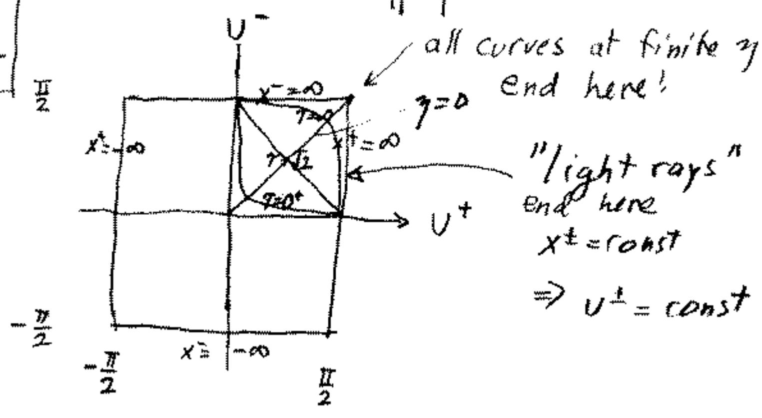
$x^+ = \tan u^+$

$x^- = \tan u^-$



$\Rightarrow \boxed{ds^2 = \dfrac{2du^+ du^-}{\cos^2 u^+ \cos^2 u^-}}$





One more conformal transformation:

$\tilde{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu} \qquad \Rightarrow \boxed{ds^2 = d\tilde{u}^+ d\tilde{u}^-}$

so we seem to be back to dx^+dx^-!

but globally different!

Ex. 2 Spherically symmetric

no $dt\,dr$

$$ds^2 = f(t,r)\,dt^2 + g(t,r)\,dr^2 + r^2 d\Omega^2$$

$$= -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\Omega^2$$

Compute $\Gamma^{\mu}_{\lambda\sigma}$, $R^{\rho}{}_{\lambda\sigma\rho}$.

(1) $R_{\mu\nu}=0$: $ds^2 = -e^{2\alpha(r)}dt^2 + \underbrace{e^{2\beta(r)}dr^2 + r^2 d\Omega^2}_{t\text{-independent!}}$

$$\boxed{ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \frac{1}{1-\frac{r_s}{r}}\,dr^2 + r^2 d\Omega^2}$$

Schw. metric

$r_s = \frac{2GM}{c^2} \sim \frac{M}{M_{Pl}^2}$

$$R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = \frac{3r_s^2}{r^6}$$

$\left(v_{escape}^2 = \frac{2GM}{R}\right)$

(2) $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\,T^{elmag}_{\mu\nu}\left(\to -F_{\mu}{}^{\alpha}F_{\nu\alpha} + \frac{1}{4}g_{\mu\nu}F^2\right)$ $\quad g^{\mu\nu}\nabla_{\mu}F_{\nu\alpha} = 0$

$$\Rightarrow \boxed{ds^2 = -\left[1-\frac{2GM}{r}+\frac{G(p^2+q^2)}{r^2}\right]dt^2 + \frac{1}{1-\frac{2GM}{r}+\frac{G(p^2+q^2)}{r^2}}\,dr^2 + r^2 d\Omega^2}$$

Reissner Nordström

$\begin{cases} F_{tr} = f(t,r) \\ F_{\theta\varphi} = g(t,r)\sin\theta \end{cases}$

$\begin{cases} q = \text{total electric charge} \\ p = \text{" magnetic "} \end{cases}$ $\quad [q] = e$ $= \text{dimless}$ $\quad \frac{G}{r^2} = \frac{1}{(M_{Pl}r)^2} = \text{dimless}$

radial electric
" magnetic
$B^r = \epsilon^{0r\mu\nu}F_{\mu\nu}$

"charged black hole"

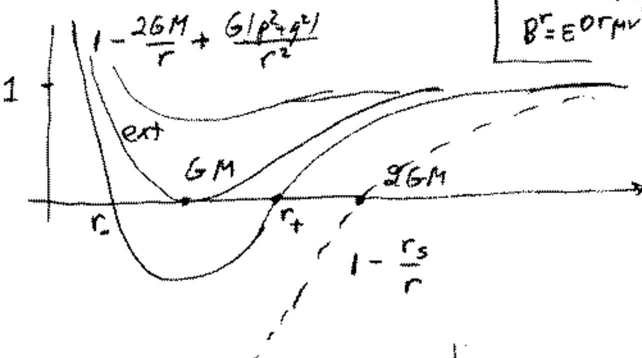
NOT very physical!

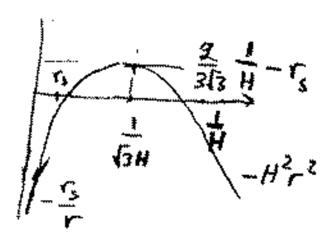
$$r_{\pm} = GM \pm \sqrt{G^2M^2 - G(p^2+q^2)}$$

Extremal: $GM^2 = p^2+q^2$

$$q \sim \frac{M}{M_{Pl}} \sim \frac{m_p \cdot N}{M_{Pl}}$$

$$\Rightarrow \frac{q}{N} = \frac{m_p}{M_{Pl}} \sim 10^{-19}$$ not much of charge!





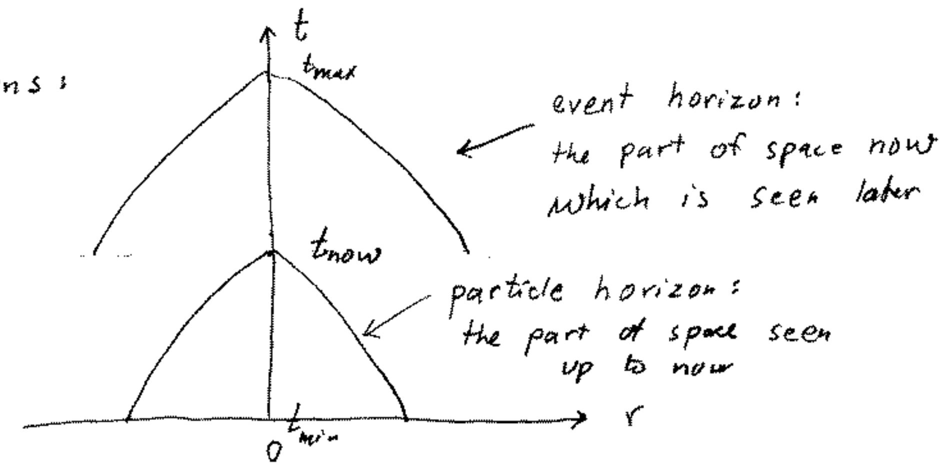
But total charge: $q_{\odot} \sim \frac{m_p}{M_{Pl}}\cdot\left(\frac{M_{Pl}}{m_p}\right)^3 \sim \left(\frac{M_{Pl}}{m_p}\right)^2 \sim 10^{38}e \sim 10^{19}C$

(3) BH in asymptotically de Sitter:

$T_{\mu\nu} = 0$

$$ds^2 = -\left[1-\frac{r_s}{r}-H^2r^2\right]dt^2 + \frac{1}{\cdots}\,dr^2 + r^2 d\Omega^2$$

Horizons:



RW:
$$ds^2 = dt^2 - R^2(t)\left[\frac{1}{1-kr^2}\,dr^2 + \cdot\right]$$

$ds^2 = 0$
$$\frac{dt}{R(t)} = \int \frac{dr}{\sqrt{1-kr^2}}$$

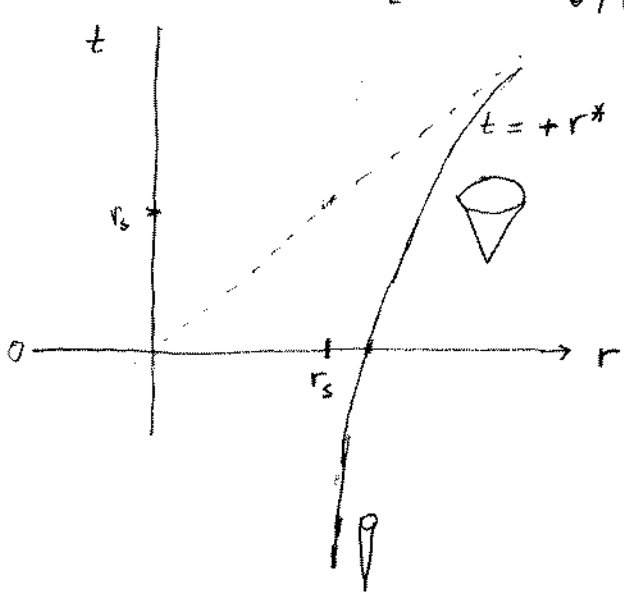
Event:
$$\int_{t_0}^{t_{max}} \frac{dt}{R(t)} = \int_0^{r_e(t)} \frac{dr}{\sqrt{1-kr^2}}$$

Particle
$$\int_{t_{min}}^{t_0} \frac{dt}{R(t)} = \int_0^{r_p(t)} \frac{dr}{\sqrt{1-kr^2}}$$

Back to BH:

$ds = 0$
$$\frac{dt}{dr} = \pm \frac{1}{1-\frac{r_s}{r}} = \pm\left[1 + \frac{r_s}{r-r_s}\right]$$

$$\Rightarrow t = \pm\left[r + r_s \log\left|\frac{r}{r_s} - 1\right|\right] + const = \pm r^* + const$$



coordinate singularity at $r = r_s$,

curvature finite

This is event horizon,
$0 < r < r_s$ cannot be seen at any time later

Special new coordinates: $\quad dt + dr\left[1 + \frac{1}{\frac{r}{r_s}-1}\right] =$

$$\begin{cases} u = t + r + r_s \log\left[\frac{r}{r_s} - 1\right] & du = dt + \frac{dr}{1-\frac{r_s}{r}} \\ v = t - r - r_s \log\left(\frac{r}{r_s} - 1\right) & dv = dt - \frac{dr}{1-\frac{r_s}{r}} \end{cases}$$

$$du\,dv = dt^2 - \frac{1}{\left(1-\frac{r_s}{r}\right)^2}dr^2$$

$\frac{1}{2}(u-v) = r + r_s \log\left(\frac{r}{r_s}-1\right)$

$$= \frac{1}{1-\frac{r_s}{r}}\underbrace{\left[\left(1-\frac{r_s}{r}\right)dt^2 - \frac{dr^2}{1-\frac{r_s}{r}}\right]}_{\text{wanted !}}$$

$$= \left(1-\frac{r_s}{r}\right)du\,dv$$

Next pull $r = r_s$ back from $u = -\infty$, $v = +\infty$ by

$$\begin{cases} \bar{u} = 2r_s e^{u/2r_s} = 2r_s\sqrt{\frac{r}{r_s}-1}\; e^{\frac{t+r}{2r_s}} \\ \bar{v} = -2r_s e^{-v/2r_s} = 2r_s\sqrt{\frac{r}{r_s}-1}\; e^{-\frac{(t-r)}{2r_s}} \end{cases}$$

$$d\bar{u}\,d\bar{v} = +e^{\frac{u-v}{2r_s}}du\,dv = +e^{\frac{r}{r_s}+\log\left(\frac{r}{r_s}-1\right)}du\,dv$$

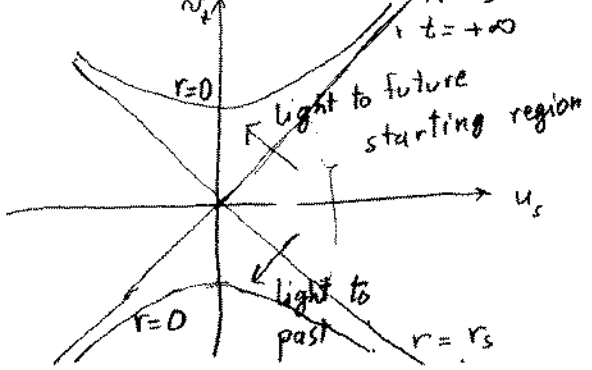
$$= +e^{\frac{r}{r_s}}\left(\frac{r}{r_s}-1\right)du\,dv$$

$$= +e^{\frac{r}{r_s}}\frac{r}{r_s}\underbrace{\left(1-\frac{r_s}{r}\right)du\,dv}_{\left[\left(1-\frac{r_s}{r}\right)dt^2 - \cdots\right]}$$

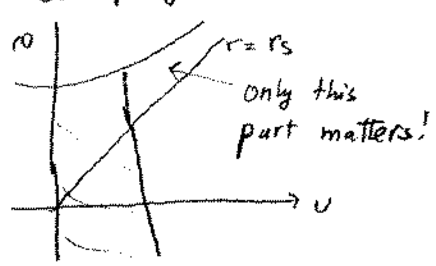
$$\Rightarrow \boxed{ds^2 = e^{-\frac{r}{r_s}}\frac{r_s}{r}\,d\bar{u}\,d\bar{v} - r^2 d\Omega_2^2}$$

In analogy with $dx^+dx^- = dt^2 - dx^2$, $\bar{U} \cong U_s - v_t$ $\quad \bar{v} \sim U_s + v_t$:

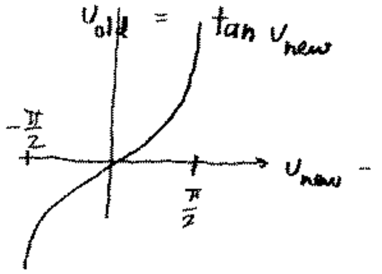
$d\bar{u}\,d\bar{v} \sim +dv^2 - du^2 - r^2 d\Omega_2^2$



Collapsing star:



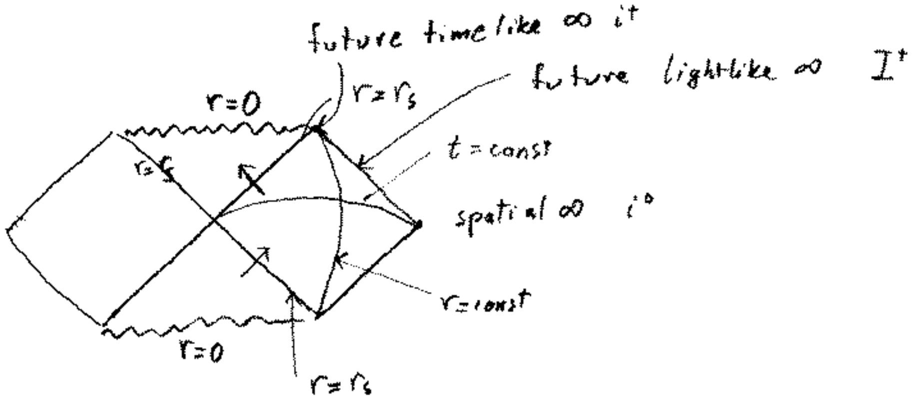
One more transformation of type



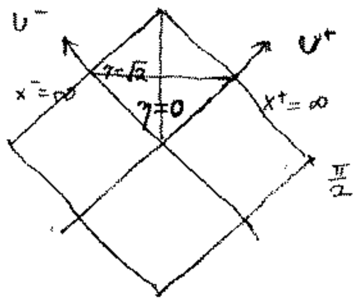
brings $\pm\infty \to \pm\frac{\pi}{2}$

and leads to a Penrose diagram:

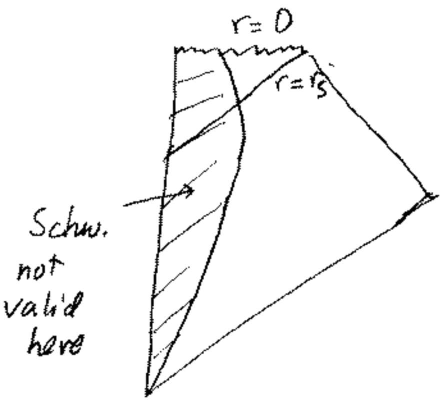
Penrose:



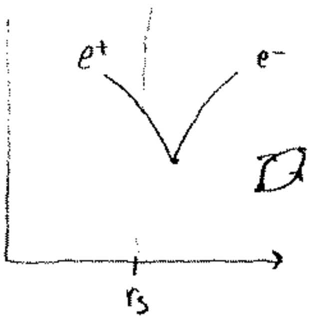
Flat space was:



Collapsing star:



Hawking radiation



Penrose diagram = ?

Convert BH to radiation